Problem Set 7: Models, compactifications, identifying spaces, characterizing subspaces

Your name:

Due: Thursday, April 21

Read the notes on the course website related to the subspace topology and quotient topology (sections 3 and 5, though you might enjoy 1 and 2 as well).

Problem 1 (8)**.** Have you done the reading?

Problem 2 (8)**.** Recall that a continuous map $i : A \to X$ satisfies *the universal property of the subspace topology* if for every continuous map $f : Y \to X$ such that $\mathrm{im}(f) \subset \mathrm{im}(i)$, there exists a unique continuous map $\hat{f} : Y \to A$ so that $i \circ \hat{f} = f$.

Prove that a map $f : A \to X$ is a homeomorphism on its image if and only if $f : A \to X$ satisfies the universal property of the subspace topology. (If you get stuck, go back and read the notes, which contains a proof of a very similar statement. Then try to prove this statement again. Repeat this procedure until you can understand and prove this result without consulting the notes.)

Problem 3 (9)**.** Let $f : X \to Y$ be a continuous surjection of topological spaces.

a. Give an example to show that f may be an open map and at the same time not be a closed map.

b. Give an example to show that f may be a closed map and at the same time not be an open map.

c. Prove that if f is either open or closed, then the topology on Y is equal to the quotient topology coming from the relation: $r, s \in X$ are equivalent iff $f(r) = f(s)$.

Problem 4 (12)**.** This problem will define the one-point compactification and ask you to prove some theorems about it. Recall that a space X is called *locally compact* if for every point $p \in X$, there exists a compact neighborhood $K \subset X$ such that $p \in K$. Note that every compact space is locally compact (prove this to yourself).

a. Give an example of space which is locally compact and not compact.

Recall that the *one-point compactification* of a space X, if it exists, consists of

- a compact Hausdorff space Y
- an embedding $i : X \hookrightarrow Y$

such that

- $Y \setminus i(X)$ is a set containing exactly one element (we call that element "∞" for convenience).
- $\overline{i(X)} = Y$.

b. Prove that if X is a space with one-point compactifications $i : X \hookrightarrow Y$ and $j : X \hookrightarrow Z$ that there is a unique homeomorphism $h : Y \to Z$ such that $h \circ i = j$.

Observe that if X has a one-point compactification, then X must be Hausdorff (if this is not clear, prove it to yourself!). We might also imagine that if X fits inside a compact space, then it can't be that far off from being compact itself.

c. Prove that if X has a one-point compactification, then X must be locally compact.

So far in this problem you have proved that if X has a one-point compactification, X must be locally compact and Hausdorff. These seem to be the only "obvious" pieces of information we can extract about X if we know it has a one-point compactification. One question to ask, then, is the following:

d. if a space X is locally compact (but not compact) and Hausdorff, does it have a one-point compactification? (Hint: try using the construction we gave in class to build one.)

Problem 5 (12)**.** For the following problem, the notation $(0, 1)$ will refer to the open interval in the real line.

a. Prove that $\text{int}\mathbb{D}^n \cong \mathbb{R}^n$.

b. Prove that $(\text{int}\mathbb{D}^n)^+ \cong S^n$.

c. Many of you argued that $\big(S^1 \times (0, 1)\big)^+$ is homeomorphic to a "pinched torus." Give a model for the pinched torus and prove that it is homeomorphic to the one-point compactification of $S^1 \times (0, 1)$.