Solutions to Problem Set 4: Connectedness

Problem 1 (8)**.** Let X be a set, and $\mathcal{T}_0$ and $\mathcal{T}_1$ topologies on X. If $\mathcal{T}_0 \subset \mathcal{T}_1$, we say that $\mathcal{T}_1$ is *finer* than $\mathcal{T}_0$ (and that $\mathcal{T}_0$ is *coarser* than $\mathcal{T}_1$).

a. Let Y be a set with topologies $\mathcal{T}_0$ and $\mathcal{T}_1$, and suppose $\mathrm{id}_Y : (Y, \mathcal{T}_1) \to (Y, \mathcal{T}_0)$ is continuous. What is the relationship between $\mathcal{T}_0$ and $\mathcal{T}_1$? Justify your claim.

b. Let Y be a set with topologies $\mathcal{T}_0$ and $\mathcal{T}_1$ and suppose that $\mathcal{T}_0 \subset \mathcal{T}_1$. What does connectedness in one topology imply about connectedness in the other?

c. Let Y be a set with topologies $\mathcal{T}_0$ and $\mathcal{T}_1$ and suppose that $\mathcal{T}_0 \subset \mathcal{T}_1$. What does one topology being Hausdorff imply about the other?

d. Let Y be a set with topologies $\mathcal{T}_0$ and $\mathcal{T}_1$ and suppose that $\mathcal{T}_0 \subset \mathcal{T}_1$. What does convergence of a sequence in one topology imply about convergence in the other?

Solution 1.

a. By definition, id_Y is continuous if and only if preimages of open subsets of $(Y, \mathcal{T}_0)$ are open subsets of $(Y, \mathcal{T}_1)$. But, the preimage of $U \subset Y$ is U itself. Hence, the map is continuous if and only if open subsets of $(Y, \mathcal{T}_0)$ are open in $(Y, \mathcal{T}_1)$, i.e. $\mathcal{T}_0 \subset \mathcal{T}_1$, i.e. $\mathcal{T}_1$ is finer than $\mathcal{T}_0$.

b. If Y is disconnected in $\mathcal{T}_0$, it can be written as $Y = U \coprod V$, where $U, V \in \mathcal{T}_0$ are non-empty. Then U and V are also open in $\mathcal{T}_1$; hence, Y is disconnected in $\mathcal{T}_1$ too. Thus, connectedness in $\mathcal{T}_1$ imply connectedness in $\mathcal{T}_0$. The other way is not true, consider discrete and indiscrete topologies on $\mathbb{R}$ for instance.

c. Hausdorff means one can seperate different points with open subsets, thus if the coarser topology $\mathcal{T}_0$ has enough opens to seperate all points, so does $\mathcal{T}_1$. Thus, if $\mathcal{T}_0$ is Hausdorff then so is $\mathcal{T}_1$. The converse is false: the example in part b provides a counter-example.

d. Convergence is preserved by continuous maps, so if a sequence/net converges in $\mathcal{T}_1$ then it converges in $\mathcal{T}_0$ by part a.

Problem 2 (8)**.** Given a space X, we define an equivalence relation on the elements of X as follows: for all $x, y \in X$,

$$x \sim y \iff \text{there is a connected subset } A \subset X \text{ with } x, y \in A.$$

The equivalence classes are called the *components* of X.

a. (0) Prove *to yourself* that the components of X can also be described as connected subspaces A of X which are as large as possible, ie, connected subspaces $A \subset X$ that have the property that whenever $A \subset A'$ for A' a connected subset of X, $A = A'$.

b. (4) Compute the connected components of $\mathbb{Q}$.

c. (4) Let X be a Hausdorff topological space, and $f, g : \mathbb{R} \to X$ be continuous maps such that for every $x \in \mathbb{Q}$, $f(x) = g(x)$. Show that $f = g$.

Solution 2. a. First, the components are connected. To show this let $A = [x] \subset U \cup V$, where U and V are open and $U \cap V \cap A = \emptyset$. Then either $x \in U \setminus V$ or $x \in V \setminus U$, assume the former. Given $y \in A$, there exist a connected set $B \subset X$, containing x and y. Clearly, $B \subset A$, so $B \subset U$ or $B \subset V$. The latter cannot happen by assumption that $x \in B$, so $y \in B \subset U$. As this holds for any $y \in A$, $A \subset U$; hence, as U and V were arbitrary A is connected. Clearly, it is maximal: if $A \subset A'$, where A' is connected, then $x \in A'$, so $A' \subset A$ as above. Thus, $A = A'$. On the other hand, maximal connected subsets are connected components of the points they contain by similar arguments.

b. Let $\emptyset \neq A \subset \mathbb{Q}$ be connected. If A contains more than one elements, say $x < y \in \mathbb{Q}$, then for a given irrational r between x and y, $(-\infty, r)$ and (r, ∞) seperate A into two disjoint, non-empty open subset. Thus it cannot be connected, so it should contain at most one element, so the connected components are $\{x\}$, for $x \in \mathbb{Q}$.

c. Hausdorff property is equivalent to closedness of the diagonal $\Delta \subset X \times X$. If this is the case, its preimage under $f \times g : \mathbb{R} \to X \times X$ is also closed. But this set is equal to $\{x \in \mathbb{R} : f(x) = g(x)\}$, which contains $\mathbb{Q}$. As $\mathbb{Q}$ is dense, this closed set is all $\mathbb{R}$, hence $f = g$. A different way would be taking rational sequences converging to a given real number and using uniqueness of the limit in Hausdorff spaces.

Problem 3 (9)**.** Prove that no pair of the following subspaces of $\mathbb{R}$ are homeomorphic:

$$(0,1), \quad (0,1], \quad [0,1].$$

Solution 3. The distinguishing property is the minimal number of connected components when we take out two points: Such a procedure will always seperate $(0,1)$ into 3 components. If we choose one of them to be a boundary point it is 2 for $(0,1]$, but not less. If we take of two boundary points from $[0,1]$, it will remain connected, hence, this number is 1 for it.

Problem 4 (8)**.** Let $(X_i)_{i \in I}$ be a family of topological spaces, and $(Y_i)_{i \in I}$ be a family of subsets $Y_i \subset X_i$. Note that the set $\prod_{i \in I} Y_i$ has two possible topologies:

- first give each Y_i the subspace topology, and then take the product topology on the product
- give the product the subspace topology as a subset of the product topology on $\prod_{i\in I} X_i$.

Are these two topologies the same? Prove or disprove using the universal properties of the subset and the product.

Solution 4. Let τ_p and τ_s denote the subset and product topologies on $Y = \prod_{i\in I} Y_i$. We will write the natural maps from one to the other, which will turn out to be identity. To write a map to a product we have to write maps to each component. Set theoretically they are just projections from $(\prod_{i\in I} Y_i, \tau_s)$ to Y_i but we need continuity. But inclusion map from $(\prod_{i\in I} Y_i, \tau_s)$ to $\prod_{i\in I} X_i$ is continuous and projection from the latter to X_i is continuous; hence, projection from $(\prod_{i\in I} Y_i, \tau_s)$ to X_i is continuous with image in Y_i. Thus the projection to Y_i is continuous and this tells us that the identity from $(\prod_{i\in I} Y_i, \tau_s)$ to $(\prod_{i\in I} Y_i, \tau_p)$ is continuous. Hence, the subset topology is finer. On the other hand $(\prod_{i\in I} Y_i, \tau_p) \to (\prod_{i\in I} X_i, \tau_{prod})$ is continuous because of the continuity of the composition $(\prod_{i\in I} Y_i, \tau_p) \to Y_i \hookrightarrow X_i$. But its image is in $(\prod_{i\in I} Y_i, \tau_s)$ so the map to subspace is also continuous, i.e. $id : (\prod_{i\in I} Y_i, \tau_p) \to (\prod_{i\in I} Y_i, \tau_s)$ is also continuous. Thus product topology is also finer, hence they are the same topologies.

Problem 5 (12 – problem seminar)**.** In this problem, we will investigate the notion of convergence in the product and box topologies on spaces of functions.

a. Let X be a space and I be a set. Recall that the set of maps X^I is also the product $\prod_{i\in I} X$, and so has a natural topology (the product topology). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of maps in X^I, and let $f \in X^I$. Show that $f_n \to f$ in X^I if and only if, for every i, $f_n(i) \to f(i)$ in X. For this reason, the product topology $\mathcal{T}_\prod$ is also called the *topology of pointwise convergence*.

b. Show that the topology of pointwise convergence on $\mathbb{R}^\mathbb{R}$ does not come from a metric.

The *topology of uniform convergence* $\mathcal{T}_\infty$ on $\mathbb{R}^\mathbb{R}$ is defined as follows: a subset $U \subset \mathbb{R}^\mathbb{R}$ belongs to $\mathcal{T}_\infty$ iff for every $f \in U$ there exists $\epsilon > 0$ such that

$$\left\{ g : \mathbb{R} \to \mathbb{R} : \sup_{x\in\mathbb{R}} |f(x) - g(x)| < \epsilon \right\} \subset U.$$

Convince yourself that this is a topology. Justify to yourself the name of $\mathcal{T}_\infty$ (by figuring out what it means for a sequence to converge in $\mathcal{T}_\infty$).

c. Show that $\mathcal{T}_\prod \subset \mathcal{T}_\infty \subset \mathcal{T}_\Box$

d. Show that $\mathcal{T}_\prod \neq \mathcal{T}_\infty$.

e. Show that the sequence of constant functions $x \mapsto \frac{1}{n+1}$ does not converge to 0 in the box topology. Conclude that $\mathcal{T}_\infty \neq \mathcal{T}_\square$.

f. Find a sequence of functions $f_n \in \mathbb{R}^\mathbb{R}$ such that $\sup_{x\in\mathbb{R}}|f(x)| \geq \frac{1}{n+1}$ and that converges to the constant function 0 in the box topology.

Solution 5. a. If $f_n \to f$, then for any $i \in I$, $p_i(f_n) \to p_i(f)$, where p_i is the projection to the i^{th} factor. This is true because continuity preserves convergence. But, $p_i(f_n) = f_n(i)$ and $p_i(f) = f(i)$; hence, $f_n(i) \to f(i)$.

On the other hand, assume $f_n(i) \to f(i)$ for any $i \in I$. This implies for any neighborhood of f, of type $p_i^{-1}(U)$, where U is a neighborhood of $f(i)$ in X, all but finitely many of f_n are in $p_i^{-1}(U)$. But, any neighborhood of f contains a finite intersection of this type of sets and this finite intersection contains all but finitely of f_n. Thus, $f_n \to f$.

b. It is easy to see that every point in a metric space has a local basis, i.e. a sequence $\{U_n\}_{n\in\mathbb{N}}$ of neighborhoods such that for any other neighborhood U there exist a $n \in \mathbb{N}$ such that $U_n \subset U$ and this property depends only on the topology. On the other hand, $\mathbb{R}^I$ has "too many" neighborhoods of any point: for instance $p_i^{-1}((-1,1))$ is a neighborhood of $(0)_{i\in I}$ for each $i \in I$, where p_i is the projection to the i^{th} component. But any neighborhood of $(0)_{i\in I}$ contains $p_{i_1}^{-1}(-\epsilon_1,\epsilon_1) \cap p_{i_2}^{-1}(-\epsilon_2,\epsilon_2) \cap \cdots \cap p_{i_k}^{-1}(-\epsilon_k,\epsilon_k)$, for some $i_1,\dots,i_k \in I$ and $\epsilon_1,\dots,\epsilon_k > 0$. Hence, it contains the a subset of the form $p_{i_1}^{-1}(0) \cap p_{i_2}^{-1}(0) \cap \cdots \cap p_{i_k}^{-1}(0)$. Thus, if $\{U_n\}$ is a countable family of neighborhoods, by choosing such a finite set of indices for each U_n we can obtain a countable set $J \subset I$ of indices such that $\bigcap_{j\in J} p_j^{-1}(0) \subset \bigcap_{n\in\mathbb{N}} U_n$. If $\{U_n\}$ were a local basis, $\bigcap_{j\in J} p_j^{-1}(0)$ would be contained in any neighborhood of $(0)_{i\in I}$, clearly implying $I = J$. Thus, if we start with an uncountable index set, such as $\mathbb{R}$ as above, this cannot happen and our topology cannot come from a metric space.

c. As the product topology is the smallest topology containing open sets of the form $p_i^{-1}(U)$, where $U \subset \mathbb{R}$ is open, it is enough to show that sets of this type are open in the uniform convergence topology, for any U and $i \in \mathbb{R}$. Let $f \in p_i^{-1}(U)$, i.e. $f(i) \in U$. Then, there exist an $\epsilon > 0$ such that $(f(i)-\epsilon, f(i)+\epsilon) \subset U$. Then, clearly $\{g : \mathbb{R} \to \mathbb{R} : \sup_{x\in\mathbb{R}}|f(x) - g(x)| < \epsilon\}$ is a subset of $p_i^{-1}((f(i)-\epsilon, f(i)+\epsilon)) \subset p_i^{-1}(U)$. Hence, $p_i^{-1}(U) \in \mathcal{T}_\infty$. This implies $\mathcal{T}_\prod \subset \mathcal{T}_\infty$.

On the other hand, let $U \in \mathcal{T}_\infty$. Given $f \in U$, there exist an $\epsilon > 0$ such that $\{g : \mathbb{R} \to \mathbb{R} : \sup_{x\in\mathbb{R}}|f(x) - g(x)| < \epsilon\} \subset U$. But then, $f \in \prod_{i\in\mathbb{R}}(f(i)-\epsilon/2, f(i)+\epsilon/2) \subset U$. Thus, U can be written as a union of boxes $\prod_{i\in\mathbb{R}}(a_i,b_i)$; thus, it is open in box topology. Hence, $\mathcal{T}_\infty \subset \mathcal{T}_\square$.

d. One way of doing this would be finding sequences of functions converging uniformly but not pointwisely. An easier way is noting $\{g : \mathbb{R} \to \mathbb{R} :$

$\sup_{x\in\mathbb{R}}|g(x)| < \epsilon\} \in \mathcal{T}_\infty$. But, for any neighborhood U of the 0 function in product topology there exist an i such that $p_i(U) = \mathbb{R}$. This clearly does not hold for the set above. Hence it is not in $\mathcal{T}_{\prod}$. Thus, $\mathcal{T}_\infty \neq \mathcal{T}_{\prod}$.

e. Box topology contains any $\prod_{i\in\mathbb{R}}(a_i, b_i)$. We can choose $a_i < 0 < b_i$ such that $sup\{a_i : i \in \mathbb{R}\} = inf\{b_i : i \in \mathbb{R}\} = 0$. Clearly, this set does not contain any of the constant functions, except 0. But it contains 0, thus the constant functions $x \mapsto \frac{1}{n+1}$ does not converge to $x \mapsto 0$. But they uniformly converge to it, hence clearly they converge in uniform convergence topology. This implies $\mathcal{T}_{\prod} \neq \mathcal{T}_{\Box}$.

f. Obviously, this cannot be continuous, hence we should look for discontinuous examples. For a very simple one, let

$$f_n(x) = \begin{cases} \frac{1}{n+1} & x = 0 \\ 0 & x \neq 0 \end{cases}$$

This sequence obviously satisfies the desired properties.