Zariski Topology

John Terilla

Fall 2014

1 The Zariski Topology

Let R be a ring (commutative, with 1). An ideal $p \subset R$ is called *prime* if $p \neq R$ and for all $xy \in p$, either $x \in p$ or $y \in p$. Let $\operatorname{spec}(R)$ denote the set of prime ideals of R. The *Zariski topology* on $\operatorname{spec}(R)$ is defining the sets

$$V(E) = \{p \in \operatorname{spec}(R) : E \subset p\}$$

for any $E \subset R$ to be closed.

1.1 A proof that the collection of $V(E)$ defines a topology

Let $E \subset R$ and let I be the ideal generated by E. Every ideal containing E contains I, therefore every prime ideal containing E contains I. So $V(E) = V(I)$. Therefore,

$$\{V(E) : E \subset p\} = \{V(I) : I \text{ is an ideal in } p\}$$

So, we need only look at sets $V(I)$ where I is an ideal in R. It's also helpful to observe that if $I \subset J$, then $V(J) \subset V(I)$.

The empty set and the entire space are closed. By definition, no prime ideal containes 1, so $V(1) = \emptyset$. Also, since 0 is in every ideal, $V(0) = \operatorname{spec}(R)$.

The union of two closed sets is closed. For any two ideals I and J, the product IJ is the ideal generated by products xy where $x \in I$ and $y \in J$. Note that $IJ \subset I$ and $IJ \subset J$, therefore $V(I) \cup V(J) \subset V(IJ)$.

Note also that $V(IJ) \subset V(I) \cup V(J)$, since if p is a prime ideal containing IJ, p contains either I or J (otherwise, you'd have an element $xy \in IJ \subset p$ with $x \notin p$ and $y \notin p$).

Thus $V(I) \cup V(J) = V(IJ)$, a closed set.

The intersection of closed sets is closed. For any collection of ideals I_α, there is a smallest ideal, denoted by $\sum I_\alpha$ containing all the I_α. (There's a construction of $\sum I_\alpha$ using sums, but it's not necessary to use the construction.)

Since $\sum I_\alpha$ contains each I_α, it follows that $V(\sum I_\alpha) \subset V(I_\alpha)$ for all α, so $V(\sum I_\alpha) \subset \cap V(I_\alpha)$. On the other hand, if p is a prime ideal containing all I_α, then p must contain $\sum I_\alpha$ since $\sum I_\alpha$ is the smallest ideal containing all the I_α. So $\cap V(I_\alpha) \subset V(\sum I_\alpha)$.

The conclusion is that $\cap V(I_\alpha) = V(\sum I_\alpha)$

1.2 Examples

A picture of $\mathrm{spec}(\mathbb{Z})$**.** The prime ideals of $\mathbb{Z}$ are the principal ideals (p) generated by prime numbers p together with the ideal $(0) = \{0\}$ (note that (0) is prime since $\mathbb{Z}$ has no zero divisors). So, here are the points of $\mathrm{spec}(\mathbb{Z})$:

$$\mathrm{spec}(\mathbb{Z}) = \{(0), (2), (3), (5), (7), (11), \ldots\}.$$

To describe the topology on $\mathrm{spec}(\mathbb{Z})$ note that the closure of any point is the set of prime ideals containing that point. So, for each prime number p, the point $(p) \in \mathrm{spec}(\mathbb{Z})$ is closed since $(p) = V(p)$. The point (0), however, is not closed; in fact, $\overline{(0)} = \mathrm{spec}(\mathbb{Z})$.

The only infinite set that is closed is the whole space. To see this, consider a closed set $V(E)$ with $E \subset \mathbb{Z}$ and suppose $V(E)$ is infinite. If $n \in E$ then $n \in (p)$ for infinitely many primes p. If $n \neq 0$, this means that $p|n$ for infinitely many primes, which is impossible. Therefore $n = 0$ and $V(E) = \mathrm{spec}(\mathbb{Z})$.

So, the topology described is similar to the cofinite topology on the set of prime numbers, except that $\mathrm{spec}(\mathbb{Z})$ has another point (0) whose closure is the whole space.

A picture of $\mathrm{spec}(\mathbb{C}[x])$**.** Since $\mathbb{C}$ is a field, the ring $\mathbb{C}[x]$ is a principal ideal domain. So every ideal of $\mathbb{C}[x]$ is generated by a single polynomial. Because polynomials with complex coefficients factor into linear factors, one can see that if $\deg(f) \geq 2$ then (f) is not prime. So, if an ideal is prime, it has the form (f) where $\deg(f) \leq 1$. Let's look at the case that the $\deg(f) = 0$. The zero ideal (0) is prime, since $\mathbb{C}[x]$ has no zero divisors but if $a \in \mathbb{C}$ is nonzero, $(a) = \mathbb{C}[x]$ isn't prime. If $\deg(f) = 1$, then $(f) = (x - a)$ for some $a \in \mathbb{C}$, which is a maximal ideal (the quotient $\mathbb{C}[x]/(f)$ is a field) hence prime. Thus the points of $\mathrm{spec}(\mathbb{C}[x])$ are the ideals $(x - a)$, where $a \in \mathbb{C}$, together with the zero ideal (0).

The picture of $\mathrm{spec}(\mathbb{C}[x])$ is similar to that of $\mathrm{spec}(\mathbb{Z})$. It looks like the cofinite topology on the complex numbers $\mathbb{C}$, together with a special point (0) (not to be confused with the point $(x - 0)$!) whose closure is the whole space.

A challenge. For an interesting challenge, describe a picture of $\operatorname{spec}(\mathbb{Z}[x])$. There's a famous picture of $\operatorname{spec}(Z[x])$ in Mumford's "red book" [2] and an interesting discussion of his picture available

`www.neverendingbooks.org/index.php/mumfords-treasure-map.html`

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References

[1] M. Atiyah and I. MacDonald. *Introduction to Commutative Algebra*. Addison-Wesley, 1969.

[2] D. Mumford. *The Red Book of Varieties and Schemes*. Springer, 1999.