Problem Set 4: Connectedness

Your name:

Due: Thursday, February 18

Problem 1 (8)**.** Let X be a set, and $\mathcal{T}_0$ and $\mathcal{T}_1$ topologies on X. If $\mathcal{T}_0 \subset \mathcal{T}_1$, we say that $\mathcal{T}_1$ is *finer* than $\mathcal{T}_0$ (and that $\mathcal{T}_0$ is *coarser* than $\mathcal{T}_1$).

a. Let Y be a set with topologies $\mathcal{T}_0$ and $\mathcal{T}_1$, and suppose $\mathrm{id}_Y : (Y, \mathcal{T}_1) \to (Y, \mathcal{T}_0)$ is continuous. What is the relationship between $\mathcal{T}_0$ and $\mathcal{T}_1$? Justify your claim.

b. Let Y be a set with topologies $\mathcal{T}_0$ and $\mathcal{T}_1$ and suppose that $\mathcal{T}_0 \subset \mathcal{T}_1$. What does connectedness in one topology imply about connectedness in the other?

c. Let Y be a set with topologies $\mathcal{T}_0$ and $\mathcal{T}_1$ and suppose that $\mathcal{T}_0 \subset \mathcal{T}_1$. What does one topology being Hausdorff imply about the other?

d. Let Y be a set with topologies $\mathcal{T}_0$ and $\mathcal{T}_1$ and suppose that $\mathcal{T}_0 \subset \mathcal{T}_1$. What does convergence of a sequence in one topology imply about convergence in the other?

Problem 2 (8)**.** Given a space X, we define an equivalence relation on the elements of X as follows: for all $x, y \in X$,

$$x \sim y \iff \text{there is a connected subset } A \subset X \text{ with } x, y \in A.$$

The equivalence classes are called the *components* of X.

a. (0) Prove *to yourself* that the components of X can also be described as connected subspaces A of X which are as large as possible, ie, connected subspaces $A \subset X$ that have the property that whenever $A \subset A'$ for A' a connected subset of X, $A = A'$.

b. (4) Compute the connected components of $\mathbb{Q}$.

c. (4) Let X be a Hausdorff topological space, and $f, g : \mathbb{R} \to X$ be continuous maps such that for every $x \in \mathbb{Q}$, $f(x) = g(x)$. Show that $f = g$.

Problem 3 (9)**.** Prove that no pair of the following subspaces of $\mathbb{R}$ are homeomorphic:

$$(0,1), \quad (0,1], \quad [0,1].$$

Problem 4 (8)**.** Let $(X_i)_{i\in I}$ be a family of topological spaces, and $(Y_i)_{i\in I}$ be a family of subsets $Y_i \subset X_i$. Note that the set $\prod_{i\in I} Y_i$ has two possible topologies:

- first give each Y_i the subspace topology, and then take the product topology on the product
- give the product the subspace topology as a subset of the product topology on $\prod_{i\in I} X_i$.

Are these two topologies the same? Prove or disprove using the universal properties of the subset and the product.

Problem 5 (12 – problem seminar)**.** In this problem, we will investigate the notion of convergence in the product and box topologies on spaces of functions.

a. Let X be a space and I be a set. Recall that the set of maps X^I is also the product $\prod_{i\in I} X$, and so has a natural topology (the product topology). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of maps in X^I, and let $f \in X^I$. Show that $f_n \to f$ in X^I if and only if, for every i, $f_n(i) \to f(i)$ in X. For this reason, the product topology $\mathcal{T}_{\prod}$ is also called the *topology of pointwise convergence.*

b. Show that the topology of pointwise convergence on $\mathbb{R}^{\mathbb{R}}$ does not come from a metric.

The *topology of uniform convergence* $\mathcal{T}_\infty$ on $\mathbb{R}^{\mathbb{R}}$ is defined as follows: a subset $U \subset \mathbb{R}^{\mathbb{R}}$ belongs to $\mathcal{T}_\infty$ iff for every $f \in U$ there exists $\epsilon > 0$ such that

$$\left\{ g : \mathbb{R} \to \mathbb{R} : \sup_{x\in\mathbb{R}} |f(x) - g(x)| < \epsilon \right\} \subset U.$$

Convince yourself that this is a topology. Justify to yourself the name of $\mathcal{T}_\infty$ (by figuring out what it means for a sequence to converge in $\mathcal{T}_\infty$).

c. Show that $\mathcal{T}_{\prod} \subset \mathcal{T}_\infty \subset \mathcal{T}_\square$

d. Show that $\mathcal{T}_{\prod} \neq \mathcal{T}_\infty$.

e. Show that the sequence of constant functions $x \mapsto \frac{1}{n+1}$ does not converge to 0 in the box topology. Conclude that $\mathcal{T}_\infty \neq \mathcal{T}_\square$.

f. Find a sequence of functions $f_n \in \mathbb{R}^{\mathbb{R}}$ such that $\sup_{x\in\mathbb{R}} |f(x)| \geq \frac{1}{n+1}$ and that converges to the constant function 0 in the box topology.