Solutions to Problem Set 5: Category theory

Read sections 1.1–1.4 and 3.1 of *Categories in Context* by Emily Riehl (link here and on course webpage) OR read sections 1.4, 2.4, 3.1, 3.3, 3.4, 5.4 of *Category theory* by Steve Awodey (link here and on course webpage).

Problem 1 (8)**.** Did you do your reading assignment?

Solution 1. Yes.

Problem 2 (8)**.**

a. What is a functor between groups, regarded as one-object categories? What is a natural transformation between a parallel pair of such functors?

b. What is a functor between preorders, regarded as categories? What is a natural transformation between a parallel pair of such functors?

Solution 2. a. Let G and H be two groups. Denote the corresponding categories by the letters. Giving a functor $F : G \to H$ is the same as giving functions between set of objects, which are one object sets in our case so the function can be only one thing, and set of morphisms compatible with compositions and the identity. This is the same as saying $F(gh) = F(g \circ h) = F(g) \circ F(h) = F(g)F(h)$ and $F(1) = 1$. So giving a functor is equivalent to giving an homomorphism of groups.

Similarly, given $F, F' : G \to H$ functors/group homomorphisms, a natural transformation $\Phi : F \to F'$ is determined by the morphism $\Phi_x : F(x) \to F'(x)$, where x is the unique object of G and $\Phi_x \in H$. Choice of such $\Phi = \Phi_x \in H$ gives a natural transformation if and only if $\Phi F(g) = F'(g)\Phi$, for all $g \in G$. Thus, a natural transformation is just a conjugation relation, together with conjugating element Φ between F and F'.

b. Let P and Q be preorders and denote the corresponding categories by the same letter. A functor induces a function $Ob(P) \to Ob(Q)$. As there is at most a unique morphism between any two objects, the functor is uniquely determined by this function between sets of objects. Moreover, a function $F_0 : Ob(P) \to Ob(Q)$ extends to a functor only if it carries pairs of objects x, x' such that $P(x, x') \neq$ to objects with the same property. This is equivalent to saying that if $x \leq x'$, then $F_0(x) \leq F_0(x')$. Clearly, this is also sufficient. Thus functors in between are just order preserving functions of preorders.

A natural transformation $F \to F'$ is an assignment of a morphism $\Phi_x : F(x) \to F'(x)$. Note, the commutativity of the squares is immediate as there is at most one morphism between any two objects(Indeed any diagram commutes). But, such there can be at most one such assignment, and its existence is equivalent to $F(x) \leq F'(x)$ for any $x \in P$, i.e. a natural transformation $F \to F'$ is a relation $F \leq F'$ on all objects.

Problem 3 (8)**.** Prove that functors carry commutative diagrams to commutative diagrams. Note: part of this exercise is to formalize the notion of commutative diagram.

Solution 3. A diagram in C, informally is a collection of objects and morphisms in between. A better way of phrasing this is functions $f_0 : X_0 \to Ob(C) =: C_0$ and $f_1 : X_1 \to Mor(C) =: C_1$, where X_0 and X_1 are sets with "domain and target maps" $X_1 \to X_0$ and the functions are compatible with them(you can picture it as a directed graph, if you wish). X_0 and X_1 can easily be completed to a category X_f by declaring $Ob(X_f) = X_0$ and morphisms $x \to x'$ to be directed paths from x to x'(where the trivial path is the identity). Then a diagram is just a functor from X_f to C. The commutativity of a diagram is that two paths around certain 2-cells giving the same composition in C(Think of triangles for simplicity). But we can declare this at a more universal level at X_f, by identifying two such paths in this category. We obtain a new category, X, and a commutative diagram is just a functor $X \to C$. After long philosophical discussion, I take this as the definition of an X-shaped diagram in C.

Then, everything is clear: If we have a commutative diagram, i.e. a functor $f : X \to C$ and another functor $F : C \to D$, then F carries the diagram to $F \circ f$, which is also a commutative diagram by definition, as the composition of functors is a functor.

Problem 4 (8)**.** Let X be a space and L_X be the category associated to the directed set $(\mathcal{T}_X, \subset)$. Convince yourself that you know what the objects and morphisms of L_X are.

a. In what way is the construction $X \mapsto L_X$ functorial? (That is, how can you make it into a functor?)

b. Let $\{X_i\}_{i \in I}$ be a family of objects of L_X. Does the product $\prod_{i \in I} X_i$ exist? Does the coproduct $\coprod_{i \in I} X_i$ exist? When?

Solution 4. It is easy to see that the objects are the open subsets and the morphism $U \to V$ is an inclusion relation $U \subset V$.

a. For each set X the power set P_X can be made a category in a similar way, and the category L_X is a full subcategory of P_X, for a space X. The construction of P_X is functorial in two ways:

i. Send the map $f : X \to Y$ to the function $P_X \to P_Y$ sending $U \subset X$ to $f(U)$. Note this is covariant.

ii. Send the map $f : X \to Y$ to the function $P_Y \to P_X$ sending $V \subset Y$ to $f^{-1}(V)$. Note this is contravariant.

The second works well for $X \mapsto L_X$ as well: just replace the words P_X by L_X. But, the first one does not: continuous functions does not even carry $L_X \subset P_X$ to P_Y. But we have a quick remedy for this: send $f : X \to Y$ to $U \mapsto int(f(U))$. Clearly, this is a function between L_X and L_Y and it preserves inclusion relation, so it is a functor $L_X \to L_Y$. It is easy to check this is compatible with compositions and the identities.

b. A product $\prod_{i \in I} X_i$ comes with the morphisms $\prod_{i \in I} X_i \to X_i$. But there are such morphisms if and only if $\prod_{i \in I} X_i \subset X_i$ for all $i \in I$, i.e. $\prod_{i \in I} X_i \subset \bigcap_{i \in I} X_i$. Clearly, this is the case if and only if $\prod_{i \in I} X_i \subset int(\bigcap_{i \in I} X_i)$. But, $int(\bigcap_{i \in I} X_i)$ is an object of L_X and there are morphisms $int(\bigcap_{i \in I} X_i) \to X_i$. The discussion tells us that any such object with morphisms to X_i, i.e. inside X_i, is a subset of $int(\bigcap_{i \in I} X_i)$, i.e. the morphisms factor(uniquely) through $int(\bigcap_{i \in I} X_i)$. Hence, $int(\bigcap_{i \in I} X_i)$ is the product, with the obvious morphisms to X_i.

A similar discussion shows that the coproduct, if exists, has to be an open set containing $\bigcup_{i \in I} X_i$. This set is already open and it is very easy to see that it is the coproduct.

Problem 5 (16)**.** Let C be a category. For a diagram $X : J \to C$, denote the colimit of the diagram (if it exists) by $(\operatorname{colim}_J(X), \{\iota_j\}_{j \in \mathrm{ob}\, J})$ where

$$\operatorname{colim}_J(X) \in \mathrm{ob}\, C$$

is the universal object of the colimit and

$$\iota_j : X(j) \to \operatorname{colim}_J(X)$$

are the universal morphisms.

A functor $F : C \to D$ is said to *preserve colimits* if for every diagram $X : J \to C$ such that the colimit of X exists, the colimit $\operatorname{colim}_J(FX)$ also exists, and the map

$$\operatorname{colim}_J(FX) \to F\left(\operatorname{colim}_J X\right),$$

which is induced by the maps $F\iota_j : F(X(j)) \to F(\operatorname{colim}_J(X))$ and the universal property of $\operatorname{colim}_J(FX)$, is an isomorphism.

a. Give an example of a functor that *does not* preserve colimits.

b. Give an example of a functor that preserves colimits.

c. Let C be a category. Prove that any functor $F : \text{Set} \to C$ that preserves colimits is determined by the image under F of any set with one-element.

d. Let C be a category. Let D be one of the following categories: $\text{Group}, \text{Vect}_{\mathbb{R}}, \text{Top}$. Is there a collection of subobjects $G \subset \text{ob}\, D$ so that any functor $F : D \to C$ that preserves colimits is determined by its values on G? What if we allow G to be a subcategory?

Solution 5. a. There can be many different examples. One is the covariant Yoneda functor $Vect(X, -)$, from $Vect$ to $Vect$, where I denote the category of (possibly infinite dimensional) vector spaces over $\mathbb{R}$ by $Vect$ and where X is infinite dimensional. Then the coproducts are the direct sums and this functor does not commute with them, for instance write $X = \bigoplus X_i$, where X_i are finite dimensional. Then the image of identity is not in $\bigoplus Vect(X, X_i)$, so it does not identify with $Vect(X, X)$.

There are other examples in the same spirit, such as the covariant Yoneda functor associated to a non-compact space or to an infinite set, etc.

b. Again, there can be many examples, such as the identity functor or the forgetful functor from Spaces to Sets. An example of the above spirit, is taking X to be finite dimensional, i.e. covariant Yoneda functor $Vect(X, -)$, from $Vect$ to $Vect$, where X is finite dimensional. It is then an easy linear algebra exercise to show this preserves colimits.

Again, Yoneda functors over Spaces and Sets corresponding to compact spaces/finite sets give us different examples.

c. This is intuitively clear: In Sets coproducts are disjoint unions and hence every set is a coproduct of one element sets. To be more precise, for every set X, $\{i_x : 1 \to X\}$ is a coproduct diagram, where 1 is a fixed one element set and i_x sends that element to x. Then we obtain a coproduct diagram $\{j_x : F(1) \to F(X)\}$(Clearly, the domains of these diagrams is the category with set of objects X and with only identity morphisms). If we have another colimit preserving functor F' such that $F(1) \cong F'(1)$, then the isomorphism induce natural isomorphisms of diagrams $\{j_x : F(1) \to F(X)\}$ and $\{j'_x : F'(1) \to F'(X)\}$; hence, isomorphisms of $F(X)$ and $F'(X)$. But as the construction of the above diagram is natural in X, so is the constructed isomorphism, so we have a natural transformation between the functors, which is an isomorphism. Thus, the image of 1 determines the functor.

d. If we allow G to be a subcategory, then the answer is trivial: just take it to be the whole category(Yet things change if we put restrictions on the cardinality of $Ob(G)$).

However, if we do not know anything about morphisms the answer is negative, i.e. there are functors on D that act the same on objects, yet that are not related by a natural isomorphism. To find an example, let us look what can go wrong. If $F, F' : D \to C$ are two functors that act the same on set of objects, then not only their action on morphisms should be different but also the objects of C should lack the class of automorphisms that could relate F and F'. So I propose the following example for $Vect_{\mathbb{R}}$(Check it though):

Take two embeddings of $\mathbb{R}$ into a field K that are not related by any (even just linear) automorphisms. Then we obtain two base change functors, i.e. $V \mapsto V \otimes_{\mathbb{R}} K$($K$ has two different $\mathbb{R}$-module structure, coming from two embeddings, hence tensoring with it is different). Those are an exact functors, clearly preserving the coproducts. Dimension of $\mathbb{R}$ turn into dimension over K, hence the images of objects under the will be isomorphic. But the action on the hom-sets will be different, and will not be related by any automorphisms as the embeddings of $\mathbb{R}$ are not related. Hence, we have a counterexample.

For groups we can imitate this, after composing with coproduct preserving abelinization functor, $G \mapsto G/[G, G]$. Note the tensor product does not preserve colimits in general and it has to be modified slightly.

For topological spaces, philosophically they are the same as a very special class of commutative algebras(with some extra structure). So we can imitate this, send $X \mapsto C(X)$, the algebra of real valued continuous functions, considered as an object of $CAlg_{\mathbb{R}}^{op}$, the opposite category of commutative algebras over $\mathbb{R}$. Then, we can again base change to same field K above(note this time it is a functor between categories of algebras) in two different ways and obtain a counterexample.