Problem Set 5: Category theory

Your name:

Due: Thursday, March 10

Read sections 1.1–1.4 and 3.1 of *Categories in Context* by Emily Riehl (link here and on course webpage) OR read sections 1.4, 2.4, 3.1, 3.3, 3.4, 5.4 of *Category theory* by Steve Awodey (link here and on course webpage).

Problem 1 (8)**.** Did you do your reading assignment?

Problem 2 (8)**.**

a. What is a functor between groups, regarded as one-object categories? What is a natural transformation between a parallel pair of such functors?

b. What is a functor between preorders, regarded as categories? What is a natural transformation between a parallel pair of such functors?

Problem 3 (8)**.** Prove that functors carry commutative diagrams to commutative diagrams. Note: part of this exercise is to formalize the notion of commutative diagram.

Problem 4 (8)**.** Let X be a space and L_X be the category associated to the directed set $(\mathcal{T}_X, \subset)$. Convince yourself that you know what the objects and morphisms of L_X are.

a. In what way is the construction $X \mapsto L_X$ functorial? (That is, how can you make it into a functor?)

b. Let $\{X_i\}_{i\in I}$ be a family of objects of L_X. Does the product $\prod\limits_{i\in I} X_i$ exist? Does the coproduct $\coprod\limits_{i\in I} X_i$ exist? When?

Problem 5 (16)**.** Let C be a category. For a diagram $X : J \to C$, denote the colimit of the diagram (if it exists) by $(\operatorname*{colim}_J(X), \{\iota_j\}_{j\in \mathrm{ob}\, J})$ where

$$\operatorname*{colim}_J(X) \in \mathrm{ob}\, C$$

is the universal object of the colimit and

$$\iota_j : X(j) \to \operatorname*{colim}_J(X)$$

are the universal morphisms.

A functor $F : C \to D$ is said to *preserve colimits* if for every diagram $X : J \to C$ such that the colimit of X exists, the colimit $\operatorname*{colim}_J(FX)$ also exists, and the map

$$\operatorname*{colim}_J(FX) \to F\left(\operatorname*{colim}_J X\right),$$

which is induced by the maps $F\iota_j : F(X(j)) \to F(\operatorname*{colim}_J(X))$ and the universal property of $\operatorname*{colim}_J(FX)$, is an isomorphism.

a. Give an example of a functor that *does not* preserve colimits.

b. Give an example of a functor that preserves colimits.

c. Let C be a category. Prove that any functor $F : \mathrm{Set} \to C$ that preserves colimits is determined by the image under F of any set with one-element.

d. Let C be a category. Let D be one of the following categories: Group, $\mathrm{Vect}_{\mathbb{R}}$, Top. Is there a collection of subobjects $G \subset \mathrm{ob}\, D$ so that any functor $F : D \to C$ that preserves colimits is determined by its values on G? What if we allow G to be a subcategory?