Problem 1 (8). Let $X$ be a set, and $T_0$ and $T_1$ topologies on $X$. If $T_0 \subset T_1$, we say that $T_1$ is finer than $T_0$ (and that $T_0$ is coarser than $T_1$).

a. Let $Y$ be a set with topologies $T_0$ and $T_1$, and suppose $\text{id}_Y : (Y, T_1) \to (Y, T_0)$ is continuous. What is the relationship between $T_0$ and $T_1$? Justify your claim.

b. Let $Y$ be a set with topologies $T_0$ and $T_1$ and suppose that $T_0 \subset T_1$. What does connectedness in one topology imply about connectedness in the other?

c. Let $Y$ be a set with topologies $T_0$ and $T_1$ and suppose that $T_0 \subset T_1$. What does one topology being Hausdorff imply about the other?

d. Let $Y$ be a set with topologies $T_0$ and $T_1$ and suppose that $T_0 \subset T_1$. What does convergence of a sequence in one topology imply about convergence in the other?

Problem 2 (8). Given a space $X$, we define an equivalence relation on the elements of $X$ as follows: for all $x, y \in X$,

$$x \sim y \iff \text{there is a connected subset } A \subset X \text{ with } x, y \in A.$$ 

The equivalence classes are called the components of $X$.

a. (0) Prove to yourself that the components of $X$ can also be described as connected subspaces $A$ of $X$ which are as large as possible, i.e., connected subspaces $A \subset X$ that have the property that whenever $A \subset A'$ for $A'$ a connected subset of $X$, $A = A'$.

b. (4) Compute the connected components of $\mathbb{Q}$.

c. (4) Let $X$ be a Hausdorff topological space, and $f, g : \mathbb{R} \to X$ be continuous maps such that for every $x \in \mathbb{Q}$, $f(x) = g(x)$. Show that $f = g$.

Problem 3 (9). Prove that no pair of the following subspaces of $\mathbb{R}$ are homeomorphic:

$$(0, 1), \quad (0, 1], \quad [0, 1].$$
Problem 4 (8). Let \((X_i)_{i \in I}\) be a family of topological spaces, and \((Y_i)_{i \in I}\) be a family of subsets \(Y_i \subset X_i\). Note that the set \(\prod_{i \in I} Y_i\) has two possible topologies:

- first give each \(Y_i\) the subspace topology, and then take the product topology on the product
- give the product the subspace topology as a subset of the product topology on \(\prod_{i \in I} X_i\).

Are these two topologies the same? Prove or disprove using the universal properties of the subset and the product.

Problem 5 (12 – problem seminar). In this problem, we will investigate the notion of convergence in the product and box topologies on spaces of functions.

- Let \(X\) be a space and \(I\) be a set. Recall that the set of maps \(X^I\) is also the product \(\prod_{i \in I} X_i\), and so has a natural topology (the product topology). Let \((f_n)_{n \in \mathbb{N}}\) be a sequence of maps in \(X^I\), and let \(f \in X^I\). Show that \(f_n \to f\) in \(X^I\) if and only if, for every \(i\), \(f_n(i) \to f(i)\) in \(X_i\). For this reason, the product topology \(T_\prod\) is also called the topology of pointwise convergence.

- Show that the topology of pointwise convergence on \(\mathbb{R}^\mathbb{R}\) does not come from a metric.

The topology of uniform convergence \(T_\infty\) on \(\mathbb{R}^\mathbb{R}\) is defined as follows: a subset \(U \subset \mathbb{R}^\mathbb{R}\) belongs to \(T_\infty\) iff for every \(f \in U\) there exists \(\epsilon > 0\) such that

\[
\left\{ g : \mathbb{R} \to \mathbb{R} : \sup_{x \in \mathbb{R}} |f(x) - g(x)| < \epsilon \right\} \subset U.
\]

Convince yourself that this is a topology. Justify to yourself the name of \(T_\infty\) (by figuring out what it means for a sequence to converge in \(T_\infty\)).

- Show that \(T_\prod \subset T_\infty \subset T_\Box\)
- Show that \(T_\prod \neq T_\infty\).

- Show that the sequence of constant functions \(x \mapsto \frac{1}{n+1}\) does not converge to 0 in the box topology. Conclude that \(T_\infty \neq T_\Box\).

- Find a sequence of functions \(f_n \in \mathbb{R}^\mathbb{R}\) such that \(\sup_{x \in \mathbb{R}} |f(x)| \geq \frac{1}{n+1}\) and that converges to the constant function 0 in the box topology.