Set Theory

Sets

A *set* is a collection of objects, called its *elements*. We write $x \in A$ to mean that x is an element of a set A, we also say that x *belongs to* A or that x is in A.

If A and B are sets, we say that B is a subset of A if every element of B is an element of A. In this case we also say that A *contains* B, and we write $B \subset A$. Two sets are considered equal iff $A \subset B$ and $B \subset A$.

Curly bracket notation

We often define sets by listing their elements, or pairing down sets we already have by describing subsets via formulas.

- The symbol $\{x, y, z\}$ describes the set whose elements are precisely x, y, and z.
- The symbol $\{x : p(x)\}$ describes the set of all x such that the sentence $p(x)$ is true.
- The symbol $\{x \in A : p(x)\}$ describes the set of all $x \in A$ such that the sentence $p(x)$ is true.

Warning. The notation $\{x : p(x)\}$ is not always meaningful. For example, assuming $\{\text{sets } A : A \notin A\}$ is a set leads to a logical contradiction (known as Russel's paradox). Axiomatic set theory has precise rules dictating when $\{x : p(x)\}$ is well-defined. If A is a set, the set $\{x \in A : p(x)\}$ is always well-defined (provided $p(x)$ is).

Examples

- The symbol $\emptyset$ denotes the set with no elements, denoted $\{\}$ in braces notation. The set $\emptyset$ is called *the empty set* and it is characterized by the property $x \notin \emptyset$ for all x.
- The following sets of numbers might be familiar: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
  - the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of natural numbers,

- the set $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ of integers,
- the set $\mathbb{Q} = \{\frac{p}{q} : p, q \in Z, q \neq 0\}$ of rational numbers,
- the set $\mathbb{R}$ of real numbers.

Exercise 1. Why do you think I avoided describing $\mathbb{R}$ by curly brackets? Can you think of a simple way to build $\mathbb{R}$ from $\mathbb{Q}$, $\mathbb{Z}$, or $\mathbb{N}$?

Operations with sets

One good thing to do with a new mathematical object is to think of ways you can produce new ones from old ones. What follows is a list of such constructions. Let A and B be sets:

- union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- More generally, if $(A_i)_{\imath \in I}$ is a family of sets indexed by a set I, we can define the union and intersection over I as follows

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}$$

$$\bigcap_{i \in I} A_i = \{x : x \in A_i \text{ for every } i \in I\}$$

- difference: $A \setminus B = \{x \in A : x \notin B\}$
- product: $A \times B = \{(x, y) \in A : x \in A \text{ and } y \in B\}$
- sum: $A \coprod B = (A \times \{0\}) \cup (B \times \{1\})$ (this is also called *disjoint union*)
- power set: $\mathcal{P}(A) = \{B : B \subset A\}$, the set of subsets of A.

Exercise 2. How might we generalize the product so that it can be indexed by a set I? The sum?

Functions

Let A and B be sets. A *map* f from A to B is a subset $f \subset A \times B$ such that, for every $a \in A$, there is a unique $b \in B$ so that the pair (a, b) is in f; this element b is often denoted $f(a)$. In other words, a map from A to B assigns to every element $a \in A$ an element $f(a) \in B$. A map is also called a function. The set A is called the *domain* or *source* of f, and the set B is called the *codomain* or *target* of f. We denote a map f from A to B by

$$\begin{aligned} f :& A \to B \\ & a \mapsto f(a). \end{aligned}$$

Examples. The following functions are each given as assignments, some of which have also been described as a subset.

- $f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^2$ is defined by the subset $\{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2\} \subset \mathbb{R} \times \mathbb{R}$, sometimes known as the *graph* of the function.
- $f : \{0, 1\} \to \{a, b\}, \quad 0 \mapsto a, 1 \mapsto b$ is defined by the subset $\{(0, a), (1, b)\} \subset \{(0, a), (0, b), (1, a), (1, b)\}$
- For every set A, there is an *identity map* $\mathrm{id}_A : A \to A, \quad a \mapsto a$.
- More generally, if $B \subset A$, there is an inclusion map $B \to A, \quad b \mapsto b$.

Exercise 3. How are the last two examples above described as subsets of the product set?

The set of all maps from A to B is denoted by B^A, and can be shown to be a set by observing that $B^A \subset \mathcal{P}(A \times B)$.

Definition 1. Let $f : A \to B$ and $g : B \to C$ be maps. The *composition* of f and g is the map

$$\begin{aligned} g \circ f :& A \to C \\ & a \mapsto g(f(a)) \end{aligned}$$

Definition 2. A map $f : A \to B$ is

- *injective* if for all $y \in B$, there is at most one $x \in A$ so that $f(x) = y$. Another way to say this is if $x, z \in A$ satisfy $f(x) = f(z)$, then in fact $x = z$.
- *surjective* if for all $y \in B$ there exists at least one $x \in A$ so that $f(x) = y$.
- *bijective* if it is injective and surjective.

Proposition 1. A map $f : A \to B$ is bijective iff there exists a map $g : B \to A$ such that

$$g \circ f = \mathrm{id}_A \quad \text{and} f \circ g = \mathrm{id}_B$$

Proof. ($\Rightarrow$) Suppose that f is bijective. Define a subset $g \subset B \times A$ by pairing each element $y \in B$ with the unique element $a \in A$ such that $f(x) = y$. The subset g is a map because f is bijective, and by its very definition we see that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

($\Leftarrow$) Suppose now that the map g exists. To see f is injective, assume that $f(x) = f(z)$ and apply the map g to obtain:

$$x = g(f(x)) = g(f(z)) = z.$$

To see that f is surjective, observe that any element $y \in B$ satisfies $f(g(y)) = y$. □

Remark. In this case, the function g is uniquely determined by the bijective function f. It is called the *inverse* of f and denoted $f^{-1}: B \to A$.

Definition 3. Let $f: A \to B$ be a map.

- If $U \subset A$, the *image* of U by f is the set $f(U) \equiv \{y \in B : y = f(x) \text{ for some } x \in U\}$.
- If $V \subset B$, the *preimage* of V by f is the set $f^{-1}(V) = \{x \in A : f(x) \in V\}$.

Warning. We use the same notation for the preimage of a function f, which is always defined, and the inverse of a function, which is defined in the case that f is bijective. Notice that when f is bijective, we have the following relationship between the preimage and the inverse of f for any element $y \in B$:

$$f^{-1}(\{y\}) = \{f^{-1}(y)\}.$$

Remark. A map $f: A \to B$ is surjective iff $f(A) = B$.

Definition 4. A set A is

- *finite* if there exists an $n \in \mathbb{N}$ and a bijection $\{1, 2, \ldots, n\} \to A$.
- *countably infinite* if there exists a bijection $\mathbb{N} \to A$
- *countable* if it is finite or countably infinite
- *uncountable* if it is not countable.

Examples. The sets

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are all countably infinite,
- $\mathbb{R}, \mathcal{P}(\mathbb{N})$ are uncountable.

Theorem 1 (Least Number Principle)**.** *Every nonempty subset $X \subset \mathbb{N}$ has a* least element*, ie, there is an $x \in X$ such that $x \leq y$ for all $y \in X$.*

Remark. As was pointed out in class, this theorem requires some version of itself as an axiom. As such, we won't try to prove it here.

Corollary 1 (Induction Principle)**.** *Let $p(n)$ be a statement depending on a natural number n. Suppose that* $p(0)$ *is true, and that $p(n) \Rightarrow p(n+1)$ for every $n \in \mathbb{N}$. Then $p(n)$ is true for all $n \in \mathbb{N}$.*

Proof. Consider the set $X \equiv \{n \in \mathbb{N} : p(n) is false\}$. By assumption, 0 is not a member of this set. If this set were nonempty, there would be a least element $x \in X$ (and it would be bigger than 0). Since $x > 0$, $x - 1$ is also a natural number, and this pair would satisfy $p(x-1)$ is true (since x is a least element of X), $p(x)$ is false (since $x \in X$). Since we have assumed $p(x-1) \Rightarrow p(x)$, no such element can exist, and X must be empty. □

Topological Spaces

Let X be a set. A *topology* τ on X is a collection of subsets of X, ie, $\tau \subset \mathcal{P}(X)$ such that

- $\emptyset \in \tau$ and $X \in \tau$
- If $(U_i)_{i\in I}$ is a family of elements of τ, then $\bigcup_{i\in I} U_i \in \tau$
- If $U, V \in \tau$, then $U \cap V \in \tau$.

A *topological space* is a pair (X, τ) where X is a set and τ is a topology on X. Elements of τ are called *open subsets* of X.

Examples. For every X, the set $\tau = \{\emptyset, X\}$ is a topology on X, called the *coarse topology.* For every set X, the set $\tau = \mathcal{P}(X)$ is a topology on X called the *discrete topology.*

In real analysis one often encounters *metric spaces.* These are sets with some notion of distance satisfying certain properties. In what follows, we will show that every metric space has a natural topology.

Definition 5. Let X be a set. A *metric* on X is a function $d : X \times X \to [0, \infty)$ such that

- for every $x, y \in X$, $d(x, y) = 0 \Leftrightarrow x = y$
- for every $x, y \in X$, $d(x, y) = d(y, x)$
- for every $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

A *metric space* is a pair (X, d) where X is a set and d is a metric on X.

Examples.

- The standard metric on $\mathbb{R}$: $d(x, y) = |x - y| = \big((x - y)^2\big)^{\frac{1}{2}}$
- Some metrics on $\mathbb{R}^2$ (note that $x = (x_1, x_2)$ and $y = (y_1, y_2)$):
  - standard metric: $d(x, y) = \big((x_1 - y_1)^2 + (x_2 - y_2)^2\big)^{\frac{1}{2}}$
  - sup metric : $d_\infty(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|)$

  - taxicab metrix: $d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|$

- Let $\mathcal{L}^\infty([a,b],\mathbb{R}) = \{f : [a,b] \to \mathbb{R} : \sup_{x\in[a,b]} |f(x)| < \infty\}$ and consider the function
$$d_\infty(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$$
. Then $(\mathcal{L}^\infty([a,b],\mathbb{R}), d_\infty)$ is a metric space.

Exercise 4. Let $\mathcal{R}([a,b],\mathbb{R}) = \{f : [a,b] \to \mathbb{R} : \int_{[a,b]} f(x)dx < \infty\}$ and consider the function $d(f,g) = \int_{[a,b]} |f(x) - g(x)|dx$. Is this function a metric?

Definition 6. Let X be a set with a map $d : X \times X \to \mathbb{R}$. For any element $x \in X$ and any $\epsilon > 0$, we define the set $B(x,\epsilon) \equiv \{y \in X : d(x,y) < \epsilon\} \subset X$, and call it the *ball of radius ϵ centered at x*.

Proposition 2. Let X be a set and $d : X \times X \to \mathbb{R}$ be a map. Define $\tau_d \subset \mathcal{P}(X)$ as follows: $U \in \tau_d$ if and only if for all $x \in U$ there exists an $\epsilon > 0$ such that $B(x,\epsilon) \subset U$. Then τ_d is a topology on X.

Proof. We need to verify that $\tau_d \subset \mathcal{P}(X)$ satisfies the axioms:

- To see $\emptyset \in \tau_d$ note that there are no elements to check the condition on, so it is satisfied vacuously. To see $X \in \tau_d$, any ϵ will do.

- Let $(U_i)_{i\in I}$ be a family of elements of τ_d. Let $x \in \bigcup_{i\in I} U_i$. Then $x \in U_i$ for some $i \in I$, so there exists an $\epsilon > 0$ such that $B(x,\epsilon) \subset U_i \subset \bigcup_{i\in I} U_i$. Hence $\bigcup_{i\in I} U_i \in \tau_d$.

- Let $U, V \in \tau_d$. Let $x \in U \cap V$. Then there exists $\epsilon, \delta > 0$ such that $B(x,\epsilon) \subset U$ and $B(x,\delta) \subset V$. Setting $\eta = \min(\epsilon,\delta)$, we see that $B(x,\eta) \subset B(x,\epsilon) \cap B(x,\delta) \subset U \cap V$. Hence $U \cap V \in \tau_d$.

□

Exercise 5. Consider the examples of metrics on $\mathbb{R}^2$ above. Prove that $\tau_d = \tau_{d_1} = \tau_{d_\infty}$. *Hint:* Draw a picture of $B(x,\epsilon)$ for each of the metrics.

Bases and neighborhoods

Definition 7. Let τ be a topology on a set X. A subset $\mathcal{B} \subset \tau$ is a *basis* for τ if every element $U \in \tau$ is a union of elements in $\mathcal{B}$.

Examples.

If (X,d) is a metric space, then $\{B(x,\epsilon) : x \in X, \epsilon > 0\}$ is a basis for τ_d.

The set $\{\{x\} : x \in X\}$ is a basis for the discrete topology on the set X.

Proposition 3. Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$. Then $\mathcal{B}$ is a basis for a topology on X iff:

1. every $x \in X$ belongs to some $U \in \mathcal{B}$
2. for every $U, V \in \mathcal{B}$ and all $x \in U \cap V$ there exists $W \in \mathcal{B}$ such that $x \in W \subset U \cap V$.

Proof. ($\Rightarrow$) Suppose $\mathcal{B}$ is a basis for a topology τ on X.

- Since X is an open set, it is a union of elements of $\mathcal{B}$.
- Let $U, V \in \mathcal{B}$. Since $U \cap V$ is open, $U \cap V = \bigcup_{i \in I} W_i$ where each $W_i \, in \mathcal{B}$. Hence there is some i so that $x \in W_i \subset U \cap V$.

($\Leftarrow$) Define $\tau_{\mathcal{B}} \subset \mathcal{P}(X)$ as follows: $U \in \tau_{\mathcal{B}}$ iff for all $x \in U$, there exists a $V \in \mathcal{B}$ such that $x \in V \subset U$. If $\tau_{\mathcal{B}}$ is a topology, then $\mathcal{B}$ is a basis for $\tau_{\mathcal{B}}$ for the same reason that open balls were a basis for the topology on a metric space. We must verify that $\tau_{\mathcal{B}}$ is a topology:

- $\emptyset \in \tau_{\mathcal{B}}$ because $\emptyset$ has no elements to verify; $X \in \tau_{\mathcal{B}}$ follows from (1).
- Let $(U_i)_{i \in I}$ be a family of elements of $\tau_{\mathcal{B}}$, we wish to prove that $\bigcup_{i \in I} U_i$ is in $\tau_{\mathcal{B}}$. In order to do that, we must show that for any element $x \in \bigcup_{i \in I} U_i$, there is a basic open set $W \in \mathcal{B}$ such that $x \in W \subset \bigcup_{i \in I} U_i$. Given an $x \in \bigcup_{i \in I} U_i$, there is an i so that $x \in U_i$. Since $U_i \in \tau_{\mathcal{B}}$, there is a basic open set W with $x \in W \subset U_i$, which implies $x \in W \subset \bigcup_{i \in I} U_i$ as required.
- Let U and V be elements of $\tau_{\mathcal{B}}$, we wish to prove that $U \cap V \in \tau_{\mathcal{B}}$. Given $x \in U \cap V$, we wish to find a basic open set around x in $U \cap V$. To do this, we observe that $x \in U$ implies there is a basic open set $x \in W_1 \subset U$ and $x \in V$ implies there is a basic open set $x \in W_2 \subset V$. Then $x \in W_1 \cap W_2$ and property (2) of the basis $\mathcal{B}$ guarantees that there is a W with $x \in W \subset W_1 \cap W_2$ which implies $x \in W \subset U \cap V$ as required.

□

Examples. Let X and Y be topological spaces.

- The collection $\mathcal{B}_{\prod} = \{U \times V \subset X \times Y : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$ is a basis for a topology on $X \times Y$ called the *product topology.*
- The collection $\mathcal{B}_{\coprod} = \{U \times \{0\} : U \text{ is open in } X\} \cup \{V \times \{1\} : V \text{ is open in } Y\} \subset X \times \{0\} \cup Y \times \{1\}$ is a basis for a topology on $X \coprod Y$ called the *sum topology.*

Definition 8. Let X be a topological space and let $x \in X$. A neighborhood of x in X is a subset $K \subset X$ such that there exists an open set U with

$$x \in U \subset K.$$

Warning. The definition in Munkres requires K itself to be open.

Proposition 4. The following are properties of neighborhoods:

1. X is a neighborhood of every point. If K is a neighborhood of x and $K \subset L$ then L is a neighborhood of x.

2. If K and L are neighborhoods of x then $K \cap L$ is a neighborhood of x.

3. If K is a neighborhood of x there exists a neighborhood L of x such that K is a neighborhood of every point of L.

4. A subset $U \subset X$ is open iff it is a neighborhood of all of its points.

Exercise 6. Prove the above properties of neighborhoods.

Remark. There is an alternative axiomatization of topological spaces based on neighborhoods. Given a set X and a collection of subsets called neighborhoods satisfying (1)–(3), there is a unique topology on X which induces this notion of neighborhood; the open sets are determined by (4).

Closed sets and limit points

Definition 9. Let X be a topological space. A subset $C \subset X$ is *closed* if its complement, $X \setminus C$, is open.

Proposition 5. Let X be a topological space. The following are properties of the closed sets of X:

1. Both $\emptyset$ and X are closed.

2. If $(U_i)_{i\in I}$ is a family of closed sets in X, then $\bigcap_{i\in I} U_i$ is closed.

3. If U, V are closed, then $U \cup V$ is closed.

Remark. There is alternative axiomatization of topological spaces based on closed sets. Given a set X and a collection of closed sets satisfying (1)–(3), there is a unique topology on X that induces this collection of closed sets. The open sets are the complements of the closed sets.

Definition 10. Let X be a topological space and let $A \subset X$ be a subset. The *interior* of A is the union of all open subsets contained in A:

$$A^\circ = \bigcup_{\substack{U\subset X \text{ open}, \\ U\subset A}} U \quad \subset A.$$

The *closure* of A is defined to be the intersection of all closed sets C containing A:

$$\bar{A} = \bigcap_{\substack{C\subset X \text{ closed}, \\ A\subset C}} C \quad \supset A.$$

We also define the boundary of A to be the difference between the closure and the interior:

$$\partial A = \bar{A} \setminus A^\circ.$$

Remark.

- A° is open in X and $\bar{A}$ is closed in X.
- The complement of the interior of A is the closure of the complement of A: $X \setminus A^\circ = \overline{X \setminus A}$, and hence $\partial A = \bar{A} \cap \overline{X \setminus A}$.

Proposition 6. Let X be a topological space and $A \subset$, and let $x \in X$. Then $x \in \bar{A}$ if and only if every neighborhood of x has nonempty intersection with A.

Proof. We will prove the equivalent statement:

$$x \notin \bar{A} \text{ iff } \text{ there exists a nbhd } U \text{ of } x \text{ such that } U \cap A = \emptyset.$$

($\Rightarrow$) Recall that the complement of $\bar{A}$ is open, so that if $x \notin \bar{A}$ then $x \in X \setminus \bar{A}$ which is open and does not intersect A.
($\Leftarrow$) Suppose there exists an open U' such that $x \in U' \subset U$ with $U \cap A = \emptyset$. Then $A \subset X \setminus U \subset X \setminus U'$. Since $X \setminus U'$ is closed, $\bar{A} \subset X \setminus U'$ and $x \notin \bar{A}$. □

Definition 11. A *sequence* in a set X is a map $x : \mathbb{N} \to X$. We usually write x_n for $x(n)$ and sometimes $(x_n)_{n \in \mathbb{N}}$ for the map x.

Definition 12. Let X be a topological space and let $x : \mathbb{N} \to X$ be a sequence. We say that x converges to $L \in X$, and write $x \to L$ if for every neighborhood U of L there exists a natural number $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in U$.

Remark. If $\mathcal{B}$ is a basis for a topology on X, to check that L is a limit of a sequence x, it suffices to check the condition on the neighborhoods of L that belong to $\mathcal{B}$.

Exercise 7. Consider $\mathbb{R}^n$ with the standard topology. Show that a sequence x converges to $L \in \mathbb{R}^n$ as above if and only if it converges to L in the familiar sense from calculus (the ϵ-δ sense).

Warning. A sequence may have zero, one, or more limits. So we typically avoid the notation $\lim_{n \to \infty} x_n$.

Examples.

- Consider $\mathcal{L}^\infty([a, b], \mathbb{R})$ with the metric topology (using the sup metric $d_\infty(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$). Then a sequence $(f_n)_{n \in \mathbb{N}} \to f$ iff f_n converges uniformly to the function f. (Later we will see there is another topology whose limits correspond to point-wise convergence.)
- Let $\mathcal{R} \subset \mathcal{L}^\infty([0, 1], \mathbb{R})$ be the subset of Riemann integrable functions, and let $d_1(f, g) = \int_0^1 |f(x) - g(x)| dx$ for $f, g \in \mathcal{R}$. Consider the sequence

$$f_n : [0, 1] \to \mathbb{R}, x \mapsto \begin{cases} n & \text{if } x < \frac{1}{2^n} \\ 0 & \text{otherwise} \end{cases}$$

Then $(f_n)_{n\in N}$ converges to 0 in the topology determined by d_1. In fact, any f such that $d_1(f,0) = \int_0^1 |f(x)|dx = 0$ is a limit of this sequence. On the other hand, this sequence has no limit in the uniform topology τ_{d_∞}.

Definition 13. A topological space (X, τ) is *first countable* if for every $x \in X$ there exists a countable set $\mathcal{N}_x$ of neighborhoods of x such that every neighborhood of x contains a neighborhood which is an element of $\mathcal{N}_x$. A space is called *second countable* if there exists a countable basis $\mathcal{B} \subset \tau$.

Remark.

- Second countable implies first countable.
- Every metric space is first countable.
- The space $\mathcal{L}^\infty([a,b],\mathbb{R})$ with the metric d_∞ is a metric space which is not second countable.

Proposition 7. Let X be a topological space, $A \subset X$, and let $L \in X$. If there exists $x : \mathbb{N} \to A$ such that $x_n \to L$ in X then $L \in \bar{A}$. The converse holds if X is first countable.

Proof. ($\Rightarrow$) Suppose $x_n \to L$ with $x_n \in A$. Then for every neighborhood U of L, by the definition of convergence, there is an element $x_k \in U$ (in fact, infinitely many). Therefore, for every neighborhood U of L, $x_k \in U \cap A \neq \emptyset$, so $L \in \bar{A}$. ($\Leftarrow$): Suppose $L \in \bar{A}$. Assuming that X is first countable means we have a countable set of neighborhoods $\mathcal{N}_L = \{U_0, U_1, U_2, \ldots\}$ such that every neighborhood of L contains some U_i. We want a guarantee that the neighborhoods in this list are in some sense "shrinking" toward L, so we define

$$U'_n = \cap_i^n U_i.$$

Then each U'_n is still a neighborhood of L, so $U_n \cap A \neq \emptyset$. For each n, choose $x_n \in U' \cap A$. The claim is that $x_n \to L$. To see this, observe that given a neighborhood U of L, there is some N so that $U_N \subset U$, and then for all $n > N$:

$$x_n \in U'_n \cap A \subset U'_n \subset U_N \subset U$$

as required. □

Definition 14. A topological space X is called *Hausdorff* (or T_2 or *separated*) if for every distinct pair of points $x, y \in X$ there exists nbhds U of x and V of y such that $U \cap V = \emptyset$.

Proposition 8. Let X be a Hausdorff topological space. Then a sequence in X has at most one limit.

Proof. Let L be a limit of $x : \mathbb{N} \to X$ and let $L' \neq L$. We want to show that L' is not a limit of x. Since X is Hausdorff, we can separate L and L' by disjoint neighborhoods: $L \in U$ and $L' \in U'$, with $U \cap U' = \emptyset$. Then because $x_n \to L$, there is some N so that for every $n > N$, $x_n \in U$. This means that $x_n \notin U'$, so L' can't be a limit of x. □

Proposition 9.

- Let (X, d) be a metric space. Then X is Hausdorff with the metric topology.
- Let $(X, <)$ be a totally ordered set. Then X is Hausdorff with the order topology.

Continuous maps

Theorem 2. *Let $f : X \to Y$ be a map between topological spaces. The following conditions are equivalent:*

1. *For every $U \subset Y$ open, $f^{-1}(U) \subset X$ is open.*
2. *For every $C \subset Y$ closed, $f^{-1}(C) \subset X$ is closed.*
3. *For every $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$.*
4. *For every $x \in X$ and every neighborhood V of $f(x)$ there exists a neighborhood U of x such that $f(U) \subset V$.*

Definition 15. A map $f : X \to Y$ is *continuous* if it satisfies the equivalent conditions of Theorem 2.

Proof. We'll prove (1) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (4) : Let V be a neighborhood of $f(x)$, so that $f(x) \in V' \subset V$ for some open V'. Let $U = f^{-1}(U)$. By (1), U is open, so U is a neighborhood of x, and $f(U) = f(f^{-1}(V')) \subset V' \subset V$.

(4) $\Rightarrow$ (3) : Let $y \in f(\bar{A})$, so $y = f(x)$ for some $x \in \bar{A}$. Given a nbhd V of y, by (4) there exists a neighborhood U of x such that $f(U) \subset V$. Since $x \in \bar{A}$, $U \cap A \neq \emptyset$, hence $\emptyset \neq f(U \cap A) \subset f(U) \cap f(A) \subset V \cap f(A)$, so that $y \in \overline{f(A)}$.

(3) $\Rightarrow$ (2) : Let $C \subset Y$ be closed. Then $\overline{f^{-1}(C)} \subset f^{-1}(f(\overline{f^{-1}(C)})) \overset{(3)}{\subset} f^{-1}(\overline{f(f^{-1}(C))}) \subset f^{-1}(\overline{C}) \subset f^{-1}(C)$. Since by definition $f^{-1}(C) \subset \overline{f^{-1}(C)}$, we have $\overline{f^{-1}(C)} = f^{-1}(C)$, ie, $f^{-1}(C)$ is closed.

(2) $\Rightarrow$ (1) : Let $V \subset Y$ be open. Then $f^{-1}(Y \backslash V)$ is closed and $X \backslash (f^{-1}(Y \backslash V)) = X \setminus (f^{-1}(Y) \setminus f^{-1}(V)) = X \setminus (X \setminus f^{-1}(V)) = f^{-1}(V)$ is open.

$\square$

Proposition 10. Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$ a map. Then f is a continuous map on the metric topologies of X and Y if and only if for every $x \in X$ and every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $w \in X$ satisfying $d_X(x, w) < \delta$, the image of w under f satisfies $d_Y(f(x), f(w)) < \epsilon$.

Exercise 8. Prove the above proposition. (*Hint*: use characterization (4) of continuity and the standard basis of a metric space.)

Examples.

- For every topological space X, the identity map $\mathrm{id}_X : X \to X$ is continuous.
- By the above proposition, the continuous maps from $\mathbb{R}$ to $\mathbb{R}$ (with the standard topology) are precisely those familiar from calculus.

Definition 16. A continuous map $f : X \to Y$ between topological spaces is a *homeomorphism* (or *isomorphism of topological spaces*) iff there is a continuous map $g : Y \to X$ such that

$$g \circ f = \mathrm{id}_X \quad \text{and} \quad f \circ g = \mathrm{id}_Y .$$

Two topological spaces are called *homeomorphic* (or *isomorphic topological spaces*) iff there exists a homeomorphism between them.

Remark. A continuous map $f : X \to Y$ is a homeomorphism if and only if it is bijective and its inverse is also continuous.

Warning. A continuous bijection is not necessarily a homeomorphism. For example, let S^1 denote the subset of $\mathbb{R}^2$ of unit distance from the origin. Then the map

$$[0, 2\pi) \to S^1, \theta \mapsto (\cos\theta, \sin\theta)$$

is a continuous bijection whose inverse is not continuous. (Why not?)

Examples.

- $f : [0, 1] \to [0, 2], x \mapsto 2x$ is a homeomorphism
- $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a homeomorphism
- A circle is homeomorphic to a square.
- $[0, 1]$ and $[0, 1)$ and $(0, 1)$ are not homeomorphic. This will be easy to prove later.
- $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ when $n \neq m$. Why not?

Definition 17. A map $f : X \to Y$ between topological spaces is *sequentially continuous* if for every sequence $x : \mathbb{N} \to X$ and every limit L of x, $f(L)$ is a limit of $f \circ x : \mathbb{N} \to Y$.

Proposition 11. A continuous map $f : X \to Y$ is sequentially continuous. The converse holds if X is first countable.

Proof. We only prove the converse, assuming X is first countable, by proving that $f(\bar{A}) \subset \overline{f(A)}$ for every $A \subset X$. Let $x \in \bar{A}$. Since X is first countable, there is a sequence $x : \mathbb{N} \to A$ such that $x_n \to x$. Then because f is sequentially continuous, $f(x_n) \to f(x)$, so $f(x) \in \overline{f(A)}$ as required. □

Proposition 12.

1. Any constant map $f : X \to Y$ is continuous.

2. If $A \subset X$ is a subset with the subspace topology, the map $i : A \to X$ is continuous.

3. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the composite $g \circ f : X \to Z$ is continuous.

4. If $f : X \to Y$ is continuous and $A \subset X$, then the map $f|_A : A \to Y$ is continuous (where A has the subspace topology).

5. If $f : X \to Y$ is continuous and $f(X) \subset B \subset Y$ then $f : X \to B$ is continuous.

Proof.

1. The inverse image under a constant map is either $\emptyset$ or X, and so is always open.

2. Let $U \in X$ be open. Then $i^{-1}(U) = A \cap U$ is open in A by the definition of the subspace topology.

3. $(g \circ f)^{-1}(U) = g^{-1}\left(f^{-1}(U)\right)$ so that continuity of f and g guarantee that the inverse image of open sets under $g \circ f$ is also open.

4. $f|_A = f \circ i$ where $i : A \to X$ is the inclusion map so this follows from (2) and (3).

5. Let $U \subset B$ be open. Then $U = V \cap B$ for some $V \subset Y$ open. Then $f^{-1}(V)$ is open, and because $f(X) \subset B$, we know that $f^{-1}(V) = f^{-1}(V \cap B) = f^{-1}(U)$ as required.

□

Proposition 13. Let X be a topological space, $C \subset X$ a closed set, and $A \subset C$. Then A is closed in C if and only if A is closed in X.

Proof. (⇐:) If A is closed in X, then then $X \setminus A$ is open in X, and $(X \setminus A) \cap C = C \setminus A$ so the complement of A is open in C. (⇒:) If A is closed in C, then $C \setminus A = U \cap C$ for some $U \subset X$ open. This means that $(X \setminus U) \cap C = A$, so A is the intersection of two closed sets in X and therefore is itself closed. □

The following theorem confirms out intuition about continuous functions: that they are entirely determined "locally." First we provide a notion of locality.

Definition 18. Let X be a topological space. A collection of subsets of X, $(U_i)_{i\in I}$ is called a *cover* if $\cup_{i\in I} U_i = X$. A cover is called *open* (*closed*) if each U_i is open (resp. closed). It is called *countable* (*finite*) if the set I is a countable (resp. finite) set.

Theorem 3. *(Pasting lemma) Let $f : X \to Y$ be a map between topological spaces.*

1. *Let $(U_i)_{i\in I}$ be an open cover of X. If $f|_{U_i} : U_i \to Y$ is continuous for all $i \in I$ then f is continuous.*

2. *Let $(C_i)_{i\in I}$ be a* finite *closed cover of X. If $f|_{C_i} : C_i \to Y$ is continuous for all $i \in I$ then f is continuous.*

Proof.

1. Let $x \in X$ and V be a nbhd of $f(x)$. Then $x \in U_i$ for some $i \in I$. Since $f|_{U_i}$ is continuous, there exists a neighborhood U of x *in* U_i so that $f(U) \subset V$. Since U_i is open in X, U is also a neighborhood of x in X, and f is continuous.

2. Let $D \subset Y$ be closed. Then $f^{-1}(D) = \cup_{i\in I} f|_{C_i}^{-1}(D)$. Since each $f|_{C_i}^{-1}(D)$ is closed, and the set I is finite, the union is closed.

□