Problem Set 9: The Fundamental Group

Your name:

Due: Thursday, May 12

Problem 1 (8)**.** A *based topological space* (X, x_0) is a topological space X together with a point $x_0 \in X$, called the basepoint. A morphism of based topological spaces from (X, x_0) to (Y, y_0) is a continuous map $f : X \to Y$ so that $f(x_0) = y_0$. Prove to yourself that this collection of objects and morphisms defines a category, which we denote Top_*.

Prove that π_1 defines a functor $\mathrm{Top}_* \to \mathrm{Set}$.

Problem 2 (8)**.** Prove that $\pi_1(\mathbb{D}^n, 0) \cong \{*\}$.

Problem 3 (8)**.** Prove that for any point $x \in S^2$,

$$\pi_1(S^2, x) \cong \{*\}.$$

You may use without proof the fact that every map $(S^1, 1) \to (S^2, x)$ is homotopic (as a loop) to a map which is not surjective.[1]

Problem 4 (B16)**.** Prove that for any point $x \in \mathbb{RP}^2$, $\pi_1(\mathbb{RP}^2, x) \ncong \{*\}$. Conclude that $\mathbb{D}^2 \ncong \mathbb{RP}^2$, $\mathbb{RP}^2 \ncong S^2$ (without any fancy theorems about curves separating the plane).

[1] For those curious about the proof, you may wish to use the Lebesgue Number Lemma.