Problem Set 1: A Set-Theory diagnostic

Solution 1.

(a)	$A \subset B$ and $B \subset C$	$\boxed{\Rightarrow}$	$A \subset (B \cup C)$	$\Rightarrow$ $\Leftarrow$ $\Leftrightarrow$	None
(b)	$A \subset B$ and $B \subset C$	$\boxed{\Rightarrow}$	$A \subset (B \cap C)$	$\Rightarrow$ $\Leftarrow$ $\Leftrightarrow$	None
(c)	$A \subset B$ or $B \subset C$	$\boxed{\text{None}}$	$A \subset (B \cup C)$	$\Rightarrow$ $\Leftarrow$ $\Leftrightarrow$	None
(d)	$A \subset B$ or $B \subset C$	$\boxed{\Leftarrow}$	$A \subset (B \cap C)$	$\Rightarrow$ $\Leftarrow$ $\Leftrightarrow$	None
(e)	$A \setminus (A \setminus B)$	$\boxed{\subset}$	B	$\subset$ $\supset$ $=$	None
(f)	$A \setminus (B \setminus A)$	$\boxed{\supset}$	$A \setminus B$	$\subset$ $\supset$ $=$	None
(g)	$A \cap (B \setminus C)$	$\boxed{=}$	$(A \cap B) \setminus (A \cap C)$	$\subset$ $\supset$ $=$	None
(h)	$A \cup (B \setminus C)$	$\boxed{\supset}$	$(A \cup B) \setminus (A \cup C)$	$\subset$ $\supset$ $=$	None
(i)	$A \subset C$ and $B \subset D$	$\boxed{\Rightarrow}$	$(A \times B) \subset (C \times D)$	$\Rightarrow$ $\Leftarrow$ $\Leftrightarrow$	None
(j)	$(A \times B) \cup (C \times D)$	$\boxed{\subset}$	$(A \cup C) \times (B \cup D)$	$\subset$ $\supset$ $=$	None
(k)	$(A \times B) \cap (C \times D)$	$\boxed{=}$	$(A \cap C) \times (B \cap D)$	$\subset$ $\supset$ $=$	None
(l)	$A \times (B \setminus C)$	$\boxed{=}$	$(A \times B) \setminus (A \times C)$	$\subset$ $\supset$ $=$	None
(m)	$(A \times B) \setminus (C \times D)$	$\boxed{\supset}$	$(A \setminus C) \times (B \setminus D)$	$\subset$ $\supset$ $=$	None

Solution 2.

(n)	$x \in \bigcup_{A \in \mathcal{A}} A$	$\boxed{\Leftrightarrow}$	$x \in A$ for at least one $A \in \mathcal{A}$	$\Rightarrow$ $\Leftarrow$ $\Leftrightarrow$	None
(o)	$x \in \bigcup_{A \in \mathcal{A}} A$	$\boxed{\Leftarrow}$	$x \in A$ for every $A \in \mathcal{A}$	$\Rightarrow$ $\Leftarrow$ $\Leftrightarrow$	None
(p)	$x \in \bigcap_{A \in \mathcal{A}} A$	$\boxed{\Rightarrow}$	$x \in A$ for at least one $A \in \mathcal{A}$	$\Rightarrow$ $\Leftarrow$ $\Leftrightarrow$	None
(q)	$x \in \bigcap_{A \in \mathcal{A}} A$	$\boxed{\Leftrightarrow}$	$x \in A$ for every $A \in \mathcal{A}$	$\Rightarrow$ $\Leftarrow$ $\Leftrightarrow$	None

Solution 3.

(r)	$g \circ f$ is injective, then f is	$\boxed{\text{inj.}}$		inj. surj. bij.	none.
(s)	$g \circ f$ is injective, then g is	$\boxed{\text{None}}$		inj. surj. bij,	none.
(t)	$g \circ f$ is surjective, then f is	$\boxed{\text{None}}$		inj. surj. bij.	none.
(u)	$g \circ f$ is surjective, then g is	$\boxed{\text{surj.}}$		inj. surj. bij.	none.
(v)	Let $A_0 \subset A$. If $A_0 = f^{-1}(B_0)$ for some $B_0 \subset B$, then $f^{-1}f(A_0)$	$\boxed{=}$	A_0	$\subset$ $\supset$ $=$	None
(w)	Let $B_0 \subset B$. If $B_0 \subset f(A)$ then $ff^{-1}(B_0)$	$\boxed{=}$	B_0	$\subset$ $\supset$ $=$	None

Solution 4. a. Define the inverse map

$$\psi : A^C \times B^C \to (A \times B)^C$$

by sending (g, h) to the unique function that maps $c \in C$ to $(g(c), h(c) \in A \times B$. Let ϕ denote the given map $(A \times B)^C \to A^C \times B^C$. Given $f \in (A \times B)^C$, $\psi(\phi(f)) = \psi(p_1 \circ f, p_2 \circ f)$ sends c to $(p_1 \circ f(c), p_2 \circ f(c)) = f(c)$, for all $c \in C$, hence it is equal to f. On the other hand given $(g, h) \in A^C \times B^C$, $\phi(\psi(g, h)) = (p_1 \circ \psi(g, h), p_2 \circ \psi(g, h))$. But $\psi(g, h)$ sends $c \in C$ to $(g(c), h(c)$; hence, $p_1 \circ \psi(g, h) = g$ and $p_2 \circ \psi(g, h) = h$. Thus $\phi(\psi(g, h)) = (g, h)$. Thus, ϕ and ψ are inverse to each other and ϕ is bijective.

b. Denote the given map by θ. We will define its inverse

$$\eta : C^A \times C^B \to C^{A \coprod B}$$

by $(g, h) \mapsto \{(x, 0) \mapsto g(x), \text{for } x \in A \text{ and } (y, 1) \mapsto h(y), \text{for } y \in B\}$. Then, to show they are inverse to each other take $f \in C^{A \coprod B}$. Clearly, $\eta(\theta(f)) = \eta(f \circ i_1, f \circ i_2)$ sends $(x, 0) \in A \times \{0\}$ to $f \circ i_1(x) = f(x, 0)$ and $(y, 1) \in B \times \{1\}$ to $f(y, 1)$, thus it is equal to f. On the other hand, given $(g, h) \in C^A \times C^B$, $\theta(\eta(g, h)) = (\eta(g, h) \circ i_1, \eta(g, h) \circ i_2)$. But $\eta(g, h) \circ i_1(x) = \eta(g, h)(x, 0) = g(x)$ for $x \in A$, by definition. Similarly for $y \in B$, $\eta(g, h) \circ i_2(y) = h(y)$. Thus $\theta(\eta(g, h)) = (g, h)$ and θ and η are inverse to each other.

Solution 5. a. By definition $h(C) = h(\bigcup_{n \in \mathbb{N}} C_n) = \bigcup_{n \in \mathbb{N}} h(C_n) = \bigcup_{n \in \mathbb{N}} C_{n+1} = \bigcup_{n \in \mathbb{N}_{\geq 1}} C_n$. Hence, $C = \bigcup_{n \in \mathbb{N}} C_n = C_0 \cup \bigcup_{n \in \mathbb{N}_{\geq 1}} C_n = C_0 \cup h(C)$.

b. $A \setminus C \subset A \setminus C_0 = Im(g)$. So given $x \in A \setminus C$ there exist a unique $y \in B$ such that $x = g(y)$. If $y \in f(C)$, then $x \in g(f(C)) = h(C) \subset C$ by part a. But this cannot happen by assumption, so $y \in B \setminus f(C)$. Thus $A \setminus C \subset g(B \setminus f(C))$. Conversely, given $y \in B \setminus f(C)$, if $g(y) \in C = C_0 \cup h(C)$, then it is in $h(C)$ as $C_0 \cap Im(g) = \emptyset$. Thus the iclusion holds the other way as well, and we have $A \setminus C = g(B \setminus f(C))$.

c. $A \setminus C \subset Im(g)$ and g is injective thus g^{-1} defines a bijection onto its image, which is $B \setminus f(C)$ by the above part. On the other hand, injectivity of implies, its restriction to C defines a bijection onto $f(C)$. This implies the map $k : A \to B$ defined by

$$\begin{cases} f(x) & x \in C \\ g^{-1}(x) & x \in A \setminus C \end{cases}$$

is a bijection.

d. The function tan defines a bijection from $(-\pi/2, \pi/2)$ to $\mathbb{R}$. On the other hand, there is a unique linear function sending a to $-\pi/2$ and b to $\pi/2$ which is a bijection. The composition of these two functions give a bijection from (a, b) to $\mathbb{R}$.

e. To use part c, we need to find injections both ways. The inclusion gives an injection from U to $\mathbb{R}$. On the other hand, the composition of a bijection from $\mathbb{R}$ to the interval contained in U, which exists by part d, with the inclusion map from the interval, gives as an injection from $\mathbb{R}$ to U. Hence by part c, there is a bijection between U and $\mathbb{R}$.