Problem Set 1: A Set-Theory diagnostic

Your name:

Due: Tuesday, February 9

Problem 1 (13)**.** What goes in the $\square$ to make the strongest possible true statement? Your choices are on the right.

(a)	$A \subset B$ and $B \subset C$	$\square$	$A \subset (B \cup C)$	$\Rightarrow$ $\Leftarrow$ $\Leftrightarrow$ None
(b)	$A \subset B$ and $B \subset C$	$\square$	$A \subset (B \cap C)$	$\Rightarrow$ $\Leftarrow$ $\Leftrightarrow$ None
(c)	$A \subset B$ or $B \subset C$	$\square$	$A \subset (B \cup C)$	$\Rightarrow$ $\Leftarrow$ $\Leftrightarrow$ None
(d)	$A \subset B$ or $B \subset C$	$\square$	$A \subset (B \cap C)$	$\Rightarrow$ $\Leftarrow$ $\Leftrightarrow$ None
(e)	$A \setminus (A \setminus B)$	$\square$	B	$\subset$ $\supset$ $=$ None
(f)	$A \setminus (B \setminus A)$	$\square$	$A \setminus B$	$\subset$ $\supset$ $=$ None
(g)	$A \cap (B \setminus C)$	$\square$	$(A \cap B) \setminus (A \cap C)$	$\subset$ $\supset$ $=$ None
(h)	$A \cup (B \setminus C)$	$\square$	$(A \cup B) \setminus (A \cup C)$	$\subset$ $\supset$ $=$ None
(i)	$A \subset C$ and $B \subset D$	$\square$	$(A \times B) \subset (C \times D)$	$\Rightarrow$ $\Leftarrow$ $\Leftrightarrow$ None
(j)	$(A \times B) \cup (C \times D)$	$\square$	$(A \cup C) \times (B \cup D)$	$\subset$ $\supset$ $=$ None
(k)	$(A \times B) \cap (C \times D)$	$\square$	$(A \cap C) \times (B \cap D)$	$\subset$ $\supset$ $=$ None
(l)	$A \times (B \setminus C)$	$\square$	$(A \times B) \setminus (A \times C)$	$\subset$ $\supset$ $=$ None
(m)	$(A \times B) \setminus (C \times D)$	$\square$	$(A \setminus C) \times (B \setminus D)$	$\subset$ $\supset$ $=$ None

Problem 2 (4)**.** Complete the following as in the previous problem, assuming the collection $\mathcal{A}$ is nonempty.

(n)	$x \in \bigcup_{A \in \mathcal{A}} A$	$\square$	$x \in A$ for at least one $A \in \mathcal{A}$	$\Rightarrow$ $\Leftarrow$ $\Leftrightarrow$ None
(o)	$x \in \bigcup_{A \in \mathcal{A}} A$	$\square$	$x \in A$ for every $A \in \mathcal{A}$	$\Rightarrow$ $\Leftarrow$ $\Leftrightarrow$ None
(p)	$x \in \bigcap_{A \in \mathcal{A}} A$	$\square$	$x \in A$ for at least one $A \in \mathcal{A}$	$\Rightarrow$ $\Leftarrow$ $\Leftrightarrow$ None
(q)	$x \in \bigcap_{A \in \mathcal{A}} A$	$\square$	$x \in A$ for every $A \in \mathcal{A}$	$\Rightarrow$ $\Leftarrow$ $\Leftrightarrow$ None

Problem 3 (6)**.** Let $f : A \to B$ and $g : B \to C$. Complete the following as in the previous problem. Note that "injective" has been abbreviated "inj." Similar abbreviations have been made for "surjective" and "bijective."

(r)	$g \circ f$ is injective, then f is			inj. surj. bij. none.
(s)	$g \circ f$ is injective, then g is			inj. surj. bij, none.
(t)	$g \circ f$ is surjective, then f is			inj. surj. bij. none.
(u)	$g \circ f$ is surjective, then g is			inj. surj. bij. none.
(v)	Let $A_0 \subset A$. If $A_0 = f^{-1}(B_0)$ for some $B_0 \subset B$, then $f^{-1}f(A_0)$		A_0	$\subset$ $\supset$ $=$ None
(w)	Let $B_0 \subset B$. If $B_0 \subset f(A)$ then $ff^{-1}(B_0)$		B_0	$\subset$ $\supset$ $=$ None

Problem 4 (24)**.** Let A and B be sets.

a. Consider the maps

$$p_1 : A \times B \to A, \quad (a, b) \mapsto a$$
$$p_2 : A \times B \to B, \quad (a, b) \mapsto b.$$

Show that for any set C, the map

$$(A \times B)^C \to A^C \times B^C, \quad f \mapsto (p_1 \circ f, p_2 \circ f)$$

is a bijection. (*Hint:* Define the inverse map.) Informally speaking, giving a map *to a product* is "the same thing" as giving a map *to each factor*. This is called the *universal property of the product*.

b. Recall that $A \coprod B \equiv (A \times \{0\}) \cup (B \times \{1\})$. Consider the maps

$$i_1 : A \to A \coprod B, \quad a \mapsto (a, 0)$$
$$i_2 : B \to A \coprod B, \quad b \mapsto (b, 1).$$

Show that for any set C, the map

$$C^{A \coprod B} \to C^A \times C^B, \quad f \mapsto (f \circ i_1, f \circ i_2)$$

is a bijection. Informally speaking, giving a map *from a sum* is "the same thing" as giving a map *from each factor*. This is called the *universal property of the sum*.

Problem 5 (23)**.** Let A and B be sets, and assume that $f : A \to B$ is injective, and $g : B \to A$ is injective. The goal of this problem is to show that this implies A and B are in bijection. For finite sets this may be intuitive (and if not, convince yourself as a warmup). This is a theorem which is commonly known as the Cantor-Schröder-Bernstein Theorem, named after a few mathematicians that contributed to its proof / dissemination.

Let $h : A \to A$ be the composite map $g \circ f$. Inductively define a sequence of subsets $C_n \subset A$ for $n \in \mathbb{N}$, as follows:

$$C_0 = A \setminus g(B), \quad C_{n+1} = h(C_n).$$

Let C be the union of all the C_ns:

$$C = \bigcup_{n \in \mathbb{N}} C_n.$$

a. Show that $C = C_0 \cup h(C)$.

b. Show that $A \setminus C = g(B \setminus f(C))$.

c. Use (b) to define a bijection between A and B.

d. Let a and b be real numbers with $a < b$. Show that there exists a bijection between $\mathbb{R}$ and the open interval $(a, b) = \{x \in \mathbb{R} : a < x < b\}$. (*Hint:* You may use trigonometry.)

e. Let U be any subset of $\mathbb{R}$ containing an open interval. Use (c) and (d) to show that there exists a bijection between U and $\mathbb{R}$.