Problem Set 2: Topological spaces

Your name:

Due: Thursday, February 11

Problem 1 (7)**.** Let (M, d) be a metric space, and let x be a point in M. Show that the subset $M \setminus \{x\}$ is open in the metric topology τ_d.

Problem 2 (12)**.** Let X be a space.

a. Suppose $(\tau_i)_{i \in I}$ is a family of topologies on X indexed by I. Prove that $\bigcap_{i \in I} \tau_i$ is a topology on X.

b. Suppose τ, τ' are topologies on X. Is $\tau \cup \tau'$ a topology on X? Justify your claim.

c. Let $\mathcal{A}$ be a basis for a topology on X, and let I be the collection of topologies τ on X such that $\mathcal{A} \subset \tau$. Prove that $\tau_{\mathcal{A}} = \bigcap_{\tau \in I} \tau$. In other words, τ_A is the *coarsest* topology that contains $\mathcal{A}$. Is this true if $\mathcal{A}$ is only a sub-basis for a topology on X?

Problem 3 (18)**.** A totally ordered set is a set X together with a subset $R \subset X \times X$ satisfying the following properties:

- for every $x \in X$, (x, x) does *not* belong to R;
- for every $x, y \in X$, exactly one of (x, y) and (y, x) belongs to R.
- for every $x, y, z \in X$, if (x, y) and (y, z) belong to R then (x, z) belongs to R.

We often write $x < y$ as an abbreviation for $(x, y) \in R$. The subset R is called the *total ordering*. Given $a, b \in X$, we define the following subsets of X:

$$\begin{aligned} (-\infty, a) &\equiv \{x \in X : x < a\} \\ (a, \infty) &\equiv \{x \in X : a < x\} \\ (a, b) &\equiv (a, \infty) \cap (-\infty, b). \end{aligned}$$

Denote by $\mathcal{B}$ the collection of all such subsets, as well as X itself.

a. Show that $\mathcal{B}$ is a basis for a topology τ_R on X.

b. Recall that $\mathbb{R}$ has a total ordering, $x < y \iff y - x$ is positive. Prove that the order topology on $\mathbb{R}$ coincides with the standard (metric) topology on $\mathbb{R}$.

c. Note that $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ has a total order defined as follows:

$$(a, b) < (c, d) \iff a < c \text{ or } (a = c \text{ and } b < d).$$

This is called the *lexicographical order* (think of how words are ordered in the dictionary). Denote this ordering $L \subset \mathbb{R}^2 \times \mathbb{R}^2$. Observe that the subset $C = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ inherits a total ordering from $\mathbb{R}^2$, which we denote by $T \subset C \times C$. This means we have *three* topologies on $[0, 1] \times [0, 1]$:

- $(\tau_L)_{[0,1]\times[0,1]}$, the subspace topology from the ordering on $\mathbb{R}^2$;
- τ_T, the topology from the induced ordering on $[0, 1] \times [0, 1]$;
- the subspace topology on $[0, 1] \times [0, 1]$ from the standard topology on $\mathbb{R}^2$.

Compute the closure of the set $A = \{(x, 0) : x \in [0, 1)\}$ in each of these topologies.

Problem 4 (18)**.** Let X and Y be topological spaces.

a. Let $\mathcal{B}_{\prod}$ be the collection of subsets of $X \times Y$ of the form $U \times V$,where U is open in X and V is open in Y. Show that $\mathcal{B}_{\prod}$ is a basis for a topology on $X \times Y$. This topology is called the *product topology.*

b. Let $\mathcal{B}_{\coprod}$ be the collection of subsets of $X \coprod Y$ of the form $U \times \{0\}$ or $V \times \{1\}$,where U is open in X and V is open in Y. Show that $\mathcal{B}_{\coprod}$ is a basis for a topology on $X \coprod Y$. This topology is called the *sum topology.*

c. Consider $\mathbb{R}$ equipped with its standard metric topology. Show that the product topology on $\mathbb{R} \times \mathbb{R}$ is the same as the standard metric topology on $\mathbb{R}^2$.

Problem 5 (20)**.** The goal of this problem is to show that there are infinitely many prime numbers. This result is known as Euclid's theorem, its first recorded proof having been published by Euclid around 300 B.C. The surprising topological proof that we will see here was discovered in 1955 by H. Furstenberg.

Recall that a natural number $p \in \mathbb{N}$ is *prime* iff $p \neq 1$ and if its only divisors are 1 and p. Let $\mathbb{P} \subset \mathbb{N}$ be the set of all prime numbers.

a. Show that every natural number $n \neq 1$ is divisible by a prime number.

b. For $x, n \in \mathbb{Z}$ let

$$x + n\mathbb{Z} = \{x + nz : z \in \mathbb{Z}\}.$$

Call a subset $U \subset \mathbb{Z}$ open iff for every $x \in U$ there exists $n \in \mathbb{N} \setminus \{0\}$ such that $x + n\mathbb{Z} \subset U$. Show that this defines a topology on $\mathbb{Z}$.

c. Show that for every $n \in \mathbb{N} \setminus \{0\}$, the set $n\mathbb{Z}$ is both open and closed in $\mathbb{Z}$.

d. Using (a), show that

$$\mathbb{Z} \setminus \{1, -1\} = \bigcup_{p \in \mathbb{P}} p\mathbb{Z}.$$

e. Conclude that $\mathbb{P}$ is infinite.