Problem Set 1: A Set-Theory diagnostic

Solution 1.

(a) \(A \subseteq B\) and \(B \subseteq C\) \(\implies\) \(A \subseteq (B \cup C)\) \(\implies\) \(\iff\) None

(b) \(A \subseteq B\) and \(B \subseteq C\) \(\implies\) \(A \subseteq (B \cap C)\) \(\implies\) \(\iff\) None

(c) \(A \subseteq B\) or \(B \subseteq C\) None \(A \subseteq (B \cup C)\) \(\implies\) \(\iff\) None

(d) \(A \subseteq B\) or \(B \subseteq C\) \(\iff\) \(A \subseteq (B \cap C)\) \(\implies\) \(\iff\) None

(e) \(A \setminus (A \setminus B)\) \(\subseteq\) \(A \setminus B\) \(\subseteq\) None

(f) \(A \cap (B \setminus A)\) \(\subseteq\) \(A \setminus B\) \(\subseteq\) None

(g) \(A \cap (B \setminus C)\) \(\subseteq\) \((A \cap C) \setminus (A \cap B)\) \(\subseteq\) None

(h) \(A \subseteq (B \setminus C)\) \(\subseteq\) \((A \cup B) \setminus (A \cup C)\) \(\subseteq\) None

(i) \(A \subseteq C\) and \(B \subseteq D\) \(\implies\) \((A \times B) \subseteq (C \times D)\) \(\implies\) \(\iff\) None

(j) \((A \times B) \cup (C \times D)\) \(\subseteq\) \((A \cup C) \times (B \cup D)\) \(\subseteq\) None

(k) \((A \times B) \cap (C \times D)\) \(\subseteq\) \((A \cap C) \times (B \cap D)\) \(\subseteq\) None

(l) \((A \times B) \setminus (C \times D)\) \(\subseteq\) \((A \times B) \setminus (A \times C)\) \(\subseteq\) None

Solution 2.

(n) \(x \in \bigcup_{A \in A} A\) \(\iff\) \(x \in A\) for at least one \(A \in A\) \(\implies\) \(\iff\) None

(o) \(x \in \bigcap_{A \in A} A\) \(\iff\) \(x \in A\) for every \(A \in A\) \(\implies\) \(\iff\) None

(p) \(x \in \bigcap_{A \in A} A\) \(\iff\) \(x \in A\) for at least one \(A \in A\) \(\implies\) \(\iff\) None

(q) \(x \in \bigcap_{A \in A} A\) \(\iff\) \(x \in A\) for every \(A \in A\) \(\implies\) \(\iff\) None

Solution 3.

(r) \(g \circ f\) is injective, then \(f\) is inj. \(\iff\) inj. surj. bij. none.

(s) \(g \circ f\) is injective, then \(g\) is None \(\iff\) inj. surj. bij. none.

(t) \(g \circ f\) is surjective, then \(f\) is None \(\iff\) inj. surj. bij. none.

(u) \(g \circ f\) is surjective, then \(g\) is surj. \(\iff\) inj. surj. bij. none.

(v) Let \(A_0 \subseteq A\). If \(A_0 = f^{-1}(B_0)\) \(\iff\) \(A_0 \subseteq \subseteq\) None

(w) for some \(B_0 \subseteq B\), then \(f^{-1}(A_0)\) \(\iff\) \(B_0 \subseteq \subseteq\) None

Solution 4.  a. Define the inverse map

\[\psi: A^C \times B^C \rightarrow (A \times B)^C\]
by sending \((g,h)\) to the unique function that maps \(c \in C\) to \((g(c), h(c)) \in A \times B\). Let \(\phi\) denote the given map \((A \times B)^C \to A^C \times B^C\). Given \(f \in (A \times B)^C\), \(\psi(\phi(f)) = \psi(p_1 \circ f, p_2 \circ f)\) sends \(c\) to \((p_1 \circ f(c), p_2 \circ f(c)) = f(c)\), for all \(c \in C\), hence it is equal to \(f\). On the other hand given \((g,h) \in A^C \times B^C\), \(\phi(\psi(g,h)) = (p_1 \circ \psi(g,h), p_2 \circ \psi(g,h))\). But \(\psi(g,h)\) sends \(c \in C\) to \((g(c), h(c))\); hence, \(p_1 \circ \psi(g,h) = g\) and \(p_2 \circ \psi(g,h) = h\). Thus \(\phi(\psi(g,h)) = (g,h)\). Thus, \(\phi\) and \(\psi\) are inverse to each other and \(\phi\) is bijective.

b. Denote the given map by \(\theta\). We will define its inverse

\[
\eta : C^A \times C^B \to C^{A \sqcup B}
\]

by \((g,h) \mapsto \{(x,0) \mapsto g(x), \text{for } x \in A \text{ and } (y,1) \mapsto h(y), \text{for } y \in B\}\). Then, to show they are inverse to each other take \(f \in C^{A \sqcup B}\). Clearly, \(\eta(\theta(f)) = \eta(f \circ i_1, f \circ i_2)\) sends \((x,0) \in A \times \{0\}\) to \(f \circ i_1(x) = f(x,0)\) and \((y,1) \in B \times \{1\}\) to \(f(y,1)\), thus it is equal to \(f\). On the other hand, given \((g,h) \in C^A \times C^B\), \(\theta(\eta(g,h)) = (\eta(g,h) \circ i_1, \eta(g,h) \circ i_2). \) But \(\eta(g,h) \circ i_1(x) = \eta(g,h)(x,0) = g(x)\) for \(x \in A\), by definition. Similarly for \(y \in B\), \(\eta(g,h) \circ i_2(y) = h(y)\). Thus \(\theta(\eta(g,h)) = (g,h)\) and \(\theta\) and \(\eta\) are inverse to each other.

**Solution 5.**

a. By definition \(h(C) = h(\bigcup_{n \in \mathbb{N}} C_n) = \bigcup_{n \in \mathbb{N}} h(C_n) = \bigcup_{n \in \mathbb{N}} C_{n+1} = \bigcup_{n \in \mathbb{N} \geq 1} C_n\). Hence, \(C = \bigcup_{n \in \mathbb{N}} C_n = C_0 \cup \bigcup_{n \in \mathbb{N} \geq 1} C_n = C_0 \cup h(C)\).

b. \(A \setminus C \subset A \setminus C_0 = \text{Im}(g)\). So given \(x \in A \setminus C\) there exist a unique \(y \in B\) such that \(x = g(y)\). If \(y \in f(C)\), then \(x \in g(f(C)) = h(C) \subset C\) by part a. But this cannot happen by assumption, so \(y \in B \setminus f(C)\). Thus \(A \setminus C \subset g(B \setminus f(C))\). Conversely, given \(y \in B \setminus f(C)\), if \(g(y) \in C = C_0 \cup h(C)\), then it is in \(h(C)\) as \(C_0 \cap \text{Im}(g) = \emptyset\). Thus the inclusion holds the other way as well, and we have \(A \setminus C = g(B \setminus f(C))\).

c. \(A \setminus C \subset \text{Im}(g)\) and \(g\) is injective thus \(g^{-1}\) defines a bijection onto its image, which is \(B \setminus f(C)\) by the above part. On the other hand, injectivity of \(g\) implies, its restriction to \(C\) defines a bijection onto \(f(C)\). This implies the map \(k : A \to B\) defined by

\[
\begin{cases}
  f(x) & x \in C \\
  g^{-1}(x) & x \in A \setminus C
\end{cases}
\]

is a bijection.

d. The function \(\tan\) defines a bijection from \((-\pi/2, \pi/2)\) to \(\mathbb{R}\). On the other hand, there is a unique linear function sending \(a\) to \(-\pi/2\) and \(b\) to \(\pi/2\) which is a bijection. The composition of these two functions give a bijection from \((a, b)\) to \(\mathbb{R}\).
e. To use part c, we need to find injections both ways. The inclusion gives an injection from $U$ to $\mathbb{R}$. On the other hand, the composition of a bijection from $\mathbb{R}$ to the interval contained in $U$, which exists by part d, with the inclusion map from the interval, gives as an injection from $\mathbb{R}$ to $U$. Hence by part c, there is a bijection between $U$ and $\mathbb{R}$. 