Practice problems

Monday May 2

Disclaimer: These problems are not meant to be a complete indicator of what to study. For example, I will ask about definitions on the exam and have not done so here.

Problem 1. Show that every compact subspace of a metric space is bounded in that metric and is closed. Find a metric space in which not every closed and bounded subspace is compact.

Problem 2. Let $f : X \to Y$ be a function, and Y be compact Hausdorff. Prove that f is continuous iff the *graph of* f,

$$G_f = \{x \times f(x) : x \in X\}$$

is closed in $X \times Y$.

Problem 3. Prove that if a space is compact, every infinite subset has a limit point.

Problem 4. Prove that if X is a metric space and every infinite subset has a limit point, then X is compact.

Problem 5. Show that if X is a Hausdorff space that is locally compact at $x \in X$, then for each neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is both a subset of U and compact.

Problem 6. Recall that a space X is called T_1 if for every point x of X, $\{x\} \subset X$ is a closed set ("points are closed"). Recall that X is called T_2 if it is Hausdorff. Show that $T_2 \Rightarrow T_1$, but not vice-versa.

Problem 7. Which of the following subsets of $\hom(\mathbb{R}, \mathbb{R})$ are pointwise bounded? Which are equicontinuous?

- $\{x + sin(nx)\}_{n\in\mathbb{N}}$
- $\{x + sin(x)\}_{n\in\mathbb{N}}$

- $\{|x|^{1/n})\}_{n\in\mathbb{N}}$

Problem 8. Recall that if X is locally compact, we have a bijection

$$\hom(X \times Y, Z) \cong \hom(Y, \hom(X, Z)).$$

Prove that if Y is Hausdorff, this bijection is a homeomorphism.