Adjunctions, the Stone-Čech compactification, the compact-open topology, the theorems of Ascoli and Arzela

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Contents

1 Adjoint functors

We assume the reader is familiar with categories, functors, and natural transformations. As David Spivak writes in his book *Category Theory for Scientists* [4] (which was published today, coincidentally!)

> *...adjoint functors are like dictionaries that translate back and forth between different categories*

I'll begin with the definition. Then I'll discuss a few examples, and then I'll discuss the concept of a free group by way of introducing how universal properties give rise to adjunctions, an idea elaborated upon in an appendix. Then, the compact-open topology is discussed first in the context of an adjunction, and then later in detail (following Hatcher's book [1], pages 529-533.)

Definition 1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An *adjunction* between $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{C}$ together with an isomorphism

$$\phi_{X,Y} : \hom_{\mathcal{D}}(FX, Y) \xrightarrow{\simeq} \hom_{\mathcal{C}}(X, UY)$$

for each objects X in $\mathcal{C}$ and each Y in $\mathcal{D}$ that is natural in both components. The functor F is called the *left adjoint* and the functor U is called the *right adjoint*. The adjunction is often denoted succinctly by

$$F\colon \mathcal{C} \rightleftarrows \mathcal{D} :U.$$

To say what it means for ϕ to be "natural in both components", one should first recognize that for each object X in $\mathcal{C}$ there are two functors $\mathcal{D} \to \mathbf{Set}$

$$\hom_{\mathcal{D}}(FX, -) \text{ and } \hom_{\mathcal{C}}(X, U-)$$

and for each object Y in $\mathcal{D}$ there are two functors $\mathcal{C}^{op} \to \mathbf{Set}$

$$\hom_{\mathcal{D}}(F-, Y), \hom_{\mathcal{C}}(-, UY).$$

To say that the isomorphism

$$\phi_{XY} : \hom_{\mathcal{C}}(FX, Y) \xrightarrow{\simeq} \hom_{\mathcal{D}}(X, UY).$$

is "natural in both coordinates" means that the isomorphism ϕ_{XY} arises from natural transformations of functors

$$\hom_{\mathcal{D}}(FX, -) \xrightarrow{\simeq} \hom_{\mathcal{C}}(X, U-) \text{ and } \hom_{\mathcal{D}}(F-, Y) \xrightarrow{\simeq} \hom_{\mathcal{C}}(-, UY).$$

2 Example: Product-Hom adjunction in Set

For any sets X, Y, and Z, there is a natural bijection between the set of functions $\{X \times Z \to Y\}$ and $\{Z \to Y^X\}$.

Given a function of two variables $f : X \times Z \to Y$, one gets a function of the first variable only by fixing a value for the second variable. That is, for each $z \in Z$, define $f_z : X \to Y$ by $f_z(x) := f(x, z)$. In this way, one obtains the function $\hat{f} : Z \to Y^X$ by $\hat{f}(z) := f_z$.

$$Y^{X \times Z} \to \left(Y^X\right)^Z$$
$$f \mapsto \hat{f}$$

On the other hand, given $g : Z \to Y^X$, one can define a function $\bar{g} : X \times Z \to Y$ by $\bar{g}(x, z) = g(z)(x)$. Clearly $\bar{\hat{f}} = f$ and $\hat{\bar{g}} = g$ so both compositions

$$Y^{X \times Z} \xrightarrow{\hat{}} \left(Y^X\right)^Z \xrightarrow{\bar{}} Y^{X \times Z} \text{ and } \left(Y^X\right)^Z \xrightarrow{\bar{}} Y^{X \times Z} \xrightarrow{\hat{}} \left(Y^X\right)^Z$$

are the identity, proving (with a little overkill, perhaps) that $Y^{X \times Z} \xrightarrow{\simeq} \left(Y^X\right)^Z$.

This isomorphism of sets $Y^{X \times Z} \xrightarrow{\simeq} \left(Y^X\right)^Z$, arises from an adjunction where $X \times - : \mathbf{Set} \to \mathbf{Set}$ is a left adjoint of the functor $\hom(X, -) : \mathbf{Set} \to \mathbf{Set}$. To see it clearly, fix a set X and define two functors

$$F = X \times - : \mathbf{Set} \to \mathbf{Set} \text{ and } U = \hom(X, -) : \mathbf{Set} \to \mathbf{Set}.$$

Then

$$\hom_{\mathbf{Set}}(FZ, Y) = Y^{X \times Z} \simeq \left(Y^X\right)^Z = \hom_{\mathbf{Set}}(Z, UY)$$

The setup F: $\mathbf{Set} \rightleftarrows \mathbf{Set}$:U is an adjunction.

3 Example: The Stone-Čech compactification

Definition 2. A compactification of a topological space is an embedding of the space as a dense subspace of a compact Hausdorff space.

Note that a compactification of a space X is an *embedding* $i : X \to Y$ of X as a dense subspace of a compact Hausdorff space Y. In May's notes (Definition 5.14 in [2]) he says "inclusion" instead of "embedding". So, a compactification of X is a compact Hausdorff space Y and a continuous injection $i : X \to Y$ with $X \simeq i(X) \subset Y$ and $\overline{X} = Y$. Requiring that a compactification be an embedding means that examples such as the identity map of $[0, 1]$ with the discrete topology into $[0, 1]$ with the ordinary topology is not a compactification.

There's the "Alexandroff" one-point compactification of a Hausdorff space (Construction 5.15 and Proposition 5.16 in [2]). There is another compactification you should be aware of that has better categorical behavior. It can be summarized as follows:

Let $\mathbf{Top}'$ be the category with objects consisting of compact Hausdorff spaces and morphisms being continuous functions. There is an obvious functor $U : \mathbf{Top}' \to \mathbf{Top}$, which is just the inclusion of compact Hausdorff spaces as a subcategory of topological spaces; the functor U is just the identity on objects and morphisms. There is a functor $\beta : \mathbf{Top} \to \mathbf{Top}'$ that is a left adjoint to U called the "Stone-Čech compactification." The construction is outlined as Construction 6.11 in [2] (and there are more details in Section 38 of [3]) but let's unwind this functorial description and see what it means. To say that β is a left adjoint of U means that for every topological space X and every compact Hausdorff space Y, we have a bijection

$$\hom(\beta(X), Y) \simeq \hom(X, Y)$$

This says that for every continuous function $f : X \to Y$ there exists a continuous function $\hat{f} : \beta(X) \to Y$ where $\beta(X)$ is a compact Hausdorff space associated to X. This universal property is stated and proved as Proposition 6.12 in [2].

Notice that the one point compactification, call it X^*, of a locally compact Hausdorff space X, while convenient to define, doesn't have good properties with respect to morphisms. It doesn't come close to satisfying the condition that

$$\hom(X^*, Y) \simeq \hom(X, Y).$$

For a simple example, consider the inclusion $i : (0,1) \to [0,1]$ which cannot be extended to a continuous function from $(0,1)^* \simeq S^1$: there's no way to define $i(*)$ since if there were $i(*) = \lim_{x\to 0^+} x = 0$ and $i(*) = \lim_{x\to 1^-} x = 1$.

4 The unit and counit of an adjunction

Suppose $F\colon \mathcal{C} \rightleftarrows \mathcal{D} :U$ is an adjunction with adjunction isomorphism

$$\phi_{X,Y} : \hom_{\mathcal{D}}(FX, Y) \xrightarrow{\simeq} \hom_{\mathcal{C}}(X, UY)$$

By setting $Y = FX$, we have an isomorphism

$$\hom_{\mathcal{D}}(FX, FX) \xrightarrow{\simeq} \hom_{\mathcal{C}}(X, UFX).$$

Via the adjunction isomorphism, the morphism id_{FX} in the category $\mathcal{D}$ corresponds to a morphism $X \to UFX$ in the category $\mathcal{C}$, which defines a natural transformation

$$\eta : \mathrm{id}_{\mathcal{C}} \to UF$$

called the *unit* of the adjunction.

Similarly, for $X = UY$, under the isomorphism

$$\phi_{UY,Y} : \hom_{\mathcal{D}}(FUY, Y) \xrightarrow{\simeq} \hom_{\mathcal{C}}(UY, UY)$$

the morphism id_{UY} in $\mathcal{C}$ corresponds to a morphism $FUY \to Y$ which defines a natural transformation

$$\epsilon : FU \to \mathrm{id}_{\mathcal{D}}$$

called the *counit* of the adjunction.

4.1 The unit of the Stone-Čech compactification

The unit of the Stone-Čech compactification adjunction

$$\beta \colon \mathbf{Top}' \rightleftarrows \mathbf{Top} : U$$

determines a morphism $e := U\beta(\mathrm{id}_X) : X \to U\beta X$, and in **Top**, $U\beta X = \beta X$. The left adjoint of U doesn't just produce a compact Hausdorff space $\beta(X)$ from any topological space X, it also produces a continuous function $e : U\beta(\mathrm{id}_X) : X \to \beta(X)$. In good cases (i.e., X is *completely regular*) the map $e : X \to \beta X$ will be a compactification of X; that is, $e : X \to \beta(X)$ is an embedding and $\overline{X} = \beta(X)$. Furthermore, the fact that $e : X \to \beta(X)$ is a compactification of X determines the functor β in the sense that if β' is any other left adjoint to U for which $U\beta'(\mathrm{id}_X) : X \to \beta'(X)$ is a compactification for every completely regular X, then $\beta(X)$ and $\beta'(X)$ are naturally homeomorphic.

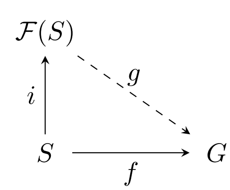
Problem 1. Describe the unit and counit of $X \times - \colon \mathbf{Set} \rightleftarrows \mathbf{Set} : \hom(X, -)$.

5 Free-Forgetful adjunction

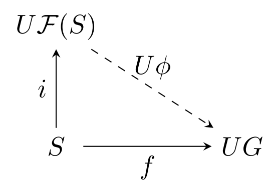
Often, a free group on a set S is defined as follows by the following universal property

> *A free group on a set S is a group $\mathcal{F}(S)$ together with a map of sets $i : S \to \mathcal{F}(S)$ satisfying the property that for any group G and any map of sets $f : S \to G$ there exists a unique group homomorphism $\phi : \mathcal{F}(S) \to G$ so that $i\phi = f$.*

It's difficult to understand the definition the first time. The picture helps:



But the picture is also confusing. After all, some objects in the picture are sets, some are groups, some arrows are set maps, some are group homomorphisms. One gets quite a bit of of clarification by observing that there is a forgetful functor $U : \mathbf{Group} \to \mathbf{Set}$. Then one may define a free group on a set S to be a group $\mathcal{F}(G)$ and a map $i : S \to U\mathcal{F}S$ with the property that for all groups G and maps $f : S \to UG$ there exists a unique map $\phi : \mathcal{F}(S) \to UG$ so that $f = U(\phi)i$. The right picture is in $\mathbf{Set}$:

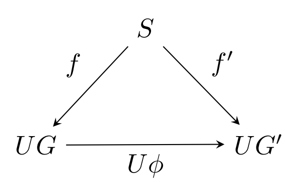


Notice that the "there exists" part of the definition of a free group says that for every group G, the map $\hom(S, UG) \to \hom(\mathcal{F}S, G)$ is surjective. The "unique" part of the definition says that $\hom(S, UG) \to \hom(\mathcal{F}S, G)$ is injective. The fact that

$$\hom(S, UG) \simeq \hom(\mathcal{F}S, G)$$

means that "free" and "forgetful" form an adjoint pair, the unit of which defines the inclusion $i : S \to UFS$.

Remark 1. The universal property defining a free group is best discussed within the context of set maps $S \to UG$ for all groups G. So, one could make a category out of this context. Let's call this category U^S. An object in U^S is a group G and a set map $f : S \to UG$. A morphism between two objects $S \xrightarrow{f} UG$ and $S \xrightarrow{f'} UG'$ is a group homomorphism $\phi : G \to G'$ so that $U\phi f = f'$. That is



Then,

> *A free group on a set S is an initial object in the category U^S.*

The context for the universal property is put into a category U^S that is built out of the undisguised material involved: the set S *and* the functor $U : \mathbf{Group} \to Set$. Then the universal object is a familiar notion (an initial object) from category theory.

6 The compact open topology: bird's eye view

If X and Y are topological spaces, the product topology makes Y^X into a topological space, but it does not use the topology on X. The first thing to do is to restrict attention to the subspace $\hom(X, Y) \subset Y^X$, this involves using the topology on X to look at a particular subset of Y^X. The next thing to do is to use the topology on X to put another topology, called *the compact-open topology*, on $\hom(X, Y)$ that has good functorial properties. A guiding principle for how to do it is to promote the Product-Hom adjunction from the category $\mathbf{Set}$ to the category $\mathbf{Top}$. That is, we'd like to put a topology on $\hom(X, Y)$ so that the functor

$$\hom(X, -) : \mathbf{Top} \to \mathbf{Top}$$

is a left adjoint to the functor

$$X \times - : \mathbf{Top} \to \mathbf{Top}.$$

With the compact-open topology, if X is locally compact and Hausdorff, we obtain a bijection of sets

$$\hom(X \times Y, Z) \simeq \hom(X, \hom(Y, Z)).$$

I prove this later in these notes (see Theorem 4).

Now, once we've put the compact-open topology on the sets $\hom(X, Y)$, the category $\mathbf{Top}$ has more structure than an ordinary category. The sets of morphisms aren't just sets, they're topological spaces. With this fact in mind, it's possible (as long as Z is Hausdorff) to promote the bijection of sets

$$\hom(X \times Y, Z) \simeq \hom(X, \hom(Y, Z)).$$

to a homeomorphism of topological spaces. I prove this later in these notes also (see Theorem 6).

That's the bird's eye-view of compact open topology. Now, some preliminaries before we move on to the frog's eye view.

7 Some useful facts about compact sets

Here are a few lemmas about compact sets. First, we note that compact sets in Hausdorff spaces are especially nice:

Lemma 1. *Let X be Hausdorff. For any point $x \in X$ and any compact set $K \subset X$ not containing x, there exist open sets U and V with $x \in U$ and $K \subset V$.*

Proof. Let $x \in X$ and let $K \subset X$ be compact. For each $y \in K$, there are open sets U_y and V_y with $x \in U_y$ and $y \in V_y$. The collection $\{V_y\}$ is an open cover of K, hence there is a finite subcover $\{V_1, \dots, V_n\}$. Let $U = U_1 \cap \cdots \cap U_n$ and $V = V_1 \cup \cdots \cup V_n$. Then U and V are disjoint open sets with $x \in U$ and $K \subset V$. $\square$

Definition 3. A topological space X is called *normal* if and only if X is T_1 and for every pair of disjoint closed set C and D there exist disjoint open sets U and V with $C \subset U$ and $D \subset V$.

Problem 2. Prove that every compact Hausdorff space is normal.

Lemma 2. *Let X be normal and $U \subset X$ be open. For every closed set C with $C \subset U$, there exists an open set V with $C \subset V \subset \overline{V} \subset U$.*

Proof. For any closed $C \subset U$, the sets C and $X \setminus U$ are closed so there exist disjoint open sets V and W with $C \subset V$ and $X \setminus U \subset W$. Then, $C \subset V \subset D \subset U$ where $D = X \setminus W$. Since $\overline{V}$ is the smallest closed set containing V, we have $\overline{V} \subset D \subset U$, giving the result. □

Problem 3. Let X be normal and suppose that $\{U_1, \ldots, U_n\}$ is an open cover of X. Then there exists a *shrinking* of this cover. That is, there exists another open cover $\{V_1, \ldots, V_n\}$ of X with $V_i \subset \overline{V_i} \subset U_i$.

Consider the product of two spaces $X \times Y$. By the definition of the topology on $X \times Y$, if $U \subset X \times Y$ is an open set and $(x, y) \in U$, then there are open sets $V \subset X$ and $W \subset Y$ with $(x, y) \subset V \times W \subset U$. Observe, however, that if we replace (x, y) with $A \times \{y\}$ for a set $A \subset X$, there need *not* be open sets V and W with $A \subset V$ and $y \in W$ with $A \times \{y\} \subset V \times W \subset U$.

Example 1. Take for example the open set

$$U := \{(x, y) \subset \mathbb{R}^2 : 0 < x < 1,\, 0 < y < x\}.$$

Here U is just the interior of the triangle with corners $(0, 0)$, $(1, 0)$, and $(1, 1)$. Consider the set $A \times \left\{\frac{1}{2}\right\}$ where A is the interval $A = \left(\frac{1}{2}, 1\right)$. Then $A \times \left(\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon\right)$ is not contained in U for any $\epsilon > 0$. But if A were compact,...

Lemma 3. *For any open set $U \subset X \times Y$ and any set $K \times \{y\} \subset U$ with $K \subset X$ compact, there exist open sets $V \subset X$ and $W \subset Y$ with $K \times \{y\} \subset V \times W \subset U$.*

Proof. For each point $(x, y) \in K \times \{y\}$, there is are open sets $V_x \subset X$ and $W_x \subset Y$ with $(x, y) \subset V_x \times W_x \subset U$. Then, $\{V_x\}_{x \in K}$ is an open cover of K; take a finite subcover $\{V_1, \ldots, V_n\}$. Then $V = V_1 \cup \cdots \cup V_n$ and $W = W_1 \cap \cdots \cap W_n$ are open sets with $K \times \{y\} \subset V \times W \subset U$. □

Consider a metric space X and an open subset U. If $x \in U$, then there is an $\epsilon > 0$ so that $B(x, \epsilon) \subset U$. This implies that for every $y \in X \setminus U$, $d(x, y) \geq \epsilon$. Now, if $A \subset U$, there may not exist an $\epsilon > 0$ so that $d(x, y) \geq \epsilon$ for every $x \in A$ and every $y \in X \setminus U$. The set $A \times \left\{\frac{1}{2}\right\} \subset U$ in Example 1 does not have distance greater than any epsilon from $\mathbb{R}^2 \setminus U$. But if A were compact,...

Lemma 4. *Let X be a metric space and let U be open. For every compact set $K \subset U$, there is an $\epsilon > 0$ so that for any $x \in K$ and any $y \in X \setminus U$, $d(x, y) > \epsilon$*

Proof. Exercise. □

8 The compact open topology: frog's eye view

Here's the definition

Definition 4. Let X and Y be topological spaces. For each compact set $K \subset X$ and each open set $U \subset Y$, define $S(K,U) = \{f \in \hom(X,Y) : f(K) \subset U\}$. The sets $S(K,U)$ form a sub-basis for a topology on $\hom(X,Y)$ called the compact-open topology.

As a matter of convention, unless declared otherwise, the topology on Y^X is the product topology, whereas the topology on $\hom(X,Y)$ is the compact-open topology.

Here's an idea of the difference between the product topology and the compact-open topology. Recall that a sequence of functions $\{f_n\}$ in $\hom([0,1],[0,1])$ converges to a limiting function f in the product topology if and only if $\{f_n\} \to f$ pointwise. The sequence $\{f_n\}$ converges to f in the compact-open if and only if $\{f_n\} \to f$ uniformly. This will be clear after considering the following, more general situation.

Suppose that X is compact and Y is a metric space. Then $\hom(X,Y)$ becomes a metric space (check this!) with the metric defined by

$$d(f,g) = \sup_{x \in X} d(f(x), g(x)).$$

Two functions $f, g \in \hom(X,Y)$ are close in this metric if their values $f(x)$ and $g(x)$ are close for all points $x \in X$. Note that a sequence $\{f_n\}$ in $\hom(X,Y)$ converges to f in this metric topoplogy if and only if for all $\epsilon > 0$ there exists an $n \in N$ so that for all $k > n$ and for all $x \in X$, $d(f_k(x), g_k(x)) < \epsilon$. Now, we have the following theorem:

Theorem 1. *Let X be compact and Y be a metric space. The compact-open topology on $\hom(X,Y)$ is the same as the metric topology.*

Proof. Let $f \hom(X,Y)$, $\epsilon > 0$, and consider $B(f,\epsilon)$. We'll find an set O, open in the compact open topology, with $f \in O \subset B(f,\epsilon)$. This will show that $B(f,\epsilon)$ is open in the compact open topology, and prove that the compact open topology is finer than the metric topology. Since X is compact, $f(X)$ is compact. The collection $\left\{B\left(y, \frac{\epsilon}{3}\right)\right\}_{y \in f(X)}$ is an open cover of $f(X)$, hence has a finite subcover

$$\left\{B\left(f(x_1), \frac{\epsilon}{3}\right), \ldots B\left(f(x_n), \frac{\epsilon}{3}\right)\right\}.$$

Define compact subsets $\{K_1, \ldots, K_n\}$ of X and open subsets $U_1, \ldots, U_n$ of Y by

$$K_i := \overline{f^{-1}\left(B\left(f(x_i), \frac{\epsilon}{3}\right)\right)} \text{ and } U_i := B\left(f(x_i), \frac{\epsilon}{2}\right).$$

Since f is continuous, $f(\overline{A}) \subset \overline{f(A)}$ for any set A. So,

$$f(K_i) \subset \overline{B\left(f(x_i), \frac{\epsilon}{3}\right)} \subset B\left(f(x_i), \frac{\epsilon}{2}\right) = U_i$$

for each $i = 1, \ldots, n$. Therefore, f is in the open set O where where

$$O := \cap_{i=1}^{n} S(K_i, U_i).$$

To see that $O \subset B(f, \epsilon)$, let $g \in O$. If $x \in K_i$ for some i, we have $f(x), g(x) \in U_i$ since $f, g \in S(K_i, U_i)$. Therefore,

$$d(f(x), g(x)) \leq d(f(x), f(x_i)) + d(f(x_i), g(x_i)) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since the balls $\left\{B\left(f(x_i), \frac{\epsilon}{3}\right)\right\}$ cover $f(X)$, the compact sets $\{K_i\}$ cover X and every point $x \in K_i$ for some i. Therefore, $d(f(x), g(x)) < \epsilon$ for every $x \in X$ and so $d(f, g) < \epsilon$ in $\hom(X, Y)$.

On the other hand, let $K \subset X$ be compact, $U \subset Y$ be open and consider $f \in S(K, U)$. From Lemma 4, we know there exists a fixed $\epsilon > 0$ so that for any $y \in f(K)$ and any $y' \in Y \setminus f(U)$, $d(y, y') \geq \epsilon$. Then, if $g \in B(f, \epsilon)$, we have $d(f(x), g(x)) < \epsilon$ for every $x \in X$. Therefore, if $x \in K$, $g(x) \in U$ and we see that $g(K) \subset U$. This proves $B(f, \epsilon) \subset S(K, U)$. If $O = S(K_1, U_1) \cap \cdots \cap S(K_n, U_n)$ is any basic open set in the compact-open topology, we have the open metric ball $B(f, \epsilon) \subset O$ where $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$. This proves that every basic open set in the compact-open topology is open in the metric topology and hence the metric-topology is finer than the compact-open topology. □

Problem 4. Compact sets can be compared to finite sets—finite sets are always compact and in many ways compact sets behave like finite sets. This idea provides a couple of ways to compare the compact-open topology to the product topology.

(a) Show that a sub-basis for the product topology on Y^X consists of sets $S(F, U) = \{f : X \to Y : f(F) \subset U\}$ where $F \subset X$ is finite and $U \subset Y$ is open. That is, the product topology could be referred to as the "finite-open" topology.

(b) If X is a set with the discrete topology and Y is any topological space then every function $f : X \to Y$ is continuous. So as sets $\hom(X, Y) = Y^X$. Prove or disprove: if X is finite, then the compact-open topology on $\hom(X, Y)$ and the product topology on Y^X are the same.

8.1 The product-hom adjunction in $\mathbf{Top}$

To get started, we use a map

$$\hom(X \times Z, Y) \to \hom(Z, \hom(X, Y)) \quad (1)$$

which, if it is defined, is simply the set-theoretic map $f \mapsto \hat{f}$. The question is if we start with a continuous f, is the function $\hat{f}$ continuous in $\hom(Z, \hom(X, Y))$. The answer, according to the next Theorem, is yes for any spaces X, Y, and Z.

Theorem 2. *For any topological spaces X, Y and Z, if $f : X \times Z \to Y$ is continuous then the set-theoretic adjoint $\hat{f} : Z \to \hom(X, Y)$ is continuous.*

Proof. Suppose $f : X \times Z \to Y$ is continuous. To show that $\hat{f}$ is continuous, consider a sub-basic open set $S(K, U)$ in $\hom(X, Y)$. We need to show that $\hat{f}^{-1}(S(K, U)) = \{z \in Z : f(K, z) \subset U\}$ is open in Z. Let $z \in \hat{f}^{-1}(S(K, U))$. So, $z \in Z$ and $f(K, z) \subset U$. Since f is continuous, we know that $f^{-1}(U) = \{(x, z) : f(x, z) \subset U\}$ is open in $X \times Z$ and contains $K \times \{z\}$. Therefore, there are open sets V and W with $K \subset V$ and $z \in W$ with $K \times \{z\} \subset V \times W \subset f^{-1}(U)$. Then, $z \in W \subset \hat{f}^{-1}(S(K, U))$ as needed. □

Theorem 2 proves the function in (1) is defined. Set theoretically, this map is injective. It turns out that the map in (1) may not be surjective (there might be continuous functions $g : \hom(Z) \to \hom(X, Y)$ for which the map $\bar{g} : X \times Z \to Y$ is not continuous. However, for locally compact Hausdorff spaces X this never happens. To prove it, we use a lemma. First, note that for any sets X and Y, there is a natural set-theoretic "evaluation function" $ev : Y^X \times X \to Y$ defined by

$$\begin{aligned} Y^X \times X &\to Y \\ (f, x) &\mapsto f(x) \end{aligned}$$

When X is locally compact and Hausdorff, this set-theoretic evaluation map is continuous.

Lemma 5. *If X is locally compact and Hausdorff, then the evaluation map ev : $\hom(X, Y) \times X \to Y$ is continuous.*

Proof. Let $(f, x) \in \hom(X, Y) \times X$ and let $U \subset Y$ be an open set containing $ev(f, x) = f(x)$. Then $f^{-1}(U)$ is an open set in X containing x. Since X is locally compact and Hausdorff, there exists an open set $V \subset X$ with $K := \overline{V}$ compact and $x \in V \subset K \subset f^{-1}(U)$. This implies that $f(x) \in f(K) \subset U$. Then $S(K, U) \times V$ is an open set in $\hom(X, Y) \times X$ with $(f, x) \in S(K, U) \times V$ and $ev(S(K, U) \times V) \subset U$. Thus the evaluation map ev is continuous at (f, x). □

Now we can prove surjectivity of (1).

Theorem 3. *Suppose that X is locally compact and Hausdorff and Y and Z are any topological spaces. Then a function $f : X \times Z \to Y$ is continuous only if the adjoint $\hat{f} : Z \to \hom(X, Y)$ is continuous.*

Proof. Consider the following diagram

$$\begin{array}{ccc} X \times Z & \xrightarrow{\text{id} \times \hat{f}} & X \times \hom(X, Y) & & \\ & & \downarrow{\scriptstyle s} & & \\ & & \hom(X, Y) \times X & \xrightarrow{ev} & Y \end{array}$$

Here $s : X \times \hom(X,Y) \to \hom(X,Y) \times X$ is the simple homeomorphism defined by $(x,f) \mapsto (f,x)$ and ev is the evaluation map, which is continuous if X is locally compact and Hausdorff. Therefore, if $\hat{f}$ is continuous, the composition $f = ev \circ s \circ (\mathrm{id} \times \hat{f})$ is continuous. $\square$

Therefore

Theorem 4. *If X is locally compact and Hausdorff, a function $f : X \times Z \to Y$ is continuous if and only if its adjoint $\hat{f} : Z \to Y^X$ is continuous.*

That is, if X is locally compact and Hausdorff and Y and Z are any topological spaces, there is a bijection (of sets)

$$\hom(X \times Z, Y) \simeq \hom(Z, \hom(X,Y))$$

8.2 Implications in homotopy theory

Theorems 2, 3, and 4 are fundamental in homotopy theory. Recall that two continuous function $f, g : X \to Y$ are homotopic provided there exists a continuous function $h : X \times [0,1] \to Y$ with $h(x,0) = f(x)$ and $h(x,1) = g(x)$. Note I is locally compact and Hausdorff. Theorem 2 says that the adjoint $\hat{h} : [0,1] \to \hom(X,Y)$ is continuous. The function $\hat{h}$ is therefore a path in the space of continuous functions connecting f to g. So, if f and g are homotopic, there is a path in the space $\hom(X,Y)$ connecting f and g. Theorem 3 says that if X is locally compact and Hausdorff, then if there is a path from f to g in $\hom(X,Y)$, then f and g are homotopic.

Corollary 1. *If X is locally compact and Hausdorff, then two functions $f, g : X \to Y$ are homotopic if and only if there is a path in $\hom(X,Y)$ connecting f to g.*

One might denote the functors $- \times I$ and $\hom(I, -)$ by $\mathcal{C}\dagger\ddagger$ and $\mathcal{P}$ respectively. So for a space X, $\mathcal{C}\dagger\ddagger(X)$ is the cylinder $X \times I$ and $\mathcal{P}(X)$ is the "path space" $\hom(I,X)$, the space of all paths in X. So, the cylinder-path adjunction $\mathcal{C}\dagger\ddagger$: $\mathbf{Top} \rightleftarrows \mathbf{Top}$:$\mathcal{P}$ is a special case of the product-hom adjunction in $\mathbf{Top}$ and says that $\hom(\mathcal{C}\dagger\ddagger(X), Y) \simeq \hom(X, \mathcal{P}(Y))$: the space of maps of the cylinder on X into a space Y is homeomorphic to the space of maps of X into the path space of Y.

Problem 5. Prove that the maps $d_0, d_1 : X \to X \times I$ defined by $d_0(x) = (x,0)$ and $d_1(x) = (x,1)$ and the projection $s : X \times I \to X$ are homotopy equivalences.

Theorem 6 in the next section goes further and says that as topological spaces $\hom(I, \hom(X,Y))$ and $\hom(X \times I), Y)$ are homeomorphic (when X is Hausdorff). So, one can, for example, identify a path in the space of paths with a homotopy between homotopies, etc...

8.3 Suspension-Loop adjunction

There's an important variation of the cylinder-path adjunction for pointed topological spaces. Before we get to pointed spaces, you should know about cones and suspension in $\mathbf{Top}$. The cone and suspension are both quotients of the cylinder $X \times I$. The cone is obtained by identifying one end to a point, and the suspension is obtained by identifying both ends to a point.

Definition 5. Let X be a topological space. The cone CX and suspension SX are defined to be

$$CX := X \times I / \sim \text{ where } (x, 0) \sim (x', 0).$$

and

$$SX = X \times I / \sim \text{ where } (x, 0) \sim (x', 0) \text{ and } (x, 1) \sim (x', 1).$$

Recall that the category of pointed topological spaces is the category $\mathbf{Top}_*$ whose objects are pairs (X, x) where X is a topological space and $x \in X$. A morphism $f \in \hom((X, x), (Y, y))$ is a continuous functions $f : X \to Y$ with $f(x) = y$. Sometimes it's convenient to drop the basepoint from the notation of a pointed space (X, x) and refer to it simply as X and just understand that every space has a basepoint. To emphasize that morphisms respect the basepoints, the notation $\hom_*(X, Y)$, or $\mathrm{Maps}_*(X, Y)$, is commonly used.

Definition 6. The reduced cone CX and reduced suspension ΣX of a pointed topological space X are defined to be the quotient

$$\Sigma X = X \times I / \sim \text{ where } (x, 1) \sim (x', 1) \text{ and } (x_1, t) \sim (x_1, s)$$

$$\Sigma X = X \times I / \sim \text{ where } (x, 0) \sim (x', 0) \text{ and } (x, 1) \sim (x', 1) \text{ and } (x_0, t) \sim (x_0, s)$$

where x_0 is the basepoint of X and the new basepoints of CX and ΣX is the class $\{x_0\} \times I$.

With the basepoint 0, the interval is a pointed space and with the basepoint 1 the circle S^1 is a pointed space. Two important mapping spaces are given names.

Definition 7. The mapping space $\mathcal{P}X = \hom_*(I, X)$ is called the *based path space* of X and $\Omega X = \hom_*(S^1, X)$ is called the *based loop space* of X.

The assignments $X \mapsto CX$, $X \mapsto \Sigma X$, $X \mapsto \mathcal{P}X$, and $X \mapsto \Omega X$ define functors $\mathbf{Top}_* \to \mathbf{Top}_*$ (work this out—you have to determine what happens to morphisms).

Finally, we can state the suspension-loop adjunction

Theorem 5. *The setup*

$$\Sigma \colon \mathbf{Top}_* \rightleftarrows \mathbf{Top}_* : \Omega$$

is an adjunction. Moreover, passing to (basepoint preserving) homotopy classes of morphisms, we obtain for every pair of pointed spaces X and Y.

$$[\Sigma X, Y] \simeq [X, \Omega Y].$$

8.4 Enrich the adjunction

Thus, if X is locally compact and Hausdorff, for any topological spaces Y and Z, the correspondence $f \leftrightarrow \hat{f}$ induces a natural isomorphism of sets

$$\hom(X \times Z, Y) \simeq \hom(Z, \hom(X, Y)) \tag{2}$$

In other words, the bijection between set-theoretic functions in $Y^{X \times Z}$ and $(Y^X)^Z$—when X is locally compact and Hausdorff and we use the compact-open topology—restricts to a bijection between the *continuous* functions $X \times Z \to Y$ and their *continuous* adjoints $Z \to Y^X$. Now, it is natrual to ask whether the bijection of sets in Equation (2) is compatible with the topologies on the two spaces. The answer to this question is, If Z is Hausdorff, then the map $\hom(X \times Z, Y) \to \hom(Z, \hom(X, Y))$ is a homeomorphism. To prove it, it will be convenient to use a smaller sub-basis for the compact-open topology that is available when the domain of a mapping space is Hausdorff—in this case, it turns out that the sets $\{S(K, U)\}$ where U only ranges over a sub-basis of the topology of the target space is still a sub-basis for the compact open topology.

Lemma 6. *If X is Hausdorff and $\mathcal{S}$ is a sub-basis for the topology on Y, then the sets $\{S(K, U) : K \subset X$ is compact and $U \in \mathcal{S}\}$ are a sub-basis for the compact-open topology on $\hom(X, Y)$*

Proof. Let $f \in S(K, U)$ for some open set $U \subset Y$. Write $U = \cup U_\alpha$ and each U_α as

$$U_\alpha = V_{\alpha,1} \cap \cdots \cap V_{\alpha,n_\alpha}.$$

The collection $\{f^{-1}(U_\alpha)\}$ is an open cover of K. Let $\{f^{-1}(U_1), \ldots, f^{-1}(U_k)\}$ be a finite subcover. Since K is compact and Hausdorff, it is normal. Therefore, (by Problem 3) there is a shrinking of the cover. Call the shrinking $\{V_1, \ldots, V_k\}$. For each $i = 1, \ldots, k$ let $K_i := \overline{V_i}$. Then $K_1, \ldots, K_k$ is a cover of K by compact sets with the property that $K_i \subset f^{-1}(U_i) \Rightarrow f(K_i) \subset U_i$. Then for each $i = 1, \ldots, k$,

$$f \in S(K_i, U_i) = S\left(K_i, \cap_{j=1}^{n_i} V_{i,j}\right) = \bigcap_{j=1}^{n_i} S(K_i, V_{i,j})$$

hence $f \in \bigcap_{i,j} S(K_i, V_{i,j})$. This proves the lemma. □

Theorem 6. *If X is locally compact and Hausdorff and Z is Hausdorff, then for any space X, the natural map*

$$\hom(X \times Z, Y) \to \hom(Z, \hom(X, Y))$$

is a homeomorphsim.

Proof. First, we show that the collection

$$\{S(A \times B, U) : A \subset X, B \subset Z \text{ are compact and } U \subset Y \text{ is open}\}$$

is a sub-basis for $\hom(X \times Z, Y))$. Let $f \in S(K, U)$ for some compact set $K \subset X \times Z$. Then $K \subset f^{-1}(U)$ and for each $p = (x, z) \in K$, we have open sets $V_p \subset X$ and $W_p \subset Z$ with $p \in V \times W \subset f^{-1}(U)$. By compactness, K is covered by finitely many $\{V_i \times W_i\}$. Now, let $K' = \pi_1(K)$ and $K'' = \pi_2(K)$ be the projections of K onto X and Z respectively. Since K' and K'' are compact and Hausdorff, they're normal. So, there are shrinkings $\{V_i'\}$ and $\{W_i'\}$ that cover K' and K'' respectively. Then $\overline{V_i'}$ and $\overline{W_i'}$ are compact sets in X and Z with $K \subset \cup_i \overline{V_i'} \times \overline{W_i'} \subset f^{-1}(U)$. Therefore, $f \in \cap_i S(\overline{V_i'} \times \overline{W_i'}, U)$.

Since $\mathcal{S} = \{S(A, U)\}$ is a sub-basis for the compact open topology on $\hom(X, Y)$ and Z is assumed to be Hausdorff, Lemma 6 says that

$$\{S(B, S(A, U)) : A \subset X, B \subset Z \text{ are compact and } U \subset Y \text{ is open}\}$$

is a sub-basis for $\hom(Z, \hom(X, Y))$. Therefore, the theorem follows from observing that the image of $(S(A \times B, U)) = S(B, S(A, U))$. $\square$

9 The theorems of Ascoli and Arzela

Now we would like to take up the question of when a set of functions $A \subset \hom(X, Y)$ is compact. It's not hard to decide when a family of functions is compact using the product topology:

Problem 6. Prove that if Y is Hausdorff, then a subset $A \subset Y^X$ is compact in the product topology if and only if A is closed in the product topology and for each $x \in X$, the set $A_x = \{f(x) \in Y : f \in A\}$ has compact closure in Y.

If one can determine families of functions for which the product topology and the compact-open topology coincide, then one has necessary and sufficient conditions for such families to be compact in the compact open topology. The following definition is important in making such a determination.

Definition 8. Let X be a topological space and (Y, d) be a metric space. A family $\mathcal{F} \subset \hom(X, Y)$ is called *equicontinuous at* $x \in X$ if and only if for every $\epsilon > 0$, there exists an open neighborhood U of x so that for every $y \in U$ and for every $f \in \mathcal{F}$, $d(f(x), f(y)) < \epsilon$. If $\mathcal{F}$ is equicontinuous for every $x \in X$, the family $\mathcal{F}$ is simply called *equicontinuous*.

The important property of equicontinuous families is that on equicontinuous families, the compact-open topology agrees with the product topology.

Lemma 7. *Let X be a topological space and (Y, d) be a metric space. If $\mathcal{F} \subset \hom(X, Y)$ is an equicontinuous family, then the compact-open topology agrees with the product topology.*

Proof. It suffices to show that if $\{f_\alpha\}$ is a net in $\mathcal{F}$ and $f_\alpha \to f$ pointwise, then $f_\alpha \to f$ in the compact open topology. And this is a good exercise. □

Now, another important property of equicontinuous families is that

Lemma 8. *If $\mathcal{F} \subset \hom(X, Y)$ is equicontinuous, then the closure of $\mathcal{F}$ in Y^X using the product topology is also equicontinuous.*

Proof. Let $\{f_\alpha\}$ be a net in $\mathcal{F}$ that converges to a function f... The rest is an exercise in the definition of equicontinuous. □

These two lemmas make it easy to prove Ascoli's theorem.

Ascoli's Theorem. *Let X be locally compact and Hausdorff, and let (Y, d) be a metric space. A family $\mathcal{F} \subset \hom(X, Y)$ has compact closure if and only if $\mathcal{F}$ is equicontinuous and for every $x \in X$, the set $F_x := \{f(x) : f \in \mathcal{F}\}$ has compact closure.*

Proof. Here we outline the proof. See section 47 of [3] for details. To prove the "if" part, let $\mathcal{G}$ be the closure of $\mathcal{F}$ in Y^X in the product topology. By Lemma 8, we know that $\mathcal{G}$ is equicontinuous. Then one shows that $\mathcal{G}$ is compact in Y^X with the product topology, which by Lemma 7 implies it's compact in the compact open topology. Then, $\overline{\mathcal{F}}$ is a closed set in the compact set $\mathcal{G}$, hence is compact.

To prove the "only if" part, one lets $\mathcal{G}$ be any compact set in $\hom(X, Y)$ containing the family $\mathcal{F}$. One can show that $\mathcal{G}$ (hence $\mathcal{F}$) is equicontinuous and that the sets $\mathcal{G}_x$ are compact (hence the sets $\mathcal{F}_x$ have compact closure). □

Problem 7. Prove

Arzela's Theorem. *Let X be compact, (Y, d) be a metric space, and $\{f_n\}$ be a sequence of functions in $\hom(X, Y)$. If $\{f_n\}$ is equicontinuous and for each $x \in X$, the set $\{f_n(x)\}$ is bounded, then $\{f_n\}$ has a subsequence that converges uniformly.*

A Universal properties and adjunctions (Optional)

A.1 Corepresentable functors

$\mathbf{Set}$ valued functors are important. In fact, they're given a name: a functor $\mathcal{C} \to \mathbf{Set}$ is called a *co-presheaf* on $\mathcal{C}$ and a functor $\mathcal{C}^{op} \to \mathbf{Set}$ is called a *presheaf* on $\mathcal{C}$. I think "co-presheaf" is a bit unwieldy with it's two prefixes, so I tend to avoid the terminology in these notes. But the sheaf terminology is common, so be aware of it. Also, the discussion that follows could be made using "presheaves" and "representable functors" instead of "co-presheaves" and "corepresentable functors."

For any category $\mathcal{C}$, and any object X of $\mathcal{C}$ there's a functor $\hom(X, -) : \mathcal{C} \to \mathbf{Set}$. This functor has been used repeatedly already, here's the definition:

- The object Y maps to the set $\hom_{\mathcal{C}}(X, Y)$
- The morphism $(Y \xrightarrow{\phi} Z)$ maps to the set-function $\left(\hom_{\mathcal{C}}(X, Y) \xrightarrow{\phi\circ} \hom_{\mathcal{C}}(X, Z)\right)$ defined by "post compose-with" ϕ.

I think the notation $\hom_{\mathcal{C}}(X, -)$ is the most descriptive and easiest to use notation for this functor, but other notation is also used. Grothendieck, for one, denoted it h^A.

Functors of the form $\hom_{\mathcal{C}}(X, -)$ are particularly co-presheaves on $\mathcal{C}$—after all, they are determined entirely by a single object X. This is more true than you might think: the Yoneda lemma says that the set of all natural transformations from any set-valued functor $F : \mathcal{C} \to \mathbf{Set}$ to $\hom_{\mathcal{C}}(X, -)$ corresponds naturally to the set $F(X)$. One might imagine, for example, that there are more natural tranformations between the functors $\hom_{\mathcal{C}}(X, -)$ and $\hom_{\mathcal{C}}(Y, -)$ than those induced from (precomposing with) morphisms $\hom(Y, X)$, but there aren't. Anyway, functors of the form $\hom_{\mathcal{C}}(X, -)$ are so nice, we have some terminology to describe when a set-valued functor is equivalent to one of them.

Definition 9. A functor $\mathbf{F} : \mathcal{C} \to \mathbf{Set}$ is corepresentable if and only if there exists an object $X \in \mathcal{C}$ and a natural equivalence $\hom(X, -) \xrightarrow{\sim} \mathbf{F}(-)$.

Consider a representable functor $\mathbf{F} : \mathcal{C} \to \mathbf{Set}$. If there is a natural equivalence $\hom(X, -) \xrightarrow{\sim} \mathbf{F}(-)$ for an object $X \in \mathcal{C}$, then in particular we have

$$\hom(X, X) \simeq \mathbf{F}(X).$$

Since there's a special morphism, namely id_X, in the set $\hom(X, X)$ there's a special element x in the set $\mathbf{F}(X)$ that corresponds to it. What's special about $x \in \mathbf{F}(X)$ is that the entire natural equivalence $\hom(X, -) \xrightarrow{\sim} \mathbf{F}(-)$ can be reconstructed from the element x. Here's how: given a morphism $\phi \in \hom_{\mathcal{C}}(X, Y)$, apply $\mathbf{F}(\phi) \in \hom(\mathbf{F}(X), \mathbf{F}(Y))$ to $x \in \mathbf{F}(X)$ to obtain an element $\mathbf{F}(\phi)(x) \in \mathbf{F}(Y)$. The assignment $\phi \mapsto \mathbf{F}(\phi)(x)$ defines a function $\hom_{\mathcal{C}}(X, Y) \to \mathbf{F}(Y)$ (check that it's a bijection!)

The terminology "a functor $\mathbf{F} : \mathcal{C} \to \mathbf{Set}$ is corepresentable" involves the existence of both an object X of $\mathcal{C}$ and a natural equivalence $\hom(X, -) \xrightarrow{\sim} \mathbf{F}(-)$. One might say the pair X and the equivalence $\hom(X, -) \xrightarrow{\sim} \mathbf{F}(-)$ "corepresents" the functor $\mathbf{F}$. But, which we just argued the natural equivalence is determined by an element $x \in \mathbf{F}(X)$, so its much more convenient to use the following terminology:

Definition 10. A pair (X, x) where X is an object of $\mathcal{C}$ and $x \in \mathbf{F}(X)$ corepresents a functor $\mathbf{F} : \mathcal{C} \to \mathbf{Set}$ if and only if for every $\phi \in \hom_{\mathcal{C}}(X, Y)$, the assignment $\phi \mapsto \mathbf{F}(\phi)(x)$ defines a natural equivalence $\hom(X, -) \xrightarrow{\sim} \mathbf{F}(-)$.

A.2 Free groups again.

Let's return to the context of set maps $f : S \to UG$. For a fixed set S, one can precompose the functor $\hom_{\mathbf{Set}}(S, -)$ with the forgetful functor $U : \mathbf{Group} \to \mathbf{Set}$

to get a functor $\hom_{\mathbf{Set}}(S, U-) : \mathbf{Group} \to \mathbf{Set}$. And naturally, for any fixed group H, there's a functor $\hom_{\mathbf{Group}}(H, -) : \mathbf{Group} \to \mathbf{Set}$.

Now, any set map $h : S \to UH$, determines a natural transformation from the functor $\hom_{\mathbf{Group}}(H, -)$ to the functor $\hom_{\mathbf{Set}}(S, U-)$ by post-composition with h:

$$\begin{array}{ccc} \hom_{\mathbf{Group}}(H, -) & \longrightarrow & \hom_{\mathbf{Set}}(S, U-) \\ H \stackrel{\phi}{\to} G & \longmapsto & S \stackrel{h}{\to} UH \stackrel{U(\phi)}{\to} UG \end{array}$$

Let's call the natural transformation η so we can refer to it

$$\eta : \hom_{\mathbf{Group}}(H, -) \longrightarrow \hom_{\mathbf{Set}}(S, U-)$$

but remember that it depends on the map $h : S \to UH$, and the whole setup depends on a fixed set S and a fixed group H.

Consider what it means for $\eta(G)$ to be surjective: it means that there exists a group homomorphism $\phi : H \to G$ so that $U(\phi) \circ h = f$. So $\eta(G)$ is surjective for all groups G if and only if the pair (H, h) satisfies the "there exists" part of the universal property of a free group. Similarly, $\eta(G)$ is for all groups G if and only if the pair (H, h) satisfies the "uniqueness" part of the universal property of a free group. Therefore,

> *The natural transformation η defines a natural equivalence if and only if the pair (H, h) is a free group on S.*

Or, in the terminology of the previous section,

> *The pair (H, h) corepresents the functor* $\hom(S, U-)$ *if and only if* (H, h) *is a free group on* S

Now, let us use the fact there is a construction for the free group on any set and consider the free groups on all different sets S together. For each S in $\mathbf{Set}$, we have a free group $(FS, i : S \to UFS)$ corepresenting the functor $\hom_{\mathbf{Set}}(S, U-) : \mathbf{Group} \to \mathbf{Set}$ establishing a bijection $\eta_S : \hom_{\mathbf{Group}}(FS, -) \stackrel{\sim}{\to} \hom_{\mathbf{Set}}(S, U-)$. This means that (F, U, η) is adjunction between $\mathbf{Set}$ and $\mathbf{Group}$.

A.3 Universal properties and adjunctions

The discussion above suggests the following quite general picture of adjunctions arising from universal properties. Consider a functor $\mathcal{C} \leftarrow \mathcal{D} : U$. Suppose that for each object X in $\mathcal{C}$, the functor $\hom_{\mathcal{C}}(X, U-) : \mathcal{D} \to \mathbf{Set}$ is corepresented by a pair $(FX \in \mathcal{D}, x \in \hom(X, UFX))$. This means that for every object $Y \in D$ and every morphism $f : X \to UY$ in $\mathcal{C}$ there exists a unique morphism $g : FX \to Y$ so that $U(g) \circ x = f$. This is a description of a universal property.

Let ϕ_X denote the bijection $\phi_X : \hom_{\mathcal{D}}(FX, -) \xrightarrow{\sim} \hom_C(X, U-)$. Thus, we have an adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : U,$$

assuming the assignment $X \to FX$ is naturally a functor so that

$$\hom_{\mathcal{D}}(F-, -) \xrightarrow{\phi_-} \hom_{\mathcal{C}}(-, U-)$$

is natural in both coordinates.

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