

Solutions to Problem Set 3: Limits and closures

Problem 1. Let X be a topological space and $A, B \subset X$.

- Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
- Show that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.
- Give an example of X , A , and B such that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.
- Let Y be a subset of X such that $A \subset Y$. Denote by \overline{A} the closure of A in X , and equip Y with the subspace topology. Describe the closure of A in Y in terms of \overline{A} and Y .

Solution 1. a. Taking closures preserve inclusion relation and A and B are subsets of $A \cup B$, so \overline{A} and \overline{B} are subsets of $\overline{A \cup B}$. Hence, $\overline{A \cup B} \subset \overline{A \cup B}$. On the other hand, $A \cup B \subset \overline{A \cup B}$ and the latter is closed; hence, $\overline{A \cup B} \subset \overline{A \cup B}$. Thus, the equality holds.

- $A \cap B \subset A$ and this implies $\overline{A \cap B} \subset \overline{A}$. Similarly, $\overline{A \cap B} \subset \overline{B}$; thus, $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.
- Let $X = \mathbb{R}$, with the standard topology, $A = \mathbb{R}_{<0}$ and $B = \mathbb{R}_{>0}$. Then, clearly $\overline{A \cap B} = \emptyset$, but $\overline{A} \cap \overline{B} = \mathbb{R}_{\leq 0} \cap \mathbb{R}_{\geq 0} = \{0\}$. So the equality fails.
- The closure of A inside of Y is equal to $\bigcap_{A \subset F \subset Y, \text{closed in subspace topology}} F$. But the set of closed subsets of Y , with respect to subspace topology, is exactly $\{F \cap Y : F \text{ is closed in } X\}$ and the set over which we take intersection is $\{F \cap Y : F \text{ is closed in } X, A \subset F\}$. Hence the above intersection is equal to $Y \cap \bigcap_{A \subset F, F \text{ is closed in } X} F = Y \cap \overline{A}$. This is the closure in Y with respect to subspace topology.

Problem 2. Let X be a set and let

$$\tau = \{U \in \mathcal{P}(X) : X \setminus U \text{ is finite, or } U = \emptyset\}.$$

- Show that τ is a topology on X . This topology is called the *cofinite topology* (or *finite complement topology*).
- Let X be a set equipped with the cofinite topology. Let $A \subset X$. Describe the boundary ∂A of A .
- Suppose $X = \mathbb{N}$. To which points does the sequence $(n)_{n \in \mathbb{N}}$ converge?

Solution 2. a. By assumption $\emptyset \in \tau$ and $X \in \tau$ as $X \setminus X = \emptyset$ is finite. All supersets of sets with finite complement, also have finite complement. In particular unions of open sets have finite complement (unless they are all empty, in which case the union is empty as well). Given two non-empty, cofinite sets $U, V \in \tau$, $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$ is finite, so $U \cap V \in \tau$.

- b.
- First the trivial case: If X is finite then the topology is the discrete topology, so everything is open and closed and boundaries are empty.
 - If X is infinite but A is finite, it is closed, so its closure is A . But there is no non-empty open set in A , so its interior is empty and its boundary is A .
 - The complement of the last case is also similar: If A is infinite with a finite complement, it is open, so its interior is itself, but the only closed set containing it is X , so its boundary is equal to $X \setminus A$.
 - If both A and its complement is infinite, then arguing as above we see that it has empty interior and its closure is X . Thus, its boundary is also X .
- c. To every point: Given $x \in \mathbb{N}$ and an open neighborhood U , all but finitely many natural numbers lie in U . Thus, the sequence eventually falls into U as well and this implies the sequence converges to x for any $x \in \mathbb{N}$.

Problem 3. Let (X, d) be a metric space. Prove that the metric topology on X is Hausdorff.

Solution 3. Given $x, y \in X$, $x \neq y$, the distance $d(x, y) > 0$. Let $r = d(x, y)/2$. Clearly $B(x, r)$ and $B(y, r)$ are open subsets containing x and y and which does not intersect each other. Thus, every such pair is separated by open sets and X is Hausdorff.

Problem 4. Let X and Y be topological spaces. A map $f : X \rightarrow Y$ is called *open* if for every open $U \subset X$, the image $f(U)$ is open in Y .

- a. Consider $X \times Y$ equipped with the product topology. Show that the map $p_1 : X \times Y \rightarrow X, (x, y) \mapsto x$ is both continuous and open.
- b. Consider $X \coprod Y$ equipped with the sum topology. Show that the map $i_1 : X \rightarrow X \coprod Y, x \mapsto (x, 0)$ is both continuous and open.

Solution 4. a. Given $U \subset X$ open, the preimage $p_1^{-1}(U)$ is equal to $U \times Y$ which is one of the basic open subsets. Hence, it is open and so, p_1 is continuous. Also as any open set in the product topology a union of sets of type $U \times V$, where U and V are open and as taking images commutes with taking unions, it is enough to check $p_1(U \times V)$ is open. But this set is equal to either U or \emptyset hence it is open. Thus, p_1 is open.

- b. The images of open sets $U \subset X$ are $U \times \{0\}$; hence are among the basis elements for the sum topology introduced last week. Hence they are open,

so i_1 is open. On the other hand, preimages of basis elements $U \times \{0\}$ are equal to U and preimages of basis elements $V \times \{1\}$ are empty; hence, they are open in any case. Thus, the preimages of unions of basis elements, i.e. of open subsets, are also open. This implies the map i_1 is continuous.

Problem 5. An *equivalence relation* on a set X is a subset $R \subset X \times X$ such that

- for each $x \in X$, $(x, x) \in R$.
- for every $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.
- for every $x, y, z \in X$ if $(x, y), (y, z) \in R$ then $(x, z) \in R$.

We write $x \sim_R y$ as an abbreviation for $(x, y) \in R$ (and sometimes just write $x \sim y$). For $x \in X$, the set

$$[x] = \{y \in X : y \sim x\}$$

is called the *equivalence class* of x . We denote by

$$X/\sim = \{[x] : x \in X\},$$

the set of equivalence classes of elements of X , called the *quotient of X by \sim* .

Suppose now that X is a topological space with an equivalence relation \sim , and consider the map

$$\pi : X \rightarrow X/\sim, \quad x \mapsto [x].$$

- a. Declare a subset $U \subset X/\sim$ to be open if $\pi^{-1}(U) \subset X$ is open. Show that this defines a topology on X/\sim , and that the map π is continuous. This topology is called the *quotient topology*.
- b. Is the map π always an open map? Justify your claim with proof or counterexample.
- c. Let Y be another topological space and let $f : X \rightarrow Y$ be a continuous map such that $f(x_1) = f(x_2)$ whenever $x_1 \sim x_2$. Show that there exists a unique map $\bar{f} : X/\sim \rightarrow Y$ such that $f = \bar{f} \circ \pi$, and show that \bar{f} is continuous. This is called the *universal property of the quotient topology*.
- d. Consider $\mathbb{R} \amalg \mathbb{R}$ with the sum topology, with the equivalence relation

$$(x, 0) \sim (y, 1) \quad \text{iff} \quad x \neq 0 \text{ and } x = y.$$

The topological space $Q = \mathbb{R} \amalg \mathbb{R}/\sim$ is called the *line with double origin*. Which points in Q are the limit of the sequence $n \mapsto [(\frac{1}{n+1}, 0)]$? Is Q a Hausdorff space?

- Solution 5.** a. The preimages of \emptyset and X/\sim are \emptyset and X respectively, hence they are open. Taking preimages commute with unions and intersections; hence, respective closedness properties on X under these operations imply the same for X/\sim . For instance, if U and V are open in X/\sim , then $\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V)$ is open in X . Hence, $U \cap V$ is open in X/\sim . It is similar for unions. Hence, this defines a topology. Preimages of opens are open by definition, hence π is continuous.
- b. No. For instance, let $X = [0, 1]$ and let $x \sim y$ if and only if $x = y$ or $x, y \in \{0, 1\}$, i.e. we are identifying 0 and 1. Let $U = (1/2, 1]$. It is open in X , but $\pi(U)$ is not: $\pi^{-1}(\pi(U)) = \{0\} \cup (1/2, 1]$ is not open in $X = [0, 1]$.
- c. If $[x_1] = [x_2]$, i.e. $x_1 \sim x_2$, then $f(x_1) = f(x_2)$, hence such a map is well defined. Uniqueness is also an easy set theoretic fact, it follows from surjectivity of π : Any $a \in X/\sim$ can be written as $\pi(x)$ and $\bar{f}(a) = f(x)$ is uniquely determined by f . To show continuity, let $V \subset Y$ be open, then $\bar{f}^{-1}(V)$ has preimage $\pi^{-1}(\bar{f}^{-1}(V)) = f^{-1}(V)$, which is open as f is continuous. Hence, \bar{f} is also continuous.
- d. For points $[(x, i)]$ with $x \neq 0$, and for $0 < \epsilon < |x|$, $B(x, \epsilon) \times \{0, 1\}$ maps to an open subset (whose preimage is itself), that contains only finitely many of the points of the sequence. On the other hand, every neighborhood of $[(0, 0)]$ and $[(0, 1)]$ pulls back to an open subset of $\mathbb{R} \amalg \mathbb{R}$ that contains $(U \times \{0\}) \setminus \{(0, 0), (0, 1)\}$, where U is a neighborhood of $0 \in \mathbb{R}$. It clearly contains all but finitely many of the points from the sequence; hence, the sequence has limits $[(0, 0)]$ and $[(0, 1)]$, i.e. the two origins. Non-uniqueness of the limit implies it is not Hausdorff.