Problem Set 2: Topological spaces

Your name:

Due: Thursday, February 11

Problem 1 (7)**.** Let (M, d) be a metric space, and let x be a point in M. Show that the subset $M \setminus \{x\}$ is open in the metric topology τ_d.

Solution 1. To check this we have to show that for any $y \in M \setminus \{x\}$, there exist an $\epsilon > 0$ such that $B(y, \epsilon) \subset M \setminus \{x\}$, i.e. $x \notin B(y, \epsilon)$. But as $y \neq x$, hence we can just take $\epsilon = d(x, y) > 0$.

Problem 2 (12)**.** Let X be a space.

a. Suppose $(\tau_i)_{i \in I}$ is a family of topologies on X indexed by I. Prove that $\bigcap_{i \in I} \tau_i$ is a topology on X.

b. Suppose τ, τ' are topologies on X. Is $\tau \cup \tau'$ a topology on X? Justify your claim.

c. Let $\mathcal{A}$ be a basis for a topology on X, and let I be the collection of topologies τ on X such that $\mathcal{A} \subset \tau$. Prove that $\tau_{\mathcal{A}} = \bigcap_{\tau \in I} \tau$. In other words, τ_A is the *coarsest* topology that contains $\mathcal{A}$. Is this true if $\mathcal{A}$ is only a sub-basis for a topology on X?

Solution 2.

a. We have to check the axioms. $X, \emptyset \in \bigcap_{i \in I} \tau_i$ as they are in all τ_i. Let $(U_j)_{j \in J}$ be a family of elements in $\bigcap_{i \in I} \tau_i$. Then it is a family of elements of any τ_i and so $\bigcup_{j \in J} U_j \in \tau_i$ for all i. Thus $\bigcup_{j \in J} U_j \in \bigcap_{i \in I} \tau_i$.The third axiom is checked similarly; given $U, V \in \bigcap_{i \in I} \tau_i$, we have $U, V \in \tau_i$ for all i; hence, $U \cap V \in \tau_i$ as τ_i is a topology. But this implies $U \cap V \in \bigcap_{i \in I} \tau_i$. Hence, $\bigcap_{i \in I} \tau_i$ is a topology.

b. The answer is no. A simple counterexample is as follows: Let $X = \mathbb{R}$, $A = \mathbb{R}_{\geq 0} \subset \mathbb{R}$ and $B = (-1, 0] \subset \mathbb{R}$. Let $\tau = \{\emptyset, A, X\}$ and $\tau' = \{\emptyset, B, X\}$, they can easily seen to be topologies. Then both A and B are in $\tau \cup \tau'$ but $A \cap B = \{0\} \notin \tau \cup \tau'$. Hence, it is not a topology. Union axiom fails as well.

c. Let $\tau_0 = \bigcap_{\tau \in I} \tau$. As $\mathcal{A} \subset \tau_{\mathcal{A}}$, $\tau_{\mathcal{A}} \in I$. Thus, $\tau_0 \subset \tau_{\mathcal{A}}$. On the other hand, given $\tau \in I$ and $U \in \tau_{\mathcal{A}}$, we can write U as a union of elements of $\mathcal{A} \subset \tau$, hence as a union of open subsets of τ. This implies U itself is in τ. Hence, $\tau_{\mathcal{A}} \subset \tau$ for all $\tau \in I$, i.e. $\tau_{\mathcal{A}} \subset \bigcap_{\tau \in I} \tau = \tau_0$. Thus, $\tau_{\mathcal{A}} = \tau$. The same holds for sub-bases $\mathcal{A}$, if we define $\tau_{\mathcal{A}}$ to be the topology with basis the set of finite intersections of $\mathcal{A}$.

Problem 3 (18)**.** A totally ordered set is a set X together with a subset $R \subset X \times X$ satisfying the following properties:

- for every $x \in X$, (x, x) does *not* belong to R;
- for every $x, y \in X$, exactly one of (x, y) and (y, x) belongs to R.
- for every $x, y, z \in X$, if (x, y) and (y, z) belong to R then (x, z) belongs to R.

We often write $x < y$ as an abbreviation for $(x, y) \in R$. The subset R is called the *total ordering*. Given $a, b \in X$, we define the following subsets of X:

$$\begin{aligned}(-\infty, a) &\equiv \{x \in X : x < a\}\\ (a, \infty) &\equiv \{x \in X : a < x\}\\ (a, b) &\equiv (a, \infty) \cap (-\infty, b).\end{aligned}$$

Denote by $\mathcal{B}$ the collection of all such subsets, as well as X itself.

a. Show that $\mathcal{B}$ is a basis for a topology τ_R on X.

b. Recall that $\mathbb{R}$ has a total ordering, $x < y \iff y - x$ is positive. Prove that the order topology on $\mathbb{R}$ coincides with the standard (metric) topology on $\mathbb{R}$.

c. Note that $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ has a total order defined as follows:

$$(a, b) < (c, d) \iff a < c \text{ or } (a = c \text{ and } b < d).$$

This is called the *lexicographical order* (think of how words are ordered in the dictionary). Denote this ordering $L \subset \mathbb{R}^2 \times \mathbb{R}^2$. Observe that the subset $C = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ inherits a total ordering from $\mathbb{R}^2$, which we denote by $T \subset C \times C$. This means we have *three* topologies on $[0, 1] \times [0, 1]$:

- $(\tau_L)_{[0,1]\times[0,1]}$, the subspace topology from the ordering on $\mathbb{R}^2$;
- τ_T, the topology from the induced ordering on $[0, 1] \times [0, 1]$;
- the subspace topology on $[0, 1] \times [0, 1]$ from the standard topology on $\mathbb{R}^2$.

Compute the closure of the set $A = \{(x, 0) : x \in [0, 1)\}$ in each of these topologies.

Solution 3. a. One needs to check that finite intersections of elements of $\mathcal{B}$ can be covered by elements of $\mathcal{B}$. But clearly intersection of two elements, is again in $\mathcal{B}$, for instance $(a, b) \cap (c, d) = (max\{a, c\}, min\{b, d\})$. Hence, $\mathcal{B}$ is a basis.

b. Let τ_d and τ_o denote the metric and order topologies respectively. Clearly, elements of $\mathcal{B}$ are open in τ_d: finite intervals are balls centered around their mid-points and infinite ones are infinite unions of finite intervals. Hence, $\tau_o \subset \tau_d$ as in the previous question. On the other hand, given $U \in \tau_d$ we can form $U^o = \bigcup_{(a,b)\subset U,\text{ finite interval}}(a,b) \subset U$ and this set is open in order topology. But given $x \in U$, there exist an $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset U$ so $(x-\epsilon, x+\epsilon) \subset U^o$. Thus $U = U^o$ is also open in order topology and $\tau_d = \tau_o$.

c.
- For any $x \in [0,1)$, $((x,0),(x,2)) \cap C$ is open in the subspace topology for the order topology on $\mathbb{R}^2$. But this set is just $\{x\} \times (0,1]$. Similarly, $\{1\} \times [0,1] = ((1,-1),(1,2)) \cap C$, where the interval is taken with respect to given order on $\mathbb{R}^2$, is open. But, these sets cover all the compliment of A in C. Hence A is closed and its closure is itself.
- Any basic open set, i.e. an open interval, in the complement would be supported in a single x-coordinate, otherwise it would intersect the elements of A with x-coordinate in between. So the basis elements in the complement are subsets of the intervals $((x,0),(x,1))$, for $0 \leq x < 1$ and subsets of the interval $((1,0),(1,1))$. Thus the complement of union of all such, i.e. the closure of A is equal to $[0,1] \times \{0,1\}$.
- A clearly accumulates to $(x,0)$ for all $x \in [0,1]$ and clearly to nothing else. Hence, the closure of A in this topology is $[0,1] \times \{0\}$.

Problem 4 (18)**.** Let X and Y be topological spaces.

a. Let $\mathcal{B}_{\prod}$ be the collection of subsets of $X \times Y$ of the form $U \times V$,where U is open in X and V is open in Y. Show that $\mathcal{B}_{\prod}$ is a basis for a topology on $X \times Y$. This topology is called the *product topology.*

b. Let $\mathcal{B}_{\coprod}$ be the collection of subsets of $X \coprod Y$ of the form $U \times \{0\}$ or $V \times \{1\}$,where U is open in X and V is open in Y. Show that $\mathcal{B}_{\coprod}$ is a basis for a topology on $X \coprod Y$. This topology is called the *sum topology.*

c. Consider $\mathbb{R}$ equipped with its standard metric topology. Show that the product topology on $\mathbb{R} \times \mathbb{R}$ is the same as the standard metric topology on $\mathbb{R}^2$.

Solution 4. a. If we have two elements $U \times V$ and $U' \times V'$ of $\mathcal{B}_{\prod}$, where U and U' are open in X and V and V' are open in Y, then their intersection is $(U \cap U') \times (V \cap V')$, which is in $\mathcal{B}_{\prod}$ as well. Hence, for any two elements of $\mathcal{B}_{\prod}$ and any $a \in X \times Y$ in their intersection, there exist an element of $\mathcal{B}_{\prod}$ that contains a and that is in the intersection, namely take the intersection itself. Hence, $\mathcal{B}_{\prod}$ is a basis.

b. Again, to check this, checking that $\mathcal{B}_{\coprod}$ is closed under finite non-empty intersections is enough. Intersections of two sets, $U \times \{0\}$ and $U' \times \{0\}$, where U and U' are open in X, is equal to $U \cap U' \times \{0\}$, which is again in $\mathcal{B}_{\coprod}$. It is similar for two sets of the type $V \times \{1\}$ in $\mathcal{B}_{\coprod}$. On the other hand, $U \times \{0\} \cap V \times \{1\} = \emptyset$, $\mathcal{B}_{\coprod}$ satisfies the desired property; hence, it is a basis.

c. The elements of $\mathcal{B}_{\prod}$ are open in the metric topology: If $(x, y) \in U \times V$, where U and V are open, then there exist $\epsilon, \epsilon' > 0$ such that $B(x, \epsilon) \subset U$ and $B(y, \epsilon') \subset V$. Then it is easy to show that $B((x, y), min\{\epsilon, \epsilon'\}) \subset U \times V$. As $(x, y) \in U \times V$ was arbitrary, this shows that the sets $U \times V$ are metric open. Thus, all the basis elements are metric open and the topology they generate is contained in τ_d, the metric topology. On the other hand, as open subsets in metric topology are unions of ball, it is enough to show that the balls are open in the product topology. Given $(x, y) = b \in B(a, \epsilon') \subset \mathbb{R}^2$, we can find $\epsilon > 0$ such that $B(b, \epsilon) \subset B(a, \epsilon')$. Then, clearly $(x-\epsilon/2, x+\epsilon/2) \times (y-\epsilon/2, y+\epsilon/2) \subset B((x, y), \epsilon) = B(b, \epsilon)$. As b, a, ϵ' was arbitrary, this shows the open balls are product open, hence metric open subsets are open in the product topology. Thus, they are the same.

Problem 5 (20)**.** The goal of this problem is to show that there are infinitely many prime numbers. This result is known as Euclid's theorem, its first recorded proof having been published by Euclid around 300 B.C. The surprising topological proof that we will see here was discovered in 1955 by H. Furstenberg.

Recall that a natural number $p \in \mathbb{N}$ is *prime* iff $p \neq 1$ and if its only divisors are 1 and p. Let $\mathbb{P} \subset \mathbb{N}$ be the set of all prime numbers.

a. Show that every natural number $n \neq 1$ is divisible by a prime number.

b. For $x, n \in \mathbb{Z}$ let
$$x + n\mathbb{Z} = \{x + nz : z \in \mathbb{Z}\}.$$
Call a subset $U \subset \mathbb{Z}$ open iff for every $x \in U$ there exists $n \in \mathbb{N} \setminus \{0\}$ such that $x + n\mathbb{Z} \subset U$. Show that this defines a topology on $\mathbb{Z}$.

c. Show that for every $n \in \mathbb{N} \setminus \{0\}$, the set $n\mathbb{Z}$ is both open and closed in $\mathbb{Z}$.

d. Using (a), show that
$$\mathbb{Z} \setminus \{1, -1\} = \bigcup_{p \in \mathbb{P}} p\mathbb{Z}.$$

e. Conclude that $\mathbb{P}$ is infinite.

Solution 5. a. Assume the converse and let $n \neq 1$ be the smallest natural number not satisfying this condition. Every natural number $p \neq 1$ dividing n should be less than than or equal to n but if $p < n$ then p and hence n is divisible by a prime number. This cannot happen by assumption; hence every such p is equal to n, i.e. n is prime. But then it is divisible by a prime, a contradiction; hence, the statement holds.

b. First notice that the family $\{x + n\mathbb{Z} : x \in \mathbb{Z}, n \in \mathbb{N} \setminus \{0\}\}$ is closed under finite non-empty intersection: Indeed, if $a \in (x + n\mathbb{Z}) \cap (y + m\mathbb{Z})$, then $(a+n\mathbb{Z}) = (x+n\mathbb{Z})$ and $(a+m\mathbb{Z}) = (y+m\mathbb{Z})$ and $(x+n\mathbb{Z}) \cap (y+m\mathbb{Z}) = (a+n\mathbb{Z}) \cap (a+m\mathbb{Z}) = a + gcd(m, n)\mathbb{Z}$, which is of the same type. Now, to check the given family of opens define a topology becomes easy: Openness of $\emptyset$ and $\mathbb{Z}$ is vacuously true. Union axiom is also obvious. Hence we only have to check this family of sets is closed under finite intersections. But if U and V are open, then given $x \in U \cap V$, we have n and m such that $x + n\mathbb{Z} \subset U$ and $x + m\mathbb{Z} \subset V$. Thus their intersection $x + gcd(m, n)\mathbb{Z}$ is a subset of $U \cap V$. This implies $U \cap V$ is open as well and given family defines a topology.

c. As $x \in y + n\mathbb{Z}$ implies $x + n\mathbb{Z} = y + n\mathbb{Z}$, the sets $x + n\mathbb{Z}$ are open. In particular, $n\mathbb{Z}$ and its complement $\bigcup_{i=1,\dots,n-1} i + n\mathbb{Z}$ are open, so $n\mathbb{Z}$ is also closed.

d. Clearly the union is a subset of $\mathbb{Z} \setminus \{1, -1\}$ for 1 or -1 are not divisible by any prime. On the other hand, given $x \in \mathbb{Z} \setminus \{1, -1\}$, part (a) implies it is divisible by some $p \in \mathbb{P}$, i.e. $x \in p\mathbb{Z}$. So the result follows.

e. If $\mathbb{P}$ were finite the union $\bigcup_{p \in \mathbb{P}} p\mathbb{Z}$ would be closed in the topology above, by part (c), i.e. as $p\mathbb{Z}$ is closed. Thus by part (d), $\{-1, 1\}$ would be open. Thus it would contain $1 + m\mathbb{Z}$, for some $m \neq 1$, yet it is finite and this is a contradiction. Thus, $\mathbb{P}$ is infinite.