Notes on categories, the subspace topology and the product topology

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Contents

1 Introduction

These notes are intended to supplement sections 1 and 2 of May's *An outline summary of basic point set topology* [4].

2 A little category theory

Category theory, now an essential framework for much of modern mathematics, was born in topology in the 1940's with work of Samuel Eilenberg and Saunders MacLane

[1].

Definition 1. A category C consists of the following data:

- A class $\mathbf{ob}(C)$ called objects,
- For every two objects $X, Y \in \mathbf{ob}(C)$, there is a set $\mathbf{mor}(X, Y)$ called morphisms,
- For every three objects $X, Y, Z \in \mathbf{ob}(C)$, there is an operation $\circ : \mathbf{mor}(X, Y) \times \mathbf{mor}(Y, Z) \to \mathbf{mor}(X, Z)$ called composition.

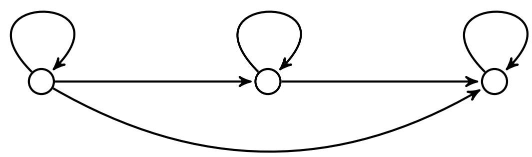
This data must satisfy the following two conditions:

- For every four objects $X, Y, Z, W \in \mathbf{ob}(C)$ and every $h \in \mathbf{mor}(X, Y)$, $g \in \mathbf{mor}(Y, Z)$, and $f \in \mathbf{mor}(Z, W)$ we have $f \circ (g \circ h) = (f \circ g) \circ h$.
- For every $X \in \mathbf{ob}(C)$ there exists a morphism $\mathrm{id}_X \in \mod (X, X)$ so that for all $Y \in \mathbf{ob}(C)$, and every $f \in \mathbf{mor}(X, Y)$ we have $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$.

Example 1. Some examples of categories are

- $Sets$: the objects are sets, the morphisms are functions, composition is composition of functions.
- $Vect_k$: the objects are vector spaces over a fixed field k, the morphisms are linear transformations, composition is composition of linear transformations.
- $RMod$: Fix a ring R. The objects are R-modules, the morphisms are R-module maps, and composition is composition of module maps.
- $Groups$: the objects are groups, the morphisms are group homomorphisms, composition is composition of homomorphisms.
- Let G be a group. Define a category C with one object $*$ and with $\mathbf{mor}(*, *) = G$ with composition defined as in the group G.
- $Sets_*$: the objects are pointed sets (sets together with a distinguished element) the morphisms are functions that respect the distinguished elements, composition is composition of functions.
- Top: the objects are topological spaces, the morphisms are continuous functions with composition being composition of functions.
- $hTop$. The objects are topological spaces, the morphisms are homotopy classes of continuous functions. If you don't know what homotopy is, don't worry about this example—it will be discussed in detail later.

- The graph below defines a category whose objects are the nodes, the morphisms are the arrows, and composition is defined in the only way it can be to satisfy the axioms.



Now, there are a few things to check to verify that the examples do define categories. For example, in $Sets$, $Vect_k$, $RMod$, $Groups$, $Sets_*$, and Top the associativity of composition is automatic since composition is defined as composition of functions, which is always associative. But, one needs to check that composition is defined. For example, for $Vect$, one has to check that if $f : V \to W$ and $g : W \to U$ are linear transformations between vector spaces, then $g \circ f : V \to U$, which a priori is only a function, is in fact a linear transformation. One also needs to check that for any vector space V, the identity function $\mathrm{id} : V \to V$ is a linear transformation.

Problem 1. Prove that Top and $hTop$ define categories.

Definition 2. Let X, Y be objects in any category. A morphism $f \in \mathbf{mor}(X, Y)$ is called an *isomorphism*, or an *equivalence*, if there exists a morphism $g \in \mathbf{mor}(Y, X)$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

The isomorphisms in the category Top are also called *homeomorphisms*. That is, if (X, τ_X) and (Y, τ_Y) are topological spaces, a function $f : X \to Y$ is called a homeomorphism if and only if f is continuous and f and has a continuous inverse $f^{-1} : Y \to X$. To have an inverse *set theoretically* means that f is bijective. In order for the inverse to be a morphism in the category Top, f^{-1} must be continuous. So, the definition of homeomorphism is often summarized as

Definition. $f : X \to Y$ is a homeomorphism iff f is continuous, f is bijective, and f^{-1} is continuous.

But I emphasize that the definition of a homeomorphism is determined categorically once the objects and morphisms in Top have been defined.

3 The subspace topology

The subspace topology is often defined as follows:

Definition 3. Let (X, τ_X) be a topological space and let Y be any subset of X. The *subspace topology* on Y is defined by $\{U \cap Y : U \in \tau_X\}$.

One checks that this definition does define a topology on Y. The properties that characterize the subspace topology are more important than the definition above. I'll give two characterizations of the subspace topology. The first one characterizes the subspace topology as the coarsest topology on Y for which the inclusion map $i : Y \to X$ is continuous. The second one is a universal property that characterizes the subspace topology on Y by characterizing which functions into Y are continuous. This is a good place to start understanding and working with universal properties.

3.1 First characterization of the subspace topology

In order to describe the first characterization, let me first illustrate a general fact. Let (X, τ_X) be a topological space and let S be any set whatsoever. Consider a function

$$f : S \to X.$$

It makes no sense to ask if f is continuous unless S is equipped with a topology. There do exist topologies on the set S that will make f is continuous, for instance the discrete topology will make f continuous. If τ_f is the intersection of all topologies on S for which f is continuous, then τ_f will be the coarsest (smallest) topology for which f is continuous. Note that τ_f has a simple explicit description as $\tau_f = \{f^{-1}(U) : U \subset X \text{ is open}\}$.

This leads to the following alternate definition of the subspace topology:

Alternate Definition. Let (X, τ_X) be a topological space and let Y be any subset of X. The *subspace topology* on Y is defined to be the coarsest topology on Y for which the canonical inclusion $i : Y \to X$ is continuous.

Since for any set $A \subset X$, $i^{-1}(A) = A \cap Y$, it is easy to see that this definition is equivalent to the first definition.

Remark 1. Let (X, τ_X) be a topological space, let S be any set, and let $f : S \to X$ be any function. Then the τ_f, the coursest topology on S that for which f is continuous, may be called *the subspace topology* on S. This is a good definition, even though the set S is not a subset of X. Here's why: since f is injective, S is isomorphic *as a set* to its image $f(S) \subset X$; and the set S with the subspace topology determined by the injection $f : S \to X$ is homeomorphic to the set $f(S) \subset X$ with the subspace topology determined by the inclusion $i : f(S) \subset X$.

If f is not injective, then the topology τ_f is not referred to as the subspace topology. Note that if f is not injective τ_f is quite different from the subspace topology on $f(S)$, for example, τ_f is never Hausdorff.

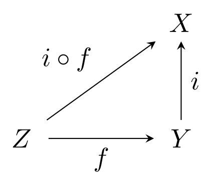
3.2 Second characterization of the subspace topology

There is a principle in mathematics that if you can understand the morphisms in a category, then you can understand the objects. Without making this principle more precise now, let me give an illustration. Suppose that you want to understand a topological space (X, τ_X). One approach is to study continuous functions $f : Z \to X$ or $f : X \to Z$, where (Z, τ_Z) is another topological space. Now, the subspace topology has an important universal property which characterizes precisely which functions $f : Z \to Y$ are continuous for all topological spaces (Z, τ_Z). This property completely determines the subspace topology on Y.

Theorem 1. *Let (X, τ_X) be a topological space, let Y be a subset of X and let $i : Y \to X$ be the natural inclusion. The subspace topology on Y is characterized by the following property:*

Universal property for the subspace topology. *For every topological space (Z, τ_Z) and every function $f : Z \to Y$, f is continuous if and only if $i \circ f : Z \to X$ is continuous.*

Here's a picture

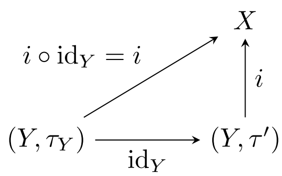


One should think of the universal property stated above as a property that may be attributed to a topology on Y. At this point, you may think that some topologies have this property and some do not. Theorem 1 means that the subspace topology on Y, as previously defined, does have this universal property. Furthermore, the subspace topology is the only topology on Y with this property. Let's prove it.

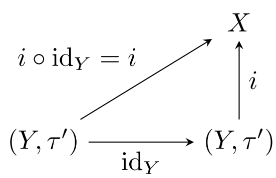
Proof. First, we prove that subspace topology on Y has the universal property. Then, we show that if Y is equipped with any topology having the universal property, then that topology must be the subspace topology.

Let τ_Y be the subspace topology on Y. Let (Z, τ_Z) be any topological space and let $f : Z \to Y$. We have to prove that $f : Z \to Y$ is continuous if and only if $i \circ f : Z \to X$ is continuous. Suppose f is continuous, then $i \circ f : Z \to X$ is continuous since the composition of continuous functions is continuous. Now suppose $i \circ f : Z \to X$ is continuous. Let U be any open set in Y. Then $U = i^{-1}(V)$ for some open $V \subset X$. Since $i \circ f$ is continuous, the set $(i \circ f)^{-1}(V) \subset Z$ is open in Z. Since $(i \circ f)^{-1}(V) = f^{-1}(U)$, we conclude that $f^{-1}(U)$ is open. This proves that $f : Z \to Y$ is continuous.

Now assume that τ' is a topology on Y and that τ' has the universal property. We have to prove that this topology τ' equals the subspace topology τ_Y. We are assuming that when Y has the topology τ', then for every topological space (Z, τ_Z) and for any function $f : Z \to Y$, f is continuous if and only if $i \circ f$ is continuous. In particular, if we let (Z, τ_Z) be (Y, τ_Y) where τ_Y is the subspace topology on Y, and let $f : Y \to Y$ be the identity function, then we have the following picture



Since we know the function $i \circ \mathrm{id}_Y = i : Y \to X$ is continuous when Y has the subspace topology τ, the universal property implies that $\mathrm{id}_Y : (Y, \tau_Y) \to (Y, \tau')$ is continuous. This implies that the subspace topology τ_Y is finer than τ'; i.e. $\tau' \subset \tau_Y$. To show that $\tau_Y \subset \tau'$, let (Z, τ_Z) be (Y, τ') and let $f = \mathrm{id}_Y : (Y, \tau') \to (Y, \tau')$. So we have the following picture



Since id_Y is continuous, we must have $i \circ \mathrm{id}_Y = i : Y \to X$ continuous. That is, τ' is a topology on Y for which the inclusion $i : Y \to X$ is continuous. Since the subspace topology τ_Y is the coarsest topology on Y for which $i : Y \to X$ is continuous, we conclude that τ_Y is coarser than τ'; i.e., $\tau_Y \subset \tau'$. The conclusion is that $\tau' = \tau_Y$. $\square$

Problem 2. Be sure to understand this argument.

4 The product topology

Let $\{X_\alpha\}_{\alpha \in A}$ be an arbitrary collection of topological spaces and let

$$X = \prod_{\alpha \in A} X_\alpha.$$

Recall that as a set

$$X = \{\text{functions } x : A \to \bigcup_{\alpha \in A} X_\alpha \text{ satisfying } x_\alpha := x(\alpha) \in X_\alpha \text{ for all } \alpha \in A\}.$$

So far, X is just a set, but we will soon define a topology on X.

Definition 4. Let $\{X_\alpha\}_{\alpha \in A}$ be an arbitrary collection of topological spaces and let $X = \prod_{\alpha \in A} X_\alpha$. The *product topology* on X is defined to be the topology generated by the basis

$$\left\{ \prod_{\alpha \in A} U_\alpha : U_\alpha \subset X_\alpha \text{ is open, and all but finitely many } U_\alpha = X_\alpha \right\}.$$

Now, I'd like to give two characterizations of this product topology. The first will be as the coarsest topology for which the projection maps are continuous. The second will be a universal property that characterizes the product topology in terms of which functions from X are continuous.

4.1 First characterization of the product topology

Observe that for each $\alpha \in A$, we have the natural projection

$$\begin{aligned} \pi_\alpha : X &\to X_\alpha \\ x &\mapsto x_\alpha \end{aligned}$$

Note that for an open set $U \subset X_\beta$, $\pi_\beta^{-1}(U) = \prod_{\alpha \in A} U_\alpha$ where $U_\alpha = X_\alpha$ for every $\alpha \neq \beta$ and $U_\beta = U$. That is, $\pi_\beta^{-1}(U)$ is just the product of all the X_α's in every place, except U is in the β place. First, note that each of these sets is open in the product topology, therefore each projection $\pi_\alpha : X \to X_\alpha$ is continuous. Second, note that that basic open sets described in the definition of the product topology consist of finite intersections of these sets $\{\pi_\alpha^{-1}(U) : U \subset X_\alpha \text{ is open }\}$.

Now, consider the coarsest topology on X for which all of the projections are continuous, let's call it τ. Then τ is the intersection of all topologies on X for which each projection $\pi_\alpha : X \to X_\alpha$ is continuous. Since the product topology is one such topology, the product topology is finer than τ. On the other hand, in any topology for which all of the projection maps are continuous, the sets $\{\pi_\alpha^{-1}(U) : U \subset X_\alpha \text{ is open }\}$ must be open. Since the finite intersection of open sets must be open and the product topology is generated by these intersections, the product topology must be coarser than τ. Therefore, we arrive at the following alternate definition of the product topology

Alternate Definition. Let $\{X_\alpha\}_{\alpha \in A}$ be an arbitrary collection of topological spaces and let $X = \prod_{\alpha \in A} X_\alpha$. The *product topology on* X is defined to be the coarsest topology on X for which all of the projections π_α are continuous.

4.2 Second characterization of the product topology

Let $\{X_\alpha\}_{\alpha \in A}$ be an arbitrary collection of topological spaces and let $X = \prod_{\alpha \in A} X_\alpha$. Now, if (Z, τ_Z) is any topological space, one can ask which kinds of functions $f : Z \to X$ are continuous. For example, if $X = \mathbb{R}^3$, one can write any function $f : Z \to X$ in terms of component functions $f = (f_1, f_2, f_3)$, where each $f_i : Z \to \mathbb{R}$. Then, one can check (you should do this—it's a good analysis exercise) that f is continuous if and only if each f_i is continuous. This is the general situation with the product topology: A function $f : Z \to X = \prod_{\alpha \in A} X_\alpha$ is continuous if and only if the component functions $f_\alpha : Z \to X_\alpha$ defined by

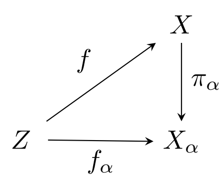
$$f_\alpha := \pi_\alpha \circ f$$

are all continuous. In fact, this situation describes a universal property that completely determines the product topology on X and could be used to give a third alternative definition of the product topology.

Theorem 2. *Let $\{X_\alpha\}_{\alpha \in A}$ be an arbitrary collection of topological spaces and let $X = \prod_{\alpha \in A} X_\alpha$. Let $\pi_\alpha : X \to X_\alpha$ denote the natural projection. The product topology on X is characterized by the following property.*

Universal property for the product topology. *For every topological space (Z, τ_Z) and every function $f : Z \to X$, f is continuous if and only if for every $\alpha \in A$, the component $f_\alpha : Z \to X_\alpha$ is continuous.*

Here is the picture:



Problem 3. Prove Theorem 2. That is, prove that $X = \prod_{\alpha \in A} X_\alpha$ with the subspace topology has this property. Then, prove that if X is equipped with any topology having this property, then that topology must be the product topology.

Problem 4. Are the subspace and product topologies are consistent with each other? Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of topological spaces and let $\{Y_\alpha\}$ be a collection of subsets; each $Y_\alpha \subset X_\alpha$. There are two things you can do to put a topology on $Y = \prod_{\alpha \in A} Y_\alpha$:

1. You can take the subspace topology on each Y_α, then form the product topology on Y.

2. You can take the product topology on X, view Y as a subset of X and equip it with the subspace topology.

Is the outcome the same either way?

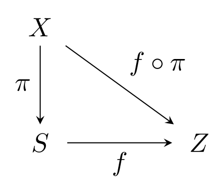
5 The quotient topology

Definition 5. Let X be a topological space, let S be a set, and let $\pi : X \to S$ be surjective. The quotient topology on S is defined to be the finest topology for which π is continuous. Equivalently, a set U in S is open in the quotient topology if and only if $\pi^{-1}(U)$ is open in X.

Remark 2. Set theoretically, if $f : X \to S$ is surjective then S is isomorphic to $X/\sim$ where $\sim$ is the equivalence relation defined by $x \sim y \Leftrightarrow \pi(x) = \pi(y)$. So, one can think of the quotient topology as being defined on the quotient of the set X by an equivalence relation. This situation of having a surjective $f : X \to S$ where the quotient topology may be defined on the set S or on the quotient $X/\sim$ is analagous to that of an injection $f : S \to X$ where the subspace topology may be defined on the set S, or on the subset $f(S) \subset X$.

Problem 5. Prove that the quotient topology on S is characterized by the following property:

Universal property for the quotient topology. *Let X be a topological space, let S be a set, and let $\pi : X \to S$ be surjective. For every topological space Z and every function $f : S \to Z$, f is continuous if and only if $f \circ \pi : X \to Z$ is continuous. Here is the picture:*



6 More Problems

6.1 Problems of categorical interest

Problem 6. For your reference, we state a theorem about sets and a definition about topological spaces. It's stated and proved on page 28 of [2], in Section 3 of [3], etc...

The Cantor-Schroeder-Bernstein Theorem. *Let X and Y be sets and let $f : X \to Y$ be injective and let $g : Y \to X$ be injective. Then there exists a bijection $h : X \to Y$.*

Definition 6. Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$. We call f *an embedding* if and only if $f' : X \to f(X)$ obtained by restricting the range of f is a homeomorphism.

(a) Prove (or read the proof) of the Cantor-Schroeder-Bernstein theorem.

(b) Show that there can be no such theorem for topological spaces. That is, give an example of two, non homeomorphic topological spaces X and Y, an embedding $f : X \to Y$, and an embedding $g : Y \to X$.

6.2 Problems about the product topology

Problem 7. Let X be a topological space.

(a) Prove that X is Hausdorff if and only if the diagonal

$$\Delta = \{(x, x) \in X \times X : x \in X\}$$

is closed in $X \times X$.

(b) Prove or disprove: X is Hausdorff if and only if

$$\{(x, x, x, x, x, \dots) \in X^{\mathbb{N}} : x \in X\}$$

is closed in $X^{\mathbb{N}}$.

(c) Let Y be a topological space and let $f : X \to Y$ be continuous. Prove or disprove: the graph of f

$$\{(x, y) \in X \times Y : f(x) = y\}$$

is closed.

Problem 8. Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces and let $X = \prod_{\alpha \in A} X_\alpha$ with the product topology. Prove that a sequence $\{f_n\}$ in X converges to $f \in X$ if and only if for every $\alpha \in A$, the sequence $\{f_n(\alpha)\}$ converges to $f(\alpha)$.

6.3 Problems about the quotient topology

Problem 9. Consider $\mathbb{R}^2$ with the usual topology. Define an equivalence relation on $\mathbb{R}^2$ by

$$(x, y) \sim (x', y') \Leftrightarrow xy = x'y'$$

and let $Y := \mathbb{R}^2 / \sim$ denote the set of equivalence classes.

(a) Prove that, as a set, Y is isomorphic to $\mathbb{R}$.

(b) Now consider Y as a topological space by equipping it with the quotient topology; i.e, the quotient topology induced by the natural projection

$$\pi : \mathbb{R}^2 \to \mathbb{R}^2 / \sim .$$

Is Y homeomorphic to $\mathbb{R}$?

Definition 7. Let X and Y be topological spaces. A function $f : X \to Y$ is called open (or closed) if and only if $f(U)$ is open (or closed) in Y whenever U is open (or closed) in X.

Problem 10. Let (X, τ_X) and (Y, τ_Y) be topological spaces and suppose $f : X \to Y$ is a continuous surjection.

(a) Give an example to show that f may be open but not closed.

(b) Give an example to show that f may be closed but not open.

(c) Prove that if f is either open or closed, then the topology τ_Y on Y is equal to τ_f, the quotient topology on Y.

Problem 11. Consider the closed disk D and the two sphere S^2:

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$
$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Consider the equivalence relation on D defined by identifying every point on the $\mathrm{bdry}(D)$. So each point in the $\mathrm{int}(D)$ is a one point equivalence class, and the entire $\mathrm{bdry}(D)$ is one equivalence class. Prove that the quotient $D/\sim$ with the quotient topology is homeomorphic to S^2.

References

[1] S. Eilenberg and S. MacLane. Relations between homology and homotopy groups of spaces. *Ann. of Math.*, 46:480–509, 1945.

[2] J. L. Kelley. *General Topology*. Van Nostrand, 1955.

[3] Seymour Lipschutz. *Schaum's Outline of Theory and Problems of General Topology*. McGraw-Hill, 1965.

[4] J.P. May. An outline summary of basic point set topology. http://www.math.uchicago.edu/ may/MISC/Topology.pdf.