Problem Set 3: Limits and closures

Your name:

Due: Thursday, February 18

Problem 1 (8)**.** Let X be a topological space and $A, B \subset X$.

a. Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

b. Show that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

c. Give an example of X, A, and B such that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

d. Let Y be a subset of X such that $A \subset Y$. Denote by $\overline{A}$ the closure of A in X, and equip Y with the subspace topology. Describe the closure of A in Y in terms of $\overline{A}$ and Y.

Problem 2 (8)**.** Let X be a set and let

$$\tau = \{U \in \mathcal{P}(X) : X \setminus U \text{ is finite, or } U = \emptyset\}.$$

a. Show that τ is a topology on X. This topology is called the *cofinite topology* (or *finite complement* topology).

b. Let X be a set equipped with the cofinite topology. Let $A \subset X$. Describe the boundary ∂A of A.

c. Suppose $X = \mathbb{N}$. To which points does the sequence $(n)_{n \in \mathbb{N}}$ converge?

Problem 3 (8)**.** Let (X, d) be a metric space. Prove that the metric topology on X is Hausdorff.

Problem 4 (8)**.** Let X and Y be topological spaces. A map $f : X \to Y$ is called *open* if for every open $U \subset X$, the image $f(U)$ is open in Y.

a. Consider $X \times Y$ equipped with the product topology. Show that the map $p_1 : X \times Y \to X, (x, y) \mapsto x$ is both continuous and open.

b. Consider $X \coprod Y$ equipped with the sum topology. Show that the map $i_1 : X \to X \coprod Y, x \mapsto (x, 0)$ is both continuous and open.

Problem 5 (12)**.** An *equivalence relation* on a set X is a subset $R \subset X \times X$ such that

- for each $x \in X, (x, x) \in R$.

- for every $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.
- for every $x, y, z \in X$ if $(x, y), (y, z) \in R$ then $(x, z) \in R$.

We write $x \sim_R y$ as an abbreviation for $(x, y) \in R$ (and sometimes just write $x \sim y$). For $x \in X$, the set

$$[x] = \{y \in X : y \sim x\}$$

is called the *equivalence class* of x. We denote by

$$X/\!\sim\; = \{[x] : x \in X\},$$

the set of equivalence classes of elements of X, called the *quotient of X by $\sim$*.

Suppose now that X is a topological space with an equivalence relation $\sim$, and consider the map

$$\pi : X \to X/\sim, \quad x \mapsto [x].$$

a. Declare a subset $U \subset X/\!\sim$ to be open if $\pi^{-1}(U) \subset X$ is open. Show that this defines a topology on $X/\!\sim$, and that the map π is continuous. This topology is called the *quotient topology.*

b. Is the map π always an open map? Justify your claim with proof or counterexample.

c. Let Y be another topological space and let $f : X \to Y$ be a continuous map such that $f(x_1) = f(x_2)$ whenever $x_1 \sim x_2$. Show that there exists a unique map $\overline{f} : X/\!\sim \to Y$ such that $f = \overline{f} \circ \pi$, and show that $\overline{f}$ is continuous. This is called the *universal property of the quotient topology.*

d. Consider $\mathbb{R} \coprod \mathbb{R}$ with the sum topology, with the equivalence relation

$$(x, 0) \sim (y, 1) \quad \text{iff} \quad x \neq 0 \text{ and } x = y.$$

The topological space $Q = \mathbb{R} \coprod \mathbb{R}/\!\sim$ is called the *line with double origin.* Which points in Q are the limit of the sequence $n \mapsto [(\frac{1}{n+1}, 0)]$? Is Q a Hausdorff space?