A free boundary problem for the localization of eigenfunctions

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Abstract. We study a variant of the Alt, Caffarelli, and Friedman free boundary problem, with many phases and a slightly different volume term, which we originally designed to guess the localization of eigenfunctions of a Schrödinger operator in a domain. We prove Lipschitz bounds for the functions and some nondegeneracy and regularity properties for the domains.

Résumé en Français. On étudie une variante du problème de frontière libre de Alt, Caffarelli, et Friedman, avec plusieurs phases et un terme de volume légèrement différent, que l'on a choisie pour deviner la localisation des fonctions propres d'un opérateur de Schrödinger dans un domaine. On démontre des estimations Lipschitziennes pour les fonctions associées à un minimiseur, et des propriétés de nondégérescence et de régularité pour les frontières libres.

Key words/Mots clés. Free boundary problem ; localization of eigenfunctions.

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Contents

1 Introduction

The initial motivation for this paper was to describe the localization of eigenfunctions for an operator $\mathcal{L}$ on a domain $\Omega \subset \mathbb{R}^n$. Let us assume that $|\Omega|$, the measure of Ω, is finite. The typical operator that we consider is the positive Laplacian $\mathcal{L} = -\Delta$, or a Schrödinger operator $\mathcal{L} = -\Delta + \mathcal{V}$, with $\mathcal{V}$ bounded and nonnegative.

In [FM], a pointwise estimate for eigenfunctions for $\mathcal{L}$ is found, which bounds them in terms of a single function w_0, namely the solution of $\mathcal{L}w_0 = 1$ on Ω, with the Dirichlet condition $w_0 = 0$ on $\mathbb{R}^n \setminus \Omega$. Our first goal is to derive an automatic way, using w_0, to find subdomains W_j, $1 \leq j \leq N$, of Ω, where the eigenfunctions of $\mathcal{L}$ are more likely to be supported. The work in [FM] indicates that, roughly speaking, one seeks a collection of disjoint $W_j \subset \Omega$, $1 \leq j \leq N$, such that w_0 is small on the boundaries of the W_j, and it is natural to try to measure "smallness" in terms of the operator $\mathcal{L}$ itself. Even though many handmade or numerical decompositions of Ω based on w_0 seem to give very good predictions of the localization of eigenfunctions, we would like to have a more systematic way to realize the decomposition.

The functional described below is intended to give such a good partition of Ω into subdomains, and it turns out to be an interesting variant of functionals introduced by Alt and Caffarelli [AC], and studied by many others. In the present paper, we shall mainly study the theoretical properties of our functional (existence and regularity of the minimizers and regularity of the corresponding free boundaries).

Let us now describe the main free boundary problem that we shall study here; the relation with w_0 and our original localization problem will be explained more in Section 2.

We are given a domain $\Omega \subset \mathbb{R}^n$, and (for instance) an operator $\mathcal{L} = -\Delta + \mathcal{V}$; assumptions on the potential $\mathcal{V}$, or other functions associated to a similar problem, will come later. We are also given an integer $N \geq 1$, and we want to cut Ω into subregions W_i, $1 \leq i \leq N$, according to the geometry associated to $\mathcal{L}$. For this, we want to define and minimize a functional J. But let us first define the set of admissible pairs $(\mathbf{u}, W)$ for which $J(\mathbf{u}, W)$ is defined.

Definition 1.1 *Given the open set $\Omega \subset \mathbb{R}^n$ and the integer $N \geq 1$, we denote by $\mathcal{F} = \mathcal{F}(\Omega)$ the set of admissible pairs $(\mathbf{u}, \mathbf{W})$, where $\mathbf{W} = (W_i)_{1 \leq i \leq N}$ is a N-uple of pairwise disjoint Borel-measurable sets $W_i \subset \Omega$, and $\mathbf{u} = (u_i)_{1 \leq i \leq N}$ is a N-uple of real-valued functions u_i, such that*

$$u_i \in W^{1,2}(\mathbb{R}^n) \tag{1.1}$$

and

$$u_i(x) = 0 \text{ for almost every } x \in \mathbb{R}^n \setminus W_i. \tag{1.2}$$

Here $W^{1,p}(\mathbb{R}^n)$, $1 \leq p < +\infty$, denotes the set of functions $f \in L^p_{loc}(\mathbb{R}^n)$ whose derivative, computed in the distribution sense, lies in $L^p(\mathbb{R}^n)$. We chose a definition for which we do not need to assume any regularity for the sets W_i, nor to give a precise meaning to the Sobolev space $W^{1,2}_0(W_i)$, but under mild assumptions, the functions u_i associated to minimizers will be continuous, and we will be able to take $W_i = \big\{x \in \Omega\,;\, u_i(x) > 0\big\}$. For the moment we took real-valued functions, but what we will say will systematically apply when some of the functions u_i are required to be nonnegative. In addition, some of our results will only work under this constraint (that $u_i \geq 0$).

Our functional J will have three main terms. The first one is the energy

$$E(\mathbf{u}) = \sum_{i=1}^{N} \int |\nabla u_i(x)|^2 dx, \tag{1.3}$$

where we denoted by ∇u_i the distributional gradient of u_i, which is an L^2 function. It does not matter whether we integrate on $\mathbb{R}^n$, Ω, or W_i, because one can check that if $u_i \in W^{1,2}(\mathbb{R}^n)$ vanishes almost everywhere on $\mathbb{R}^n \setminus W_i$, then $\nabla u_i = 0$ almost everywhere on $\mathbb{R}^n \setminus W_i$. Indeed, by the Rademacher-Calderón theorem, u_i is differentiable at almost every point of $\mathbb{R}^n$, with a differential that coincides almost everywhere with the distribution Du_i. It is then easy to check that $Du_i(x) = 0$ when x is a point of Lebesgue differentiability of $\mathbb{R}^n \setminus W_i$, hence, for almost every $\mathbb{R}^n \setminus W_i$.

The second term of our functional will be

$$M(\mathbf{u}) = \sum_{i=1}^{N} \int [u_i(x)^2 f_i(x) - u_i(x) g_i(x)] dx, \tag{1.4}$$

where the f_i and the g_i, $1 \leq i \leq N$, are given functions on Ω that we may choose, depending on our problem. We are slightly abusing notation here, because $M(\mathbf{u})$ also depends on the W_i through the choice of functions we integrate against u_i^2 or u_i, at least if the f_i and g_i depend on i. The convergence of the integrals in (1.4) will follow from our assumptions on the f_i and the g_i. For our initial localization problem, all the f_i will be equal to the potential $\mathcal{V}$, and $g_i = 2$ for all i.

We put a negative sign before $u_i(x)g_i(x)$ so that the f_i and g_i will be nonnegative in the most interesting cases, but we won't always need both assumptions. When this happens, the functions u_i will also naturally be nonnegative, which will allow some specific arguments.

The last term of the functional is a function $F(\mathbf{W})$ that depends on the sets W_i. We do not need to be too specific for the moment, but our main example is to take a continuous function of the Lebesgue measures $|W_i|$ of the W_i. Thus our functional is

$$\begin{aligned} J(\mathbf{u}, \mathbf{W}) &= E(\mathbf{u}) + M(\mathbf{u}) + F(\mathbf{W}) \\ &= \sum_{i=1}^{N} \int |\nabla u_i(x)|^2 dx + \sum_{i=1}^{N} \int [u_i(x)^2 f_i(x) - u_i(x) g_i(x)] dx + F(W_1, \dots W_N). \end{aligned} \tag{1.5}$$

We add the term $F(\mathbf{W})$ to the functional to avoid some trivial solutions (typically, where $W_1 = \Omega$ and all the other W_i are empty). For some of our main results we shall put some monotonicity assumptions on F, but for the moment let us merely say that a typical choice would be

$$F(\mathbf{W}) = \sum_{i=1}^{N} a|W_i| + b|W_i|^{1+\alpha} \tag{1.6}$$

for some $\alpha > 0$ and suitable positive constants a and b, where the second term tends to make us choose roughly equal volumes when we minimize the functional (see Section 13) and the linear one ensures a certain non-degeneracy (see, e.g., a discussion after (2.9)).

The reader may also think about the case when we choose nonnegative bounded functions q_i, $1 \leq i \leq N$, and we set

$$F(\mathbf{W}) = \sum_{i=1}^{N} \int_{W_i} q_i \tag{1.7}$$

which is the typical choice that people use for the functionals of [AC] and [ACF].

We shall first prove that if the Lebesgue measure of Ω is finite, the f_i are bounded and nonnegative, and the g_i lie in L^2, there exist minimizers for J in the class $\mathcal{F}$. See Section 3.

But the main goal of the paper is to consider a minimizer $(\mathbf{u}, \mathbf{W})$ for the functional and establish, under slightly stronger assumptions, some regularity results for u (for instance, Lipschitz bounds), and even the sets W_i (all the way up to C^1-regularity almost everywhere when we are very lucky). We demonstrate that all subregions satisfying a certain non-degeneracy condition have locally Ahlfors-regular and uniformly rectifiable boundaries. Moreover, if, for instance, all involved indices are non-degenerate, any point of the free boundary can belong to at most two subregions, that is, the problem locally reduces to one or two phases. Under further additional conditions we also manage to show that these subregions are asymptotically flat (almost everywhere when dimension is larger than 3). Please see the details below.

Maybe we should say right away that although many of our results also hold when $n = 1$ (but may not be interesting), we shall concentrate our attention on $n \geq 2$ and can't even promise that all our statements will make sense when $n = 1$.

Variants of our functional J have been studied extensively, starting with the very important papers [AC] and [ACF], where the authors wanted to study the regularity of the boundary of the positive set $\{u > 0\}$ of some PDE solution; see also [CJK] for further results on these free boundary problems. Similar problems were also raised, for instance to study optimal shapes; see [HP], [BV].

There is an interesting difference between the present context and the situation of the aforementioned papers, which is that we want to allow a number $N \geq 3$ of domains. When $N = 2$ and the u_i are nonnegative, as in most of the aforementioned papers, we can regroup u_1 and u_2 into one single real-valued function $u_1 - u_2$, and this is very convenient to produce competitors for u, for instance by taking the harmonic extension of the restriction of $u_1 - u_2$ to a sphere. This trick will not be available here, and will force us to be slightly more imaginative in the construction of our competitors. See our description of harmonic competitors in Section 6 for a more detailed discussion. Nonetheless, we shall still be able to use a monotonicity formula, which was first formulated in [ACF], and which is a major tool in the aforementioned papers. This will allow us to prove that u is Lipschitz, for instance, and try to follow the same route as in these papers to prove some regularity results for the W_i.

Let us describe the plan of this paper and the results that it contains. In Section 2 we try to explain why we introduced the functional J above, what choices of the parameters seem more interesting to us, and what the regularity results may mean in the context of localization of eigenfunctions.

Recall that the existence of minimizers is proved in Section 3, in a quite general setting; the main point of the proof is the fact that, by Poincaré's inequalities, the energy term $E(\mathbf{u})$ controls the second term $M(\mathbf{u})$. When $F(\mathbf{W})$ is a continuous function of the volumes $|W_i|$, as in Theorem 3.1, we extract from a minimizing sequence a subsequence for which these volumes converge, and we get a minimizer rather easily (without having to make the sets W_i themselves converge); the proof also works when $F(\mathbf{W})$ is a continuous and nondecreasing function of the W_i, (hence, for instance, when F is as in (1.7) with nonnegative integrable functions q_i). See Corollary 3.5.

After a short Section 4 where we derive simple consequences of the Poincaré inequalities, we check in Section 5 that if $|\Omega| < +\infty$, the f_i are nonnegative, and the f_i and the g_i lie in $L^p(\Omega)$ for some $p > n/2$, and $(\mathbf{u}, \mathbf{W})$ is a minimizer for J in the class $\mathcal{F}$, then the u_i are bounded. The proof uses simple bounds on the fundamental solution of $-\Delta$. We put his intermediate result here because it makes it easier to estimate various error terms later.

Many of our subsequent estimates will be obtained by comparing $(\mathbf{u}, \mathbf{W})$ with two competitors that we introduce in Section 6. The first one is simply obtained by multiplying some of the u_i by a cut-off function that vanishes in a ball; the interest is that we may save on the energy or volume terms. The second one is our substitute for the harmonic extension. We want to define a smooth competitor with the same values of $\mathbf{u}$ on a sphere, and since we cannot extend harmonically each component u_i, we cut them off as above, except one for which we can use a harmonic extension because we just created some space.

The competitors of Section 6 are used in Section 7 to prove that the u_i are locally Hölder-continuous on Ω if in addition to the assumptions of Section 5, F is a Hölder-continuous function of the W_i, with an exponent $\beta > \frac{n-2}{n}$. Here and below, the distance between $\mathbf{W}$ and $\mathbf{W}'$ is defined as the sum of the measures of the symmetric differences $W_i \Delta W_i'$; see (7.1). The proof is arranged like Bonnet's monotonicity argument for the Mumford-Shah functional, but the estimates are not sharp and we only get a very small Hölder exponent.

We show in Section 8 that if Ω is smooth (but in fact much less is needed), $\mathbf{u}$ is also Hölder-continuous all the way to the boundary $\partial\Omega$.

Then we turn to the monotonicity formula. From now on, let us assume that the f_i and the g_i are bounded, and that the f_i are nonnegative. This formula concerns products of two functions Φ_φ that are defined as follows. Choose an origin x_0, and denote by I the set of pairs $\varphi = (i, \varepsilon)$, where $i \in [1, N]$ and $\varepsilon \in \{-1, 1\}$. To each $\varphi \in I$ we associate the function $v_\varphi = (\varepsilon u_i)_+ = \max(0, \varepsilon u_i)$ (we shall often call v_φ a phase of $\mathbf{u}$) and the function Φ_φ defined by

$$\Phi_\varphi(r) = \frac{1}{r^2} \int_{B(x_0,r)} \frac{|\nabla v_\varphi|^2}{|x - x_0|^{n-2}} \, dx \tag{1.8}$$

for $r > 0$. For $\varphi \neq \psi \in I$, set $\Phi_{\varphi,\psi}(r) = \Phi_\varphi(r)\Phi_\psi(r)$. This is just a minor variant of the functional introduced by Alt, Caffarelli, and Friedman in [ACF], and we show that $\Phi_{\varphi,\psi}$ is nearly nondecreasing near the origin when $(\mathbf{u}, \mathbf{W})$ minimizes J in $\mathcal{F}$, at least if we assume that $|\Omega| < +\infty$, and F is Hölder-continuous as in (7.1). See Theorem 9.1 for a precise statement. The proof consists in checking that the functions v_φ and v_ψ satisfy the assumptions of a near monotonicity formula that was established in [CJK], and for this the Hölder estimates of Section 7 are useful. This result is also valid when x_0 lies on $\partial\Omega$ or close to $\partial\Omega$, if Ω is smooth (as in (8.1)), and then we use the results of Section 8 to check the assumptions of [CJK].

Once we have a control on the functionals Φ, we can use a bootstrap argument to show that $\mathbf{u}$ is locally Lipschitz. For local Lipschitz bounds in Ω, we just need to assume that (in addition to the previous assumptions) F is a Lipschitz function of the W_i (as in (10.2)); See Theorem 10.1.

If in addition Ω is bounded and has a $C^{1+\alpha}$ boundary for some $\alpha > 0$, we show in Section 11 that $\mathbf{u}$ is also Lipschitz near $\partial\Omega$.

Notice that in general, we do not expect the sets W_i, $1 \leq i \leq N$, to cover, or almost cover Ω. In fact, we will often make sure that $F(\mathbf{W})$ is a sufficiently large, or increasing function of each W_i, so that if some part of W_i is not really useful to make $E(\mathbf{u}) + M(\mathbf{u})$ small, we may as well remove it and save more on the $F(\mathbf{W})$ term. In Section 12, we take the opposite approach and find conditions on F and the g_i that imply that the W_i associated to a minimizer $(\mathbf{u}, \mathbf{W})$ almost cover Ω and that $\mathbf{u} \neq 0$ almost everywhere. Typically, this means some decay for F (so that adding a missing piece to the W_i does not cost anything) and the positivity of g_i. See Propositions 12.3 and 12.4 in particular.

In Section 13 we show that if the f_i are bounded and nonnegative, the g_i lie in L^2 and at least one of them is nonzero, and F is given by (1.6) with $\alpha > \frac{2}{n}$, b is large enough, and $a \geq 0$ small enough, then the mininizers for J are such that $\mathbf{u} \neq 0$ and $|W_i| < |\Omega|/10$ for $1 \leq i \leq N$.

In Section 14 we show that under mild conditions on F (where we say that volume is not too cheap), the number of indices i for which $|W_i| > 0$ is bounded, even if we allowed much more components by taking N very large.

We then return to a general scheme in the study of free boundary problems. In Section 15, we consider for instance $u_{1,+}$, the positive part of u_1, and we want to show that under suitable non degeneracy conditions, it behaves roughly like the distance to the free boundary $\partial_1 = \partial\big\{x\,;\, u_1(x) > 0\big\}$. The nondegeneracy condition that we will use is that, as far as the volume term $F(\mathbf{W})$ is concerned, we can always sell small parts of W_1 and get a proportional profit. That is, if $A \subset W_1$ has a small enough measure, we can remove A from W_1, and maybe distribute some part of it to the W_i, $i \neq 1$, in such a way as to make $F(\mathbf{W})$ smaller by at least $\lambda|A|$ for some fixed $\lambda > 0$. See (15.1) for the precise condition. For instance, if F is given by (1.7), we get this condition as soon as $q_1(x) \geq \lambda + \min\big\{0, q_2(x), \cdots, q_N(x)\big\}$ almost-everywhere on Ω (see (15.4)); when we have (1.6), we just need to take $a \geq \lambda$ and $b \geq 0$.

If we add this nondegeneracy condition to the other assumptions above, we get the desired rough behavior of $u_{1,+}$; see (15.6) and (15.7) in Theorem 15.1, (15.10) in Theorem 15.2, and (15.40) in Theorem 15.3. We also get that the complementary region $\big\{u_1 \leq 0\big\}$ is not too thin near a point of ∂_1; see Theorem 15.4.

In Section 16 we show that if W_1 satisfies our nondegeneracy condition (we'll also say that $\Omega_1 = \big\{u_1 > 0\big\}$ is a good region), then $\partial_1 = \partial\Omega_1$ is locally Ahlfors-regular and (uniformly) rectifiable. The argument goes nearly as in [AC], and is based on the fact that when $(\mathbf{u}, \mathbf{W})$ is a minimizer, $\Delta u_1 + C$ is a positive measure (that we can also estimate). We use the non degeneracy results of Section 15 to show that the restriction of this measure to ∂_1 is locally Ahlfors-regular (Proposition 16.1), compare it to the total variation measure $D\mathbb{1}_{\Omega_1}$, show that Ω_1 is a set of finite perimeter, and get the rectifiability of ∂_1 almost for free (as the reduced boundary of Ω_1). We also deduce from this a representation formula for $\Delta u_{1,+}$

in terms of the density of that measure with respect to the restriction of $\mathcal{H}^d$ to ∂_1; see Proposition 16.2. Finally we check that $\partial\Omega_1$ is locally uniformly rectifiable, with "Condition B" and big pieces of Lipschitz graphs, because this follows rather easily from the results of Section 15. See Proposition 16.3.

The Lipschitz bounds in Sections 10 and 11 allow us to define the blow-up limits of $(\mathbf{u}, \mathbf{W})$ at a point, and the non degeneracy results of Section 15 will often be the best way to make sure that these limits are nontrivial. Before we really get to that, we need a theorem about limits. We take care of this in Section 17. We introduce a notion of local minimizer for a functional J in an open set $\mathcal{O}$, and prove that under reasonable assumptions, if we have pairs $(\mathbf{u}_k, \mathbf{W}_k)$ of local minimizers in $\mathcal{O}$ of functionals J_k (associated to domains Ω_k, and defined as J above), and if the $\mathbf{u}_k$ converge to a limit $\mathbf{u}$, then we can find $\mathbf{W}$ such that $(\mathbf{u}, \mathbf{W})$ is a local minimizer in $\mathcal{O}$ of the natural limit functional J. See Theorem 17.1 and Corollary 17.5.

We use this in Section 18 to prove that if $(\mathbf{u}, \mathbf{W})$ is a minimizer for J and $\mathbf{u}_\infty$ is a blow-up limit of $\mathbf{u}$ at some point x_0 such that $\mathbf{u}(x_0) = 0$, then (under some reasonable smoothness assumptions, in particular on the volume term F and on Ω if $x_0 \in \Omega$) we can find $\mathbf{W}_\infty$ such that $(\mathbf{u}_\infty, \mathbf{W}_\infty)$ is a local minimizer (in an infinite domain, which is $\mathbb{R}^n$ when x_0 is an interior point of Ω, and otherwise is a blow-up limit of Ω) of a simpler functional J_∞. The functional J_∞ is simpler because it does not have an M-term, and its F term is like (1.7), with constant functions q_i. See Theorem 18.1 or Corollary 18.3.

In Sections 19 and 20 we find various situations where the blow-up limits of $\mathbf{u}$ at a point are given by a simple formula with affine functions. The main result of Section 19 is Corollary 19.4, where we get such an expression as soon as we can find two phases $\varphi \neq \psi \in I$ (as above) such that the Alt-Caffarelli-Friedman functional $\Phi_{\varphi,\psi}(r)$ that we use in Section 9 has a nonzero limit at $r = 0$. This is the case, for instance, if the nondegeneracy condition of Section 15 is satisfied for (the indices i that come from) φ and ψ, and the origin lies in $\partial\big\{v_\varphi > 0\big\} \cap \partial\big\{v_\psi > 0\big\}$. The proof is not surprising (all the ingredients were prepared in the previous section) and is based on a careful study of the case of equality in the monotonicity theorem of [ACF]; see Theorem 19.3.

The main result of Section 20 is Corollary 20.3, which concerns the case when we cannot find φ and ψ as above, but the origin lies in $\partial\big\{v_\varphi > 0\big\}$ for some $\varphi \in I$ that satisfies the nondegeneracy condition of Section 15. Then we use a result of [CJK2] to show that, in dimensions $n \leq 3$, some blow-up limits of $\mathbf{u}$ are composed of just one phase v, which is the positive part of an affine function.

We also show (in any dimension) that the conclusion of Corollary 19.4 or Corollary 20.3 holds when the origin is a point of $\partial_\varphi = \partial\big\{v_\varphi > 0\big\}$ where ∂_φ has a tangent (and the nondegeneracy condition holds). By Proposition 16.3, this happens almost everywhere on ∂_φ. See Proposition 20.5 and Remark 20.6.

In Section 21 we summarize the situation: when all the regions satisfy the nondegeneracy condition of Section 15, for instance, a given point $x_0 \in \Omega$ can only lie in at most two sets ∂_φ (at most one if $x_0 \in \partial\Omega$, and so there is a small neighborhood of x_0 where $(\mathbf{u}, \mathbf{W})$ is a minimizer of a variant of the Alt-Caffarelli-Friedman functional with only one or two phases.

See Lemmas 21.1, 21.4, and 21.5. While we were preparing this manuscript, we learned about a recent result of Bucur and Velichkov [BV], with a version of Lemma 21.1, obtained by a completely different method (in particular a monotonicity formula with three phases). We discuss this a little more in Section 21.

At the end of Section 21, we are left with a series of good sufficient conditions for $\mathbf{u}$ to have blow-up limits composed of affine functions. These conditions imply the asymptotic flatness of the free boundaries ∂_φ, which holds everywhere inside Ω under nondegeneracy conditions and if $n \leq 3$, and only almost everywhere when $n > 3$; see Proposition 19.5 and Lemma 20.4. We do not continue the regularity study of the ∂_φ, that would normally lead to local $C^{1,\alpha}$-regularity, probably under additional nondegeneracy assumptions, but Lemma 21.1 says that this now is a problem about functionals with at most 2 phases.

Section 22 can be seen as an appendix. We complete some of the proofs of Sections 19 and 20 with a standard computation of first variation, which we only do for the affine blow-up limits of our problem, and then use the representation formula of Proposition 16.2 to show that, at points where $\mathbf{u}$ has a nice blow-up limit, there are Euler-Lagrange relations between the values of the normal derivative of $\mathbf{u}$ on both sides of the free boundary, and the multipliers that comes from the derivatives of F in the corresponding directions. These formulas are deduced from the corresponding formulas for the blow-up limits, which we derived with the first variation argument. See Proposition 22.1.

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Frequently used notation.

$B(x,r)$ is the open ball centered at x, with radius r

$|A|$ denotes the Lebesgue measure of a set A

C and c are positive constants that change from line to line; usually C is large and c is small.

$W^{1,2}(\mathbb{R}^n)$ is the Sobolev space (one derivative in L^2)

$J(\mathbf{u},\mathbf{W}) = E(\mathbf{u}) + M(\mathbf{u}) + F(\mathbf{W})$ is our main functional; see (1.3), (1.4), (1.5)

$\mathcal{F} = \mathcal{F}(\Omega)$ is our class of acceptable pairs $(\mathbf{u},\mathbf{W})$; see Definition 1.1

$\mathcal{F}(\mathcal{O},\Omega)$ is its variant for local minimizers in a domain $\mathcal{O}$; see the start of Section 17

$S_r = \partial B(0,r)$

$\mathcal{W}(\Omega)$ is the set of acceptable N-uples $\mathbf{W}$; see the beginning of Section 12

$A\Delta B$ is the symmetric difference $(A \setminus B) \cup (B \setminus A)$
$\Omega_i = \big\{x \in \mathbb{R}^n\,;\, u_i(x) > 0\big\}$ is usually strictly smaller than Ω and may be smaller than W_i
$I = [1, N] \times \{-1, 1\}$ denotes our set of phases, and we set
$v_\varphi = [\varepsilon u_i]_+ = \max(0, \varepsilon u_i)$ and $\Omega_\varphi = \big\{x \in \mathbb{R}^n\,;\, v_\varphi(x) > 0\big\}$ for $\varphi = (i, \varepsilon) \in I$
Φ_j and Φ define our Alt-Caffarelli-Friedman functional; see (8.4)
Φ^0_φ and $\Phi^0_{\varphi_1,\varphi_2}$ are their new names in Sections 19 and later; see (19.11) and (19.17)
$\Phi_{\varphi,k}$ and $\Phi_{\varphi_1,\varphi_2,k}$ are the same ones, along a blow-up sequence; see (19.13) and (19.18).

2 Motivation for our main functional

In this section we try to explain the connection between the functional defined in (1.5) and our initial localization problem. Let us start with a domain $\Omega \subset \mathbb{R}^n$, which we may assume to be bounded, and the operator $\mathcal{L} = -\Delta + \mathcal{V}$, with $\mathcal{V}$ bounded and nonnegative. For simplicity of exposition, let us also assume that Ω is smooth (only for the duration of this section: the relevant regularity conditions on Ω, when necessary, are carefully tracked in the remainder of the paper). Let w_0 be the solution of $\mathcal{L}w_0 = 1$ on Ω, with the Dirichlet condition $w_0 = 0$ on $\mathbb{R}^n \setminus \Omega$. It is shown in [FM] that the eigenfunctions of $\mathcal{L}$ can be estimated pointwise by the single function w_0, so we can expect w_0 to give useful information on the localization of the eigenfunctions.

Given an integer $N \geq 2$, we want to to split Ω into N disjoint subregions W_j, preferably with a nice boundary $\Gamma = \cup_j \partial W_j$, so that the values of w_0 on Γ give the best control (on w_0 and the eigenfunctions), and for this we want the restriction of w_0 to Γ to be as small as possible, in a sense that will be discussed soon. Numerical experiments suggest that when we do this, the eigenfunctions tend to be localized inside the regions W_j, even though the precise mechanism why this happens is not clear to us. See [FM].

One interesting way to require this smallness is be to minimize something like $\int_\Gamma w_0$, plus the same volume term $F(\mathbf{W})$ as in the present paper. We shall not do this here, and instead we shall encode smallness in a slightly more subtle and perhaps more natural way: via the energy corresponding to the governing operator $\mathcal{L}$. Let us associate, to any function $w \in W^{1,2}(\mathbb{R}^n)$ such that $w(x) = 0$ almost everywhere on $\mathbb{R}^n \setminus \Omega$, the energy

$$E_0(w) = \int_\Omega |\nabla w|^2 + \mathcal{V}\, w^2. \tag{2.1}$$

Given the closed set $\Gamma \subset \Omega$, we want to define

$$\begin{aligned} J_0(\Gamma) = \inf \big\{ E_0(w)\,;\, w \in W^{1,2}(\mathbb{R}^n), w(x) = 0 &\text{ almost everywhere on } \mathbb{R}^n \setminus \Omega, \\ &\text{and } w = w_0 \text{ on } \Gamma \big\}, \end{aligned} \tag{2.2}$$

and for the moment let us assume that Γ is composed of smooth hypersurfaces. Then it is possible to define the traces on Γ of the functions w and w_0 (just because they lie in $W^{1,2}(\mathbb{R}^n)$), and thus give a sense to the phrase "$w = w_0$ on Γ".

Of course $J_0(\emptyset) = 0$ (take $w = 0$ in (2.2)), and so $J_0(\Gamma)$ can be seen as the minimal amount of energy that we need to pay, when we add the constraint that $w = w_0$ on Γ to the basic condition that $w(x) = 0$ a.e. on $\mathbb{R}^n \setminus \Omega$.

It does not make sense to minimize $J_0(\Gamma)$ without any constraint on Γ, because the infimum would be when $\Gamma = \emptyset$, so we add a term $F(\mathbf{W})$ to our functional, that depends on Γ (typically through the volumes of the components of $\Omega \setminus \Gamma$), just to compensate and avoid this case. Then we try to minimize $J_0(\Gamma) + F(\mathbf{W})$. Using a reasonably smooth term F that depends on the volumes seems to be the mildest way to avoid degeneracy; for instance requiring that the connected components of $\Omega \setminus \Gamma$ have prescribed volumes looks a little too violent, even though it gives an interesting (but harder) mathematical problem. The definition of $F(\mathbf{W})$ will be discussed later. Assuming that this procedure can be justified and that a minimizer exists, we denote the latter by w.

Anyway, it is expected that picking sets Γ on which w_0 is small helps making $J_0(\Gamma)$ small, and our definition of smallness of w_0 on Γ will be through the smallness of $J_0(\Gamma)$. An apparently simpler choice of $E_0(w)$ would have been $\int_\Omega |w|^2$ (after all, we started the discussion with pointwise estimates on the functions themselves), but the constraint that $w = w_0$ on Γ does not really make sense with the weaker L^2 norm, or in other words the infimum in (2.2) (say, on smooth functions w) would be zero. Working in the class $W^{1,2}(\mathbb{R}^n)$ and with the energy in (2.1) then seems to be the simplest reasonable choice, with an obvious connection with our operator $-\Delta + \mathcal{V}$.

Next we continue with $\mathcal{L} = -\Delta + \mathcal{V}$, and show how to restate the minimization problem for J_0 more simply, in terms of the difference $v = w_0 - w$. Roughly speaking, the idea is that viewing w_0 as a solution to $\mathcal{L}w_0 = 1$ on Ω, with the Dirichlet condition $w_0 = 0$ on $\mathbb{R}^n \setminus \Omega$ and, trivially, $w_0 = w_0$ on Γ, and viewing w as a solution to $\mathcal{L}w = 0$ on Ω, with $w = 0$ on $\mathbb{R}^n \setminus \Omega$ and $w = w_0$ on Γ, we would have $\mathcal{L}v = 1$ on Ω, with the Dirichlet condition $v = 0$ on $\mathbb{R}^n \setminus \Omega$ and $v = 0$ on Γ. The latter problem is more natural to formalize and address in our context, but let us first convert this reasoning to the language of minimization.

By (2.1),

$$E_0(w) = E_0(w_0 - v) = \int_\Omega |\nabla(w_0 - v)|^2 + V(w_0 - v)^2 \\ = E_0(w_0) + E_0(v) - 2\int_\Omega \langle \nabla v, \nabla w_0 \rangle - 2\int_\Omega \mathcal{V}\, v w_0, \tag{2.3}$$

where for us $E_0(w_0)$ is just a constant. Let us integrate by parts brutally; some justifications will come soon. This yields

$$-2\int_\Omega \langle \nabla v, \nabla w_0 \rangle = 2\int_\Omega v \Delta w_0 - 2\int_{\partial\Omega} v \frac{\partial w_0}{\partial n} = 2\int_\Omega \mathcal{V}\, v w_0 - 2\int_\Omega v \tag{2.4}$$

because the boundary term $\int_{\partial\Omega} v \frac{\partial w_0}{\partial n}$ vanishes since $v = 0$ on $\mathbb{R}^n \setminus \Omega$, and where we use the fact that $(-\Delta + \mathcal{V})w_0 = \mathcal{L}w_0 = 1$ on Ω. Then (2.3) yields

$$E_0(w) = E_0(w_0) + E_0(v) - 2\int_\Omega v. \tag{2.5}$$

Concerning the integration by parts in (2.4), one way not to do it is to use the variational definition of the function w_0, i.e., the fact that it is the function $f \in W^{1,2}(\mathbb{R}^n)$ such that $f(x) = 0$ almost everywhere on $\mathbb{R}^n \setminus \Omega$, and which minimizes $\int_\Omega |\nabla f|^2 + \mathcal{V} f^2 - 2f$ under these constraints. Since for all $\lambda \in \mathbb{R}$, the function $w_0 + \lambda v$ also satisfies these constraints, we get that

$$\int_\Omega |\nabla w_0|^2 + \mathcal{V} w_0^2 - 2w_0 \leq \int_\Omega |\nabla(w_0 + \lambda v)|^2 + \mathcal{V}(w_0 + \lambda v)^2 - 2(w_0 + \lambda v), \tag{2.6}$$

which implies the result of (2.4) (just compute the derivative at $\lambda = 0$).

So $J_0(\Gamma)$ is the same, modulo adding the constant $E_0(w_0)$, as

$$\begin{aligned} J_1(\Gamma) \;=\; \inf \Big\{ & \int_\Omega |\nabla v|^2 + \mathcal{V} v^2 - 2v\,;\, v \in W^{1,2}(\mathbb{R}^n), \\ & v(x) = 0 \text{ almost everywhere on } \mathbb{R}^n \setminus \Omega, \text{ and } v = 0 \text{ on } \Gamma \Big\}. \end{aligned} \tag{2.7}$$

We can further simplify this and define $J_1(\Gamma)$ without our smoothness assumption on Γ (so far implicit in the understanding of what $v = 0$ on Γ means). Denote by W_i, $i \in I$, the connected components of $\Omega \setminus \Gamma$, set $u_i = \mathbb{1}_{W_i} v$. If Ω and Γ are smooth and I is finite, then the functions u_i have traces from both sides of Γ, and, by a simple welding lemma (and the definition of the traces), our constraint that $v = 0$ on Γ implies that $u_i \in W^{1,2}(\mathbb{R}^n)$, i.e., that the distribution derivative of u_i does not catch an extra piece along Γ. See for instance Chapters 10-13 in [D], or rather the discussion near (4.13)-(4.18) where we do similar manipulations near spheres.

Conversely, if the $u_i \in W^{1,2}(\mathbb{R}^n)$ are such that $u_i = 0$ almost everywhere on $\mathbb{R}^n \setminus W_i$, it is easy to see that $v = \sum_i u_i$ satisfies the constraints in (2.7) (in order to prove that $v = 0$ on Γ, just compute the trace of u_i from the side that does not lie in W_i). In other words,

$$J_1(\Gamma) = \inf \Big\{ \sum_i \int_{W_i} |\nabla u_i|^2 + \mathcal{V} u_i^2 - 2u_i\,;\, u_i \in W^{1,2}(\mathbb{R}^n), u_i(x) = 0 \text{ a.e. on } \mathbb{R}^n \setminus W_i \}. \tag{2.8}$$

This is exactly the term $E(\mathbf{u}) + M(\mathbf{u})$ of our functional (see (1.3) and (1.4)), with $f_i = \mathcal{V}$ and $g_i = 2$ for $i \in I$. There is a small difference in the class of competitors. Here the W_i, $i \in I$, are the connected components of $\Omega \setminus \Gamma$, while in our description of the functional, we are allowed to regroup some of them into a single set. In principle, we shall take functions $F(\mathbf{W})$ that are convex, in such a way that regrouping two regions makes the functional larger, because $E(\mathbf{u}) + M(\mathbf{u})$ does not change, but $F(\mathbf{W})$ increases. So the only case when we expect two components of $\Omega \setminus \Gamma$ to be merged is when I has more than N elements, and we need to regroup some of them because we added the constraint that we do not use more than N sets. This difference should not disturb us much. If $\Omega \setminus \Gamma$ has more than N components, this may just be a sign that we chose N a little too small, and anyway putting a constraint on the number of sets seems less brutal than putting a constraint on the number of components (which we could have done instead).

The advantage of this new definition is that we don't need to know that Γ is smooth to define $J_1(\Gamma)$. This is why we used this definition in the introduction, and intend to keep it in this paper. We still hope that, under mild conditions on our data, the minimizers will provide smooth enough boundaries Γ so that J_0 also makes sense, but we shall not attempt to check this and do the backward translation.

Notice that in the present situation we can restrict to nonnegative functions: our special function w_0 is nonnegative, and all the interesting competitors for w or the u_i are nonnegative, because replacing u_i with its positive part always makes our functionals smaller.

Recall that we also want to add a volume term $F(\mathbf{W})$, that for instance depends on the volumes $|W_i|$. Our main goal for this is to encourage the functional to choose nontrivial minimizers for which the W_i have roughly comparable volumes, and in this respect the precise choice of F should mostly be a matter of experience. But we have no good reason to make F complicated, or to treat one of the component differently, so choosing

$$F(\mathbf{W}) = \sum_{i=1}^{N} f(|W_i|) \tag{2.9}$$

looks like a good idea. We may even brutally decide to use (1.6) for simplicity.

It is probably a good idea to choose f convex, so that the functional will prefer to choose sets W_i with roughly comparable volumes. At this time, we also prefer to choose $f(t) \geq at$ for some $a > 0$; this way, the non degeneracy assumptions of Sections 14 and 15 are automatically satisfied for every i, so we get an upper bound on the number of nonzero functions u_i (even if we took $N = +\infty$) and a better description of the sets $\Omega_i = \big\{x \in \Omega\,;\, u_i(x) > 0\big\}$. Recall that here the u_i are nonnegative, so we do not care for $\big\{u_i < 0\big\}$, and that Ω_i is open because $\mathbf{u}$ is continuous when $(\mathbf{u}, \mathbf{W})$ is a minimizer. For instance, we get that the $\partial\Omega_i$ are uniformly rectifiable (as in Section 16), and that a given point x_0 cannot lie on more than two sets $\partial\Omega_i$ (one if $x_0 \in \partial\Omega$). In dimension $n \leq 3$, we even expect the boundary $\partial\Omega_i$ to be smooth, but we only show that $\partial\Omega_i$ has flat blow-up limits at every point.

More directly, choosing f such that $f(t) \geq at$ is a way of making sure that if a piece of W_i is not so useful to make $E(\mathbf{u}) + M(\mathbf{u})$ smaller, we may as well remove it and save on the term $F(\mathbf{W})$. In terms of localization of eigenfunctions, this means that we introduce a black zone $\Omega \setminus \bigcup_i W_i$, where $\mathbf{u} = 0$, and we bet that the eigenfunctions will not live in that zone.

Another option would be to try to force the W_i to cover Ω, for instance by choosing f decreasing. Then, by the results of Section 12, we expect no black zone. If f is strictly convex, the regions W_i that do not have the smallest volume satisfy the degeneracy assumptions of Section 15, and we get a better description for them. But not for the regions with the smallest volume, just as in the previous case we do not get a good description of the black zone.

We should insist on the fact that the regularity results that we get for the free boundaries $\partial\big\{x \in \Omega\,;\, \pm u_i(x) > 0\big\}$ are mostly a consequence of our choice of function F, not a corollary of any localization property for the eigenfunctions. However the first evidence that we have suggests that the eigenfunctions have a tendency to live in the regions computed by the functional, and away from the black zone.

Let us also comment on the choice of N. If for instance $F(\mathbf{W}) = \sum_i f(|W_i|)$ of some f such that $f(t) \geq at$ (as above), the non degeneracy assumptions of Section 15 are satisfied, and we even get that there is a constant $\tau > 0$, that does not depend on N, such that $|W_i| \geq \tau$ when $|W_i| > 0$; see Proposition 14.1. Thus there is a bounded number of nontrivial sets W_i, regardless of our initial choice of N. In other words, even though we select N in advance in our theory, as soon as N is large enough (depending in particular on a above and $|\Omega|$), the minimizers will no longer depend on N and will not have too many pieces.

There are some sort of Euler-Lagrange conditions on the minimizers, that we could obtain with a (heuristic) computation of first variation of the domains. For instance, along a smooth boundary between Ω_1 and Ω_2, the normal derivatives $\frac{\partial u_1}{\partial n}$ and $\frac{\partial u_2}{\partial n}$ would need to satisfy the relation

$$\left(\frac{\partial u_1}{\partial n}\right)^2 - \left(\frac{\partial u_2}{\partial n}\right)^2 = f'(|W_1|) - f'(|W_2|). \tag{2.10}$$

We do not do this first variation computation here, but in Section 22, we do it for the nice blow-up limits of u, and get an almost-everywhere variant of (2.10) and similar formulas. See near (22.14) (where the fact that $\lambda_i = f'(|W_i|)$ comes from (18.11) and (2.9)).

Starting with the next section, we shall forget about $\mathcal{L}$ and the localization of eigenfunctions and return to a more general expression for J, but of course the regularity results that we shall obtain can be seen as an encouragement for the use of our functional in this context.

3 Existence of minimizers

In this section we prove the existence of minimizers for our functional J in the class $\mathcal{F}$, under the following mild assumptions. First we shall assume that

$$\Omega \text{ is a borel set, with } |\Omega| < +\infty, \tag{3.1}$$

where we use the notation $|E|$ for the Lebesgue measure of any Borel set $E \subset \mathbb{R}^n$. Thus we don't need Ω to be bounded, or even open. We also assume that for $1 \leq i \leq N$,

$$f_i \in L^\infty(\Omega), f_i(x) \geq 0 \text{ almost-everywhere, and } g_i \in L^2(\Omega). \tag{3.2}$$

We could be a little more general and only assume that f_i and g_i are in slightly larger L^p-spaces, and also allow f_i to be slightly negative, but we don't expect to use that generality, and also we shall soon have more restrictive conditions on the g_i. See Remark 3.4 at the end of the section. Concerning the volume term, let us first assume that

$$F(W_1, \ldots, W_N) = \widetilde{F}(|W_1|, \ldots, |W_N|), \text{ where } \widetilde{F} : [0, |\Omega|]^N \to \mathbb{R} \text{ is continuous.} \tag{3.3}$$

We shall see at the end of the section how to deal with other types of functions F, including the more classical definition of $F(\mathbf{W})$ by (1.7); see Corollary 3.5. Now define $\mathcal{F}$ and J as in Definition 1.1 and (1.3)-(1.5).

Theorem 3.1 *Under the assumptions (3.1)-(3.3), we can find* $(\mathbf{u}_0, W_0) \in \mathcal{F}$ *such that*

$$J(\mathbf{u}_0, \mathbf{W}_0) \leq J(\mathbf{u}, \mathbf{W}) \text{ for all } (\mathbf{u}, \mathbf{W}) \in \mathcal{F}. \tag{3.4}$$

Proof. The proof will rather easily follow from the compactness of some Sobolev injections. Let Ω, F, and the f_i be as in the statement, and set

$$m = \inf_{(\mathbf{u}, \mathbf{W}) \in \mathcal{F}} J(\mathbf{u}, \mathbf{W}). \tag{3.5}$$

We shall see soon that m is bounded, but let us first observe that $m < +\infty$: just pick $\mathbf{u} = 0$ and any acceptable disjoint collection $\mathbf{W}$ of subsets, and observe that $J(\mathbf{u}, \mathbf{W}) < +\infty$. For $(\mathbf{u}, \mathbf{W}) \in \mathcal{F}$, set

$$E(\mathbf{u}) = \sum_{i=1}^{N} \int_\Omega |\nabla u_i|^2 = \sum_{i=1}^{N} \int_{W_i} |\nabla u_i|^2 \tag{3.6}$$

as in (1.3), where the last identity comes from the fact that $u_i = 0$ a.e. outside of W_i, hence $\nabla u_i = 0$ a.e. on $\Omega \setminus W_i$ too. The next lemma will show that $E(\mathbf{u})$ controls the potentially negative terms $\int_\Omega u_i(x) g_i(x)$.

Lemma 3.2 *If E is a measurable set such that $|E| < +\infty$ and $u \in W^{1,2}(\mathbb{R}^n)$ is such that $u(x) = 0$ a.e. on $\mathbb{R}^n \setminus E$, then*

$$\int_E u^2 \leq C|E|^{2/n} \int_E |\nabla u|^2, \tag{3.7}$$

where C depends only on n.

Proof. The lemma is a fairly easy consequence of the standard Poincaré's inequality (on balls and where we subtract mean values); for the sake of completeness, we shall add a short Section 4 where we recall this inequality and prove Lemma 3.2 and its analogue on a sphere. In the mean time let us refer to [HP], Lemma 4.5.3. ∎

We are ready to check that $m > -\infty$. For $(\mathbf{u}, \mathbf{W}) \in \mathcal{F}$ and $1 \leq i \leq N$, observe that

$$\Big| \int_\Omega u_i(x) g_i(x) dx \Big| \leq ||u_i||_2 ||g_i||_2 \leq C|\Omega|^{1/n} \Big\{ \int_\Omega |\nabla u|^2 \Big\}^{1/2} ||g_i||_2 \tag{3.8}$$

by Lemma 3.2, and then

$$\Big| \sum_i \int_\Omega u_i(x) g_i(x) dx \Big| \leq C E(\mathbf{u})^{1/2}, \tag{3.9}$$

where C now depends on the data Ω and the $||g_i||_2$. Since $F(\mathbf{W})$ is bounded by (3.3), we get that

$$
\begin{aligned}
J(\mathbf{u},\mathbf{W}) &= E(\mathbf{u}) + \sum_i \int [u_i(x)^2 f_i(x) - u_i(x) g_i(x)]dx + F(\mathbf{W}) \\
&\geq E(\mathbf{u}) - \Big| \sum_i \int u_i(x) g_i(x) dx \Big| - C \geq E(\mathbf{u}) - CE(\mathbf{u})^{1/2} - C, \qquad (3.10)
\end{aligned}
$$

because $f_i \geq 0$. This is bounded from below; so $m > -\infty$.

Return to the proof of Theorem 3.1. Let $\{(\mathbf{u}_k, \mathbf{W}_k)\}$, $k \geq 0$, be a minimizing sequence in $\mathcal{F}$, which means that

$$\lim_{k\to+\infty} J(\mathbf{u}_k, \mathbf{W}_k) = m. \qquad (3.11)$$

We want to extract a converging subsequence and show that the limit is a minimizer in $\mathcal{F}$. The following compactness lemma will help.

Lemma 3.3 *Let Ω be a measurable set, with finite measure $|\Omega|$, and denote by U_Ω the set of functions $u \in W^{1,2}(\mathbb{R}^n)$ such that $u(x) = 0$ a.e. on $\mathbb{R}^n \setminus \Omega$ and $\int_\Omega |\nabla u|^2 \leq 1$. Then U_Ω is contained in a compact subset of $L^2(\mathbb{R}^n)$.*

Proof. We start with the case when Ω is bounded, because then we can use the compactness of the Sobolev injection, as stated for instance in [Z], Theorem 2.5.1. Let B be a ball that contains Ω; it is well known, and easy to check, that $U_\Omega \subset W_0^{1,2}(2B)$. By Theorem 2.5.1 in [Z], it is contained in a compact subset of $L^2(\mathbb{R}^n)$, and the result follows.

Let us now treat the general case when $|\Omega| < +\infty$, which is probably almost equally well known. Since $L^2(\mathbb{R}^n)$ is complete, it is enough to show that U_Ω is completely bounded in $L^2(\mathbb{R}^n)$. That is, for each given $\varepsilon > 0$ we want to show that U_Ω is contained in the union of a finite number of L^2-balls of radius ε.

Let $R \geq 1$ be large, to be chosen soon (depending on ε), and let φ be a smooth cut-off function such that $0 \leq \varphi(x) \leq 1$ for $x \in \mathbb{R}^n$, $\varphi(x) = 1$ for $0 \leq |x| \leq R$, $\varphi(x) = 0$ for $|x| \geq 3R$, and $|\nabla\varphi(x)| \leq 2/R$ everywhere. Then write any function $u \in U_\Omega$ as $u = \varphi u + (1-\varphi)u$. Set $v = (1-\varphi)u$, and notice that $v \in W^{1,2}(\mathbb{R}^n)$, with $\nabla v = (1-\varphi)\nabla u - u\nabla\varphi$ (as a distribution), so that $\int_\Omega |\nabla v|^2 \leq 2\int_\Omega |\nabla u|^2 + 2\int_\Omega |u|^2|\nabla\varphi|^2 \leq 2\int_\Omega |\nabla u|^2 + 8R^{-2}\int_\Omega |u|^2$. By Lemma 3.2 and because $u \in U_\Omega$, we get that $\int_\Omega |u|^2 \leq C\int_\Omega |\nabla u|^2$ (where C depends on Ω), and hence $\int_\Omega |\nabla v|^2 \leq 3\int_\Omega |\nabla u|^2$ if R is large enough.

But now $v = 0$ almost everywhere on $B(0,R)$, so Lemma 3.2 applies with $E = \Omega \setminus B(0,R)$ and yields $\int_\Omega |v|^2 \leq 3C|\Omega \setminus B(0,R)|^{2/n} \leq \varepsilon^2/4$ if R is large enough. That is, $||(1-\varphi)u||_2 = ||v||_2 \leq \varepsilon/2$.

Because of this, it is now enough to cover the set $U' = \big\{u\varphi\,;\, u \in U_\Omega\big\}$ by a finite number of L^2-balls of radius $\varepsilon/2$. But $u\varphi \in W^{1,2}(\mathbb{R}^n)$, with $\int_\Omega |\nabla(u\varphi)|^2 \leq 3$ by the same proof as above, and in addition $u \in W_0^{1,2}(B(0,4R))$, so Theorem 2.5.1 in [Z] says that U' is relatively compact in $L^2(\mathbb{R}^n)$, hence totally bounded, and the relative compactness of U_Ω follows. ∎

Observe that by (3.10), $E(\mathbf{u}_k)$ stays bounded along our minimizing sequence, so by Lemma 3.3 the components $u_{i,k}$, $1 \leq i \leq N$, of $\mathbf{u}_k$ stay in a compact subset of $L^2(\mathbb{R}^n)$. This allows us to replace $\{(\mathbf{u}_k, \mathbf{W}_k)\}$ with a subsequence, which for convenience we still denote by $\{(\mathbf{u}_k, \mathbf{W}_k)\}$, for which $\mathbf{u}_k$ converges to a limit $\mathbf{u}$ in $L^2(\mathbb{R}^n)$. This just means that for $1 \leq i \leq N$, the component $u_{i,k}$ converges to u_i in L^2. By extracting a new subsequence if needed, we can also assume that $u_{i,k}$ converges to u_i pointwise almost everywhere. This is more pleasant, because it will help in the definition of the W_i. In the mean time, observe that u_i automatically satisfies our constraint that $u_i(x) = 0$ a.e. on $\mathbb{R}^n \setminus \Omega$. Similarly, if in our definition of F we added the constraint that some u_i be nonnegative, this stays true as well for our limit. Notice that

$$\begin{aligned} E(\mathbf{u}) &= \sum_i \int_\Omega |\nabla u_i|^2 \leq \sum_i \liminf_{k\to+\infty} \int_{\mathbb{R}^n} |\nabla u_{i,k}|^2 \\ &\leq \liminf_{k\to+\infty} \sum_i \int_{\mathbb{R}^n} |\nabla u_{i,k}|^2 = \liminf_{k\to+\infty} E(\mathbf{u}_k) \end{aligned} \tag{3.12}$$

by the lower semicontinuity of $\int_{\mathbb{R}^n} |\nabla u|^2$ when u converges (even weakly in L^2) to some limit. More easily,

$$\sum_i \int_\Omega u_i^2 f_i = \sum_i \lim_{k\to+\infty} \int_\Omega u_{i,k}^2\, f_i \tag{3.13}$$

because $f_i \in L^\infty$ and the $u_{i,k}$ converge in L^2, and

$$\sum_i \int_\Omega u_i g_i = \sum_i \lim_{k\to+\infty} \int_\Omega u_{i,k}\, g_i \tag{3.14}$$

because $g_i \in L^2$, so, with the notation of (1.4),

$$M(\mathbf{u}) = \lim_{k\to+\infty} M(\mathbf{u}_k). \tag{3.15}$$

We also need to take care of the volumes. We don't know how to make the characteristic functions $\mathbb{1}_{W_{i,k}}$ converge, but at least we can replace $\{(\mathbf{u}_k, \mathbf{W}_k)\}$ with a subsequence for which each $V_{i,k} = |W_{i,k}|$ converges to a limit l_i. We want to associate to $\mathbf{u}$ a collection of sets W_i such that

$$|W_i| = \lim_{k\to+\infty} V_{i,k} = l_i \tag{3.16}$$

because then we will get that

$$F(\mathbf{W}) = \widetilde{F}(|W_1|, \dots, |W_N|) = \lim_{k\to+\infty} \widetilde{F}(V_{1,k}, \dots, V_{N,k}) = \lim_{k\to+\infty} F(\mathbf{W}_k) \tag{3.17}$$

because $\widetilde{F}$ is continuous.

Let Z denote the set of $x \in \Omega$ for which $\mathbf{u}(x)$ is not the limit of the $\mathbf{u}_k(x)$, or for some choice of k and i, $x \in \Omega \setminus W_{i,k}$ but $u_{i,k}(x) \neq 0$; then $|Z| = 0$ by definitions; it will be more convenient to work in $\Omega' = \Omega \setminus Z$.

Also set $W_i' = \big\{x \in \Omega' \,;\, u_i(x) \neq 0\big\}$ and similarly, for each k, $W_{i,k}' = \big\{x \in \Omega' \,;\, u_{i,k}(x) \neq 0\big\}$. By definition of Ω', $W_{i,k}' \subset W_{i,k}$, and hence the $W_{i,k}'$, $1 \leq i \leq N$, are disjoint (recall that $(\mathbf{u}_k, \mathbf{W}_k) \in \mathcal{F}$).

If $x \in W_i'$, then $u_i(x) \neq 0$ and hence $u_{i,k}(x) \neq 0$ and $x \in W_{i,k}'$ for k large. In particular,

$$\text{the } W_i',\ 1 \leq i \leq N, \text{ are disjoint,} \tag{3.18}$$

because the $W_{i,k}'$ are disjoint for each k. Also, $\mathbb{1}_{W_i'} \leq \liminf_{k\to+\infty} \mathbb{1}_{W_{i,k}'}$ everywhere, and by Fatou

$$|W_i'| = \int \mathbb{1}_{W_i'} \leq \liminf_{k\to+\infty} \int \mathbb{1}_{W_{i,k}'} = \liminf_{k\to+\infty} |W_{i,k}'| \leq \lim_{k\to+\infty} |W_{i,k}| = l_i. \tag{3.19}$$

Notice that $\sum_i l_i \leq |\Omega|$ because for each k, $\sum_i V_{i,k} \leq |\Omega|$ (recall that the $W_{i,k}$ are disjoint when $(\mathbf{u}_k, \mathbf{W}_k) \in \mathcal{F}$). Hence we can add disjoint measurable sets to the W_i' if needed, and get new sets W_i, $1 \leq i \leq N$, such that $W_i' \subset W_i \subset \Omega$, the W_i are still disjoint (see (3.18)), and $|W_i| = l_i$, as required for (3.16).

We now check that with this choice of $\mathbf{W} = (W_1, \ldots, W_N)$, the pair $(\mathbf{u}, \mathbf{W})$ lies in $\mathcal{F}$. The fact that $u_i \in W^{1,2}(\mathbb{R}^n)$ comes from the convergence of $u_{i,k}$ in L^2, and our uniform bound for $E(\mathbf{u}_k)$. We have that $u_i(x) = 0$ for a.e. $x \in \mathbb{R}^n \setminus \Omega$ by pointwise convergence, and then $u_i(x) = 0$ for a.e. $x \in \mathbb{R}^n \setminus W_i$ by definition of $W_i \supset W_i'$. So $(\mathbf{u}, \mathbf{W})$ lies in $\mathcal{F}$. Since in addition

$$J(\mathbf{u}, \mathbf{W}) \leq \liminf_{k\to+\infty} J(\mathbf{u}_k, \mathbf{W}_k) \leq m \tag{3.20}$$

by (3.12), (3.15), (3.17), and (3.11), $(\mathbf{u}, \mathbf{W})$ is the desired minimizer. ∎

Remark 3.4 *In Theorem 3.1, we can replace the assumption (3.2) with the following weaker set of assumptions: for $1 \leq i \leq N$,*

$$f_i \in L^r(\Omega) \quad \textit{and } g_i \in L^p(\Omega), \tag{3.21}$$

as soon as we pick $r > n/2$ and $p > \frac{2n}{n+2}$ if $n \geq 2$, and $r \geq 1$ and $p \geq 1$ if $n = 1$, and (instead of our assumption that $f_i \geq 0$)

$$||f_{i,-}||_{L^r(\Omega)} \leq \varepsilon(r, n, \Omega, F) \tag{3.22}$$

with the same r, where $f_{i,-}$ denotes the negative part of f_i, and where $\varepsilon(r, n, \Omega, F)$ is a small positive constant that depends only on n, r, F, and $|\Omega|$.

Proof. We just sketch the proof because this remark is not so important (and probably (3.22) is hard to use). For the different exponents, we just want to use better Sobolev embeddings. We claim that Lemmas 3.2 and 3.3 still hold when we replace $||u||_2$ with $||u||_q$, as long as we take $q < \frac{2n}{n-2}$ when $n \geq 2$ and $q \leq +\infty$ when $n = 1$; we just need to know that the Sobolev embedding is still compact with such exponents, and follow the proof above.

Thus, in the argument above, we can get the u_k to converge in L^q. We can choose q to be the conjugate exponent of p, and this gives enough control on the integrals $\int u_{i,k} g_i$ (as in (3.14)). Or we choose $q/2$ to be the conjugate exponent of r, and we get a good control on $\int u_{i,k}^2 f_i$, as in (3.13). The other estimates are as as above, except that we also need to replace (3.10) because (3.22) is weaker than our earlier assumption that $f_i \geq 0$. So we keep $q/2 = r'$, the conjugate exponent of r, and say that

$$\begin{aligned}
J(\mathbf{u}, \mathbf{W}) &= E(\mathbf{u}) + \sum_i \int [u_i(x)^2 f_i(x) - u_i(x) g_i(x)] dx + F(\mathbf{W}) \\
&\geq E(\mathbf{u}) - \sum_i \int u_i^2(x) f_{i,-}(x) dx - \Big| \sum_i \int u_i(x) g_i(x) dx \Big| - C \\
&\geq E(\mathbf{u}) - C \sum_i ||u_i^2||_{r'} ||f_{i,-}||_r - C E(\mathbf{u})^{1/2} - C \\
&= E(\mathbf{u}) - C \sum_i ||u_i||_q^2 ||f_{i,-}||_r - C E(\mathbf{u})^{1/2} - C \qquad (3.23) \\
&\geq E(\mathbf{u}) - C \sum_i ||\nabla u_i||_2^2 \, ||f_{i,-}||_r - C E(\mathbf{u})^{1/2} - C \\
&\geq E(\mathbf{u}) - C E(\mathbf{u}) \sup_i ||f_{i,-}||_r - C E(\mathbf{u})^{1/2} - C \geq \frac{1}{2} E(\mathbf{u}) - C E(\mathbf{u})^{1/2} - C
\end{aligned}$$

if $||f_{i,-}||_r$ is small enough; then we can continue the argument as above, and the remark follows. ∎

We mentioned earlier that Theorem 3.1 is enough for the application that we have in mind, but since we feel bad about the case when our volume term $F(\mathbf{W})$ is given by the more standard formula (1.7), let us mention the following variant. We want to replace (3.3) with the monotonicity condition

$$F(\mathbf{W}) \leq F(\mathbf{W}') \text{ whenever } W_i \subset W_i' \text{ for } 1 \leq i \leq N, \qquad (3.24)$$

whose effect is that we should try to take the W_j as small as we can (all things being equal otherwise). Notice that (3.24) is satisfied when $F(\mathbf{W}) = \sum_i \int_{W_i} q_i$, as in (3.24), as soon as the q_i are nonnegative and integrable on Ω. The integrability of the q_j also gives the continuity of F, which we define as follows.

Denote by $A \Delta B = (A \setminus B) \cup (B \setminus A)$ the symmetric difference between two sets A and B; if $\mathbf{W} = (W_1, \ldots, W_N)$ and $\mathbf{W}' = (W_1', \ldots, W_N')$ are N-uples of disjoint subsets of Ω, set

$$\mathbf{W} \Delta \mathbf{W}' = \bigcup_i W_i \Delta W_i' = \bigcup_i [W_i \setminus W_i'] \cup [W_i' \setminus W_i], \qquad (3.25)$$

and then

$$\delta(\mathbf{W}, \mathbf{W}') = |\mathbf{W} \Delta \mathbf{W}'| \simeq \sum_i |W_i \Delta W_i'|. \tag{3.26}$$

We say that the sequence $\{\mathbf{W}_k\}$ tends to $\mathbf{W}$ when $\lim_{k \to +\infty} \delta(\mathbf{W}, \mathbf{W}_k) = 0$, and then we say that F is continuous at $\mathbf{W}$ when $F(\mathbf{W}) = \lim_{k \to +\infty} F(\mathbf{W}_k)$ when $\{\mathbf{W}_k\}$ tends to $\mathbf{W}$. In other words, F is continuous when

$$\lim_{k \to +\infty} F(\mathbf{W}_k) = F(\mathbf{W}) \text{ whenever } \lim_{k \to +\infty} |W_i \Delta W_{i,k}| = 0 \text{ for } 1 \leq i \leq N. \tag{3.27}$$

Here is the variant of Theorem 3.1 that we want to prove.

Corollary 3.5 *Assume (3.1), (3.2), (3.24), and that F is continuous (as in (3.27)). Then we can find $(\mathbf{u}_0, W_0) \in \mathcal{F}$ such that*

$$J(\mathbf{u}_0, \mathbf{W}_0) \leq J(\mathbf{u}, \mathbf{W}) \textit{ for all } (\mathbf{u}, \mathbf{W}) \in \mathcal{F}. \tag{3.28}$$

Proof. We proceed as in the proof of Theorem 3.1, except that instead of completing the W_i' to get sets W_i such that (3.18) holds, we just keep $W_i = W_i'$. The sets $W_{i,k} \cap W_i'$ converge to W_i' (that is, each $|(W_{i,k} \cap W_i') \Delta W_i'| = |W_i' \setminus W_{i,k}|$ tends to 0), because every $x \in W_i'$ lies in $W_{i,k}$ for k large, and then

$$F(W_1', \dots, W_N') = \lim_{k \to +\infty} F(W_{1,k} \cap W_i', \dots, W_{N,k} \cap W_i') \leq \liminf_{k \to +\infty} F(W_{1,k}, \dots, W_{N,k}) \tag{3.29}$$

by (3.27) and (3.24). The rest of the proof is the same. ∎

4 Poincaré inequalities and restriction to spheres

We record here some easy properties of functions in the Sobolev spaces $W^{1,p}$, $1 \leq p < +\infty$. In particular, some consequences of the Poincaré inequalities (like Lemma 3.2 above), the fact that the restriction of $u \in W^{1,p}$ to almost every sphere $S_r = \partial B(0,r)$ lies in $W^{1,p}(S_r)$, and a way to glue two Sobolev functions along a sphere.

We first recall the standard Poincaré inequalities. We shall find it convenient to use the notation

$$m_B u = \fint_B u = \frac{1}{|B|} \int_B u \tag{4.1}$$

for the average of a function u on a set B, which will often be a ball. With this notation, the standard Poincaré inequality for a ball says that if $B = B(x,r) \subset \mathbb{R}^n$ and $u \in W^{1,p}(B)$ for some $p \in [1, +\infty)$, then $u \in L^p(B)$ and

$$\fint_B |u - m_B u|^p \leq C_p r^p \fint_B |\nabla u|^p. \tag{4.2}$$

We shall also use Poincaré inequalities on spheres, for which we want to use similar notation. On the spheres $S_r = \partial B(0,r)$, we shall systematically use the surface measure, which we denote by σ; we could also have used the equivalent Hausdorff measure $\mathcal{H}^{n-1}$, but σ will be simpler and in particular we won't need to worry about normalization in the Fubini-like formula

$$\int_{B(0,r)} f(x)dx = \int_0^r \int_{S_t} f(x)d\sigma(x)dt, \tag{4.3}$$

which we shall use from time to time. We will use the notation $\sigma(E) = |E|_\sigma$ for the surface measure of a measurable set $E \subset S_r$, and

$$m_E^\sigma u = \fint_E u d\sigma = \frac{1}{|E|_\sigma} \int_E u(x)d\sigma(x) \tag{4.4}$$

when $E \subset S_r$ is such that $\sigma(E) > 0$ and u is a measurable function at least defined on E.

It is easy to define the Sobolev spaces $W^{1,p}(S_r)$, $1 \leq p < +\infty$, for instance by making smooth local changes of variable that sent surface disks in S_r to disks in $\mathbb{R}^{n-1}$. We can then define the distribution gradient $\nabla_t u$ for $u \in W^{1,p}(S_r)$, in such a way as to coincide with the usual notion when u is smooth. We shall use the Poincaré inequalities on surface disks, which says that if $1 \leq p < +\infty$, $u \in W^{1,p}(S_r)$, and $D = B(x,s) \cap S_r$ for some choice of $x \in S_r$ and $0 < s \leq 2r$, then

$$\fint_D |u - m_B u|^p d\sigma \leq C_p s^p \fint_D |\nabla_t u|^p d\sigma. \tag{4.5}$$

This can be proved just like (4.2) above; here C_p does not depend on r, x, s, or u.

We shall now prove an analogue of Lemma 3.2 on the sphere $\partial B(0,r)$. The proof will also apply on $\mathbb{R}^n$, and give a proof of Lemma 3.2. We state the result for all p, but we are mostly interested in $p = 1$ or 2.

Lemma 4.1 *Set $S_r = \partial B(0,r)$, let $E \subset S_r$ be a measurable set such that $\sigma(S_r \setminus E) > 0$, and suppose $u \in W^{1,p}(S_r)$ for some $p \in [1,+\infty)$ is such that $u(x) = 0$ for σ-a.e. $x \in S_r \setminus E$. Then*

$$\int_E |u|^p \leq C_p r^p \frac{\sigma(S_r)}{\sigma(S_r \setminus E)} \int_E |\nabla_t u|^p \tag{4.6}$$

and, in the special case when $\sigma(E) < \frac{1}{2}\sigma(S_r)$,

$$\int_E |u|^p \leq C_p \sigma(E)^{\frac{p}{n-1}} \int_E |\nabla_t u|^p. \tag{4.7}$$

Here C_p depends only on n and p.

Proof. We start with the easier (4.6). Notice that $|u(x) - m_B^\sigma u|^p = |m_B^\sigma u|^p$ almost everywhere on $S_r \setminus E$ (just because $u(x) = 0$). We average and get that

$$
\begin{aligned}
|m_B^\sigma u|^p &= \sigma(S_r \setminus E)^{-1} \int_{S_r \setminus E} |u(x) - m_B^\sigma u|^p d\sigma(x) \\
&\leq \frac{\sigma(S_r)}{\sigma(S_r \setminus E)} \fint_{S_r} |u - m_B^\sigma u|^p d\sigma \leq C_p r^p \frac{\sigma(S_r)}{\sigma(S_r \setminus E)} \fint_{S_r} |\nabla_t u|^p d\sigma
\end{aligned}
\tag{4.8}
$$

by (4.5); then

$$
\begin{aligned}
\int_E |u|^p &\leq C_p \int_{S_r} \left[|u - m_B^\sigma u|^p + |m_B^\sigma u|^p\right] \leq C_p' r^p \int_{S_r} |\nabla_t u|^p + C_p |m_B^\sigma u|^p \sigma(S_r) \\
&\leq C_p'' r^p \frac{\sigma(S_r)}{\sigma(S_r \setminus E)} \int_{S_r} |\nabla_t u|^p d\sigma
\end{aligned}
\tag{4.9}
$$

by (4.5) and (4.8); (4.6) follows.

Now we prove (4.7). The proof will also work for Lemma 3.2. We shall use a covering. Let $x \in E$ be given, and consider the density ratio $d(x,t) = \sigma(E \cap B(x,t))/\sigma(S_r \cap B(x,t))$. Notice that $d(x, 2r) = \sigma(E)/\sigma(S_r) < 1/2$ if $\sigma(E) < \sigma(S_r)/2$. If x is a Lebesgue density point of E (on the sphere), $\lim_{t \to 0} d(x,t) = 1$, and by continuity we can find $t = t(x) \in (0, 2r)$ such that $d(x,t) = 1/2$. Then use the Besicovitch covering lemma (see for instance [F]) to find a set X such that the $B(x, r(x))$, $x \in X$, cover the set of Lebesgue points of E (and hence, σ-almost all E), but $\sum_{x \in X} \mathbb{1}_{B(x,r(x))} \leq C$.

Fix $x \in X$ and set $D = B(x, t(x)) \cap S_r$ The point of choosing $t(x)$ as we did is that, as in (4.8),

$$
\begin{aligned}
|m_D^\sigma u|^p &= \sigma(D \setminus E)^{-1} \int_{D \setminus E} |u - m_D^\sigma u|^p \\
&= 2\sigma(D)^{-1} \int_{D \setminus E} |u - m_D^\sigma u|^p \leq 2 \fint_D |u - m_D^\sigma u|^p \leq 2C_p t(x)^p \fint_D |\nabla_t u|^p
\end{aligned}
\tag{4.10}
$$

because $u(x) = 0$ almost everywhere on $S_r \setminus E$, and by (4.5). Then, as in (4.9),

$$
\int_D |u|^p \leq C_p \int_D |u - m_D^\sigma u|^p + C_p |m_D^\sigma u|^p \sigma(D) \leq C_p' t(x)^p \int_D |\nabla_t u|^p \tag{4.11}
$$

by (4.5) and (4.10). Observe that $t(x)^{n-1} \leq C\sigma(D) = C\sigma(S_r \cap B(x,t)) = 2C\sigma(E \cap B(x,t)) \leq 2C\sigma(E)$ by various definitions. We now sum over $x \in X$ and get that

$$
\begin{aligned}
\int_E |u|^p &\leq \sum_{x \in X} \int_{B(x,t(x)) \cap S_r} |u|^p \leq C_p \sum_{x \in X} t(x)^p \int_{B(x,t(x)) \cap S_r} |\nabla_t u|^p \\
&\leq C_p \sigma(E)^{\frac{p}{n-1}} \sum_{x \in X} \int_{S_r} \mathbb{1}_{B(x,t(x))} |\nabla_t u|^p \leq C_p \sigma(E)^{\frac{p}{n-1}} \int_{S_r} |\nabla_t u|^p
\end{aligned}
\tag{4.12}
$$

because the $B(x, t(x))$, $x \in X$, have bounded covering. This completes our proof of (4.7); Lemma 4.1 follows. ∎

Next we talk about the restriction of $u \in W^{1,p}$ to almost every sphere S_r. Let $u \in W^{1,p}_{loc}(\mathbb{R}^n)$ be given, with $1 \leq p < +\infty$. It is possible to define the trace of u on every sphere S_r; this looks fine, but the trace is a less regular object that will be difficult to control. What we shall do instead is use arguments based on Fubini, and control the restriction of u on almost every sphere, with often a better control.

The following description is not hard to get, for instance by changing variables (so that spheres become hyperplanes) and using Fubini; see for instance Chapter 10-13 in [D]. First, for almost every half line L through the origin, u is equal almost everywhere on L to an absolutely continuous, i.e., the integral of some locally integrable function on L.

Let us modify u on a set of measure zero, so that it becomes absolutely continuous (and hence, continuous) on almost every half line through the origin. This way, for each $r > 0$, we have a radial limit description of the restriction, as

$$u(x) = \lim_{t \to 1} u(tx) \;\text{ for } \sigma\text{-almost every } x \in S_r. \tag{4.13}$$

Next, for almost every $r > 0$,

$$\text{the restriction of } u \text{ to } S_r \text{ lies in } W^{1,p}(S_r), \tag{4.14}$$

and with tangential partial derivatives that coincide almost everywhere on S_r with (or we should rather say, can be naturally computed in terms of) the restriction of the derivative Du to S_r. This will be pleasant, for instance because we immediately get, by (4.3) and Fubini, that

$$\int_{\rho=0}^{r} \int_{S_r} |\nabla_t u|^p d\sigma d\rho = \int_{B(0,r)} |\pi_t(\nabla u)|^p \leq \int_{B(0,r)} |\nabla u|^p, \tag{4.15}$$

where for the sake of the argument we introduced the tangential part $\pi_t(\nabla u)$ of ∇u.

It will also be useful to glue two Sobolev functions along a sphere. Suppose we have two functions $u_1 \in W^{1,p}(B(0,r))$ and $u_2 \in W^{1,p}(B(0,2r) \setminus \overline{B}(0,r))$. Again modulo changing u_1 and u_2 on sets of measure zero, and by the same proof as for (4.13), we have the existence of boundary values

$$\overline{u}_1(x) = \lim_{t \to 1^-} u_1(tx) \;\text{ and } \overline{u}_2(x) = \lim_{t \to 1^+} u_2(tx). \tag{4.16}$$

for σ-almost every $x \in S_r$. Now suppose that

$$\overline{u}_1(x) = \overline{u}_2(x) \;\text{ for } \sigma\text{-almost every } x \in S_r. \tag{4.17}$$

Then set $u(x) = u_1(x)$ for $x \in B(0,r)$ and $u(x) = u_2(x)$ for $x \in B(0,2r) \setminus B(0,r)$. It is not hard to check that

$$u \in W^{1,p}(B(0,2r)), \tag{4.18}$$

and of course its derivative can be computed locally, so it coincides with the derivative of u_i on the corresponding domain. That is, because of the absence of jump, the distribution derivative of u does not have an extra piece on S_r. The proof is not hard: for the existence of the radial derivative, for instance, we integrate against a test function, use Fubini to integrate on rays, write u as the integral of its radial derivative, and on each good ray do a soft integration by parts (using Fubini). Again, see [D] for details.

Let us record a last estimate where we mix the values in $B(0,r)$ and on S_r.

Lemma 4.2 *Suppose $u \in W^{1,1}(B(0,r))$, and let $\overline{u}$ denote the boundary values of u on S_r, defined σ-almost everywhere on S_r as in (4.16). Then $\overline{u} \in L^1(S_r)$, and*

$$|m^\sigma_{S_r}\overline{u} - m_{B(0,r)}u| \leq Cr \fint_{B(0,r)} |\nabla u|. \tag{4.19}$$

Proof. Change u on a set of measure 0 so that u is absolutely continuous along almost all rays. Then, for almost every $y \in S_r$, we get that for every $\rho \in (1/2, 1)$

$$|\overline{u}(y) - u(\rho y)| \leq r \int_{t=\rho}^1 |\nabla u|(ty)dt \leq r \int_{1/2<t<1} |\nabla u(ty)|dt \tag{4.20}$$

and hence, setting $m = m_{B(0,r)}u$,

$$|\overline{u}(y) - m| \leq |u(\rho y) - m| + r \int_{1/2<t<1} |\nabla u(ty)|dt. \tag{4.21}$$

We average this over ρ and get that

$$|\overline{u}(y) - m| \leq 2 \int_{1/2<t<1} |u(ty) - m|dt + r \int_{1/2<t<1} |\nabla u(ty)|dt. \tag{4.22}$$

Then we average on $y \in S_r$ and obtain

$$\begin{aligned} \fint_{S_r} |\overline{u}(y) - m|d\sigma(y) &= \fint_{y\in S_r} \int_{1/2<t<1} 2|u(ty) - m| + r|\nabla u(ty)| \\ &\leq C \fint_{z\in B(0,r)} |u(z) - m| + r|\nabla u(z)|, \end{aligned} \tag{4.23}$$

where the constant C comes from a Jacobian, but which we control because we restricted to $t > 1/2$. We apply Poincaré's inequality (4.2), get that

$$\fint_{S_r} |\overline{u}(y) - m|d\sigma(y) \leq Cr \fint_{B(0,r)} |\nabla u|, \tag{4.24}$$

and use the triangle inequality to conclude from there. ∎

5 Minimizers are bounded

In this section we use a variant of the maximum principle to show that $\mathbf{u}$ is bounded when $(\mathbf{u}, \mathbf{W}) \in \mathcal{F}$ is a minimizer for the functional J and the following assumptions hold:

$$|\Omega| < +\infty, \tag{5.1}$$

and, for $1 \leq i \leq N$,

$$f_i \in L^p(\Omega) \ \text{ for some } p > n/2, \tag{5.2}$$

$$f_i(x) \geq 0 \text{ almost everywhere}, \tag{5.3}$$

and

$$g_i(x) \in L^p(\Omega) \text{ for some } p > n/2. \tag{5.4}$$

When $n = 1$, let us not allow exponents smaller than 1. That is, let us understand that (5.2) and (5.4) mean that the f_i and the g_i lie in $L^1(\Omega)$ when $n = 1$.

In this section we shall not be able to escape from using Sobolev exponents entirely, so we decided to choose our exponents a little in the spirit of Remark 3.4. The reader may assume that the f_i are bounded and (in dimensions $n \leq 3$) the g_i lie in L^2; this will simplify some estimates (as in Section 3) but unfortunately not all.

We keep the same conditions on the f_i as in Remark 3.4, and this way we can compute the terms $\int u_i^2 f_i$ because $u_i \in W^{1,2}(\mathbb{R}^n)$ and $u_i = 0$ a.e. on $\mathbb{R}^n \setminus \Omega$; see Remark 3.4. Our condition on the g_i is stronger now because $\frac{n}{2} \geq \frac{2n}{n+2}$ when $n \geq 2$. Thus we can also compute the $\int u_i g_i$.

The new constraint that now $g_i \in L^p$ for some $p > n/2$ is about right: we expect Theorem 5.1 to fail when $g_i \notin L^{n/2}$; see Remark 5.2 at the end of this section.

Here we shall only assume that F is bounded; then we may not have an existence theorem, but this does not matter. We only need to assume that F is bounded because we want a bound on $E(\mathbf{u})$ (see below); otherwise we don't need information on F because we shall not modify the W_i in the proof.

Theorem 5.1 *Assume that (5.1)-(5.4) hold, that the volume functional F is bounded, and that $(\mathbf{u}, \mathbf{W})$ is a minimizer for J in $\mathcal{F}$ (see Definition 1.1 and (1.3)-(1.5)). Then $\mathbf{u}$ is bounded, and we have bounds on the $||u_i||_\infty$ that depend only on n, N, p, a bound for F, and the constants that arise in (5.1) and(5.4).*

Proof. First we should observe that we have bounds on the energy $E(\mathbf{u})$ of (3.6), which depend only on n, N, p, a bound for F, and the constants that arise in (5.1) and (5.4). This follows from (3.8) (with $f_{i,-} = 0$ and where we never use estimates on the size of f_i), and a trivial bound for $J(\mathbf{u}, \mathbf{W})$ that we obtain by testing the function $\mathbf{u} = 0$. We included the assumption that F is bounded precisely for this; otherwise, our estimates would also depend on $E(\mathbf{u})$.

We shall first do the proof in dimensions larger than 2, because of complications with the sign of the fundamental solution of $-\Delta$ in dimension 2; the proof will have to be modified when $n = 2$; we will take care of that near (5.21).

So let us assume that $n \geq 3$. We intend to show that each u_i is bounded by comparing it with a function v that we shall construct by hand. Fix i, and choose $\rho \in L^p(\mathbb{R}^n)$, with $||\rho||_p \leq ||g_i||_p$ and such that $\rho(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$ but $\rho(x) > |g_i(y)|/2$ almost everywhere on Ω (we lose a factor 2 but we win a strict inequality). Then set

$$v(x) = (\rho * G)(x) = \int_{\mathbb{R}^n} G(x-y)\rho(y)dy, \tag{5.5}$$

where G is the fundamental solution of $-\Delta$. Here $n \geq 3$ so $G(z) = c_n|z|^{2-n}$ for some positive constant c_n. In dimension 2, we would get a logarithm, and we would not like that as much because it takes negative values.

Let us first check that the integral converges, and even that for all $x \in \mathbb{R}^n$,

$$0 \leq v(x) = C|\Omega|^{\frac{2p-n}{np}}||g_i||_p. \tag{5.6}$$

Let q denote the conjugate exponent of p and $r > 0$ be such that $|\Omega| = |B(0,r)|$. Observe that since G is radial and decaying,

$$\int_\Omega G(x-y)^q dy \leq \int_{B(0,r)} G(z)^q dz = C\int_{B(0,r)} |z|^{-q(n-2)}dz. \tag{5.7}$$

Recall that $p > n/2$, so $p^{-1} < 2/n$, $q^{-1} > (n-2)/n$, $q < n/(n-2)$, $q(n-2) < n$, and the integral converges. We also get that $\int_\Omega G(x-y)^q dy \leq Cr^{n-q(n-2)}$. Then by Hölder

$$||v||_\infty \leq ||G||_{L^q(B(0,r))}||\rho||_p \leq Cr^{\frac{n}{q}-(n-2)}||\rho||_p = C|\Omega|^{\frac{1}{q}-\frac{n-2}{n}}||\rho||_p, \tag{5.8}$$

which implies (5.6) because $||\rho||_p \leq ||g_i||_p$. We may as well assume that $p < n$, because the result is easier otherwise. We shall need to know that

$$\Delta v = -\rho \ \text{ and } \ \nabla^2 v \in L^p(\mathbb{R}^n) \tag{5.9}$$

and

$$\nabla v \in L^r(\mathbb{R}^n), \text{ with } r = \frac{n}{n-p}. \tag{5.10}$$

We start with (5.9); when φ is smooth and compactly supported, $\Delta(G * \varphi) = -\varphi$ (by our choice of G), and $\nabla^2(G*\varphi)$ is obtained from $-\varphi$ by applying second order Riesz transforms. These are known to be bounded on L^p because $p > 1$ and we assumed that $p < n < +\infty$ (for this fact and the proof of (5.10), see [S]), so we get that $||\nabla^2(G * \varphi)||_p \leq C||\varphi||_p$. Then (5.9) follows by writing ρ as a limit in L^p of test functions φ.

So v lies in the Sobolev space $W^{2,p}(\mathbb{R}^n)$, hence $\nabla v \in W^{1,p}(\mathbb{R}^n) \subset L^r$, where $1/r = 1/p - 1/n$, and (5.10) holds. See for instance Theorem 1 on page 119 of [S].

Return to the proof of Theorem 5.1, still when $n \geq 3$. We shall show that $u_i \leq v$ almost everywhere on Ω (hence on $\mathbb{R}^n$, since $u_i = 0$ on $\mathbb{R}^n \setminus \Omega$) and the proof will also show that $u_i \geq -v$ (either change the sign of u_i and g_i in the functional, or modify slightly the proof). Theorem 5.1 will follow because v is bounded by (5.6). Set

$$w = \min(u_i, v), h = u_i - w \geq 0, \tag{5.11}$$

and, for $0 < t < 1$,

$$u_{t,i} = u_i - th = (1-t)u_i + tw. \tag{5.12}$$

We want to see whether replacing u_i with $u_{t,i}$ yields a better competitor. First observe that for almost every $x \in \mathbb{R}^n \setminus W_i$, $u_i(x) = 0$ and hence $w(x) = h(x) = u_{t,i}(x) = 0$. Next, $v \in W^{1,2}_{loc}(\mathbb{R}^n)$, by (5.10) and because $\frac{n}{n-p} \geq 2$ when $n/2 \leq p < n$. Since both u_i and v lie in $W^{1,2}_{loc}(\mathbb{R}^n)$, so do w, h, and $u_{t,i}$. Since their derivative vanishes a.e. on $\mathbb{R}^n \setminus W_i$, we get that

$$w,\ h,\ \text{and } u_{t,i} \text{ lie in } W^{1,2}(\mathbb{R}^n). \tag{5.13}$$

Notice also that if $u_i \geq 0$, then $w \geq 0$ too, and (by (5.12) and because $0 < t < 1$), $u_{t,i} \geq 0$. Because of all this, if we replace u_i with $u_{t,i}$, leave the other u_j as they are, and also keep the same sets W_j, we get a new pair $(\mathbf{u}_t, \mathbf{W})$ that still lies in $\mathcal{F}$. By minimality, $J(\mathbf{u}_t, \mathbf{W}) \geq J(\mathbf{u}, \mathbf{W})$. We compute

$$\begin{aligned} J(\mathbf{u}_t, \mathbf{W}) - J(\mathbf{u}, \mathbf{W}) &= E(\mathbf{u}_t) - E(\mathbf{u}) + M(\mathbf{u}_t) - M(\mathbf{u}) \\ &= \int_\Omega |\nabla u_{t,i}|^2 - |\nabla u_i|^2 + [u_{t,i}^2 - u_i^2] f_i - [u_{t,i} - u_i] g_i\,; \end{aligned} \tag{5.14}$$

then we use (5.12) and compute the derivative

$$\begin{aligned} \frac{d}{dt} J(\mathbf{u}_t, \mathbf{W})_{|t=0} &= -2\int_\Omega \langle \nabla u_i, \nabla h\rangle - 2\int_\Omega h u_i f_i + \int_\Omega h g_i \\ &\leq -2\int_\Omega \langle \nabla u_i, \nabla h\rangle + \int_\Omega h g_i \end{aligned} \tag{5.15}$$

because h and f_i are nonnegative, and $u_i(x) > v(x) \geq 0$ when $h(x) \neq 0$. Notice that our assumption (5.2) that f_i lie in L^p for some $p > n/2$ was useful to define $\int [u_{t,i}^2 - u_i^2] f_i$ (see Remark 3.4); now we used the main assumption that $f_i \geq 0$ and we can forget about (5.2) (and in fact f_i altogether).

The derivative is nonnegative (because $J(\mathbf{u}_t, \mathbf{W}) \geq J(\mathbf{u}, \mathbf{W})$), hence

$$2\int_\Omega \langle \nabla u_i, \nabla h\rangle \leq \int_\Omega h g_i. \tag{5.16}$$

We may now use the fact that $\Delta v = -\rho$ (by (5.9)), which means that

$$\int_{\mathbb{R}^n} \langle \nabla v, \nabla \varphi \rangle = -\int_{\mathbb{R}^n} \Delta v \varphi = \int_{\mathbb{R}^n} \rho\varphi \tag{5.17}$$

for every test function φ. Let us check that this remains true with $\varphi = h$. Denote by r' the conjugate exponent of $r = \frac{n}{n-p}$ in (5.10); thus $r' = \frac{n}{p} < 2$. Recall from (5.13) that $h \in W^{1,2}(\mathbb{R}^n)$; thus $h \in W^{1,r'}(\mathbb{R}^n)$ (recall that h and ∇h are supported on $W_i \subset \Omega$), and we can write h as a limit in $W^{1,2} \cap W^{1,r'}$ of test functions φ_k. Then the left-hand side of (5.17) converges to $\int_{\mathbb{R}^n} \langle \nabla v, \nabla h \rangle$, because $\nabla v \in L^r(\Omega)$ by (5.10).

For the left-hand side, we know that $\rho \in L^p$, so it is enough to show that the φ_k converge to h in $L^{p'}$. But $p' < \frac{n}{n-2}$ because $p > \frac{n}{2}$, so the desired convergence follows from the convergence of the φ_k in $W^{1,2}$, because the Sobolev exponent for $W^{1,2}$ is $\frac{2n}{n-2} > \frac{n}{n-2}$, and by the proof of Lemma 3.2. So (5.17) holds with $\varphi = h$, i.e.,

$$\int_{\mathbb{R}^n} \langle \nabla v, \nabla h \rangle = \int_{\mathbb{R}^n} \rho h. \tag{5.18}$$

Set $Z = \{x \in \mathbb{R}^n \,;\, h(x) > 0\}$. By (5.11), $w(x) \neq u_i(x)$ and hence $w(x) = v(x)$ on Z. Since $\nabla h = 0$ almost everywhere on $\mathbb{R}^n \setminus Z$, we get that

$$\begin{aligned}\int_{\mathbb{R}^n} \rho h &= \int_{\mathbb{R}^n} \langle \nabla v, \nabla h \rangle = \int_Z \langle \nabla v, \nabla h \rangle = \int_Z \langle \nabla w, \nabla h \rangle \\ &= \int_{\mathbb{R}^n} \langle \nabla w, \nabla h \rangle = \int_{\mathbb{R}^n} \langle \nabla (u_i - h), \nabla h \rangle \leq \int_{\mathbb{R}^n} \langle \nabla u_i, \nabla h \rangle\end{aligned} \tag{5.19}$$

(by (5.11) again, and because $\int |\nabla h|^2 \geq 0$). By (5.19) and (5.16)

$$\int_{\mathbb{R}^n} 2\rho h \leq 2\int_{\mathbb{R}^n} \langle \nabla u_i, \nabla h \rangle = 2\int_{\Omega} \langle \nabla u_i, \nabla h \rangle \leq \int_{\Omega} h g_i. \tag{5.20}$$

Recall that we chose ρ such that $\rho = O$ on $\mathbb{R}^n \setminus \Omega$ and $\rho > |g_i|/2$ everywhere on Ω; since $h \geq 0$ by (5.11), we deduce from (5.20) that $h(x) = 0$ almost everywhere on Ω, which precisely means that $u_i \leq v$ almost everywhere on Ω. A minor modification of the argument would show that $-u_i \leq v$ too, and we have seen that this proves Theorem 5.1 when $n \geq 3$.

We now turn to $n = 2$. A fundamental solution of $-\Delta$ is now $\log(1/|x|)$, but since it becomes negative for $|x|$ large, we shall need to localize the argument above. Let i be given, and also choose r so that $|B(0,r)| = 2|\Omega|$. We will find a constant C_0 (that depends only on $p > 1$, $|\Omega|$ and $||g_i||_p$) such that $u_i \leq C_0$ almost everywhere on $B(0, r/2)$. The same proof would also yield $-u_i \leq C_0$, and since we can choose the origin arbitrarily, this will give the desired L^∞ bound. We still choose $\rho \in L^p$ such that $||\rho||_p \leq ||g_i||_p$, $\rho = 0$ on $\mathbb{R}^n \setminus \Omega$, and $2\rho > |g_i|$ on Ω, and set

$$v(x) = \int_{B(0,2r)} G(x-y)\rho(y)dy, \text{ with } G(x) = c\log(10r/|x|), \tag{5.21}$$

where c is still chosen so that $\Delta G = -\delta_0$ (a Dirac mass); we added a constant to the logarithm to make sure that $v \geq 0$ in $B(0,2r)$. Denote by q the conjugate exponent of p and observe that for $x \in B(0,2r)$,

$$(5.22) \qquad \int_{B(0,2r)} |G(x-y)|^q dy \leq \int_{z \in B(0,4r)} |G(z)|^q dz \leq \int_{B(0,4r)} \log^q(10r/|x|) \leq C,$$

where C depends on p and r, so by Hölder

$$(5.23) \qquad |v(x)| \leq C||\rho||_p \leq C||g_i||_p \text{ for } x \in B(0,2r).$$

Now we want to choose a radius $s \in (r, 2r)$ where we will do some surgery. We modify u_i on a set of measure zero so that (4.13) (the continuity of u_i along almost all rays) holds. We want the restriction of u_i to S_s to be in $W^{1,2}$; this is true for almost every s, because $u_i \in W^{1,2}(\mathbb{R}^n)$ (compare with (4.14)). We also require that

$$(5.24) \qquad \int_{S_s} |\nabla_t u_i|^2 d\sigma \leq 10r^{-1} \int_{\mathbb{R}^2} |\nabla u_i|^2,$$

which by (4.15) and Chebyshev is true except for a set of measure at most $r/10$ of radii s. In addition, we chose s such that

$$(5.25) \qquad u_i(y) = 0 \text{ for } \sigma\text{-almost every } y \in S_s \setminus \Omega$$

(also true for almost every s, because $u_i = 0$ almost everywhere on $\mathbb{R}^n \setminus W_i$) and

$$(5.26) \qquad \sigma(S_s \setminus \Omega) \geq 10^{-1} \sigma(S_s)$$

which by Fubini holds for most $s \in (r, 2r)$ because

$$\begin{aligned} \int_{s=r}^{2r} \sigma(S_s \setminus \Omega) ds &= \Big| [B(0,2r) \setminus B(0,r)] \setminus \Omega \Big| \\ (5.27) \qquad &\geq |B(0,2r) \setminus B(0,r)| - |\Omega| \geq \frac{1}{2} |B(0,2r) \setminus B(0,r)| \end{aligned}$$

by (4.3) and because $|B(0,r)| = 2|\Omega|$. So we choose s with all these properties.

Here things will be simpler because $n = 2$; we could make the argument work in higher dimensions (instead of adding a constant below, add the harmonic extension of the restriction of $|u_i|$ to S_s), but let us not do that. By (5.24), we have that for almost all choices of $x, y \in S_s$,

$$(5.28) \qquad |u_i(x) - u_i(y)| \leq \int_{S_s} |\nabla_t u_i| d\sigma \leq C s^{1/2} \Big\{ \int_{S_s} |\nabla_t u_i|^2 d\sigma \Big\}^{1/2} \leq C ||\nabla u_i||_2.$$

We apply this with some $y \in S_s \setminus \Omega$ and get that

$$(5.29) \qquad |u_i(x)| < C_1 \text{ for } \sigma\text{-almost every } x \in S_s,$$

where C_1 depends on $||\nabla u_i||_2$.

We may now define a competitor for $(\mathbf{u}, \mathbf{W})$, as we did near (5.11). First set

$$
\begin{aligned}
w(x) &= \; u_i(x) && \text{for } x \in \mathbb{R}^2 \setminus B(0,s) \\
w(x) &= \; \min(u_i(x), v(x) + C_1) && \text{for } x \in B(0,s),
\end{aligned}
\tag{5.30}
$$

where v is as in (5.21) and C_1 as in (5.29). We want to show that

$$
w \in W^{1,2}(\mathbb{R}^2),
\tag{5.31}
$$

and we shall first check that

$$
v \in W^{1,2}(B(0,2r)) \;\text{ with }\; \nabla v = \nabla G * (\mathbb{1}_{B(0,2r)}\rho).
\tag{5.32}
$$

Recall from (5.21) that $v = G * \psi$, with $\psi = \mathbb{1}_{B(0,2r)}\rho \in L^p$.

Let r be such that $1/r = 1/p - 1/2 < 1/2$; since $|\nabla G|$ is a Riesz potential of order 1, the Hardy-Littlewood-Sobolev theorem of fractional integration on page 119 of [S] says that for any $\psi \in L^p(\mathbb{R}^n)$, $|\nabla G| * |\psi| \in L^r$, with

$$
\big|\big||\nabla G| * |\psi|\big|\big|_r \le C||\psi||_p.
\tag{5.33}
$$

Returning to $\psi = \mathbb{1}_{B(0,2r)}\rho$, we see that $\nabla G * \psi \in L^r$ and, since $r > 2$, $\nabla G * \psi \in L^2(B(0,2r))$ as well.

We still need to check that $\nabla G * \psi$ is the distribution derivative of v in $B(0,2r)$. If ψ were a test function, or just bounded with compact support, we could brutally differentiate $v = G * \psi$ under the integral, using the fact that $|\nabla G|$ is integrable near the origin, and get that $\nabla v = \nabla G * \psi$ (a continuous derivative). Unfortunately, ψ is not bounded, but we can write it as the limit in L^p of a sequence of test functions ψ_k. Then set $v_k = G * \psi_k$; we just observed that $\nabla v_k = \nabla G * \psi_k$, and (5.33) shows that ∇v_k tends to $\nabla G * \psi$ in L^r, hence in $L^2(B(0,2r))$. But v_k tends to v in $L^2(B(0,2r))$, more brutally because $|G|$ is integrable near the origin, and this implies that $v \in W^{1,2}(B(0,2r))$, with $\nabla v = \nabla G * \psi$ (pair v against a the gradient of test function).

So (5.32) holds, and by (5.30) this implies that $w \in W^{1,2}(B(0,r))$, because the minimum of two functions of $W^{1,2}$ lies in $W^{1,2}$.

We also have that $w \in W^{1,2}(\mathbb{R}^2 \setminus \overline{B}(0,r))$, again by (5.30) and because $u_i \in W^{1,2}(\mathbb{R}^2)$.

We want to glue the two pieces, so we consider the radial limits of w on S_s. From $\mathbb{R}^2 \setminus B(0,s)$, this limit is equal to the restriction of u_i to S_s (because u_i is continuous along almost all rays). From $B(0,s)$, let us modify v so that it is also continuous along almost all rays; then $v + C_1$ has limits on S_r that are larger than the values of u_i, by (5.29) and because $v \ge 0$ on $B(0,2r)$. Then the radial limit of w from $B(0,s)$ is the same as for u_i, the gluing condition (4.17) holds, and (4.18) says that $w \in W^{1,2}(\mathbb{R}^2)$, as needed for (5.31).

Obviously $w \le u_i$ (just by (5.30)), so $h = u_i - w$ is nonnegative, and null outside of B_s. Now we follow the same argument as when $n \ge 3$, starting below (5.12). We still have (5.16) for the same reasons, but we need to compute its left-hand side $2\int_{\mathbb{R}^n}\langle \nabla u_i, \nabla h\rangle$ differently. We want to replace (5.17) with the fact that

$$\int_{\mathbb{R}^n} \langle \nabla v, \nabla \varphi\rangle = \int_{\mathbb{R}^n} \rho \mathbb{1}_{B(0,2r)}\varphi \tag{5.34}$$

for every test function φ, which we expect to hold because $v = G * \rho\mathbb{1}_{B(0,2r)}$ should yield $\Delta v = -\rho\mathbb{1}_{B(0,2r)}$ in the sense of distributions. And indeed (5.32) yields

$$\begin{aligned}\int_{\mathbb{R}^n} \langle \nabla v, \nabla \varphi\rangle &= \int_{\mathbb{R}^n} \langle \nabla G * \rho\mathbb{1}_{B(0,2r)}, \nabla\varphi\rangle \\ &= \int_{x\in\mathbb{R}^n}\int_{y\in B(0,2r)} \rho(y)\langle \nabla G(x-y), \nabla\varphi(x)\rangle dxdy,\end{aligned} \tag{5.35}$$

where the double integral converges absolutely by (5.33). We apply Fubini, use the fact that

$$\int \langle \nabla G(x-y), \nabla\varphi(x)\rangle dx = \int \langle \nabla G(x), \nabla\varphi(x+y)\rangle = -\langle \Delta G, \varphi(\cdot+y)\rangle = \varphi(y) \tag{5.36}$$

by definition of G (and because φ is a test function), and get (5.34). Then we wish to replace φ with h in (5.34), i.e., get that

$$\int_{\mathbb{R}^n} \langle \nabla v, \nabla h\rangle = \int_{B(0,2r)} \rho h. \tag{5.37}$$

as in (5.18). Recall that $h = u_i - w$, so $h \in W^{1,2}(\mathbb{R}^n)$. In addition, $h = 0$ on $\mathbb{R}^n \setminus B^s$, so the Sobolev embedding theorem says that in lie in L^q for every $q < +\infty$. We choose for q the dual exponent of p (recall that $p > n/2 = 1$). Then we write h as a limit of test functions φ_k, so that the $\nabla\varphi_k$ converge to ∇h in L^2 and the φ_k converge to h in L^q. The identity (5.34) for φ_k goes to the limit, because $\nabla v \in L^2$ by (5.32) and $\rho \in L^p$ by definition. So (5.37) holds.

Set $Z = \big\{x \in \mathbb{R}^n\,;\, h(x) > 0\big\}$ as before, and notice that $Z \subset B_s$ (modulo a negligible set). This time the definition of h yields $w(x) = v(x) + C_1$ on Z, and

$$\begin{aligned}\int_{\mathbb{R}^n} \rho h &= \int_{\mathbb{R}^n} \langle \nabla v, \nabla h\rangle = \int_Z \langle \nabla v, \nabla h\rangle = \int_Z \langle \nabla (w - C_1), \nabla h\rangle \\ &= \int_{\mathbb{R}^n} \langle \nabla w, \nabla h\rangle = \int_{\mathbb{R}^n} \langle \nabla (u_i - h), \nabla h\rangle \le \int_{\mathbb{R}^n} \langle \nabla u_i, \nabla h\rangle\end{aligned} \tag{5.38}$$

by (5.37), as in (5.19), and because $\nabla C_1 = 0$. So the conclusion of (5.19) still holds, and we may end the argument as before. This completes our proof of Theorem 5.1 when $n = 2$.

We are left with the easier case when $n = 1$. Recall that we have a bound on $E(\mathbf{u})$ (see the very beginning of the proof, and recall that we could take $p = r = 1$ in Remark 3.4); then u_i is Hölder continuous, with bounds that depend only $E(\mathbf{u})$, and since $u_i = 0$ almost everywhere on $\mathbb{R} \setminus \Omega$, we get the desired L^∞ bounds on the u_i because $|\Omega| < +\infty$. ∎

Remark 5.2 *The constraint that $p > n/2$ in (5.4) is not so far from optimal, in the sense that the exponent $n/2$ cannot be made smaller.*

Let us just consider the case when $N = 1$, $f_1 = 0$, $F = 0$, and $\Omega = B(0,1)$. It is not too hard to check that the minimizer for J is the pair (u, Ω), where u is the solution of $\Delta u = -g_1/2$, with the usual Dirichlet constraint $u = 0$ on $\mathbb{R}^n \setminus \Omega$ (see (9.6) below for the equation). Locally u behaves like $G * g_1$, which in general is not bounded (even locally) when $g_1 \notin L^{n/2}$; for instance, take $g_1(x) = |x|^{-2}$ near the origin and observe that $G * g_1(x)$ tends to $G * g_1(0) = +\infty$ when x tends to 0.

6 Two favorite competitors

We shall soon start for good our study of the local regularity of minimizers for the functional J, and in this section we present constructions of competitors that we shall often use to obtain information on such minimizers. In particular, the second one (harmonic competitors) will sometimes be a good replacement for the main competitor that people use when $N \leq 2$.

We are given a pair $(\mathbf{u}, \mathbf{W}) \in \mathcal{F}$ (see Definition 1.1), and a ball B, and we want to define other pairs $(\mathbf{u}^*, \mathbf{W}^*) \in \mathcal{F}$ by modifying $\mathbf{u}$ and $\mathbf{W}$ in B (with $u = u^*$ and $W = W^*$ in $\Omega \setminus B$), and then compare them with $(\mathbf{u}, \mathbf{W}) \in \mathcal{F}$ to get valuable information if $(\mathbf{u}, \mathbf{W})$ is a minimizer.

The first pair will be called the <u>cut-off competitor</u>. We may as well suppose that $B = B(0,r)$ (by translation invariance of our problems), we give ourselves a number $a \in (0,1)$ (often close to 1), and we choose a smooth cut-off function φ such that

$$\begin{array}{lr} & \varphi(t) = 0 \qquad \text{for } 0 \leq t \leq ar \\ (6.1) & 0 \leq \varphi(t) \leq 1 \qquad \text{for } ar \leq t \leq r \\ & \varphi(t) = 1 \qquad \text{for } t \geq r \\ & 0 \leq \varphi'(t) \leq 2(1-a)^{-1} r^{-1} \qquad \text{everywhere.} \end{array}$$

Then we pick any collection I of indices $1 \leq i \leq N$, and set

$$(6.2) \qquad \begin{aligned} u_i^*(x) &= \varphi(|x|) u_i(x) \quad \text{when } i \in I \\ u_i^*(x) &= u(x) \quad \text{when } i \notin I. \end{aligned}$$

Here the simplest is to keep $\mathbf{W}^* = \mathbf{W}$, because $u_i^*(x) = 0$ as soon as $u_i(x) = 0$, but the fact that $u_i^* = 0$ in $B(0, ar)$ allows us to modify some of the $W_i \cap B(0, ar)$ in some arguments. Anyway, it is easy to see that $(\mathbf{u}^*, \mathbf{W}) \in \mathcal{F}$; the only thing left to check is that $u_i \in W^{1,2}(\mathbb{R}^n)$ for $i \in I$. But the definition of a distribution derivative yields

$$(6.3) \qquad \nabla u_i^*(x) = \varphi(|x|) \nabla u_i(x) + u_i(x) \varphi'(|x|) \frac{x}{|x|}$$

which lies in L^2 because φ' is bounded and $u \in L^2(B)$ (recall that $u \in W^{1,2}(B)$ and use Poincaré's inequality (4.2)). We shall often choose a small number $\tau > 0$, and apply the fact that $(A+B)^2 = A^2 + B^2 + 2AB \leq (1+\tau)A^2 + (1+\tau^{-1})B^2$ for $A, B \geq 0$ to get that for $x \in B$,

$$|\nabla u_i^*(x)|^2 \leq (1+\tau)|\nabla u_i(x)|^2 + 4(1-a)^{-2}(1+\tau^{-1})r^{-2}|u_i(x)|^2. \tag{6.4}$$

This will typically be useful when we know that, by some application of Poincaré's inequality, $\int_B |u_i|^2$ is small. Of course $\nabla u_i^*(x) = 0$ when $x \in B(0, ar)$ (by (6.1), (6.2), and because $i \in I$). We now integrate and get that for $i \in I$,

$$\int_{B(0,r)} |\nabla u_i^*|^2 \leq (1+\tau) \int_{B(0,r)\setminus B(0,ar)} |\nabla u_i|^2 + 4(1-a)^{-2}(1+\tau^{-1})r^{-2} \int_{B(0,r)\setminus B(0,ar)} |u_i|^2. \tag{6.5}$$

Let us also record trivial estimates for the difference in the terms of $M(\mathbf{u})$; for $i \in I$,

$$\begin{aligned} \Big| \int_\Omega (u_i^*)^2 f_i - \int_\Omega u_i^2 f_i \Big| &= \Big| \int_{B(0,r)} (1 - \varphi(|x|)^2) u_i(x)^2 f_i(x) \Big| \\ &\leq \int_{B(0,r)} |u_i^2 f_i| \leq C r^{n-\frac{n}{p}} ||u_i||_\infty^2 ||f_i||_p \end{aligned} \tag{6.6}$$

and

$$\begin{aligned} \Big| \int_\Omega u_i^* g_i - \int_\Omega u_i g_i \Big| &= \Big| \int_{B(0,r)} (1 - \varphi(|x|)) u_i(x) g_i(x) \Big| \\ &\leq \int_{B(0,r)} |u_i g_i| \leq C r^{n-\frac{n}{p}} ||u_i||_\infty ||g_i||_p. \end{aligned} \tag{6.7}$$

The next competitors that we want to introduce are obtained by extending the values of $\mathbf{u}$ on $S_r = \partial B(0,r)$. Recall from the discussion near (4.13) that modulo modifying $\mathbf{u}$ on a set of measure zero, we can assume that $\mathbf{u}$ is continuous along almost every ray, and (as in (4.14)) that the restriction of u to S_r is itself in $W^{1,2}(S_r)$ for almost every r, with partial derivatives that correspond to the restriction of the derivative Du to S_r.

It will be easier to define our competitors for these radii r, because estimates for the tangential gradient $\nabla_t u$ on the sphere will often be useful to control the extension.

The simplest description is when $N = 1$ (and $u = u_1$ is allowed to be real-valued). We assume that

$$\text{the restriction of } u \text{ to } S_r \text{ lies in } W^{1,2}(S_r) \tag{6.8}$$

(this holds for almost all r, by the discussion above), and also that

$$B(0,r) \subset \Omega \quad \text{(modulo a set of vanishing Lebesgue measure)}. \tag{6.9}$$

Denote by $\overline{u}$ the restriction of u to S_r, and by u^* the harmonic extension of $\overline{u}$ to $B(0,r)$, obtained by convolution of $\overline{u}$ with the Poisson kernel. Also set $u^* = u(x)$ for $x \in \mathbb{R}^n \setminus B(0,r)$. In this simple case the harmonic competitor (for the pair $(u_1, W_1) \in \mathcal{F}$, in the ball $B(0,r)$)

is just the pair (u^*, W^*), where $W^* = W_1 \cup (\Omega \cap B(0,r))$. Let us check that $(u^*, W^*) \in \mathcal{F}$. It is well known that when $\overline{u} \in W^{1,2}(S_r)$, $u^* \in W^{1,2}(B(0,r))$, and even

$$\int_{B(0,r)} |\nabla u^*|^2 = \inf \Big\{ |\nabla v|^2 \, ; \, v \in W^{1,2}(B(0,r)) \text{ and } v = \overline{u} \text{ on } S_r \Big\}, \tag{6.10}$$

where by $v = \overline{u}$ on S_r we mean that the radial limits of v on S_r (which exist as in (4.13) and (4.16)) coincide with $\overline{u}$ almost everywhere on S_r. See for instance [D], Chapter 15 (and use the maximum principle for the uniqueness of the harmonic extension u^*). Also, u^* itself satisfies $u^* = \overline{u}$ on S_r and it is the unique minimizer in (6.10).

The gluing condition (4.17) on S_r holds, because we kept $u^* = u$ on $\mathbb{R}^n \setminus B(0,r)$, so (4.18) implies that $u^* \in W^{1,2}(\mathbb{R}^n)$. Since we added the constraint (6.9), almost every point of $B(0,r)$ lies in W^*, and hence $u^* = 0$ almost everywhere on $\mathbb{R}^n \setminus W^*$ (because $u = 0$ almost everywhere on $\mathbb{R}^n \setminus W_1$). If in addition we required that $u_1 \geq 0$, we still get $u^* \geq 0$. So $(u^*, W^*) \in \mathcal{F}$, and it is often a very good competitor to use, because of (6.10) and the fact that u^* is as smooth as possible in $B(0,r)$.

When $N = 2$, but we only consider nonnegative functions u_i, there is a nice trick that allows us to define the harmonic competitor. We start from $\mathbf{u} = (u_1, u_2)$, and set $u = u_1 - u_2$, which is now a real-valued function in $W^{1,2}(\mathbb{R}^n)$. For r as above (i.e. satisfying (6.8) and (6.9)), we define $\overline{u}$ and u^* as above. Then we set $\mathbf{u}^* = (u_1^*, u_2^*)$, where u_1^* is the positive part of u^* and u_2^* is its negative part. The sets $Z_i = \big\{ x \in B(0,r) \, ; \, u_i^* > 0 \big\}$ are disjoint, so we may set $W_i^* = [W_i \setminus B(0,r)] \cup [Z_i \cap \Omega]$, and get disjoint subsets of Ω. This gives a pair $(\mathbf{u}^*, \mathbf{W}^*)$, which lies in $\mathcal{F}$ as before (notice in particular that $u_i^* = 0$ almost everywhere on $\mathbb{R}^n \setminus W_i^*$, again by (6.9)). This trick has been used extensively in the literature, often implicitly by setting the problem directly in terms of $u = u_1 - u_2$.

Unfortunately, this trick is not available when $N \geq 3$ (or when $N = 2$ and we use real-valued functions). If we just extend the restrictions to S_r of the u_i, we get functions with overlapping supports, and we will not be able to find sets W_i^* for which $(\mathbf{u}^*, \mathbf{W}^*) \in \mathcal{F}$. We shall be able to circumvent this problem at a price; we shall decide that the main contribution to our functional comes from one of the u_i (say, u_1), and we'll make the other ones vanish brutally on a slightly smaller ball. Of course this will only be useful in special situations, where for some reason there is a dominant component u_1. Let us do this; the competitor that we will define now will be still referred to as the <u>harmonic competitor</u> of $(\mathbf{u}, \mathbf{W})$ in $B(0,r)$. Suppose as before that $B(0,r) \subset \Omega$ (as in (6.9)), and that (6.8) holds. Pick $a \in (0,1)$ (rather close to 1), and define φ as in (6.1). For $i \geq 2$, set

$$\begin{aligned} u_i^*(x) &= \varphi(|x|) \overline{u}_i(rx/|x|) && \text{when } x \in B(0,r) \\ u_i^*(x) &= u_i(x) && \text{when } x \in \mathbb{R}^n \setminus B(0,r). \end{aligned} \tag{6.11}$$

This is not exactly the same formula as in the first part of (6.2), because here we only use the values of u_i on S_r to do the extension, and this is why we prefer to have (6.8) (and often some bounds on the norm in $W^{1,2}(S_r)$). Notice that

$$u_i^*(x) = 0 \text{ when } i \geq 2 \text{ and } x \in B(0, ar), \tag{6.12}$$

which will allow u_1 to be nonzero on the whole $B(0,ar)$.

For u_1 we shall use a harmonic extension. Denote by $\overline{u}_1$ the restriction of u_1 to S_r, and then by v_1 the harmonic extension of $\overline{u}_1$ to $B(0,r)$, obtained by convolution of $\overline{u}_1$ with the Poisson kernel. By (6.8), even though we do not know whether $\overline{u}_1$ is continuous, we still have that $v_1 \in W^{1,2}(B(0,r))$, and even that

$$\int_{B(0,r)} |\nabla v_1|^2 = \inf\Big\{ \int_{B(0,r)} |\nabla v|^2 \,;\, v \in W^{1,2}(B(0,r)) \text{ and } v = \overline{u}_1 \text{ on } S_r \Big\}, \tag{6.13}$$

with the same definition of the boundary condition $v = \overline{u}_1$ as in (6.10). We set

$$\begin{array}{lll} u_1^*(x) & = u_1(x) & \text{when } x \in \mathbb{R}^n \setminus B(0,r), \\ u_1^*(x) & = \overline{u}_1(rx/|x|) & \text{when } x \in B(0,r) \setminus B(a,r) \\ u_1^*(x) & = v_1(a^{-1}x) & \text{when } x \in B(0,ar). \end{array} \tag{6.14}$$

Let us now define $\mathbf{W}^*$ so that, with this definition of the u_i^*,

$$(\mathbf{u}^*, \mathbf{W}^*) \in \mathcal{F}. \tag{6.15}$$

First we need to know that $u_i^* \in W^{1,2}(\mathbb{R}^n)$. By construction, it is easy to see that $u_i^* \in W^{1,2}(\mathcal{O})$ for $\mathcal{O} = \mathbb{R}^n \setminus \overline{B}(0,r)$, $\mathcal{O} = B(0,r) \setminus \overline{B}(0,ar)$, and $\mathcal{O} = B(0,ar)$, but we also need to check the gluing condition (4.17) on the spheres S_{ar} and S_r, to make sure that Du_i^* does not have any extra piece on them. But this is the case, because we chose u_i^* with equal radial limits from both sides of these spheres.

We shall also assume that for $1 \leq i \leq N$,

$$u_i(x) = 0 \text{ for } \sigma\text{-almost every } x \in S_r \setminus W_i. \tag{6.16}$$

This is true for almost every $r > 0$, because $u_i(x) = 0$ a.e. on $\mathbb{R}^n \setminus W_i$, so this extra assumption will not cost us anything. Now set

$$W_1^* = [W_1 \setminus B(0,r)] \cup \big\{x \in \Omega \cap B(0,r) \setminus B(0,ar); x/|x| \in W_1\big\} \cup [\Omega \cap B(0,ar)] \tag{6.17}$$

and, for $i \geq 2$,

$$W_i^* = [W_i \setminus B(0,r)] \cup \big\{x \in \Omega \cap B(0,r) \setminus B(0,ar); x/|x| \in W_i\big\}. \tag{6.18}$$

It is easy to see that these sets are disjoint and contained in Ω. Let us also check that $u_i^*(x) = 0$ for almost every $x \in \mathbb{R}^n \setminus W_i^*$. If $x \in \mathbb{R}^n \setminus B(0,r)$, this comes from the corresponding property for u_i (we did not change anything there). When $x \in B(0,r) \setminus B(0,ar)$, $u_i^*(x) = 0$ because $u_i(rx/|x|) = 0$, which itself holds almost everywhere by (6.16). When $x \in B(0,ar)$, either $i \geq 2$ and $u_i^*(x) = 0$ by (6.11) and because $\varphi(|x|) = 0$, or else $i = 1$, but then x lies in the set $B(0,ar) \setminus \Omega$, which has zero measure by (6.9). As usual, if some of the u_i are nonnegative, so are the corresponding u_i^*. This proves (6.15).

Let us also give a first estimate for $\int_{B(0,r)} |\nabla \mathbf{u}^*|^2$. For $i \geq 2$ and $x \in B(0,r) \setminus B(0,ar)$, the definition (6.11) says that ∇u_i^* has a tangential gradient $\nabla_t u_i^*$ such that $|\nabla_t u_i^*(x)|^2 =$

$\varphi(|x|)^2\big(r/|x|\big)^2|\nabla_t u_i(rx/|x|)|^2 \leq \big(r/|x|\big)^2|\nabla_t u_i(rx/|x|)|^2$, and a radial gradient $\nabla_r u_i^*$ such that $|\nabla_r u_i^*(x)|^2 = \varphi'(|x|)^2 u_i(rx/|x|)|^2 \leq 4(1-a)^{-2}r^{-2}u_i(rx/|x|)|^2$. Thus

$$\begin{aligned}
\int_{B(0,r)} |\nabla u_i^*|^2 &= \int_{ar}^{r}\int_{\partial B(0,t)} |\nabla u_i^*|^2 \\
&\leq \int_{ar}^{r}\int_{\partial B(0,t)} \Big[(r/t)^2|\nabla_t u_i(rx/|x|)|^2 + 4(1-a)^{-2}r^{-2}u_i(rx/|x|)|^2\Big]d\sigma(x)dt \\
&= \int_{ar}^{r}\int_{S_r} \Big[(r/t)^{2-n}|\nabla_t u_i(\xi)|^2 + 4(1-a)^{-2}r^{-2}(r/t)^{-n}u_i(\xi)|^2\Big]d\sigma(\xi)dt \qquad (6.19)\\
&\leq (1-a)ra^{2-n}\int_{S_r}|\nabla_t u_i|^2 + 4(1-a)^{-1}ra^{-n}\int_{S_r}|r^{-1}u_i|^2.
\end{aligned}$$

This will be acceptably small (with $a = 1/2$) when $\int_{S_r}|\nabla_t u_i|^2$ and $\int_{S_r}|r^{-1}u_i|^2$ are small (the second often following from the first one and Poincaré), or even (with a close to 1) if $\int_{S_r}|\nabla_t u_i|^2$ is not too large but $\int_{S_r}|r^{-1}u_i|^2$ is very small (which typically follows from Poincaré if $W_i \cap S_r$ is very small).

For $i = 1$, the estimate for ∇u_i^* on $B(0,r)\setminus B(0,ar)$ is simpler, because it only has a tangential gradient, and $|\nabla u_1^*(x)|^2 = |\nabla_t u_1^*(x)|^2 = \big(r/|x|\big)^2|\nabla_t u_1(rx/|x|)|^2$ (as above), so

$$\int_{B(0,r)\setminus B(0,ar)} |\nabla u_1^*|^2 \leq (1-a)ra^{2-n}\int_{S_r}|\nabla_t u_1|^2. \qquad (6.20)$$

The remaining part is

$$\begin{aligned}
\int_{B(0,ar)}|\nabla u_1^*|^2 &= a^{-2}\int_{B(0,ar)}|\nabla v_1(a^{-1}x)|^2 = a^{n-2}\int_{B(0,r)}|\nabla v_1|^2 \\
&= a^{n-2}\inf\Big\{\int_{B(0,r)}|\nabla v|^2\,;\, v\in W^{1,2}(B(0,r)) \text{ and } v = u_1 \text{ on } S_r\Big\}, \\
&= a^{n-2}\inf\Big\{\int_{B(0,r)}|\nabla v|^2\,;\, v\in W^{1,2}(\mathbb{R}^n) \text{ and } v = u_1 \text{ a.e. on } \mathbb{R}^n\setminus B(0,r)\Big\}, \qquad (6.21)
\end{aligned}$$

by (6.13), and where we mentioned the last infimum because its definition no longer involves radial limits; but this is the same by the gluing property (4.18).

Notice that $||u_i^*||_\infty \leq ||u_i||_\infty$ because the Poisson kernel is nonnegative and sends the constant 1 to 1; then the proof of (6.6) and (6.7) also yields

$$\Big|\int_\Omega (u_i^*)^2 f_i - \int_\Omega u_i^2 f_i\Big| \leq C r^{n-\frac{n}{p}}||u_i||_\infty^2||f_i||_p \qquad (6.22)$$

and

$$\Big|\int_\Omega u_i^* g_i - \int_\Omega u_i g_i\Big| \leq C r^{n-\frac{n}{p}}||u_i||_\infty||g_i||_p. \qquad (6.23)$$

Remark 6.1 *When the u_i are nonnegative, we could use the same trick as when $N = 2$ to define a variant of the harmonic competitor that we just defined, but where we select two main functions, say u_1 and u_2, and get rid of the other ones by the same cut-off argument. That is, we would use the definition (6.11) for $i \geq 3$, and we would define u_1^* and u_2^* as follows. We would group u_1 and u_2 as the real-valued $u = u_1 - u_2$, denote by $\overline{u}$ the restriction of u to S_r, call v the harmonic extension of $\overline{u}$ to $B(0, r)$, define u^* by a formula like (6.14), and then cut it into its positive part u_1^* and its negative part u_2^*; we could still define W_1^* and W_2^* so that $(\mathbf{u}^*, \mathbf{W}^*) \in \mathcal{F}$ (with the other W_i^* defined as in (6.18)). The estimates (6.20)-(6.23) would have analogues too. But we do not seem to need this trick in the present paper.*

7 Hölder-continuity of u inside Ω

In this section we keep the same assumptions (5.1)-(5.4) as in Section 5, also assume that the function F in the volume term is Hölder-continuous with an exponent $\beta > \frac{n-2}{n}$, and prove that if $(\mathbf{u}, \mathbf{W})$ is a minimizer for our functional J, then $\mathbf{u}$ is Hölder-continuous on the interior of Ω. We only see this as a first step towards interior regularity, which will allow us to be more relaxed about the definition of $\{u_i > 0\}$ (see Remark 7.2 below), and more importantly to use results of [CJK] in later sections. But we intend to get more regularity later on (under stronger assumptions). Also, we shall discuss the Hölder-continuity of u near $\partial\Omega$ in the next section. Precisely, our Hölder condition on F is that for some $\beta > \frac{n-2}{n}$,

$$\Big|F(W_1, W_2, \ldots, W_N) - F(W_1', W_2', \ldots, W_N')\Big| \leq C \sum_{i=1}^{N} |W_i \Delta W_i'|^\beta \tag{7.1}$$

for some $C \geq 0$ and all choices of N-uples (W_i) and $(W_i') \subset \Omega^N$, of disjoint sets, and where $A \Delta B$ still denotes the symmetric difference $(A \setminus B) \cup (B \setminus A)$.

Theorem 7.1 *Assume that (5.1)-(5.4) and (7.1) hold. There is an exponent $\alpha > 0$, that depends only on n, N, β and p (from (5.2) and (5.4)), and a constant $C_0 \geq 0$, that also depends on $|\Omega|$ and the bounds in (5.2), (5.4), and (7.1), such that if $(\mathbf{u}, \mathbf{W})$ is a minimizer for J in $\mathcal{F}$ (see Definition 1.1 and (1.3)-(1.5)), $x_0 \in \Omega$ and $0 < r_0 \leq 1$ are such that $B(x_0, r_0) \subset \Omega$, then (possibly after modifying $\mathbf{u}$ on a set of measure zero)*

$$|\mathbf{u}(x) - \mathbf{u}(y)| \leq C|x - y|^\alpha \quad \text{for } x, y \in B(x_0, r_0/2), \tag{7.2}$$

with $C = C_0 + C_0 r_0^{1-\alpha} \left\{ \fint_{B(0,r_0)} |\nabla \mathbf{u}|^2 \right\}^{1/2}$.

Observe that we do not require Ω to be nice (or even open), but we compensate by requiring that $B(x_0, r_0) \subset \Omega$ (and in fact, we only need this modulo a set of vanishing Lebesgue measure, since modifying Ω on a set of measure zero does not change the problem). We required $r_0 \leq 1$ in order to obtain a constant C in (7.2) that depends on r_0 as stated. Of course the fact that C_0 does not depend on the specific choice of data, or of the minimizer, but only on the various constants in our assumptions, is more important. Finally, the fact that α depends on β and p hides an important defect of Theorem 7.1: even with $p = +\infty$ and $\beta = 1$, our proof will only give a very small exponent $\alpha > 0$, while much more regularity is expected.

Our proof of Theorem 7.1 will follow the same rough outline as a proof of monotonicity for the normalized energy that A. Bonnet gave in the context of the Mumford-Shah functional [Bo]. But let us first observe that when $n = 1$, Theorem 7.1 holds with $\alpha = 1/2$, just because $|\mathbf{u}(x) - \mathbf{u}(y)| = \left| \int_{[x,y]} \nabla\mathbf{u} \right| \leq |x-y|^{1/2} \left\{ \int_{[x,y]} |\nabla\mathbf{u}|^2 \right\}^{1/2}$ by Hölder's inequality. So we may assume that $n \geq 2$.

We fix x_0 (without loss of generality we shall immediately assume that $x_0 = 0$), and we want to prove a differential inequality on the function E, where

$$E(r) = \int_{B(x_0,r)} |\nabla\mathbf{u}|^2 = \int_{B(0,r)} |\nabla\mathbf{u}|^2 = \int_{B(0,r)} \sum_{i=1}^{N} |\nabla u_i|^2. \tag{7.3}$$

By Fubini (and as in (4.3)),

$$E(r) = \int_{t=0}^{r} \int_{y\in S_r} |\nabla\mathbf{u}|^2 d\sigma(y) dt, \tag{7.4}$$

where we still use the notation $S_r = \partial B(0,r)$. Hence the derivative $E'(r)$ exists for almost every $r > 0$,

$$E'(r) = \int_{S_r} |\nabla\mathbf{u}|^2 d\sigma \quad \text{for almost every } r > 0, \tag{7.5}$$

and E is the indefinite integral of E'. Our main goal is to estimate $E(r)$ in terms of $E'(r)$, and then integrate in r to get a good upper bound for $E(r)$ for r small; the Hölder estimate (7.2) will then follow easily.

Notice that $E(r)$ is a nice quantity to work with, because the minimality of $(\mathbf{u}, \mathbf{W})$ gives an estimate on $E(r)$ each time we build a competitor for $(\mathbf{u}, \mathbf{W})$ in $B(0,r)$. So let us do this. We restrict to $r \leq r_0$ (and this way we get that $B(0,r) \subset \Omega$, as in (6.9)), and also assume that (6.8), (6.16), and the conclusion of (7.5) hold (they all hold for almost all r, and we'll only need almost all r to integrate the differential inequality).

We shall distinguish between two cases. Let $\varepsilon > 0$ be small, to be chosen later; we start with the case when $W_1 \cap \partial B(0,r)$ is very large, more precisely

$$\sigma(S_r \setminus W_1) \leq \varepsilon\sigma(S_r), \tag{7.6}$$

and we use the harmonic competitor $(\mathbf{u}^*, \mathbf{W}^*)$ defined near (6.11). Since $J(\mathbf{u}, \mathbf{W}) \leq J(\mathbf{u}^*, \mathbf{W}^*)$, we get that

$$E(r) = \int_{B(0,r)} |\nabla\mathbf{u}|^2 \leq \int_{B(0,r)} |\nabla\mathbf{u}^*|^2 + |M(\mathbf{u}^*) - M(\mathbf{u})| + |F(\mathbf{W}^*) - F(\mathbf{W})| \tag{7.7}$$

where M and F are as in (1.3)-(1.5). Obviously W_i^* coincides with W_i on $\mathbb{R}^n \setminus B(0,r)$, so $|W_i^* \Delta W_i| \leq Cr^n$, and (7.1) yields

$$|F(\mathbf{W}^*) - F(\mathbf{W})| \leq C \sum_i |W_i^* \Delta W_i|^\beta \leq Cr^{\beta n}. \tag{7.8}$$

For the M-terms, we use (6.22) and (6.23) and get that

$$\begin{aligned} |M(\mathbf{u}^*) - M(\mathbf{u})| &\leq \sum_i \Big| \int_\Omega (u_i^*)^2 f_i - \int_\Omega u_i^2 f_i \Big| + \Big| \int_\Omega u_i^* g_i - \int_\Omega u_i g_i \Big| \\ &\leq Cr^{n-\frac{n}{p}} ||u_i||_\infty^2 ||f_i||_p + Cr^{n-\frac{n}{p}} ||u_i||_\infty ||g_i||_p \leq Cr^{n-\frac{n}{p}}, \end{aligned} \tag{7.9}$$

where we just used Theorem 5.1. Recall that Theorem 5.1 also says that the $||u_i||_\infty^2$ are bounded in terms of the various constants in the assumptions of Theorem 7.1; hence C only depends on these constants. This remark will also apply to the other constants C in the computations that follow.

We now use the energy estimates (6.19)-(6.20), plus the first part of (6.21), and get that

$$\begin{aligned} \int_{B(0,r)} |\nabla \mathbf{u}^*|^2 \leq \sum_{i=1}^N (1-a) r a^{2-n} \int_{S_r} |\nabla_t u_i|^2 \;&+\; \sum_{i\geq 2} 4(1-a)^{-1} r a^{-n} \int_{S_r} |r^{-1} u_i|^2 \\ &+\; a^{n-2} \int_{B(0,r)} |\nabla v_1|^2 \end{aligned} \tag{7.10}$$

where v_1 (defined below (6.11)), is the harmonic extension of the restriction $\overline{u}_1$ of u_1 to S_r. Thus, by (7.7)-(7.10) and because $a^{n-2} \leq 1$,

$$E(r) \leq \int_{B(0,r)} |\nabla v_1|^2 + A, \tag{7.11}$$

with

$$\begin{aligned} A &\leq Cr^{\beta n} + Cr^{n-\frac{n}{p}} + \sum_{i=1}^N (1-a) r a^{2-n} \int_{S_r} |\nabla_t u_i|^2 + \sum_{i\geq 2} 4(1-a)^{-1} r a^{-n} \int_{S_r} |r^{-1} u_i|^2 \\ &\leq Cr^{\beta n} + Cr^{n-\frac{n}{p}} + C(1-a) r \int_{S_r} |\nabla_t \mathbf{u}|^2 + C(1-a)^{-1} r^{-1} \sum_{i\geq 2} \int_{S_r} |u_i|^2 \end{aligned} \tag{7.12}$$

because we shall take $a \geq 1/2$. In addition, for $i \geq 2$ (6.8) says that (the restriction of) $\overline{u}_i$ lies in $W^{1,2}(S_r)$, and (6.16) says that $u_i(x) = 0$ almost everywhere on $S_r \setminus W_i$. Set $E = S_r \setminus W_1$; if $x \in S_r \setminus E$, then $x \in W_1$, hence (by disjointness) $x \in S_r \setminus W_i$, and almost always $u_i(x) = 0$. This allows us to apply Lemma 4.1 (with $p = 2$) and get that

$$\int_{S_r} |u_i|^2 = \int_E |u_i|^2 \leq C_2 \sigma(E)^{\frac{2}{n-1}} \int_E |\nabla_t u_i|^2 \leq C_2 (\varepsilon \sigma(S_r))^{\frac{2}{n-1}} \int_E |\nabla_t u_i|^2 \tag{7.13}$$

by (7.6) and (4.7). Recall also that we restrict to r such that the conclusion of (7.5) holds, so

$$\sum_i \int_{S_r} |\nabla_t u_i|^2 \leq \sum_i \int_{S_r} |\nabla u_i|^2 = \int_{S_r} |\nabla \mathbf{u}|^2 = E'(r); \tag{7.14}$$

then (7.12) yields

$$A \leq Cr^{\beta n} + Cr^{n-\frac{n}{p}} + C(1-a)rE'(r) + C(1-a)^{-1}\varepsilon^{\frac{2}{n-1}} rE'(r). \tag{7.15}$$

Let $\tau > 0$ be small, to be chosen later. We choose a close to 1 (depending on τ), and then ε very small (depending also on a), so that (7.15) yields

$$A \leq Cr^{\beta n} + Cr^{n-\frac{n}{p}} + \tau rE'(r), \tag{7.16}$$

and now we concentrate on $\int_{B(0,r)} |\nabla v_1|^2$. We shall just need an estimate on the norm of the Poisson extension, from $W^{1,2}(S_r)$ to $W^{1,2}(B_r)$, but since $\int_{B(0,r)} |\nabla v_1|^2$ is minimal (by (6.13) or (6.21)), it will be enough to control the energy of some extension. Call u the restriction of $u_1 - m^\sigma_{S_r} u_1$ to S_r, and define v on $B(0,r)$ by

$$v(ty) = tu(y) \text{ for } y \in S_r \text{ and } 0 \leq t \leq 1. \tag{7.17}$$

[We will not really lose much, because if we were to find the optimal extension, it would happen to have the largest extension norm on spherical harmonics of degree 1, which happen to have homogeneous extensions of degree 1.] Anyway, it is easy to see that $v \in W^{1,2}(B(0,r))$, with a gradient that we compute now. In fact, since v is homogeneous of degree 1, its gradient is homogeneous of degree 0. So we compute it at $y \in S_r$. Its radial part is $r^{-1}u(y)$, and its tangential part is just $\nabla_t u(y)$. Then $|\nabla v(y)|^2 = r^{-2}u(y)^2 + |\nabla_t u(y)|^2$, and

$$\begin{aligned} \int_{B(0,r)} |\nabla v|^2 &= \int_{t\in(0,r)} \int_{S_t} |\nabla v|^2 d\sigma dt = \int_{t\in(0,r)} (t/r)^{n-1} \int_{S_r} |\nabla v|^2 d\sigma dt \\ &= \frac{r^n}{nr^{n-1}} \int_{S_r} |\nabla v|^2 d\sigma = \frac{r}{n} \int_{S_r} r^{-2}u^2 + |\nabla_t u|^2 \end{aligned} \tag{7.18}$$

by homogeneity. We use Poincaré's inequality on the sphere (exceptionally, with the right constant!), which says that

$$\int_{S_r} r^{-2}|u|^2 \leq \frac{1}{n-1} \int_{S_r} |\nabla_t u|^2 \tag{7.19}$$

because $\int_{S_r} u d\sigma = 0$. See for instance Exercise 76.21 in [D]. Then (7.18) and (7.19) yield $\int_{B(0,r)} |\nabla v|^2 \leq \frac{r}{n}\frac{n}{n-1} \int_{S_r} |\nabla_t u|^2$ and, since $v + m^\sigma_{S_r} u_1$ has boundary values on S_r equal to $u + m^\sigma_{S_r} u_1 = u_1$, the minimizing property of v_1 yields

$$\begin{aligned} \int_{B(0,r)} |\nabla v_1|^2 &\leq \int_{B(0,r)} |\nabla (v + m^\sigma_{S_r} u_1)|^2 = \int_{B(0,r)} |\nabla v|^2 \\ &\leq \frac{r}{n-1} \int_{S_r} |\nabla_t u(y)|^2 = \frac{r}{n-1} \int_{S_r} |\nabla_t u_1(y)|^2 \leq \frac{r}{n-1} E'(r) \end{aligned} \tag{7.20}$$

by (7.14). We combine with (7.11) and (7.16) and get that

$$E(r) \leq \int_{B(0,r)} |\nabla v_1|^2 + A \leq Cr^{\beta n} + Cr^{n-\frac{n}{p}} + \Big(\tau + \frac{1}{n-1}\Big) rE'(r). \tag{7.21}$$

This is our first differential inequality, valid at almost every $r \leq r_0$ such that (7.6) holds. If (7.6), with $i = 1$ replaced by some other index i, holds, we do the same argument with u_1 replaced by u_i, and we still get the conclusion of (7.21).

When (7.6) fails for all indices, i.e., if $\sigma(S_r \setminus W_i) \geq \varepsilon\sigma(S_r)$ for all i, we use another competitor to get a similar differential inequality. This time we pick a very small $\gamma > 0$, to be chosen later, and we set

$$u_i^*(ty) = t^\gamma u_i(y) \text{ for } y \in S_r \text{ and } 0 \leq t < 1. \tag{7.22}$$

On $\mathbb{R}^n \setminus B(0,r)$, we keep $u_i^* = u_i$, as usual. It is easy to see, using again the gluing condition (4.17) that $u_i^* \in W^{1,2}(\mathbb{R}^n \setminus \{0\})$. Its gradient is now homogeneous of degree $\gamma - 1$, and we compute it at $y \in S_r$. The tangential gradient is still $\nabla_t u_i(y)$, and the radial derivative is $\gamma r^{-1} u_i(y)$. Thus $|\nabla u_i^*(y)|^2 = |\nabla_t u_i(y)| + \gamma^2 r^{-2} u_i(y)^2$. The same computation as in (7.18) yields

$$\begin{aligned}
\int_{B(0,r)} |\nabla u_i^*|^2 &= \int_{t\in(0,r)} \int_{S_t} |\nabla u_i^*|^2 d\sigma dt = \int_{t\in(0,r)} (t/r)^{n-1}(t/r)^{2\gamma-2} \int_{S_r} |\nabla u_i^*|^2 d\sigma dt \\
&= \frac{r^{n+2\gamma-2}}{(n+2\gamma-2)r^{n+2\gamma-3}} \int_{S_r} |\nabla u_i^*|^2 d\sigma = \frac{r}{(n+2\gamma-2)} \int_{S_r} \gamma^2 r^{-2} u_i^2 + |\nabla_t u_i|^2
\end{aligned} \tag{7.23}$$

In particular the integral converges (recall that $n \geq 2$), and it is not hard to show that $u_i^* \in W^{1,2}$ near the origin. For instance, we can approximate t^γ in (7.22) with functions that vanish near 0, and take a limit.

Since (7.6) fails for i, we can apply Lemma 4.1 to u_i, with $E = W_i \cap S_r$ (by (6.16)). We get that

$$\int_{S_r} |u_i|^2 = \int_E |u_i|^2 \leq Cr^2 \frac{\sigma(S_r)}{\sigma(S_r \setminus W_i)} \int_E |\nabla_t u_i|^2 \leq Cr^2 \varepsilon^{-1} \int_{S_r} |\nabla_t u_i|^2 \tag{7.24}$$

by (4.6). Set

$$\lambda = \frac{1}{(n+2\gamma-2)} + \frac{C\gamma^2}{(n+2\gamma-2)\varepsilon}, \tag{7.25}$$

with the same constant C as in (7.24). Then (7.23) yields

$$\int_{B(0,r)} |\nabla u_i^*|^2 \leq \lambda r \int_{S_r} |\nabla_t u_i|^2. \tag{7.26}$$

We sum over i and get that

$$\int_{B(0,r)} |\nabla \mathbf{u}^*|^2 \leq \lambda r \int_{S_r} |\nabla_t \mathbf{u}|^2 \leq \lambda r E'(r) \tag{7.27}$$

with $\mathbf{u}^* = (u_1^*, \dots, u_N^*)$, and by (7.14).

We now complete the definition of our competitor by choosing sets $W_i^* \subset \Omega$, so that $(\mathbf{u}^*, \mathbf{W}^*) \in \mathcal{F}$. We keep $W_i^* \setminus B(0,r) = W_i \setminus B(0,r)$, and we take for $W_i^* \cap B(0,r)$ the intersection of Ω with the cone over $W_i \cap S_r$. The W_i^* are disjoint because the W_i are disjoint, and $u_i^* = 0$ almost everywhere on $\mathbb{R}^n \setminus W_i$ because Ω is almost contained in $B(0,r)$, and $u_i(y) = 0$ almost everywhere on $S_r \setminus W_i$ (by (6.16)). As usual, $u_i^* \geq 0$ if $u_i \geq 0$, so $(\mathbf{u}^*, \mathbf{W}^*) \in \mathcal{F}$.

We complete the estimate with the F and M-terms. As before, $||u_i^*||_\infty \leq ||u_i||_\infty$, so

$$|M(\mathbf{u}^*) - M(\mathbf{u})| \;\leq\; Cr^{n-\frac{n}{p}} ||u_i||_\infty^2 ||f_i||_\infty + Cr^{n-\frac{n}{p}} ||u_i||_\infty ||g_i||_p \leq Cr^{n-\frac{n}{p}}, \tag{7.28}$$

as in (7.9); also, $|W_i^* \Delta W_i| \leq Cr^n$, so (7.1) yields

$$|F(\mathbf{W}^*) - F(\mathbf{W})| \leq C \sum_i |W_i^* \Delta W_i|^\beta \leq Cr^{\beta n}. \tag{7.29}$$

as in (7.8). Since $J(\mathbf{u}, \mathbf{W}) \leq J(\mathbf{u}^*, \mathbf{W}^*)$, we get (as in (7.7)) that

$$\begin{aligned} E(r) &= \int_{B(0,r)} |\nabla \mathbf{u}|^2 \leq \int_{B(0,r)} |\nabla \mathbf{u}^*|^2 + |M(\mathbf{u}^*) - M(\mathbf{u})| + |F(\mathbf{W}^*) - F(\mathbf{W})| \\ &\leq \int_{B(0,r)} |\nabla \mathbf{u}^*|^2 + Cr^{n-\frac{n}{p}} + Cr^{\beta n} \leq \lambda r E'(r) + Cr^{n-\frac{n}{p}} + Cr^{\beta n} \end{aligned} \tag{7.30}$$

by (7.27). This is our alternative differential inequality, which holds for almost every r such that (7.21) fails for each i.

Let us now check that we can choose δ such that

$$n - 2 < \delta < \min\Big(n\beta, n - \frac{n}{p}, \Big(\tau + \frac{1}{n-1}\Big)^{-1}, \lambda^{-1}\Big). \tag{7.31}$$

We know that $n\beta > n - 2$ (see the definition of β near (7.1)), and $n - \frac{n}{p} > n - 2$ because $p > n/2$ (see (5.4)). Next choose τ so small that $\tau + \frac{1}{n-1} < \frac{1}{n-2}$ (no condition if $n = 2$); then $n - 2 < \Big(\tau + \frac{1}{n-1}\Big)^{-1}$, and this too leaves some room for δ. This choice of τ forces a to be chosen close enough to 1, and ε small enough, but this is all right. We still need to check that $(n-2)\lambda < 1$; rewrite (7.25) as

$$(n-2)\lambda = \frac{n-2}{(n + 2\gamma - 2)}\Big(1 + \frac{C\gamma^2}{\varepsilon}\Big) = \Big(1 + \frac{2\gamma}{n-2}\Big)^{-1}\Big(1 + \frac{C\gamma^2}{\varepsilon}\Big); \tag{7.32}$$

even if we chose ε very small, this expression becomes smaller than 1 when γ is small enough. So we choose γ small, depending on ε, and we can choose δ as in (7.31). Notice that a, then

ε, γ, and finally δ depend on N, because (we think that) C in (7.15) depends on N. With all this new notation, (7.21) and (7.30) yield

$$\delta E(r) \leq rE'(r) + Cr^{n-\frac{n}{p}} + Cr^{\beta n} \text{ for almost every } r \leq r_0, \tag{7.33}$$

where C also depends on constants like δ, but not on r or r_0.

Now we want to integrate this differential inequality between $r \in (0, r_0)$ and r_0. Set $f(r) = r^{-\delta}E(r)$; then f is differentiable almost everywhere on $(0, r_0)$, and

$$f'(r) = -\delta r^{-\delta-1}E(r) + r^{-\delta}E'(r) \geq -Cr^{-\delta-1}[r^{n-\frac{n}{p}} + r^{\beta n}] \tag{7.34}$$

almost everywhere, by (7.33). We want to integrate this between $r \in (0, r_0)$ and r_0 and get that

$$\begin{aligned} f(r) &= f(r_0) - \int_r^{r_0} f'(t)dt \leq f(r_0) + C\int_r^{r_0} t^{-\delta-1}[t^{n-\frac{n}{p}} + t^{\beta n}]dt \\ &\leq f(r_0) + Cr^{-\delta}[r^{n-\frac{n}{p}} + r^{\beta n}] \end{aligned} \tag{7.35}$$

(recall that by (7.31), the final exponents are negative). So we have to justify the first equality. Write $g(r) = r^{-\delta}$ to simplify the algebra, and recall from (7.4) and (7.5) that $E(t) = E(r) + \int_r^t E'(s)ds$ for $r < t < r_0$. Then set $I = [r, r_0]$ and compute

$$\begin{aligned} \int_r^{r_0} f'(t)dt &= \int_I E(t)g'(t) + E'(t)g(t)dt \\ &= \Big(\int_I E(r)g'(t) + \int_{I\times I} E'(s)g'(t)\mathbb{1}_{s<t}\Big) + \Big(\int_I E'(t)g(r) + \int_{I\times I} E'(t)g'(s)\mathbb{1}_{s<t}\Big) \\ &= E(r)[g(r_0) - g(r)] + g(r)[E(r_0) - E(r)] + \int_{I\times I} E'(t)g'(s)dsdt \\ &= E(r)[g(r_0) - g(r)] + g(r)[E(r_0) - E(r)] + [E(r_0) - E(r)][g(r_0) - g(r)] \\ &= E(r_0)g(r_0) - E(r)g(r) = f(r_0) - f(r) \end{aligned} \tag{7.36}$$

where Fubini can be used because E' and g' are both nonnegative (or integrable). So (7.35) holds. We multiply it by r^δ and get that

$$E(r) = r^\delta f(r) \leq r^\delta f(r_0) + C[r^{n-\frac{n}{p}} + r^{\beta n}] = (r/r_0)^\delta E(r_0) + C[r^{n-\frac{n}{p}} + r^{\beta n}]. \tag{7.37}$$

Since we shall use it a few times in the future, let us record what we just got: under the general assumptions of Theorem 7.1, we just proved that

$$\int_{B(x_0,r)} |\nabla \mathbf{u}|^2 \leq (r/r_0)^\delta \int_{B(x_0,r_0)} |\nabla \mathbf{u}|^2 + C[r^{n-\frac{n}{p}} + r^{\beta n}] \text{ for } 0 < r < r_0 \tag{7.38}$$

as soon as $B(x_0, r_0) \subset \Omega$ (recall that we immediately assumed that $x_0 = 0$, and see (7.3) for the definition of $E(r)$).

Return to the proof; (7.37) is the energy estimate that we wanted, but we also want its analogue for other centers. And indeed we can do the proof of (7.37), but with any other origin $x \in B(0, 2r_0/3)$, and with radii $0 < r \leq r_0/3$, and we get that

$$\int_{B(x,r)} |\nabla \mathbf{u}|^2 \leq (r/r_0)^\delta E(r_0) + C[r^{n-\frac{n}{p}} + r^{\beta n}] \tag{7.39}$$

for $x \in B(0, 2r_0/3)$ and $0 < r \leq r_0/3$. Here C depends on the usual constants, i.e., n, N, p, $|\Omega|$, the $||f_i||_\infty$ and $||g_i||_p$, and the two constants in (7.1). Incidentally, these constants also control $E(r_0)$, by (3.10) (modified as in Remark 3.4 if $p < 2$) and because they easily control $J(\mathbf{u}, \mathbf{W})$. But we may lose some information if we use this remark.

Anyway, let us rewrite (7.39) as

$$r\Big\{ \fint_{B(x,r)} |\nabla \mathbf{u}|^2 \Big\}^{1/2} \leq C\theta(r), \quad \text{with } \theta(r) = r^{\frac{\delta-n+2}{2}}[r_0^{-\delta}E(r_0)]^{1/2} + r^{\frac{2p-n}{2p}} + r^{\frac{2+\beta n-n}{2}} \tag{7.40}$$

and notice that by (7.31), the smallest exponent in (7.40) is

$$\alpha = \frac{\delta - n + 2}{2} > 0. \tag{7.41}$$

We shall now check that (7.2) follows from (7.40), with this exponent α. This will be a standard consequence of the Poincaré inequalities.

Fix $i \in [1, N]$, set $u = u_i$ (to save notation), and define $u(x,r) = \fint_{B(x,r)} u$ for $x \in B(0, 2r_0/3)$ and $0 < r \leq r_0/3$. Then

$$\begin{aligned} |u(x,r/2) - u(x,r)| &= \Big| \fint_{B(x,r/2)} u - u(x,r) \Big| \leq \fint_{B(x,r/2)} |u - u(x,r)| \\ &\leq 2^n \fint_{B(x,r)} |u - u(x,r)| \leq Cr \fint_{B(x,r)} |\nabla u| \\ &\leq Cr \Big\{ \fint_{B(x,r)} |\nabla u|^2 \Big\}^{1/2} \leq C\theta(r) \end{aligned} \tag{7.42}$$

by Poincaré (see (4.2)), Hölder, and (7.40). It follows from iterations of (7.42) that $\overline{u}(x) = \lim_{k \to +\infty} u(x, 2^{-k})$ exists for all $x \in B(0, 2r_0/3)$, and that

$$|\overline{u}(x) - u(x, 2^{-k})| \leq C\theta(2^{-k}) \quad \text{when } 2^{-k} \leq r_0/3 \tag{7.43}$$

(also use the special form of θ to sum three geometric series). Since $\overline{u}(x) = u(x)$ for every point of Lebesgue differentiability for u, we see that replacing u with $\overline{u}$ on $B(0, 2r_0/3)$ will only change its values on a set of measure 0, so it is now enough to prove that

$$|\overline{u}(x) - \overline{u}(y)| \leq C_1 |x - y|^\alpha \tag{7.44}$$

for $x, y \in B(x_0, r_0/2)$, and with the announced value of

$$C_1 = C_0 + C_0 r_0^{1-\alpha} \Big\{ \fint_{B(0,r_0)} |\nabla \mathbf{u}|^2 \Big\}^{1/2}, \tag{7.45}$$

where C_0 depends on the usual constants but not on $r_0 \leq 1$. It is even enough to prove this when $|x-y| \leq r_0/10$ (just use a short chain of points, and maybe multiply C by 5). Choose k such that $2^{-k-2} \leq |x-y| \leq 2^{-k-1}$; then $2^{-k} \leq r_0/3$, and

$$\begin{aligned} |\overline{u}(x) - \overline{u}(y)| &\leq |\overline{u}(x) - u(x, 2^{-k-1})| + |u(x, 2^{-k-1}) - u(y, 2^{-k})| + |u(y, 2^{-k}) - \overline{u}(y)| \\ &\leq |u(x, 2^{-k-1}) - u(y, 2^{-k})| + C\theta(2^{-k}) \\ &\leq |u(x, 2^{-k-1}) - u(y, 2^{-k})| + C\theta(|x-y|). \end{aligned} \tag{7.46}$$

But, as in (7.42)

$$\begin{aligned} |u(x, 2^{-k-1}) - u(y, 2^{-k})| &= \Big| \fint_{B(x, 2^{-k-1})} u - u(y, 2^{-k}) \Big| \leq \fint_{B(y, 2^{-k})} |u - u(y, 2^{-k})| \\ &\leq 2^n \fint_{B(y, 2^{-k})} |u - u(y, 2^{-k})| \leq C 2^{-k} \fint_{B(y, 2^{-k})} |\nabla u| \\ &\leq C 2^{-k} \Big\{ \fint_{B(y, 2^{-k})} |\nabla u|^2 \Big\}^{1/2} \leq C\theta(2^{-k}) \leq C\theta(|x-y|) \end{aligned} \tag{7.47}$$

because $B(x, 2^{-k-1}) \subset B(y, 2^{-k})$. But

$$\theta(r) = r^\alpha [r_0^{-\delta} E(r_0)]^{1/2} + r^{\frac{2p-n}{2p}} + r^{\frac{2+\beta n - n}{2}} \leq r^\alpha [r_0^{-\delta} E(r_0)]^{1/2} + 2r^\alpha \tag{7.48}$$

by (7.40), (7.41), and because $r \leq r_0 \leq 1$ and δ was the smallest exponent (by (7.31)), and

$$[r_0^{-\delta} E(r_0)]^{1/2} = \Big[r_0^{-2\alpha - n + 2} \int_{B(0,r_0)} |\nabla \mathbf{u}|^2 \Big]^{1/2} = r_0^{1-\alpha} \Big[\fint_{B(0,r_0)} |\nabla \mathbf{u}|^2 \Big]^{1/2}, \tag{7.49}$$

so (7.46) and (7.47) yield

$$|\overline{u}(x) - \overline{u}(y)| \leq C\theta(|x-y|) \leq C|x-y|^\alpha + C|x-y|^\alpha r_0^{1-\alpha} \Big[\fint_{B(0,r_0)} |\nabla \mathbf{u}|^2 \Big]^{1/2}, \tag{7.50}$$

which proves (7.44) with C_1 as in (7.45). This completes our proof of Theorem 7.1. ∎

Remark 7.2 *Once we have Theorem 7.1, we can be a little more relaxed about the definition of the W_i. Suppose Ω is open and that the assumptions of Theorem 7.1are satisfied. We know that there is a locally Hölder continuous function $\widetilde{\mathbf{u}}$ that coincides with $\mathbf{u}$ almost everywhere on Ω (the local continuous functions provided by applications of the theorem on small balls $B \subset \Omega$ can easily be glued). Then we can work with the open sets*

$$\Omega_i = \big\{ x \in \Omega \,;\, \widetilde{u}_i(x) > 0 \big\}; \tag{7.51}$$

it is easy to see that the Ω_i are disjoint, and the constraints that $W_i \subset \Omega$ and $u_i = 0$ almost everywhere on $\mathbb{R}^n \setminus W_i$ just mean that $\Omega_i \subset W_i$, modulo a set of vanishing measure. After the next section, and under additional regularity assumptions on Ω, we will also know that (if we set $\mathbf{u} = 0$ on $\mathbb{R}^n \setminus \Omega$) $\widetilde{\mathbf{u}}$ is also continuous across $\partial\Omega$, and we will feel free to replace $\mathbf{u}$ with $\widetilde{\mathbf{u}}$ without saying.

8 Hölder-continuity of u on the boundary

In this section we keep the assumptions of Section 7, add a smoothness assumption on Ω, and prove that $\mathbf{u}$ is Hölder-continuous on the whole $\mathbb{R}^n$ when $(\mathbf{u}, \mathbf{W})$ is a minimizer for J. For the main statement, let us be brutal and just assume that

$$\Omega \text{ is a bounded open set with } C^1 \text{ boundary.} \tag{8.1}$$

But we shall see that the result holds under somewhat weaker assumptions; see Remark 8.3 at the end of the section.

Theorem 8.1 *Assume that (5.1)-(5.4), (7.1), and (8.1) hold. There is an exponent $\alpha > 0$, that depends only on n, N, β and p (from (5.4)), such that if $(\mathbf{u}, \mathbf{W})$ is a minimizer for J in $\mathcal{F}$ (see Definition 1.1 and (1.3)-(1.5)) there is a constant $C \geq 0$ such that (possibly after modifying $\mathbf{u}$ on a set of measure zero)*

$$|\mathbf{u}(x) - \mathbf{u}(y)| \leq C|x-y|^\alpha \quad \text{for } x, y \in \mathbb{R}^n \text{ such that } |x-y| \leq 1. \tag{8.2}$$

As for Theorem 7.1, we even get that $C = C_0 + C_0 r_0^{1-\alpha}\left\{ \fint_{B(x,r_0)} |\nabla \mathbf{u}|^2 \right\}^{1/2}$, where C_0 depends only on n, N, β, p, and $|\Omega|$.

Again the main difficulty for the proof will be to find $\delta > n-2$ such that

$$\int_{B(x,r)} |\nabla \mathbf{u}|^2 \leq C_0 r^\delta \tag{8.3}$$

for $x \in \mathbb{R}^n$ and $r > 0$; the conclusion, with $\alpha = \frac{1}{2}(\delta - n + 2)$ will follow by the same argument as in Section 7, near (7.41).

We would still be happy to prove that for $x \in \mathbb{R}^n$, $E(r) = \int_{B(x,r)} |\nabla u|^2$ often satisfies a differential inequality, and there will be one main new case, when $x \in \partial\Omega$. We cannot repeat the argument of Section 7 as it is, because we want to make sure that the function $\mathbf{u}^*$ that we build still vanishes on Ω. There is a special case where we can still use our second competitor, the one where we used a homogeneous extension of $\mathbf{u}$, and this is when $\Omega \cap B(x,r)$ is a cone centered at x. In the next proposition, which is the main ingredient for Theorem 8.1, we will assume that Ω looks like a cone near the origin, and then we shall get some decay for $E(r)$.

Again let us center our balls at the origin. Let $r_0 \in (0,1]$ be given. We shall assume that

$$0 \in \partial\Omega \tag{8.4}$$

and that there is a (measurable) cone Γ, centered at the origin, such that

$$\sigma(S_1 \setminus \Gamma) \geq \varepsilon \sigma(S_1), \tag{8.5}$$

as well as a mapping $\Phi : B(0, 2r_0) \to \mathbb{R}^n$, which is $(1+\eta)$-bilipschitz in the sense that

$$(1+\eta)^{-1}|x-y| \leq |\Phi(x) - \Phi(y)| \leq (1+\eta)|x-y| \text{ for } x, y \in B(0, 2r_0), \tag{8.6}$$

and for which

$$\Phi(0) = 0, \tag{8.7}$$

$$\Phi(B(0, 2r_0) \cap \Omega) \subset \Gamma, \tag{8.8}$$

$$\Phi(B(0, 2r_0) \setminus \Omega) \subset \mathbb{R}^n \setminus \Gamma, \tag{8.9}$$

and

$$\Phi(B(0, 2r_0)) \supset B(0, 3r_0/2). \tag{8.10}$$

Here $\eta > 0$ is a small constant, that will be chosen in terms of ε, and then (8.10) is quite probably a consequence of the other, but we are too lazy to prove this.

This will be our main additional assumption of approximation by a good cone. When Ω is a C^1 domain, as in the statement of Theorem 8.1, and $0 \in \partial\Omega$, this property holds for r_0 small enough, and we can even take for Γ an open half space. In addition, by compactness of $\partial\Omega$, the same property holds with the origin replaced by any point $x \in \partial\Omega$, and for $0 < r_0 \leq R$, where R does not depend on x. We now state the main decay estimate.

Proposition 8.2 *For each $\varepsilon > 0$, we can find $\eta \in (0, 1/2)$ and $\delta > n-2$, that depend only on n, N, β, p, and ε, such that if $(\mathbf{u}, \mathbf{W})$ satisfies the assumptions of Theorem 8.1, and in addition (8.4)-(8.10) hold for some choice of $r_0 \leq 1$, Γ and Φ, then*

$$\int_{B(0,r)} |\nabla \mathbf{u}|^2 \leq 2(r/r_0)^\delta \int_{B(0,2r_0)} |\nabla \mathbf{u}|^2 + C_1 r^{n-\frac{n}{p}} + C_1 r^{\beta n} \tag{8.11}$$

for $0 < r < r_0$. Here C_1 depends on the various constants in the assumptions, but not on r_0.

This will be our analogue of (7.38). For the proof we intend to conjugate by Φ to simplify the geometry, and then copy the the proof of (7.30)-(7.38) in Section 7.

Observe that because of (8.10) and (8.6), we can define an inverse mapping $\psi = \Phi^{-1} : B(0, 3r_0/2) \to B(0, 2r_0)$. This allows us to define $\mathbf{v}$ on $B(0, 3r_0/2)$ by

$$\mathbf{v}(y) = \mathbf{u}(\psi(y)) \text{ for } y \in B(0, 3r_0/2). \tag{8.12}$$

Notice that

$$v \in W^{1,2}(B(0, 3r_0/2) \tag{8.13}$$

because $u \in W^{1,2}(\mathbb{R}^n)$ and Φ is bilipschitz. See for instance [Z]. Because of the geometry, it will be preferable to work with the function $\mathbf{v}$, and prove appropriate differential inequalities on the energy

$$E(r) = \int_{B(0,r)} |\nabla \mathbf{v}|^2 = \sum_i \int_{B(0,r)} |\nabla v_i|^2. \tag{8.14}$$

So let $(\mathbf{u}, \mathbf{W})$ and r_0 satisfy the assumptions of the proposition, and let $r \leq r_0$ be such that the restriction of $\mathbf{v}$ to S_r lies in $W^{1,2}(S_r)$ (as in (6.8)), with derivatives that can be computed from the restriction of $D\mathbf{v}$, and that

$$v_i(x) = 0 \text{ for } \sigma\text{-almost every } x \in S_r \setminus \Phi(W_i), \tag{8.15}$$

as in (6.16). These properties hold for the same reason as before (and by (8.13)). Let us also assume that

$$E'(r) = \int_{S_r} |\nabla \mathbf{v}|^2 \tag{8.16}$$

(which again holds a.e. as in (7.5)). We may now define $\mathbf{v}^*$ by

$$\mathbf{v}^*(z) = \mathbf{v}(z) \text{ for } z \in B(0, 3r_0/2) \setminus B(0, r), \tag{8.17}$$

and

$$\mathbf{v}^*(tz) = t^\gamma \mathbf{v}(z) \text{ for } z \in S_r \text{ and } 0 \leq t < 1, \tag{8.18}$$

where the small γ will be chosen later, depending on ε. And then we set

$$\mathbf{u}^*(x) = \mathbf{u}(x) \text{ for } x \in \mathbb{R}^n \setminus B(0, 5r/4), \tag{8.19}$$

and

$$\mathbf{u}^*(x) = \mathbf{v}^*(\Phi(x)) \text{ for } x \in B(0, 4r/3), \tag{8.20}$$

which is defined because $\Phi(x) \in B(0, 3r/2) \subset B(0, 3r_0/2)$ (by (8.6) and (8.7)); the two definitions coincide when $x \in B(0, 4r/3) \setminus B(0, 5r/4)$, because $\Phi(x) \in B(0, 3r/2) \setminus B(0, r)$, and by (8.17)).

We deduce from (8.13) that $\mathbf{v}^* \in W^{1,2}(B(0, 3r_0/2))$ (as we did near (7.22)-(7.23)), and then $\mathbf{u}^* \in W^{1,2}(\mathbb{R}^n)$ (because we have a whole gluing region $B(0, 4r/3) \setminus B(0, 5r/4)$). As always, $u_i^* \geq 0$ everywhere when $u_i \geq 0$ everywhere. We now need to define sets W_i^* such that

$$(\mathbf{u}^*, \mathbf{W}^*) \in \mathcal{F}. \tag{8.21}$$

We keep

$$W_i^* \setminus \psi(B(0,r)) = W_i \setminus \psi(B(0,r)) \tag{8.22}$$

(where $\psi = \Phi^{-1}$ as before) and set

$$W_i^* \cap \psi(B(0,r)) = \psi(H_i), \quad \text{with } H_i = \big\{ty\,;\, y \in S_r \cap \Phi(W_i) \text{ and } 0 < t < 1\big\}. \tag{8.23}$$

The W_i^* are disjoint, because the W_i are disjoint and $\psi : B(0,r) \to \psi(B(0,r))$ is injective. Next let us check that $W_i^* \subset \Omega$. Pick $x \in W_i^*$. If $x \in W_i^* \setminus \psi(B(0,r))$, then $x \in W_i \subset \Omega$ by (8.22) and the definition of $\mathcal{F}$. Otherwise, $x \in \psi(H_i)$, so there exist $y \in S_r \cap \Phi(W_i)$ and $0 < t < 1$ such that $x = \psi(ty)$. But $y = \Phi(z)$ for some $z \in W_i$, $z \in B(0, 3r/2)$ by (8.6) and (8.7), $y \in \Gamma$ because $z \in W_i \subset \Omega$ and by (8.8), $ty \in \Gamma \cap B(0,r)$ because Γ is a cone, and finally $x = \psi(ty) \in \Omega$ by (8.9). So $W_i^* \subset \Omega$.

Finally, we claim that $u_i^*(x) = 0$ for almost every $x \in \mathbb{R}^n \setminus W_i^*$. Start when $x \in \psi(B(0,r))$. Write $x = \psi(z)$, with $z = \Phi(x) \in B(0,r)$. Further write $z = ty$, with $y \in S_r$ and $t < 1$; then $y \in S_r \setminus \Phi(W_i)$, because otherwise $z \in H_i$ and $x \in W_i^*$. If $v_i(y) \neq 0$, then y lies in the σ-negligible set from (8.15) (we just saw that $y \in S_r \setminus \Phi(W_i)$); then $z = ty$ lies in a negligible set too, and so does $x = \psi(z)$. Hence $v_i(y) = 0$ for almost every x, and so $v_i^*(z) = 0$ by (8.18) and $u_i^*(x) = v_i^*(\Phi(x)) = v_i^*(z) = 0$ by (8.20).

We are left with the case when $x \in \mathbb{R}^n \setminus \psi(B(0,r))$, and then $x \in \mathbb{R}^n \setminus W_i$ by (8.22). But we claim that

$$\mathbf{u}^*(x) = \mathbf{u}(x) \quad \text{for } x \in \mathbb{R}^n \setminus \psi(B(0,r)). \tag{8.24}$$

Indeed, if $x \in B(0, 5r/4)$, $u_i^*(x) = v_i^*(\Phi(x)) = v_i(\Phi(x)) = u_i(x)$ by (8.20), (8.17), and (8.12). Otherwise, $x \in \mathbb{R}^n \setminus B(0, 5r/4)$ and $u_i^*(x) = u_i(x)$ directly by (8.19). This completes our proof of (8.24), and we deduce from (8.24) that $u_i^*(x) = u_i(x) = 0$ for almost every $x \in [\mathbb{R}^n \setminus W_i^*] \cap [\mathbb{R}^n \setminus \psi(B(0,r))]$. In turn (8.21) follows.

We now need to estimate various terms. Since all our functions are bounded and $\mathbf{u}^* = \mathbf{u}$ on $\mathbb{R}^n \setminus B(0, 5r/4)$, we have the same estimates on the F and M terms as in (7.28) and (7.29). We still have that $J(\mathbf{u}, \mathbf{W}) \leq J(\mathbf{u}^*, \mathbf{W}^*)$ and hence, as in (7.30))

$$\begin{aligned}\int_{B(0,5r/4)} |\nabla \mathbf{u}|^2 &\leq \int_{B(0,5r/4)} |\nabla \mathbf{u}^*|^2 + |M(\mathbf{u}^*) - M(\mathbf{u})| + |F(\mathbf{W}^*) - F(\mathbf{W})| \\ &\leq \int_{B(0,5r/4)} |\nabla \mathbf{u}^*|^2 + Cr^{n-\frac{n}{p}} + Cr^{\beta n}.\end{aligned} \tag{8.25}$$

By (8.24), the energy contributions of $B(0, 5r/4) \setminus \psi(B(0,r))$ cancel and we get that

$$\int_{\psi(B(0,r))} |\nabla \mathbf{u}|^2 \leq \int_{\psi(B(0,r))} |\nabla \mathbf{u}^*|^2 + Cr^{n-\frac{n}{p}} + Cr^{\beta n}. \tag{8.26}$$

We now change use our bilipschitz mapping to change variables. By (8.20) and (8.6),

$$\int_{\psi(B(0,r))} |\nabla \mathbf{u}^*|^2 = \int_{\psi(B(0,r))} |\nabla (\mathbf{v}^* \circ \Phi)|^2 \leq (1+\eta)^{n+2} \int_{B(0,r)} |\nabla \mathbf{v}^*|^2. \tag{8.27}$$

Then the proof of (7.23) (or (7.18)) yields

$$\int_{B(0,r)} |\nabla v_i^*|^2 = \frac{r}{(n+2\gamma-2)} \int_{S_r} \gamma^2 r^{-2} v_i^2 + |\nabla_t v_i|^2. \tag{8.28}$$

We claim that $v_i = 0$ almost everywhere on $S_r \setminus \Gamma$. Indeed, if $y \in S_r \setminus \Gamma$ and $x = \psi(y)$, then $x \in B(0,2r) \setminus \Omega$ by (8.8), hence x lies out of W_i and $y = \Phi(x)$ lies out of $\Phi(W_i)$. Almost always, $v_i(y) = 0$, by (8.15). This, (8.5), and our assumption that $\mathbf{v} \in W^{1,2}(S_r))$ allows us to apply (4.6) and get

$$\int_{S_r} |v_i|^2 = \int_{S_r \cap \Phi(W_i)} |v_i|^2 \leq C r^2 \varepsilon^{-1} \int_{S_r} |\nabla_t v_i|^2 \tag{8.29}$$

as in (7.24). We return to (8.28), sum over i (the pieces are still orthogonal because of disjoint supports), use (8.29), and obtain as in (7.27), and with the same λ as in (7.25), that

$$\int_{B(0,r)} |\nabla \mathbf{v}^*|^2 = \sum_i \int_{B(0,r)} |\nabla v_i^*|^2 \leq \lambda r \sum_i \int_{S_r} |\nabla_t v_i|^2 = \lambda r \int_{S_r} |\nabla_t \mathbf{v}|^2. \tag{8.30}$$

We complete the estimate with a change of variable in the other direction:

$$\begin{aligned} E(r) &= \int_{B(0,r)} |\nabla \mathbf{v}|^2 = \int_{B(0,r)} |\nabla(\mathbf{u}\circ\psi)|^2 = (1+\eta)^{n+2} \int_{\psi(B(0,r))} |\nabla \mathbf{u}|^2 \\ &\leq (1+\eta)^{n+2} \int_{\psi(B(0,r))} |\nabla \mathbf{u}^*|^2 + C r^{n-\frac{n}{p}} + C r^{\beta n} \\ &\leq (1+\eta)^{2n+4} \int_{B(0,r)} |\nabla \mathbf{v}^*|^2 + C r^{n-\frac{n}{p}} + C r^{\beta n} \\ &\leq (1+\eta)^{2n+4} \lambda r \int_{S_r} |\nabla_t \mathbf{v}|^2 + C r^{n-\frac{n}{p}} + C r^{\beta n} \\ &\leq (1+\eta)^{2n+4} \lambda r E'(r) + C r^{n-\frac{n}{p}} + C r^{\beta n} \end{aligned} \tag{8.31}$$

by (8.12), (8.6), (8.26), by (8.27), (8.30), and (8.16). This is our analogue of (7.30), with the only difference that we have the extra term $(1+\eta)^{2n+4}$. Also, we do not need to care about (7.21) (there is no first case). Anyway, $(n-2)\lambda < 1$ if γ is small enough, depending on ε (see (7.32)), so we can choose η so small that $(n-2)\lambda(1+\eta)^{2n+4} < 1$, and then choose δ such that

$$n - 2 < \delta < \min\left(n\beta, n - \frac{n}{p}, (1+\eta)^{-2n-4}\lambda^{-1}\right) \tag{8.32}$$

(see (7.31)), and (8.31)) becomes

$$\delta E(r) \leq r E'(r) + C r^{n-\frac{n}{p}} + C r^{\beta n}. \tag{8.33}$$

This is the same as (7.33). We integrate this as we did before and get the analogue of (7.38):

$$\int_{B(x,r)} |\nabla \mathbf{v}|^2 \leq (r/r_0)^\delta E(r_0) + C[r^{n-\frac{n}{p}} + r^{\beta n}]. \tag{8.34}$$

We complete this by a last change of variable: we are interested in

$$\begin{aligned}\int_{x\in B(x,r)} |\nabla \mathbf{u}(x)|^2 &\leq (1+\eta)^{n+2} \int_{y\in\Phi(B(x,r))} |\nabla \mathbf{v}(y)|^2 \\ &\leq (1+\eta)^{n+2} \int_{y\in B(0,(1+\eta r))} |\nabla \mathbf{v}(y)|^2 \\ &\leq (1+\eta)^{n+3}(r/r_0)^\delta E(r_0) + 2C[r^{n-\frac{n}{p}} + r^{\beta n}]\end{aligned} \tag{8.35}$$

by (8.12), (8.6), (8.7), if $(1+\eta)r \leq r_0$, and by (8.34). Since $E(r_0) \leq (1+\eta)^{n+2}\int_{B(0,2r_0)} |\nabla \mathbf{u}|^2$ by the usual change of variable, (8.35) implies (8.11) when $(1+\eta)r \leq r_0$. The other case is trivial (recall that δ is very small). This completes our proof of Proposition 8.2. ∎

We are now ready to prove Theorem 8.1. Let Ω satisfy (8.1). That is, $\partial\Omega$ is a compact C^1 embedded submanifold if codimension 1, and we can assume that Ω is locally on one side of $\partial\Omega$; otherwise, remove the set of points of $\partial\Omega$ which have Ω on both sides (this set is open and closed in $\partial\Omega$), without changing the problem. By compactness, we can find $r_0 \in (0,1]$ such that, for each $x \in \partial\Omega$, the set $\Omega_x = \Omega - x$ satisfies the geometrical assumptions (8.4)-(8.10) of Proposition 8.2. We can even take for Γ a half space (and so $\varepsilon = 1/2$).

Now let $(\mathbf{u}, \mathbf{W})$ be a minimizer, as in the statement of Theorem 8.1, and let $x \in \partial\Omega$ be given. We can apply Proposition 8.2, with the value of r_0 that we just found, to a translation by $-x$ of Ω, $(\mathbf{u}, \mathbf{W})$, and the data f_i and g_i. We get that (8.11) holds, so

$$\int_{B(x,r)} |\nabla \mathbf{u}|^2 \leq 2(r/r_0)^\delta \int_{B(x,2r_0)} |\nabla \mathbf{u}|^2 + C_1 r^{n-\frac{n}{p}} + C_1 r^{\beta n} \tag{8.36}$$

for $0 < r \leq r_0$. The constant C_1 depends only on the various constants that show up in the assumptions.

This shall take care of balls $B(x,r)$ centered on $\partial\Omega$. Now consider $x \in \mathbb{R}^n \setminus \partial\Omega$, set $d(x) = \operatorname{dist}(x, \partial\Omega)$, and choose $y \in \partial\Omega$ such that $|x-y| = d(x)$.

Let $r > 0$ be given, with $r \leq r_0/3$. We shall need to discuss cases. If $r \geq d(x)/2$, just observe that

$$\begin{aligned}\int_{B(x,r)} |\nabla \mathbf{u}|^2 &\leq \int_{B(y,r+d(x))} |\nabla \mathbf{u}|^2 \leq \int_{B(y,3r)} |\nabla \mathbf{u}|^2 \\ &\leq 2(3r/r_0)^\delta \int_{B(x,2r_0)} |\nabla \mathbf{u}|^2 + 3^n C_1 r^{n-\frac{n}{p}} + 3^n C_1 r^{\beta n} \\ &\leq C(r/r_0)^\delta \int_{B(x,2r_0)} |\nabla \mathbf{u}|^2 + C[r^{n-\frac{n}{p}} + r^{\beta n}]\end{aligned} \tag{8.37}$$

by (8.36), and where we do not need to keep track of the dependence in C_1. So we may assume that $r \leq d(x)/2$, and the only interesting case is when $x \in \Omega$, because otherwise $B(x,r) \subset \mathbb{R}^n \setminus \Omega$ and $\int_{B(x,r)} |\nabla \mathbf{u}|^2 = 0$.

Let us first assume that $d(x) \leq r_0/3$. The proof of (8.37), with $r = d(x)$, yields

$$\int_{B(x,d(x))} |\nabla \mathbf{u}|^2 \leq C(d(x)/r_0)^{\delta} \int_{B(x,2r_0)} |\nabla \mathbf{u}|^2 + Cd(x)^{n-\frac{n}{p}} + Cd(x)^{\beta n}. \tag{8.38}$$

Let us use the proof of Theorem 7.1, applied as usual after translating everything by $-x$, and with $r_0 = d(x)$, so that $B(x,r_0) = B(x,d(x)) \subset \Omega$. In fact, we are only interested by (7.38), which implies that

$$\begin{aligned}
\int_{B(x,r)} |\nabla \mathbf{u}|^2 \;&\leq\; C(r/d(x))^{\delta} \int_{B(x,d(x))} |\nabla \mathbf{u}|^2 + C[r^{n-\frac{n}{p}} + r^{\beta n}] \\
&\leq\; C(r/r_0)^{\delta} \int_{B(x,2r_0)} |\nabla \mathbf{u}|^2 + C(r/d(x))^{\delta}[d(x)^{n-\frac{n}{p}} + d(x)^{\beta n}] + C[r^{n-\frac{n}{p}} + r^{\beta n}] \\
&\leq\; C(r/r_0)^{\delta} \int_{B(x,2r_0)} |\nabla \mathbf{u}|^2 + Cr^{\delta} \qquad (8.39)
\end{aligned}$$

by (8.38) and because $\delta \leq \min(n - \frac{n}{p}, \beta n)$ and $d(x) \leq r_0/3 \leq 1/3$.

In the last case when $d(x) > r_0/3$, we also use the proof of Theorem 7.1, but with the radius $r_0/3$ (which is all right because $B(x, r_0/3) \subset \Omega$), and deduce directly from (7.38) that

$$\begin{aligned}
\int_{B(x,r)} |\nabla \mathbf{u}|^2 \;&\leq\; C(3r/r_0)^{\delta} \int_{B(x,2r_0/3)} |\nabla \mathbf{u}|^2 + C[r^{n-\frac{n}{p}} + r^{\beta n}] \\
&\leq\; C(r/r_0)^{\delta} \int_{B(x,2r_0)} |\nabla \mathbf{u}|^2 + C[r^{n-\frac{n}{p}} + r^{\beta n}] \qquad (8.40)
\end{aligned}$$

because $r \leq r_0/3$ by assumption.

Thus in all the cases we get the same conclusion as in (8.39), or better, which now holds for all balls of radius $r \leq r_0/3$. This is not exactly as good as in (7.38), because the error term Cr^{δ} is a little larger. In fact if we want to really get (8.2) later, let us observe that we can get that

$$\int_{B(x,r)} |\nabla \mathbf{u}|^2 \leq C(r/r_0)^{\delta} \int_{B(x,2r_0)} |\nabla \mathbf{u}|^2 + Cr^{\delta'} \tag{8.41}$$

for some $\delta' > \delta$ (we just use a constant a little larger than δ in the estimates above that lead to (8.39)). The conclusion of Theorem 8.1, namely (8.2), now follows by the same proof as for (7.2); see (7.41)-(7.50). ∎

Remark 8.3 *The regularity assumption (8.1) that we put in Theorem 8.1 is really far from optimal. First, we just use the boundedness of Ω to have some uniformity in the approximation by cones. But more importantly, we do not need to have a good approximation by cones* *at all scales $r \le r_0$. It would be more than enough, for instance, if for some choice of $R > 0$ and $C \ge 0$ and all $x \in \partial\Omega$, the set $\Omega - x$ had the approximation condition (8.4)-(8.10) for all $r_0 \le R$, except perhaps for r_0 in an exceptional set $Z(x)$ such that $\int_{Z(x)} dr/r \le C$. But many other conditions would probably work as well.*

9 The monotonicity formula

The Hölder-continuity of $\mathbf{u}$ (Theorems 7.1 and 8.1) will allow us to apply a near monotonicity result of [CJK] that will be very useful, in particular for proving that $\mathbf{u}$ is Lipschitz inside Ω and controlling blow-up limits.

We shall need slightly stronger assumptions on the data. We still assume that $|\Omega| < +\infty$ (as in (3.1)), that F is Hölder-continuous with exponent $\beta > \frac{n-2}{n}$, as in (7.1), and that

$$f_i \in L^p(\Omega), \text{ for some } p > \frac{n}{2} \text{ and } f_i(x) \ge 0 \text{ almost everywhere on } \Omega, \tag{9.1}$$

(see (5.2)-(5.3)), but this time we shall also require

$$g_i \in L^\infty(\Omega) \tag{9.2}$$

(and not just L^p for some $p > n/2$); this is probably not optimal, but we should probably at least require $p > n$. See Remark 9.2 When we consider balls that meet $\partial\Omega$, we shall also assume Ω to be a bounded open set with a C^1 boundary, as in (8.1).

A consequence of these assumptions is that we can change $\mathbf{u}$ on a set of zero measure to make it Hölder-continuous. We shall always assume that this modification has been done, which will allow us to talk about the open sets $\Omega_i = \big\{x \in \Omega\,;\, u_i(x) > 0\big\}$. See Remark 7.2.

Fix $x_0 \in \mathbb{R}^n$, two indices $i_1, i_2 \in [1, N]$ and two signs $\varepsilon_1, \varepsilon_2 \in \{-1, +1\}$. Then define functions v_1 and v_2 by

$$v_j(x) = [\varepsilon_j u_{i_j}(x)]_+ = \max(0, \varepsilon_j u_{i_j}(x)) \in [0, +\infty) \tag{9.3}$$

for $j = 1, 2$ and $x \in \mathbb{R}^n$. We always take different pairs (i_j, ε_j), so typical choices of the two v_j would be $v_1 = (u_1)_+$ and $v_2 = (u_1)_-$, or $v_1 = (u_1)_+$ and $v_2 = (u_2)_+$. Our complicated notation is designed to accommodate both cases. Finally set

$$\Phi_j(r) = \frac{1}{r^2}\int_{B(x_0,r)} \frac{|\nabla v_j|^2}{|x - x_0|^{n-2}}\, dx \ \text{ for } j = 1, 2 \ \text{ and } \Phi(r) = \Phi_1(r)\Phi_2(r) \tag{9.4}$$

for $r > 0$. This is the function which will be nearly monotone. As we shall see later, the integrals often converge because of (7.38) or (8.11).

Theorem 9.1 *Assume that (3.1), (7.1), (9.1), and (9.2) hold. Let $(\mathbf{u}, \mathbf{W})$ be a minimizer of the functional J, and let x_0 and $r_0 > 0$ be given. If $B(x_0, r_0)$ is not contained in Ω, also assume (8.1). Then for all choices of $(i_1, \varepsilon_1) \neq (i_2, \varepsilon_2)$ as above, and $0 < r \le r_0$,*

$$\Phi(r) \le C\Big(r_0^2 ||g_{i_1}||_\infty^2 + r_0^2 ||g_{i_2}||_\infty^2 + \Phi_1(r_0) + \Phi_2(r_0)\Big)^2, \tag{9.5}$$

with a constant C that depends only on n.

Proof. Our proof will mostly consist in checking that v_1 and v_2 satisfy the assumptions of Theorem 1.3 in [CJK], which is perfectly fit for our situation. This result is in the same spirit as in the initial monotonicity formula in [ACF]. It looks a little less nice because (9.5) is less precise than saying that Φ is nondecreasing, but this allows more general situations (as here), and will give almost as good consequences.

The first assumption of [CJK], that the v_j be continuous, is a consequence of Theorems 7.1 or 8.1, and this is why we include (3.1), L^p bounds on the f_i, the Hölder assumption (7.1), and sometimes (8.1), which will not show up in the estimates. They also satisfy the exclusion relation $v_1 v_2 = 0$, just because $(\mathbf{u}, \mathbf{W}) \in \mathcal{F}$ (and $\mathbf{u}$ is continuous).

Next we want to show that for each i, u_i satisfies the equation

$$\Delta u_i = f_i u_i - \frac{1}{2} g_i \tag{9.6}$$

in the open set $\Omega_i = \big\{x \in \mathbb{R}^n\,; u_i(x) \neq 0\big\} \subset \Omega$. Here we restrict to Ω_i because in other places we may not modify u_i freely as we do in the proof below. Otherwise we proceed in the most usual way. For each test function φ with compact support in Ω_i, we observe that if we replace u_i with $u_i + t\varphi$, $t \in \mathbb{R}$ small, and otherwise change nothing, we get a new competitor $(\mathbf{u}_t, \mathbf{W})$. Thus $J(\mathbf{u}_t, \mathbf{W}) \geq J(\mathbf{u}, \mathbf{W})$ for t small. But $J(\mathbf{u}_t, \mathbf{W})$ has a derivative at $t = 0$, which is

$$\frac{\partial J(\mathbf{u}_t, \mathbf{W})}{\partial t}(0) = 2\int \langle \nabla u_i, \nabla \varphi \rangle + 2\int f_i u_i \varphi - \int g_i \varphi \tag{9.7}$$

(see (1.5) and recall that only u_i changes). This derivative vanishes, so by definition of the distribution Δu_i,

$$0 = 2\int \langle \nabla u_i, \nabla \varphi \rangle + 2\int f_i u_i \varphi - \int g_i \varphi = \langle -2\Delta u_i + 2 f_i u_i - g_i, \varphi \rangle. \tag{9.8}$$

This holds for every test function φ, and this gives (9.6). As an immediate consequence of (9.6) and the definitions,

$$\Delta v_j = \varepsilon_j \Delta u_{i_j} = \varepsilon_j f_i u_{i_j} - \frac{1}{2}\varepsilon_j g_{i_j} \geq -\frac{1}{2}||g_{i_j}||_\infty \tag{9.9}$$

in the sense of distributions, in the open set $\Omega(j) = \big\{x \in \mathbb{R}^n\,;\, v_j(x) > 0\big\} \subset \Omega$.

We want to take advantage of the normalization in [CJK], so we don't apply the result directly to the v_j, but to $w_j(x) = \lambda_j v_j(x_0 + r_0 x)$, which are defined on the unit ball and such that $\Delta w_j(x) = \lambda_j r_0^2 \Delta v_j(x_0 + r_0 x) \geq -1$ if $\lambda_j r_0^2 ||g_{i_j}||_\infty \leq 2$. A brutal, but acceptable choice will be to take

$$\lambda_1 = \lambda_2 = r_0^{-2}(\tau + ||g_{i_1}||_\infty + ||g_{i_2}||_\infty)^{-1}, \tag{9.10}$$

with a very small $\tau > 0$ that will tend to 0 soon.

Notice that the w_j satisfy all the assumptions of Theorem 1.3 in [CJK], in particular because Remark 1.4 in [CJK] says that since w_j is nonnegative and continuous and $\Delta w_j \geq -1$ on $\{w_j > 0\}$, we get that $\Delta w_j \geq -1$ (as a distribution and on the whole $\mathbb{R}^n$). Set

$$\widetilde{\Phi}_j(\rho) = \frac{1}{\rho^2} \int_{B(x_0,\rho)} \frac{|\nabla w_j|^2}{|x|^{n-2}}\, dx \quad \text{and} \quad \widetilde{\Phi}(\rho) = \Phi_1(\rho)\Phi_2(\rho) \tag{9.11}$$

for $0 < \rho \leq 1$; then by [CJK]

$$\widetilde{\Phi}(\rho) \leq C\big(1 + \widetilde{\Phi}_1(1) + \widetilde{\Phi}_2(1)\big)^2. \tag{9.12}$$

Also, a change of variable yields $\widetilde{\Phi}_j(\rho) = \lambda_j^2 r_0^2 \Phi_j(r_0\rho)$ for $0 < r \leq 1$, and now

$$\begin{aligned} \Phi(r) &= r_0^{-4}\lambda_1^{-2}\lambda_2^{-2}\widetilde{\Phi}(r/r_0) \leq C r_0^{-4}\lambda_1^{-2}\lambda_2^{-2}\big(1 + \widetilde{\Phi}_1(1) + \widetilde{\Phi}_2(1)\big)^2 \\ &\leq C r_0^{-4}\lambda_1^{-2}\lambda_2^{-2}\big(1 + \lambda_1^2 r_0^2 \Phi_1(r_0) + \lambda_2^2 r_0^2 \Phi_2(r_0)\big)^2 \\ &= \big(\lambda_1^{-2} r_0^{-2} + \Phi_1(r_0) + \Phi_2(r_0)\big)^2 \end{aligned} \tag{9.13}$$

because $\lambda_1 = \lambda_2$ by (9.10). Since $\lambda_1^{-2} r_0^{-2} = r_0^2(\tau + ||g_{i_1}||_\infty + ||g_{i_2}||_\infty)^2$ by (9.10), we get that

$$\Phi(r) \leq C\big(r_0^2(\tau + ||g_{i_1}||_\infty + ||g_{i_2}||_\infty)^2 + \Phi_1(r_0) + \Phi_2(r_0)\big)^2. \tag{9.14}$$

We now let τ tend to 0 and get (9.5); Theorem 9.1 follows. ∎

Remark 9.2 *Our previous assumption than $g_i \in L^p$ for some $p > n/2$ is no longer enough. In the simple case when $N = 1$ and $f_1 = 0$, we get a solution u that satisfies $\Delta u = -\frac{1}{2} g_1$ locally (see (9.6)), and that looks like $G * g_1$, where G is the fundamental solution of $-\Delta$ (as in Section 5). Then ∇u looks like $\nabla G * g_1$ (a Riesz transform of order 1). If we want to make sure that u behaves like a Lipschitz function (this is what is suggested by the normalization in (9.5)), we should probably require that $\Delta u = -\frac{1}{2} g_1 \in L^p$, where $p > n$ is larger than the Sobolev exponent.*

10 Interior Lipschitz bounds for u

In this section we make our assumptions just a bit stronger than before (we do not want error terms much larger than r^n), and show that u is locally Lipschitz inside Ω when $(\mathbf{u}, \mathbf{W})$ is a minimizer for J. We shall take care of the Lipschitz regularity near $\partial\Omega$ in the next section.

We now assume that for $1 \leq i \leq n$,

$$f_i \geq 0 \text{ a.e. on } \Omega,\ f_i \in L^\infty(\Omega), \text{ and } g_i \in L^\infty(\Omega), \tag{10.1}$$

and we also require F to be a Lipschitz function of $\mathbf{W}$, i.e., that

$$\Big| F(W_1, W_2, \ldots, W_N) - F(W_1', W_2', \ldots, W_N') \Big| \leq C \sum_{i=1}^{N} |W_i \Delta W_i'| \tag{10.2}$$

for some $C \geq 0$ and all choices of $W_i, W_i' \subset \Omega$, $1 \leq i \leq N$. As usual Δ denotes a symmetric difference.

Theorem 10.1 *Assume that $|\Omega| < +\infty$ (as in (3.1)), and that (10.1) and (10.2) hold. Let $(\mathbf{u}, \mathbf{W})$ be a minimizer of the functional J, and let x_0 and $r_0 \in (0,1]$ be such that $B(x_0, 2r_0) \subset \Omega$. Then*

$$|\mathbf{u}(x) - \mathbf{u}(y)| \leq C_2 \Big(1 + \fint_{B(x_0,2r_0)} |\nabla \mathbf{u}|^2\Big)^{1/2} |x-y| \quad \text{for } x, y \in B(x_0, r_0), \tag{10.3}$$

with a constant C_2 that depends only on n, N, $|\Omega|$, and the constants in (10.1) and (10.1).

The main ingredient for the proof of Theorem 10.1 is Theorem 9.1, which we shall use to say that when $\int_{B(x,r)} |\nabla \mathbf{u}|^2$ is very large, then one of the $\int_{B(x,r)} |\nabla u_i|^2$ is much larger than the other ones, which will allow us to use the harmonic competitor described in Section 6.

Before we start with the proof itself, let us describe a small decoupling trick that will allow us to simplify our notation.

Lemma 10.2 *It is enough to prove Theorem 10.1 when, in the definition of $\mathcal{F}$, all the functions u_i are required to be nonnegative.*

Proof. Let J be our initial functional (for which we want to prove Theorem 10.1); we want to construct new functional $\widetilde{J}$, defined on a new set $\widetilde{\mathcal{F}}$ of competitors, so that the minimization of J on $\mathcal{F}$ is equivalent to the minimization of $\widetilde{J}$ on $\widetilde{\mathcal{F}}$. Let I denote the set of indices i for which u_i is not required to be nonnegative (in $\mathcal{F}$). For each $i \in I$, decouple i as two indices i_+ and i_-; for $i \in [1, N] \setminus I$, just keep the same index i. This gives a new set of indices, which we call I'.

Define $\widetilde{\mathcal{F}}$ as in Section 1, but with the new set I' of indices, and the constraint that all u_i, $i \in I'$, are nonnegative. For the M-term of the functional, keep the f_i and g_i, $i \in [1, N] \setminus I$, as they were, and for $i \in I$, set $f_{i,+} = f_{i,-} = f_i$ and $g_{i,+} = g_{i,-} = g_i$. Also define $\widetilde{F}$ by setting $W_i = W_{i,+} \cup W_{i,-}$ for each $i \in I$, and then substituting W_i in the definition of F. That is, if $\widetilde{\mathbf{W}}$ is indexed by I', we define $\mathbf{W}$ indexed by $[1, N]$ by the rule above, and set $\widetilde{F}(\widetilde{\mathbf{W}}) = F(\mathbf{W})$.

All this gives a new functional $\widetilde{J}$ defined on $\widetilde{\mathcal{F}}$. If $(\mathbf{u}, \mathbf{W}) \in \mathcal{F}$, we define a pair $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{W}}) \in \widetilde{\mathcal{F}}$ in the natural way: we keep u_i as it is when $i \in [1, N] \setminus I$, and when $i \in I$ we set $u_{i,\pm} = \max(0, \pm u_i)$ (the positive and negative part). We keep W_i when $i \in [1, N] \setminus I$, and otherwise we set $W_{i,+} = \big\{x \in W_i\,;\, u_i(x) \geq 0\big\}$ and $W_{i,-} = \big\{x \in W_i\,;\, u_i(x) < 0\big\}$. Of course we could have sent part of the set $\big\{x \in W_i\,;\, u_i(x) = 0\big\}$ in $W_{i,-}$, but this will not matter. It is easy to see that this gives a pair $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{W}}) \in \widetilde{\mathcal{F}}$, and that $J(\widetilde{\mathbf{u}}, \widetilde{\mathbf{W}}) = J(\mathbf{u}, \mathbf{W})$.

Conversely, given $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{W}}) \in \widetilde{\mathcal{F}}$, we construct a pair $(\mathbf{u}, \mathbf{W})$ by setting $W_i = W_{i,+} \cup W_{i,-}$ and $u_i = u_{i,+} - u_{i,-}$ when $i \in I$, and changing nothing otherwise. It is easy to see that $(\mathbf{u}, \mathbf{W}) \in \mathcal{F}$ and $J(\widetilde{\mathbf{u}}, \widetilde{\mathbf{W}}) = J(\mathbf{u}, \mathbf{W})$.

Now, if we prove Theorem 10.1 for $\widetilde{J}$ in $\widetilde{\mathcal{F}}$, it immediately follows for J on $\mathcal{F}$, as needed for the lemma. ∎

Of course Lemma 10.2 does not change the nature of our problem, it will just allow us to simplify our notation. Notice however that things would not have been so easy if F was required to be a strictly convex function of the volume in each variable, since the new function $\widetilde{F}$ is not.

Return to the proof of Theorem 10.1. Now assume that all the u_i are required to be nonnegative. We shall try to control quantities like

$$E(r) = \int_{B(0,r)} |\nabla \mathbf{u}|^2 = \sum_i \int_{B(x_1,r)} |\nabla u_i|^2 \tag{10.4}$$

and for the interior regularity, we shall concentrate on the case when $0 < r \leq r_0$ for some r_0 such that $B(0, r_0) \subset \Omega$. Let us also define, for $1 \leq i \leq N$ and $0 < r \leq r_0$,

$$E_i(r) = \int_{B(0,r)} |\nabla u_i|^2. \tag{10.5}$$

Let us now record what we get when we apply Theorem 9.1.

Lemma 10.3 *Suppose $B(0, r_0) \subset \Omega$ and $r_0 \leq 1$. Then*

$$r^{-2n} E_i(r) E_j(r) \leq C_3 \Big(1 + \fint_{B(0,r_0)} |\nabla \mathbf{u}|^2\Big)^2 \tag{10.6}$$

for $0 < \rho \leq r_0$ and $1 \leq i \neq j \leq N$. The constant C_3 depends only on the usual constants, i.e., n, N, $|\Omega|$, the $||f_i||_\infty$ (we would even get away with bounds on $||f_i||_p$ for some $p > n/2$), the $||g_i||_\infty$, and the Lipschitz constant in (10.2).

Proof. Set

$$\Phi_i(r) = r^{-2} \int_{B(0,r)} \frac{|\nabla u_i|^2}{|x|^{n-2}} \tag{10.7}$$

for $1 \leq i \leq N$ and $0 < \rho \leq r_0$, observe that

$$r^{-n} E_i(r) \leq \Phi_i(r) \tag{10.8}$$

because $|x| \leq r$ in the integral, and that $\Phi_i(r)$ is the same number that we called $\Phi_j(r)$ in (9.4), if we take $x_0 = 0$ there and $(i_j, \varepsilon_j) = (i, +1)$. Thus Theorem 9.1 says that

$$r^{-2n} E_i(r) E_j(r) \leq \Phi_i(r)\Phi_j(r) \leq C\Big(r_0^2 ||g_{i_1}||_\infty^2 + r_0^2 ||g_{i_2}||_\infty^2 + \Phi_i(r_0) + \Phi_j(r_0)\Big)^2 \tag{10.9}$$

for $i \neq j$ and $0 < r \leq r_0$. We also need bounds on the right-hand side of (10.9), and indeed

$$\begin{aligned}
\Phi_i(r_0) &= \frac{1}{r_0^2}\int_{B(0,r_0)} \frac{|\nabla u_i|^2}{|x|^{n-2}}\,dx \leq \sum_{k\geq 0}\frac{1}{r_0^2}\int_{B(0,2^{-k}r_0)\setminus B(0,2^{-k-1}r_0)} (2^{-k-1}r_0)^{2-n}|\nabla u_i|^2 \\
&\leq Cr_0^{-n}\sum_{k\geq 0}2^{k(n-2)}\int_{B(0,2^{-k}r_0)}|\nabla \mathbf{u}|^2 \\
&\leq Cr_0^{-n}\sum_{k\geq 0}2^{k(n-2)}\Big\{2^{-k\delta}\int_{B(0,r_0)}|\nabla \mathbf{u}|^2 + [(2^{-k}r_0)^{n-\frac{n}{p}} + (2^{-k}r_0)^{\beta n}]\Big\} \qquad (10.10)\\
&= Cr_0^{-n}\sum_{k\geq 0}2^{k(n-2)}\Big\{2^{-k\delta}\int_{B(0,r_0)}|\nabla \mathbf{u}|^2 + (2^{-k}r_0)^{n}\Big\}
\end{aligned}$$

by (10.7) and (7.38), and because with our new assumptions (10.1) and (10.2), we now have $p = +\infty$ and $\beta = 1$. We do not really need this additional information here, but it simplifies the formulas. Recall from (7.31) that $\delta > n-2$. Thus the sum over k converges geometrically, and

$$\Phi_i(r_0) \leq Cr_0^{-n}\int_{B(0,r_0)}|\nabla \mathbf{u}|^2 + C. \qquad (10.11)$$

Then (10.6) follows from (10.9) and (10.11). ∎

Notice that we can get upper bounds for $\int_{\mathbb{R}^n}|\nabla \mathbf{u}|^2$ in terms of the usual constants, by (3.10), but when we cannot find a large ball $B(0,r_0) \subset \Omega$, we may need to content ourselves with a small r_0, and get a large lower bound in (10.6).

We now state the main decay estimate in the proof of Theorem 10.1, which concerns the case when $\mathbf{u}(0) = 0$ (the notion makes sense because $\mathbf{u}$ is Hölder continuous inside Ω).

Lemma 10.4 *We can find $\tau \in (0, 10^{-1})$, that depends on n and N, and C_4, that also depends on $|\Omega|$ and the constants in (10.1) and (10.2), such that if $\mathbf{u}(0) = 0$, $0 < r_0 \leq 1$ and $B(0,r_0) \subset \Omega$,*

$$\fint_{B(0,\tau\rho)}|\nabla \mathbf{u}|^2 \leq C(\tau, r_0) + \frac{1}{10}\fint_{B(0,\rho)}|\nabla \mathbf{u}|^2 \qquad (10.12)$$

for $0 < \rho \leq r_0$, with $C(\tau, r_0) = C_4\big(1 + \fint_{B(0,r_0)}|\nabla \mathbf{u}|^2\big)$.

We see this as decay because in the most unpleasant situation when $\rho^{-n}\int_{B(0,\rho)}|\nabla \mathbf{u}|^2$ is very large (10.12) will say that $\fint_{B(0,\tau\rho)}|\nabla \mathbf{u}|^2 \leq \frac{1}{2}\fint_{B(0,\rho)}|\nabla \mathbf{u}|^2$. We keep for later the case of balls that are not centered on the set $\{\mathbf{u} = 0\}$.

Proof. Let $\rho \leq r_0$ be given. Let $M \geq 0$ be a very large number, to be chosen later. Let us first treat the easy case when

$$\fint_{B(0,\rho)}|\nabla \mathbf{u}|^2 \leq M. \qquad (10.13)$$

In this case, we just need to say that $\fint_{B(0,\tau\rho)} |\nabla \mathbf{u}|^2 \leq \tau^{-n} \fint_{B(0,\rho)} |\nabla \mathbf{u}|^2 \leq \tau^{-n} M$, and so (10.12) holds if we choose

$$C(\tau, r_0) \geq \tau^{-n} M. \tag{10.14}$$

So we may now assume that (10.13) fails. Select i so that $E_i(\rho)$ is largest; without loss of generality, we may assume that $i = 1$. Since $\int_{B(0,\rho)} |\nabla \mathbf{u}|^2 = E(\rho) = \sum_{j=1}^{N} E_j(\rho)$, we deduce from the failure of (10.13) that for the largest term

$$\rho^{-n} E_1(\rho) \geq N^{-1} \rho^{-n} \int_{B(0,\rho)} |\nabla \mathbf{u}|^2 \geq C^{-1} M \tag{10.15}$$

and hence, by (10.6), that

$$\rho^{-n} E_i(\rho) \leq \Lambda \ \text{ for } i > 1, \tag{10.16}$$

with

$$\Lambda = C M^{-1} C_3 \Big(1 + \fint_{B(0,r_0)} |\nabla \mathbf{u}|^2 \Big)^2. \tag{10.17}$$

Next we want to choose a radius $r \in (\rho/2, \rho)$, with a few good properties that will help us define and use the harmonic competitor of Section 6. First we want the restriction of each u_i to S_r to lie in $W^{1,2}(S_r)$, with tangential derivatives that can be computed from the restriction of $\nabla \mathbf{u}$ to S_r. This is easy to arrange, because it is true for almost every $r \in (\rho/2, \rho)$ (see the discussion near (4.14)). Next,

$$u_i(x) = 0 \ \text{ for } 1 \leq i \leq N \text{ and } \sigma\text{-almost every } x \in S_r \setminus W_i \tag{10.18}$$

(as in (6.16)), which is also true for almost all r. Finally, we choose r so that

$$\int_{S_r} |\nabla_t u_i|^2 \leq 2N\rho^{-1} \int_{B(0,\rho)} |\nabla u_i|^2 \leq 2N\rho^{-1} E(\rho) \tag{10.19}$$

for $1 \leq i \leq N$. The first inequality is easy to arrange by Chebyshev (use (4.15) with $p = 2$), and the second one is trivial (see (10.4)). When $i > 1$, we deduce from (10.16) and the first part of (10.19) that

$$\int_{S_r} |\nabla_t u_i|^2 \leq 2N\rho^{-1} E_i(\rho) \leq 2N\rho^{n-1} \Lambda, \tag{10.20}$$

which is a much better estimate if we choose M large enough.

Let $(\mathbf{u}^*, \mathbf{W}^*)$ denote the harmonic competitor that was defined near (6.11). The construction has a parameter $a \in (0, 1)$, which will be chosen soon, close to 1. All the prerequisites that were mentioned before (6.11) are satisfied, and in particular (6.9) holds because

$B(0,r) \subset B(0,r_0) \subset \Omega$. Since $(\mathbf{u}^*, \mathbf{W}^*) \in \mathcal{F}$ (by construction, see (6.15)) and $(\mathbf{u}, \mathbf{W})$ is a minimizer, we get that

$$J(\mathbf{u}, \mathbf{W}) \leq J(\mathbf{u}^*, \mathbf{W}^*). \tag{10.21}$$

The M-term of the functional is estimated as usual: by Theorem 5.1, $||u_i||_\infty \leq C$, with a constant C that depends only on the usual constants, and by construction (and the maximum principle, or rather the corresponding properties of the Poisson kernel, for the harmonic extension), $||u_i^*||_\infty \leq ||u_i||_\infty \leq C$. Then

$$|M(\mathbf{u}) - M(\mathbf{u}^*)| \leq C(||u_i||_\infty^2 ||f_i||_\infty + ||u_i||_\infty ||g_i||_\infty) \, |B(0,r)| \leq Cr^n, \tag{10.22}$$

because $u_i^* = u_i$ outside of $B(x,r)$, and where again C depends on the usual constants (see the definition (1.4)). Similarly, the W_i^* only differ from the W_i in the ball $B(0,r)$, so (10.2) yields

$$|F(\mathbf{W}) - F(\mathbf{W}^*)| \leq Cr^n \tag{10.23}$$

and (10.21) implies that

$$\int_{B(0,r)} |\nabla \mathbf{u}|^2 \leq \int_{B(0,r)} |\nabla \mathbf{u}^*|^2 + Cr^n \tag{10.24}$$

(compare with the definitions (1.5) and (1.3)). Recall from (6.19)-(6.21) that

$$\begin{aligned} \int_{B(0,r)} |\nabla \mathbf{u}^*|^2 &= \sum_{i\geq 2} \int_{B(0,r)} |\nabla u_i^*|^2 + \int_{B(0,r)\setminus B(0,ar)} |\nabla u_1^*|^2 + \int_{B(0,ar)} |\nabla u_1^*|^2 \\ &\leq (1-a) r a^{2-n} \sum_{i=1}^N \int_{S_r} |\nabla_t u_i|^2 + 4(1-a)^{-1} r a^{-n} \sum_{i=2}^N \int_{S_r} |r^{-1} u_i|^2 \\ &\qquad + a^{n-2} \int_{B(0,r)} |\nabla v_1|^2 \end{aligned} \tag{10.25}$$

where v_1 still denotes the harmonic extension (to $B(0,r)$) of the restriction of u_1 to S_r; see above (6.13). For the first term, we just use (10.19) and get that

$$(1-a) r a^{2-n} \sum_{i=1}^N \int_{S_r} |\nabla_t u_i|^2 \leq 2N(1-a) a^{2-n} r \rho^{-1} E(\rho) \leq C(1-a) E(\rho) \tag{10.26}$$

because $r \leq \rho$ and we will choose $a > 1/2$. In fact, we will take a very close to 1 to make this term small. We shall only be able to continue our estimates with the present competitor when

$$\sigma(S_r \setminus W_i) \geq \varepsilon \sigma(S_r) \ \text{ for } 2 \leq i \leq N, \tag{10.27}$$

where the small $\varepsilon > 0$ will be chosen later. In the remaining case when (10.27) fails, we shall use a different harmonic competitor; this will be somewhat easier, but will be done later, near (10.56).

Our assumption (10.27) allows us to use the Poincaré estimate (4.6), with $p = 2$ and $E = S_r \cap W_i$ (recall (10.18)). We get that

$$\int_{S_r} |r^{-1}u_i|^2 = r^{-2} \int_{S_r \cap W_i} |u_i|^2 \leq C\varepsilon^{-1} \int_{S_r} |\nabla_t u_i|^2. \tag{10.28}$$

For our second term, we use (10.28) and (10.20), and get that

$$\begin{aligned} 4(1-a)^{-1} r a^{-n} \sum_{i=2}^{N} \int_{S_r} |r^{-1}u_i|^2 &\leq C(1-a)^{-1} r \varepsilon^{-1} \sum_{i=2}^{N} \int_{S_r} |\nabla_t u_i|^2 \\ &\leq C(1-a)^{-1} \varepsilon^{-1} \rho^n \Lambda. \end{aligned} \tag{10.29}$$

For this term, the two large constants $(1-a)^{-1}$ and ε^{-1} will be neutralized by taking M large enough and hence Λ very small. We drop a^{n-2} in the last term of (10.25), and get that

$$\int_{B(0,r)} |\nabla \mathbf{u}^*|^2 \leq \int_{B(0,r)} |\nabla v_1|^2 + C\alpha_1 \tag{10.30}$$

with

$$\alpha_1 = (1-a)E(\rho) + (1-a)^{-1}\varepsilon^{-1}\rho^n \Lambda. \tag{10.31}$$

Now recall from (6.13) that v_1 is the minimizer of $\int_{B(0,r)} |\nabla v|^2$, among functions $v \in W^{1,2}(B(0,r))$ that coincide with u_1 on S_r. Of course we can take $v = u_1$ in the definition, and even $v_t = v_1 + t(u_1 - v_1)$. Notice that $\int_{B(0,r)} |\nabla v_t|^2$ is a quadratic function of t, whose derivative at $t = 0$ vanishes by minimality. We compute this derivative and get that $\int_{B(0,t)} \langle \nabla v_1, \nabla(u_1 - v_1)\rangle = 0$. Hence, by Pythagorus,

$$\int_{B(0,r)} |\nabla u_1|^2 = \int_{B(0,r)} |\nabla v_1|^2 + \int_{B(0,r)} |\nabla (u_1 - v_1)|^2. \tag{10.32}$$

We may now deduce from (10.24) and (10.30) that

$$\begin{aligned} \int_{B(0,r)} |\nabla(u_1 - v_1)|^2 + \int_{B(0,r)} |\nabla v_1|^2 &= \int_{B(0,r)} |\nabla u_1|^2 \leq \int_{B(0,r)} |\nabla \mathbf{u}|^2 \\ &\leq \int_{B(0,r)} |\nabla \mathbf{u}^*|^2 + Cr^n \\ &\leq \int_{B(0,r)} |\nabla v_1|^2 + C\alpha_1 + C\rho^n. \end{aligned} \tag{10.33}$$

We subtract $\int_{B(0,r)} |\nabla v_1|^2$ from both sides and get that

$$\int_{B(0,r)} |\nabla(u_1 - v_1)|^2 \leq C\alpha_1 + C\rho^n. \tag{10.34}$$

Next we need to find ways to say that ∇v_1 is small near the origin. We claim that

$$r^{-2} \int_{B(0,r)} |u_1 - v_1|^2 \leq C \int_{B(0,r)} |\nabla(u_1 - v_1)|^2 \leq C\alpha_1 + C\rho^n. \tag{10.35}$$

The second part follows from(10.34). The most direct way to see the first part is to notice that if we extend $u_1 - v_1$ by setting $(u_1 - v_1)(x) = 0$ for $x \in \mathbb{R}^n \setminus B(0,r)$, we get a function of $W^{1,2}(\mathbb{R}^n)$ (see (4.18)) . Then by Poincaré's inequality (for instance, apply (4.2) to $B(x, 2r)$ and use it to control $m_{B(x,2r)}(u_1 - v_1)$), we get the claim. Or we can use Lemma 4.2 to control $m_{B(0,r}(u_1 - v_1)$ and the apply Poincaré's inequality (4.2) on $B(0,r)$.

Next we want to use the (estimates that lead to the) Hölder-continuity of $\mathbf{u}$ to control $\mathbf{u}(x) - \mathbf{u}(0)$ near the origin. Recall from (7.38) that for $0 < s \leq r$,

$$\begin{aligned} \int_{B(0,s)} |\nabla u_1|^2 &\leq \int_{B(0,s)} |\nabla \mathbf{u}|^2 \leq (s/r)^\delta \int_{B(0,r)} |\nabla \mathbf{u}|^2 + C[s^{n-\frac{n}{p}} + s^{\beta n}] \\ &\leq (s/r)^\delta E(\rho) + C[s^{n-\frac{n}{p}} + s^{\beta n}] = (s/r)^\delta E(\rho) + Cs^n, \end{aligned} \tag{10.36}$$

where C depend on the usual constants, and the last equality comes from the fact that $p = +\infty$ and $\beta = 1$ with our new assumptions (10.1) and (10.2). Set $m(s) = \fint_{B(0,s)} u_1$ for $s < r$; the same proof as in (7.42) yields

$$\begin{aligned} |m(s/2) - m(s)| &= \Big| \fint_{B(0,s/2)} u_1 - m(s) \Big| \leq \fint_{B(0,s/2)} |u_1 - m(s)| \\ &\leq 2^n \fint_{B(0,s)} |u_1 - m(s)| \leq Cs \fint_{B(0,s)} |\nabla u_1| \\ &\leq Cs \Big\{ \fint_{B(0,s)} |\nabla u_1|^2 \Big\}^{1/2} \leq C \Big\{ s^{2-n} \int_{B(0,s)} |\nabla u_1|^2 \Big\}^{1/2} \\ &\leq C \Big\{ s^{2-n} (s/r)^\delta E(\rho) + s^2 \Big\}^{1/2}. \end{aligned} \tag{10.37}$$

By (7.31), the exponent $\delta + 2 - n$ is positive, so $\sum_{k \geq 0} |m(2^{-k-1}s) - m(2^{-k}s)| < +\infty$ and

$$m(s) \leq \sum_{k \geq 0} |m(2^{-k-1}s) - m(2^{-k}s)| \leq C \Big\{ s^{2-n}(s/r)^\delta E(\rho) + s^2 \Big\}^{1/2} \tag{10.38}$$

because u_1 is continuous and $\mathbf{u}(0) = 0$, by (10.37), and after summing a geometric series whose main term is for $k = 0$. Let $\tau \in (0,1)$ be small, to be chosen soon, and apply this with $s = \tau\rho$. This yields

$$m(\tau\rho)^2 \leq C\tau^{\delta+2-n}\rho^{2-n}E(\rho) + C\tau^2\rho^2. \tag{10.39}$$

We are interested in

$$v_1(0) = \fint_{B(0,\tau\rho)} v_1 = \fint_{S_r} u_1, \tag{10.40}$$

where both identities hold because v_1 is the Poisson integral of the restriction of u_1 to S_r. Notice that

$$v_1(0) = \fint_{B(0,\tau\rho)} v_1 \leq \fint_{B(0,\tau\rho)} u_1 + \fint_{B(0,\tau\rho)} |u_1 - v_1| = m(\tau\rho) + \fint_{B(0,\tau\rho)} |u_1 - v_1| \tag{10.41}$$

and that

$$\begin{aligned} \fint_{B(0,\tau\rho)} |u_1 - v_1| &\leq \Big\{ \fint_{B(0,\tau\rho)} (u_1 - v_1)^2 \Big\}^{1/2} \leq C\Big\{ (\tau\rho)^{-n} \int_{B(0,\tau\rho)} (u_1 - v_1)^2 \Big\}^{1/2} \\ &\leq C\Big\{ (\tau\rho)^{-n} \int_{B(0,r)} (u_1 - v_1)^2 \Big\}^{1/2} \leq C\Big\{ (\tau\rho)^{-n} r^2 [\alpha_1 + \rho^n] \Big\}^{1/2} \end{aligned} \tag{10.42}$$

by (10.35). Hence

$$\begin{aligned} v_1(0)^2 &\leq 2m(\tau\rho)^2 + 2\Big\{ \fint_{B(0,\tau\rho)} |u_1 - v_1| \Big\}^2 \\ &\leq C\tau^{\delta+2-n}\rho^{2-n}E(\rho) + C\tau^2\rho^2 + C(\tau\rho)^{-n}r^2[\alpha_1 + \rho^n] \\ &\leq C\tau^{\delta+2-n}\rho^{2-n}E(\rho) + C\tau^{-n}\rho^{2-n}\alpha_1 + C\tau^{-n}\rho^2 \end{aligned} \tag{10.43}$$

by (10.41), (10.39), and (10.42).

Recall that v_1 is the Poisson integral of the restriction of u_1 to S_r, and its derivative is simply obtained by differentiating under the integral sign. In addition, the derivative of the Poisson kernel, say, from the unit sphere to the unit ball, is uniformly bounded in $B(0, 1/2)$. This and the obvious invariance under dilations yield

$$||\nabla v_1||_{L^\infty(B(0,r/2))} \leq Cr^{-1} \fint_{S_r} |v_1| = Cr^{-1} \fint_{S_r} |u_1|. \tag{10.44}$$

But $u_1 \geq 0$ (this was our initial reduction, and if we did not do it we would just have been working with $u_{1,+}$), so $\fint_{S_r} |u_1| = \fint_{S_r} u_1 = v_1(0)$, by (10.40). Therefore

$$\begin{aligned} \fint_{B(0,\tau\rho)} |\nabla v_1|^2 &\leq ||\nabla v_1||^2_{L^\infty(B(0,r/2))} \leq C\rho^{-2} v_1(0)^2 \\ &\leq C\tau^{\delta+2-n}\rho^{-n}E(\rho) + C\tau^{-n}\rho^{-n}\alpha_1 + C\tau^{-n} \end{aligned} \tag{10.45}$$

by (10.43). We add this to (10.34) and get that

$$\begin{aligned} \int_{B(0,\tau\rho)} |\nabla u_1|^2 &\leq 2\int_{B(0,\tau\rho)} |\nabla v_1|^2 + 2\int_{B(0,\tau\rho)} |\nabla (u_1 - v_1)|^2 \\ &\leq C\tau^n\rho^n \fint_{B(0,\tau\rho)} |\nabla v_1|^2 + 2\int_{B(0,r)} |\nabla (u_1 - v_1)|^2 \\ &\leq [C\tau^{\delta+2}E(\rho) + C\alpha_1 + C\rho^n] + [C\alpha_1 + C\rho^n] \\ &\leq C\tau^{\delta+2}E(\rho) + C\alpha_1 + C\rho^n. \end{aligned} \tag{10.46}$$

We also need to estimate the contribution of the other indices $i > 1$, but fortunately

$$\begin{aligned}
\sum_{i>1} \int_{B(0,\tau\rho)} |\nabla u_i|^2 &\leq \sum_{i>1} \int_{B(0,r)} |\nabla u_i|^2 = \int_{B(0,r)} |\nabla \mathbf{u}|^2 - \int_{B(0,r)} |\nabla u_1|^2 \\
&\leq \int_{B(0,r)} |\nabla \mathbf{u}^*|^2 + Cr^n - \int_{B(0,r)} |\nabla u_1|^2 \\
&\leq \int_{B(0,r)} |\nabla v_1|^2 + C\alpha_1 + Cr^n - \int_{B(0,r)} |\nabla u_1|^2 \leq C\alpha_1 + Cr^n \qquad (10.47)
\end{aligned}$$

by (10.24), (10.30), and the minimization property of v_1 (see (6.13) or (10.32)). Altogether

$$\int_{B(0,\tau\rho)} |\nabla \mathbf{u}|^2 \leq C\tau^{\delta+2} E(\rho) + C\alpha_1 + C\rho^n \qquad (10.48)$$

by (10.46) and (10.47). We divide this by $(\tau\rho)^n$ and get that

$$\begin{aligned}
\fint_{B(0,\tau\rho)} |\nabla \mathbf{u}|^2 &\leq C\tau^{\delta+2-n}\rho^{-n} E(\rho) + C\tau^{-n}\alpha_1\rho^{-n} + C\tau^{-n} \\
&= C\tau^{\delta+2-n} \fint_{B(0,\rho)} |\nabla \mathbf{u}|^2 + C\tau^{-n}\alpha_1\rho^{-n} + C\tau^{-n}. \qquad (10.49)
\end{aligned}$$

We need to compare this with the desired conclusion (10.12). To take care of the first term, we just choose τ so small that

$$C\tau^{\delta+2-n} \leq 30^{-1} \qquad (10.50)$$

(recall from (7.31) that $\delta > n - 2$). The second term is

$$\begin{aligned}
C\tau^{-n}\alpha_1\rho^{-n} &= C\tau^{-n}\rho^{-n}[(1-a)E(\rho) + (1-a)^{-1}\varepsilon^{-1}\rho^n\Lambda] \\
&= C(1-a)\tau^{-n}\rho^{-n}E(\rho) + C\tau^{-n}(1-a)^{-1}\varepsilon^{-1}\Lambda \\
&\leq C(1-a)\tau^{-n} \fint_{B(0,\rho)} |\nabla \mathbf{u}|^2 + C\tau^{-n}(1-a)^{-1}\varepsilon^{-1}M^{-1}C_3\Big(1 + \fint_{B(0,r_0)} |\nabla \mathbf{u}|^2\Big)^2 \qquad (10.51)
\end{aligned}$$

by (10.31) and (10.17). We shall choose a so close to 1 (depending on τ, but not on M) that

$$C(1-a)\tau^{-n} \leq 30^{-1}. \qquad (10.52)$$

We do not choose a yet, because there will be a third case where a similar condition on a will arise (in (10.67)).

For the second piece of (10.51), recall that $M \leq \fint_{B(0,\rho)} |\nabla \mathbf{u}|^2$ because we are in the interesting case when (10.13) fails; hence the second piece is at most

$$C\tau^{-n}(1-a)^{-1}\varepsilon^{-1}M^{-2}C_3\Big(1 + \fint_{B(0,r_0)} |\nabla \mathbf{u}|^2\Big)^2 \fint_{B(0,\rho)} |\nabla \mathbf{u}|^2. \qquad (10.53)$$

We promise that we shall choose ε soon, and that it will not depend on M. Then we shall choose M so large, depending on τ, a, and ε, that

$$C\tau^{-n}(1-a)^{-1}\varepsilon^{-1}M^{-2}C_3\Big(1+\fint_{B(0,r_0)}|\nabla \mathbf{u}|^2\Big)^2 \leq 30^{-1}. \tag{10.54}$$

With all these choices, the two first terms of (10.49) are dominated by $10^{-1}\fint_{B(0,\rho)}|\nabla \mathbf{u}|^2$. Then, in the present case, (10.12) holds as soon as we choose

$$C(\tau, r_0) \geq C\tau^{-n}. \tag{10.55}$$

We are not finished yet, because we still have to deal with the case when (10.27) fails, i.e., when we can find $i \geq 2$ such that

$$\sigma(S_r \setminus W_i) < \varepsilon\sigma(S_r). \tag{10.56}$$

In this case, we shall also use the harmonic competitor defined in Section 6, but with u_i (rather that u_1) as our preferred variable, and with the same parameter a as above (just to simplify the discussion). Denote by $(\mathbf{u}^*, \mathbf{W}^*)$ this new competitor. We still have (10.21)-(10.24) for the same reasons (all our functions are bounded and we change nothing outside of $B(0,r)$), but we estimate $\int_{B(0,r)}|\nabla \mathbf{u}^*|^2$ differently. For $j \neq i$, including $j = 1$, we use (6.19), which says that

$$\int_{B(0,r)}|\nabla u_j^*|^2 \leq (1-a)ra^{2-n}\int_{S_r}|\nabla_t u_j|^2 + 4(1-a)^{-1}ra^{-n}\int_{S_r}|r^{-1}u_j|^2. \tag{10.57}$$

By (10.18), u_j is essentially supported in the small set $W_j \cap S_r \subset S_r \setminus W_i$, and (4.7) yields

$$\int_{S_r}|r^{-1}u_j|^2 \leq Cr^{-2}\sigma(S_r \setminus W_i)^{\frac{2}{n-1}}\int_{S_r}|\nabla_t u_j|^2 \leq C\varepsilon^{\frac{2}{n-1}}\int_{S_r}|\nabla_t u_j|^2. \tag{10.58}$$

Hence

$$\begin{aligned}\int_{B(0,r)}|\nabla u_j^*|^2 &\leq [(1-a)a^{2-n} + C(1-a)^{-1}a^{-n}\varepsilon^{\frac{2}{n-1}}]\, r\int_{S_r}|\nabla_t u_j|^2\\ &\leq C[(1-a) + (1-a)^{-1}\varepsilon^{\frac{2}{n-1}}]\, E(\rho)\end{aligned} \tag{10.59}$$

because we shall choose a close to 1, and by (10.19). Next we use (6.20) to say that

$$\int_{B(0,r)\setminus B(0,ar)}|\nabla u_i^*|^2 \leq (1-a)ra^{2-n}\int_{S_r}|\nabla_t u_i|^2 \leq C(1-a)E(\rho) \tag{10.60}$$

by (10.19) again. Finally, by (6.21),

$$\int_{B(0,ar)}|\nabla u_i^*|^2 = a^{n-2}\inf\Big\{\int_{B(0,r)}|\nabla v|^2\,;\, v \in W^{1,2}(B(0,r)) \text{ and } v = u_i \text{ on } S_r\Big\}. \tag{10.61}$$

A simple choice of v is given by the following extension. Set $m = \fint_{S_r} u_i$, and then

$$v(ty) = m + t(u_i(y) - m) \quad \text{for } y \in S_r \text{ and } 0 \le t \le 1. \tag{10.62}$$

This is the same extension that we used near (7.17), and the same computations as above yield

$$\begin{aligned} \int_{B(0,r)} |\nabla v|^2 &= \int_{B(0,r)} |\nabla (v-m)|^2 \le Cr \int_{S_r} \big[r^{-2}(u_i - m)^2 + |\nabla u_i|^2\big] \\ &\le Cr \int_{S_r} |\nabla u_i|^2 \le C\rho^n \Lambda \end{aligned} \tag{10.63}$$

(see (7.18) and then use (4.2) and (10.20)). We combine this with (10.59), (10.60), and (10.61), and get that

$$\begin{aligned} \int_{B(0,r)} |\nabla \mathbf{u}^*|^2 &\le C\big[(1-a) + (1-a)^{-1}\varepsilon^{\frac{2}{n-1}}\big]E(\rho) + \int_{B(0,r)} |\nabla u_i^*|^2 \\ &\le C\big[(1-a) + (1-a)^{-1}\varepsilon^{\frac{2}{n-1}}\big]E(\rho) + C\rho^n\Lambda =: \alpha_2, \end{aligned} \tag{10.64}$$

where the last equality is a definition of α_2. Then, by (10.24),

$$\int_{B(0,r)} |\nabla \mathbf{u}|^2 \le \int_{B(0,r)} |\nabla \mathbf{u}^*|^2 + Cr^n \le \alpha_2 + Cr^n \tag{10.65}$$

We keep the same τ that we chose in (10.50), and notice that

$$\begin{aligned} \fint_{B(0,\tau\rho)} |\nabla \mathbf{u}|^2 &\le C\tau^{-n}\rho^{-n} \int_{B(0,r)} |\nabla \mathbf{u}|^2 \le C\tau^{-n}\rho^{-n}\alpha_2 + C\tau^{-n} \\ &= C\big[(1-a) + (1-a)^{-1}\varepsilon^{\frac{2}{n-1}}\big]\tau^{-n}\rho^{-n}E(\rho) + C\tau^{-n}\Lambda + C\tau^{-n} \\ &\le C\big[(1-a) + (1-a)^{-1}\varepsilon^{\frac{2}{n-1}}\big]\tau^{-n} \fint_{B(0,\rho)} |\nabla \mathbf{u}|^2 + C\tau^{-n}\Lambda + C\tau^{-n}. \end{aligned} \tag{10.66}$$

We may now chose a, satisfying the old constraint (10.52) and in addition

$$C(1-a)\tau^{-n} \le 30^{-1} \tag{10.67}$$

(with C as in (10.66)). Then we choose ε, depending on τ and a, so that

$$C(1-a)^{-1}\varepsilon^{\frac{2}{n-1}}\tau^{-n} \le 30^{-1}. \tag{10.68}$$

Next

$$\begin{aligned} C\tau^{-n}\Lambda &= C\tau^{-n}M^{-1}C_3\Big(1 + \fint_{B(0,r_0)} |\nabla \mathbf{u}|^2\Big)^2 \\ &\le C\tau^{-n}M^{-2}C_3\Big(1 + \fint_{B(0,r_0)} |\nabla \mathbf{u}|^2\Big)^2 \fint_{B(0,\rho)} |\nabla \mathbf{u}|^2 \end{aligned} \tag{10.69}$$

by (10.17) and because (10.13) fails, and now we choose M so large, depending on τ, a, and ε, that

$$C\tau^{-n}M^{-2}C_3\Big(1+\fint_{B(0,r_0)}|\nabla\mathbf{u}|^2\Big)^2 < 30^{-1}, \tag{10.70}$$

in addition to the earlier similar condition (10.54). With all these choices, (10.66) implies that $\fint_{B(0,\tau\rho)}|\nabla\mathbf{u}|^2 \leq 10^{-1}\fint_{B(0,\rho)}|\nabla\mathbf{u}|^2 + C\tau^{-n}$, and (10.12) holds if we choose $C(\tau,r_0)\geq C\tau^{-n}$. We had a similar constraint in (10.55), but our main constraint on $C(\tau,r_0)$ comes from (10.14), which demands that $C(\tau,r_0)\geq C\tau^{-n}M$. In view of our constraints (10.54) and (10.70) on M, we see that we can choose $C(\tau,r_0)$ such that

$$C(\tau,r_0)\leq C\tau^{-3n/2}(1-a)^{-1/2}\varepsilon^{-1/2}\Big(1+\fint_{B(0,r_0)}|\nabla\mathbf{u}|^2\Big)\leq C\big(1+\fint_{B(0,r_0)}|\nabla\mathbf{u}|^2\big). \tag{10.71}$$

This completes our proof of (10.12), with the announced bound on $C(\tau,r_0)$; Lemma 10.4 follows. ■

Corollary 10.5 *Let $B(0,r_0)$ satisfy the assumptions of Lemma 10.4; then*

$$\fint_{B(0,r)}|\nabla\mathbf{u}|^2\leq C\big(1+\fint_{B(0,r_0)}|\nabla\mathbf{u}|^2\big) \text{ for } 0<r\leq r_0, \tag{10.72}$$

with a constant C that depends on the usual constants.

Proof. Define a sequence $\{\omega_k\}$ by

$$\omega_k=(\tau^k r_0)^{-n}\int_{B(0,\tau^k r_0)}|\nabla\mathbf{u}|^2 \text{ for } k\geq 0, \tag{10.73}$$

and let $C(\tau,r_0)$ be as in Lemma 10.4. Let us show by induction that $\omega_k\leq 2C(\tau,r_0)+\omega_0$. This is obviously true for $k=0$. Now suppose that it is true for k; Lemma 10.4 says that

$$\omega_{k+1}\leq 10^{-1}\omega_k+C(\tau,r_0)\leq 10^{-1}(2C(\tau,r_0)+\omega_0)+C(\tau,r_0)<2C(\tau,r_0)+\omega_0, \tag{10.74}$$

which proves our claim. That is,

$$\omega_k\leq 2C(\tau,r_0)+\omega_0\leq C\big(1+\fint_{B(0,r_0)}|\nabla\mathbf{u}|^2\big) \tag{10.75}$$

for all k. The corollary follows easily, by comparing any $r\leq r_0$ with a slightly larger $\tau^k r_0$. ■

We shall now consider balls that do not meet the set $\{\mathbf{u}=0\}$. They will be easier to deal with, because we can use the equation (9.6).

Lemma 10.6 *Suppose that $0 < r_0 \leq 1$, $B(0,r_0) \subset \Omega$ and $\mathbf{u}(x) \neq 0$ for $x \in B(0,r_0)$. Then*

$$|\nabla \mathbf{u}(x)| \leq C r_0 + 2^n \fint_{B(0,r_0)} |\nabla \mathbf{u}| \quad \textit{for } x \in B(0,r_0/2), \tag{10.76}$$

with a constant C that depends only on the usual constants.

Proof. Since $\mathbf{u}(0) \neq 0$, we can find i such that $u_i(0) \neq 0$. Then we claim that $u_i > 0$ on $B(0,r_0)$. Indeed, the set $V = \big\{x \in B(0,r_0)\,;\, u_i(x) > 0\big\}$ contains 0 (recall that $u_i \geq 0$ since Lemma 10.2), and is open in $B(0,r_0)$. But $u_j(z) = 0$ for $j \neq i$ and $z \in V$, because $u_i u_j = 0$ everywhere (by (1.2) because $\mathbf{u}$ is continuous), so $u_j(z) = 0$ when $j \neq i$ and z lies in the closure of V in $B(0,r_0)$, and since $\mathbf{u}(z) \neq 0$, this forces $u_i(z) \neq 0$. So V is closed too, $V = B(0,r_0)$, and this proves our claim.

So we can use (9.6), which says that

$$\Delta u_i = f_i u_i - \frac{1}{2} g_i \ \text{ in } B(0,r_0) \tag{10.77}$$

Set $h = 1_{B(0,r_0)}[f_i u_i - \frac{1}{2} g_i]$; by Theorem 5.1, $h \in L^\infty(\mathbb{R}^n)$, with bounds that depend only on the usual constants. Set $w = G * h$, where G is the fundamental solution of $-\Delta$; from (10.77) we deduce that $u_i - w$ is harmonic in $B(0,r_0)$. At the same time, $\nabla w = (\nabla G) * h$, so $\nabla w \in L^\infty$, and even

$$|\nabla w(x)| \leq \int_{B(0,r_0)} |\nabla G(x-y)||h(y)|dy \leq C||h||_\infty \int_{B(0,r_0)} |x-y|^{1-n} dy \leq C||h||_\infty r_0 \tag{10.78}$$

for $x \in \mathbb{R}^n$. If $x \in B(0,r_0/2)$,

$$\begin{aligned} |\nabla u_i(x)| &\leq |\nabla(u_i - w)(x)| + |\nabla w(x)| \leq |\nabla(u_i - w)(x)| + C||h||_\infty r_0 \\ &\leq \fint_{B(x,r_0/2)} |\nabla(u_i - w)| + C r_0 \leq 2^n \fint_{B(x,r_0)} |\nabla(u_i - w)| + C r_0 \\ &\leq 2^n \fint_{B(x,r_0)} |\nabla u_i| + 2^n ||\nabla w||_\infty + C r_0 \leq 2^n \fint_{B(x,r_0)} |\nabla u_i| + C r_0 \end{aligned} \tag{10.79}$$

because $u_i - w$ is harmonic in $B(x,r_0/2) \subset B(0,r_0)$; (10.76) and the lemma follow because all the other u_j vanish on $B(0,r_0)$. ∎

After all these lemmas, we are now ready to prove Theorem 10.1. Let $B(x_0,r_0)$ be as in the theorem, and let x be any point of $B(x_0,r_0)$. We distinguish between cases, depending on the size of $d(x) = \operatorname{dist}(x,Z)$, where $Z = \big\{z \in \mathbb{R}^n\,;\, \mathbf{u}(z) = 0\big\}$.

If $d(x) \geq r_0/4$, we notice that $\mathbf{u}(x) \neq 0$ on $B(x,r_0/4)$ and that $B(x,r_0/4) \subset B(x_0,2r_0) \subset \Omega$, apply Lemma 10.6 to a translation by $-x$ of the minimizer $(\mathbf{u},\mathbf{W})$, and to the radius $r_0/4$, and get that for y near x,

$$|\nabla \mathbf{u}(y)| \leq C r_0 + 2^n \fint_{B(x,r_0/4)} |\nabla \mathbf{u}| \leq C + C \fint_{B(x_0,2r_0)} |\nabla \mathbf{u}|. \tag{10.80}$$

If $0 < d(x) < r_0/4$, choose $z \in Z$ such that $|z-x| = d(x)$, notice that $B(z, r_0/2) \subset B(0, 2r_0) \subset \Omega$, and apply Corollary 10.5 to the translation of $(\mathbf{u}, \mathbf{W})$ by $-z$ and with the radius $r_0/2$; then

$$(10.81)\qquad \fint_{B(z,2d(x))} |\nabla \mathbf{u}|^2 \leq C\big(1 + \fint_{B(z,r_0/2)} |\nabla \mathbf{u}|^2\big) \leq C + C \fint_{B(x_0,2r_0)} |\nabla \mathbf{u}|^2.$$

We now use the fact that $\mathbf{u} \neq 0$ on $B(x, d(x)) \subset B(z, 2d(x)) \subset \Omega$, apply Lemma 10.6 to the translation by $-x$ of $(\mathbf{u}, \mathbf{W})$ and with the radius $d(x)$, and get that for y near x,

$$\begin{aligned} |\nabla \mathbf{u}(y)| &\leq Cd(x) + 2^n \fint_{B(x,d(x))} |\nabla \mathbf{u}| \leq C + C \fint_{B(z,2d(x))} |\nabla \mathbf{u}| \\ (10.82)\qquad &\leq C + C\Big\{\fint_{B(z,2d(x))} |\nabla \mathbf{u}|^2\Big\}^{1/2} \leq C + C\Big\{\fint_{B(x_0,2r_0)} |\nabla \mathbf{u}|^2\Big\}^{1/2} \end{aligned}$$

by (10.81). In both cases, by the proof of Lemma 10.6, u is even C^1 near x, so we don't need to worry about the definition of $\nabla \mathbf{u}(x)$.

We are left with the case when $d(x) = 0$. We may observe that $\mathbf{u}(x) = 0$ on the corresponding set, and $\nabla \mathbf{u} = 0$ almost everywhere on that set. But we can also restrict our attention to the case when x is a Lebesgue density point for $\nabla \mathbf{u}$, and use Corollary 10.5 (applied to the translation of $(\mathbf{u}, \mathbf{W})$ by $-x$ and with the radius r_0) to get that $|\nabla \mathbf{u}(x)|^2 = \lim_{r\to 0} \fint_{B(x,r)} |\nabla \mathbf{u}|^2 \leq C + C \fint_{B(x,r_0)} |\nabla \mathbf{u}|^2 \leq C + C \fint_{B(x_0,2r_0)} |\nabla \mathbf{u}|^2$.

Anyway, we get that $|\nabla \mathbf{u}(x)| \leq C + C\big(\fint_{B(x_0,2r_0)} |\nabla \mathbf{u}|^2\big)^{1/2}$ almost everywhere on $B(x_0, r_0)$. The Lipschitz bound in (10.2), and then Theorem 10.1, follow. ∎

11 Global Lipschitz bounds for u when Ω is smooth

In this section we prove that if Ω is a bounded open set with $C^{1+\alpha}$ boundary and the assumptions of Section 10 are satisfied, then $\mathbf{u}$ is Lipschitz.

Theorem 11.1 *Assume that for some $\alpha > 0$, Ω is a bounded open set with a $C^{1+\alpha}$ boundary, and that (10.1) and (10.2) hold. Then for each minimizer $(\mathbf{u}, \mathbf{W})$ of the functional J, $\mathbf{u}$ is C-Lipschitz, with a constant C that depends only on n, N, $|\Omega|$, α, the constants in (10.1) and (10.2), and the $C^{1+\alpha}$ constants for $\partial\Omega$.*

In fact, the theorem also holds if we replace our $C^{1+\alpha}$ assumption with the weaker assumption that $\partial\Omega$ is a Lyapunov-Dini surface, i.e., that it is C^1 with a Dini condition on the modulus of continuity on the unit normal. As we shall see in the proof, this is just because we want to know to know that if f is a smooth function on $\partial\Omega$, its harmonic extension to Ω is Lipschitz.

Remark 11.2 *Theorem 11.1 may fail if Ω is merely C^1.*

Proof. To see this, consider the simple case when $n = 2$, Ω is a simply connected domain in the plane, $N = 1$, and $f_1 = 0$. Also take $F(\mathbf{W}) = 0$. Let $(\mathbf{u}, \mathbf{W})$ be a minimizer for J; since $N = 1$, we may write $(\mathbf{u}, \mathbf{W}) = (u, W)$, with $u = u_1$ and $W = W_1$. Since making W larger does not make $F(W)$ larger, we may assume that $W = \Omega$. Let us not put any constraint on the sign of u; then by the proof of (9.6), we get that $\Delta u = -g/2$ on Ω. Now we do not expect solutions of this equation, with Dirichlet boundary values on $\partial\Omega$, to be Lipschitz if Ω is merely C^1.

In fact, suppose that $0 \in \partial\Omega$; even if $\Delta u = 0$ in $\Omega \cap B(0, r)$ for some $r > 0$, we do not expect u to be Lipschitz near 0, so the regularity of g is not an issue. To turn these considerations into a counterexample, we shall use conformal mappings.

Let Ω be a C^1, simply connected domain in $\mathbb{R}^2 \simeq \mathbb{C}$, with $0 \in \partial\Omega$, and let Ψ be a conformal mapping from Ω to the unit ball. A result of Caratheodory says that Ψ has a continuous extension to $\overline{\Omega}$; for this and other information on conformal mappings that we shall use, we refer to [P]. It is also known that we can choose Ω so that Ψ is not Lipschitz in $\Omega \cap B(0, r)$, where $B(0, r)$ is a small ball centered at 0. We can further arrange that Ψ is Lipschitz on a neighborhood $\Omega \setminus B(0, r)$, simply by taking $\partial\Omega$ smooth enough on $\mathbb{R}^2 \setminus B(0, r/2)$ (the regularity of Ψ is a local notion). Denote by Γ_1 a small arc of $\partial\Omega$ that contains $\partial\Omega \cap B(x, 2r)$ and by Γ_2 the rest of $\partial\Omega$. We can make sure that Γ_2 is not empty, and we have a neighborhood V_2 of Γ_2 such that Ψ is Lipschitz on $V_2 \cap \overline{\Omega}$.

Now compose Ψ with a conformal mapping Φ, from the unit disk to the upper half disk $\big\{z \in B(0, 1)\,;\, Im(z) > 0\big\}$, which we choose so that $\Phi \circ \Psi(0) = 0$ and $\Phi \circ \Psi(\Gamma_1) \subset [-1/2, 1/2]$. This last condition is easy to obtain, by composing Ψ first with a Möbius transform that sends $\Psi(\Gamma_1)$ to a small enough arc of circle, before we apply a standard conformal mapping to the half disk.

Let v denote the imaginary part of $\Phi \circ \Psi$; this is a harmonic function on Ω, with vanishing boundary values near 0, and yet it is not Lipschitz near Γ_1 because its gradient is not bounded (if it were, by Cauchy-Riemann's equation the complex derivative of $\Phi \circ \Psi$ would be too, which is false by construction). To make v into an acceptable solution u, we multiply it by $h = \varphi \circ \Phi \circ \Psi$, where φ is a smooth radial function such that $\varphi(z) = 1$ for $|z| \leq 1/2$ and $\varphi(z) = 0$ for $|z| \leq 2/3$. Now $u = vh$ is continuous on $\overline{\Omega}$ and vanishes on $\partial\Omega$. It also lies in $W^{1,2}(\Omega)$, because $v \in W^{1,2}(\Omega)$ (for these counterexamples, ψ is barely not Lipschitz), and if we extend u by setting $u = 0$ on $\mathbb{R}^2 \setminus \Omega$, we get that $u \in W^{1,2}(\mathbb{R}^2)$ by the proof of (4.18). Also, $g = -2\Delta u$ is a bounded function: it vanishes near Γ_1 because $h = 1$ there, and otherwise we compute $\Delta(vh)$ by the chain rule, and use the fact that $\Delta v = 0$, ∇v is bounded, and h is smooth.

Finally let $\widetilde{u}$ denote (the first component of) a minimizer for J, with the data g; we know that $\Delta\widetilde{u} = -g/2 = \Delta u$ on Ω (the reader may check that $\Delta u = -g/2$ also as a distribution), then $\widetilde{u} - u$ is harmonic in Ω, continuous on $\overline{\Omega}$ (by Theorem 8.1 to be lazy), and null on the boundary. By the maximum principle, $\widetilde{u} = u$ and $\widetilde{u}$ is not Lipschitz.

We are even so lucky that $u \geq 0$, so things do not get better if we restrict to nonnegative functions. We expect that this lack of regularity is the general rule for C^1 domains, even though our suggested example was fairly special. ∎

The main new ingredient for the proof of Theorem 11.1 will be the following simple estimate, obtained by the maximum principle.

Lemma 11.3 *Let $(\mathbf{u}, \mathbf{W})$ be as in the theorem; then there is a constant $C \geq 0$ such that*

$$|\mathbf{u}(x)| \leq C \operatorname{dist}(x, \partial\Omega) \quad \text{for } x \in \Omega. \tag{11.1}$$

The constant C depends only on the $C^{1+\alpha}$ constants for Ω, its diameter, and the $||g_i||_\infty$.

Proof. Assume, for the sake of normalization, that $0 \in \Omega$. Set $h_1(x) = -C_1|x|^2$, where $C_1 = \frac{1}{2n} \max_{1 \leq i \leq N} ||g_i||_\infty$, and then let h be the harmonic extension of the restriction of h_1 to Ω. That is, h is continuous on $\overline{\Omega}$, $h = h_1$ on $\partial\Omega$, and h is harmonic in Ω. The existence of h is classical, and since h_1 is smooth and $\partial\Omega$ is $C^{1+\alpha}$ for some $\alpha > 0$, we can apply Theorem 2.4 on page 23 of [Wi] to get that f_2 is Lipschitz on $\overline{\Omega}$, with estimates that depend only on the $C^{1+\alpha}$ constants for Ω, its diameter, and C_1. The same theorem also applies, and gives the same result, when $\partial\Omega$ is a Lyapunov-Dini surface, as defined on page 18 of [Wi]. Now set $w = h_1 - h$; by construction, $w = 0$ on $\partial\Omega$ and $\Delta w = -2nC_1$ on Ω. By the maximum principle, $w \geq 0$ on Ω.

Let us compare this with what happens for $u_{i,+} = \max(0, u_i)$, $1 \leq i \leq N$. We know from (9.6) that $\Delta u_i = f_i u_i - \frac{1}{2} g_i$ on the open set $\Omega_{i,+} = \big\{x \in \Omega\,;\, u_i(x) > 0\big\}$; Since $w - u_{i,+} = w \geq 0$ on $\partial\Omega_{i,+}$ and $\Delta(w - u_{i,+}) = \Delta w - \Delta u_{i,+} \leq 0$ on $\Omega_{i,+}$, we get that $u_{i,+} \leq w$ on $\Omega_{i,+}$, and hence trivially on Ω. That is, $u_{i,+}(x) \leq w(x) \leq C \operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega$, just because w is Lipschitz and vanishes on Ω. Of course the other inequality $u_{i,-}(x) \leq w(x) \leq C \operatorname{dist}(x, \partial\Omega)$ would be proved the same way, and the lemma follows. ∎

Next we use Lemma 11.3 to prove a decay estimate for balls centered on $\partial\Omega$. Still let $(\mathbf{u}, \mathbf{W})$ be a minimizer, as in Theorem 11.1.

Lemma 11.4 *We can find $C \geq 0$, that depend on the same constants as in the statement of the theorem, such that if $0 \in \partial\Omega$, then*

$$\fint_{B(0,r)} |\nabla \mathbf{u}|^2 \leq C \quad \text{for } 0 \leq r \leq 1. \tag{11.2}$$

Proof. It is time for us to start using our first favorite competitor, the cut-off competitor of Section 6. Let $r \in (0, 1]$ be given, and denote by $(\mathbf{u}^*, \mathbf{W}^*)$ the competitor given by (6.1)-(6.2), with the simplest choice $a = 1/2$ and $I = [1, N]$ (we cut of all the u_i). Let $\tau > 0$ be small, to be chosen soon, and recall from (6.5) that for $1 \leq i \leq N$,

$$\begin{aligned}
\int_{B(0,r)} |\nabla u_i^*|^2 &\leq (1+\tau) \int_{B(0,r) \setminus B(0,r/2)} |\nabla \mathbf{u}_i|^2 + 4(1-a)^{-2}(1+\tau^{-1}) r^{-2} \int_{B(0,r) \setminus B(0,r/2)} |u_i|^2 \\
&\leq (1+\tau) \int_{B(0,r) \setminus B(0,r/2)} |\nabla \mathbf{u}_i|^2 + C\tau^{-1} r^{-2} \int_{B(0,r)} |u_i|^2 \\
&\leq (1+\tau) \int_{B(0,r) \setminus B(0,r/2)} |\nabla \mathbf{u}_i|^2 + C\tau^{-1} r^n
\end{aligned} \tag{11.3}$$

because $0 \in \partial\Omega$ and Lemma 11.3 says that $|\mathbf{u}_i| \leq Cr$ on $B(0,r)$. We also deduce from (6.6) and (6.7) (with $p = +\infty$) that

$$\Big| \int_\Omega (u_i^*)^2 f_i - \int_\Omega u_i^2 f_i \Big| \leq C r^n ||u_i||_\infty^2 ||f_i||_\infty \leq C r^n \tag{11.4}$$

and

$$\Big| \int_\Omega u_i^* g_i - \int_\Omega u_i g_i \Big| \leq C r^n ||u_i||_\infty ||g_i||_\infty \leq C r^n. \tag{11.5}$$

Recall also that we did not change the sets W_i, so the volume term will not interfere here. We sum over i and use the fact that $(\mathbf{u}, \mathbf{W})$ minimizes J to get that

$$\begin{aligned}
0 \ &\leq \ J(\mathbf{u}^*, \mathbf{W}^*) - J(\mathbf{u}, \mathbf{W}) = \int_{B(0,r)} |\nabla \mathbf{u}^*|^2 - \int_{B(0,r)} |\nabla \mathbf{u}|^2 + M(\mathbf{u}^*) - M(\mathbf{u}) \\
&= \int_{B(0,r)} |\nabla \mathbf{u}^*|^2 - \int_{B(0,r)} |\nabla \mathbf{u}|^2 + \sum_i \int_\Omega [(u_i^*)^2 f_i + u_i^* g_i - u_i^2 f_i - u_i^* g_i] \\
&\leq \int_{B(0,r)} |\nabla \mathbf{u}^*|^2 - \int_{B(0,r)} |\nabla \mathbf{u}|^2 + C r^n \qquad (11.6) \\
&\leq \tau \int_{B(0,r) \setminus B(0,r/2)} |\nabla \mathbf{u}|^2 - \int_{B(0,r/2)} |\nabla \mathbf{u}|^2 + C\tau^{-1} r^n
\end{aligned}$$

by (1.5), (11.4) and (11.5), and then (11.3). We move $\int_{B(0,r/2)} |\nabla \mathbf{u}|^2$ to the other side and get that

$$\int_{B(0,r/2)} |\nabla \mathbf{u}|^2 \leq \tau \int_{B(0,r) \setminus B(0,r/2)} |\nabla \mathbf{u}|^2 + C\tau^{-1} r^n \leq \tau \int_{B(0,r)} |\nabla \mathbf{u}|^2 + C\tau^{-1} r^n. \tag{11.7}$$

Then we choose $\tau = 2^{-n-1}$ and obtain

$$\fint_{B(0,r/2)} |\nabla \mathbf{u}|^2 \leq \frac{1}{2} \fint_{B(0,r)} |\nabla \mathbf{u}|^2 + C_0 \tag{11.8}$$

for some C_0.

Now this holds for $0 < r \leq 1$. When $r = 1$, we start with $\fint_{B(0,1)} |\nabla \mathbf{u}|^2 \leq C \fint_\Omega |\nabla \mathbf{u}|^2 \leq C_1$. Then we easily prove by induction that $\fint_{B(0,2^{-k})} |\nabla \mathbf{u}|^2 \leq C_1 + 2C_0$ for $k \geq 0$; (11.2) and Lemma 11.4 follow easily. ■

We may now complete the proof of Theorem 11.1 a little as we did with the Hölder estimate near (8.36). We want to prove that

$$\fint_{B(x,r)} |\nabla \mathbf{u}|^2 \leq C \tag{11.9}$$

for $x \in \mathbb{R}^n$ and $0 < r \leq 1$. By invariance under translations, it follows from Lemma 11.4 that (11.9) holds for $x \in \partial\Omega$ and $0 \leq r \leq 1$. It also holds when $r \geq 10^{-3}$, just because

$$\fint_{B(x,r)} |\nabla \mathbf{u}|^2 \leq C \int_{B(x,r)} |\nabla \mathbf{u}|^2 \leq C \int_{\Omega} |\nabla \mathbf{u}|^2 \leq C, \tag{11.10}$$

as before. So we can suppose that $r \leq 10^{-3}$ and $x \in \mathbb{R}^n \setminus \partial\Omega$.

Let $x \in \mathbb{R}^n \setminus \partial\Omega$ be given, set $d(x) = \operatorname{dist}(x, \partial\Omega)$ and choose $z \in \partial\Omega$ such that $|z - x| = d(x)$. Also set $d'(x) = \min(d(x), 1)$.

If $r \geq 10^{-2} d(x)$, notice that $B(x,r) \subset B(z, d(x) + r)$ and $d(x) + r \leq 101r \leq 1$, so

$$\fint_{B(x,r)} |\nabla \mathbf{u}|^2 \leq C \fint_{B(z,d(x)+r)} |\nabla \mathbf{u}|^2 \leq C \tag{11.11}$$

because the radii are comparable and we already know (11.9) for $B(z, d(x) + r)$ So we can assume that $r \leq 10^{-2} d(x)$. Let us check that

$$\fint_{B(x,d'(x))} |\nabla \mathbf{u}|^2 \leq C. \tag{11.12}$$

When $d'(x) \geq 10^{-2}$, this follows from (11.10). Otherwise, the proof of (11.11) yields $\fint_{B(x,d'(x))} |\nabla \mathbf{u}|^2 \leq C \fint_{B(z,2d'(x))} |\nabla \mathbf{u}|^2 \leq C$, as needed for (11.12).

Recall that we are left with the case when $r \leq 10^{-2} d(x)$ and $r \leq 10^{-3}$; thus $r \leq 10^{-2} d'(x)$ (because $d'(x) \neq d(x)$ implies that $d(x) \geq 1$). We distinguish new cases, depending on the value of $d_1(x) = \operatorname{dist}(x, Z)$, where Z still denotes the zero set of $\mathbf{u}$. Notice that $0 \leq d_1(x) \leq d(x)$, choose $z_1 \in Z$ such that $|z_1 - x| = d_1(x)$, and also set $d_1'(x) = \min(d_1(x), 1) \leq d'(x)$.

If $r \geq 10^{-1} d_1(x)$, notice that $B(x,r) \subset B(z_1, d_1(x) + r)$ and $d_1(x) + r \leq 11r \leq d'(x)/4$ (because $r \leq 10^{-2} d'(x)$). In addition, $B(z_1, d'(x)/4) \subset B(x, d'(x)) \subset \Omega$ because $|z_1 - x| = d_1(x) \leq 10r \leq d'(x)/10$, so

$$\begin{aligned}
\fint_{B(x,r)} |\nabla \mathbf{u}|^2 &\leq 11^n \fint_{B(z_1,d_1(x)+r)} |\nabla \mathbf{u}|^2 \leq C\big(1 + \fint_{B(z_1,d'(x)/4)} |\nabla \mathbf{u}|^2\big) \\
&\leq C\big(1 + \fint_{B(x,d'(x))} |\nabla \mathbf{u}|^2\big) \leq C
\end{aligned} \tag{11.13}$$

by Corollary 10.5 and (11.12). Next we check that

$$\fint_{B(x,d_1'(x))} |\nabla \mathbf{u}|^2 \leq C. \tag{11.14}$$

If $d_1(x) \leq 10^{-2} d'(x)$, we can apply (11.13) with $r = d_1(x) = d_1'(x)$ (precisely because then $r \leq 10^{-2} d'(x)$), and we get that $\fint_{B(x,d_1'(x))} |\nabla \mathbf{u}|^2 = \fint_{B(x,d_1(x))} |\nabla \mathbf{u}|^2 \leq C$. If $d_1(x) \geq 10^{-2} d'(x)$, then $d_1'(x) \geq 10^{-2} d'(x)$ (because $d'(x) \leq 1$ anyway) and we immediately get that

$\fint_{B(x,d_1'(x))} |\nabla \mathbf{u}|^2 \leq C \fint_{B(x,d'(x))} |\nabla \mathbf{u}|^2 \leq C$, because $d_1'(x) \leq d'(x)$ and by (11.12). So (11.14) holds in both cases.

In our remaining case when $r \leq 10^{-1} d_1(x)$, we notice that $\mathbf{u} \neq 0$ in $B(x, d_1'(x))$, so we can apply Lemma 10.6 and get that

$$\fint_{B(x,r)} |\nabla \mathbf{u}|^2 \ \leq \ ||\nabla \mathbf{u}||^2_{L^\infty(B(x,r))} \leq C + \fint_{B(x,d_1'(x))} |\nabla \mathbf{u}|^2 \leq C \tag{11.15}$$

by (11.14). This completes our list of cases, and we get (11.9).

Once we have (11.9), we also get that $|\nabla \mathbf{u}| \leq C$ almost everywhere (for instance at Lebesgue density points for ∇u), and then $\mathbf{u}$ is Lipschitz, as desired. This completes our proof of Theorem 11.1. ∎

Remark 11.5 *Our $C^{1+\alpha}$, or Dini condition, is just here to get Lipschitz bounds on harmonic functions on Ω that are smooth on $\partial\Omega$; we clearly want to forbid corners pointing inside (for instance, if Ω is the union of two half spaces through the origin), but a $C^{1+\alpha}$ domain with a few corners pointing outside would be all right.*

12 A sufficient condition for $|\mathbf{u}|$ to be positive

We shall now start adding assumptions to our regularity assumptions, that will yield some form of non-degenerescence results for minimizers of the functional J. Thus special properties (mostly partial monotonicity properties) of our volume functional F will start playing a role, in this section and the next ones.

In this section we try to get minimizers $(\mathbf{u}, \mathbf{W})$ for which the W_i almost cover the set Ω, and for which $u_i > 0$ almost everywhere on W_i. The first property will be easy to get, if F is decreasing (or not increasing) in some directions. The second one will be more interesting, and we will give two conditions that imply that $|\mathbf{u}(x)| > 0$ almost everywhere on Ω when $(\mathbf{u}, \mathbf{W})$ is a minimizer, both involving the positivity of some g_i and the fact that F is non increasing in some directions. See Propositions 12.3 and 12.4.

We shall start with a (trivial) sufficient condition for the existence of a minimizer $(\mathbf{u}, \mathbf{W})$ such that the W_i almost cover Ω.

Some notation will be useful. Denote by $\mathcal{W}(\Omega)$ the class of N-uples $\mathbf{W} = (W_1, \ldots, W_N)$ such that the W_i are disjoint Borel subsets of Ω; thus our functional F is defined on $\mathcal{W}(\Omega)$. Then let $\mathbf{W} = (W_1, \ldots, W_N)$ and $\mathbf{W}' = (W_1', \ldots, W_N')$ be given; we say that $\mathbf{W} \preceq \mathbf{W}'$ when $W_i \subset W_i'$ for $1 \leq i \leq N$.

Also, we say that $\mathbf{W} \in \mathcal{W}(\Omega)$ fills Ω when $\left|\Omega \setminus \bigcup_{i=1}^N W_i\right| = 0$. The following lemma will not be a surprise.

Lemma 12.1 *Assume that for each $\mathbf{W} \in \mathcal{W}(\Omega)$, we can find $\mathbf{W}' \in \mathcal{W}(\Omega)$ such that $\mathbf{W} \preceq \mathbf{W}'$, $\mathbf{W}'$ fills Ω, and $F(\mathbf{W}') \leq F(\mathbf{W})$. Then for every minimizer $(\mathbf{u}, \mathbf{W})$ for J in $\mathcal{F}$ (see Section 1 for the definitions), we can find $\mathbf{W}' \in \mathcal{W}(\Omega)$ such that $\mathbf{W} \preceq \mathbf{W}'$, $\mathbf{W}'$ fills Ω, and $(\mathbf{u}, \mathbf{W}')$ is a minimizer for J in $\mathcal{F}$.*

Notice that the sufficient condition is satisfied if F is a non increasing function of some variable W_i (when the other variables are fixed, and subject to the constraint $\mathbf{W} \in \mathcal{W}(\Omega)$). When F is given by (1.7), it is satisfied when for each $x \in \Omega$, we can find i such that $q_i(x) \leq 0$. The lemma is obvious, because if $\mathbf{W}'$ is the N-uple associated to $\mathbf{W}$ by the sufficient condition, it is clear that $(\mathbf{u}, \mathbf{W}') \in \mathcal{F}$ and $J(\mathbf{u}, \mathbf{W}') = J(\mathbf{u}, \mathbf{W}) + F(\mathbf{W}') - F(\mathbf{W}) \leq J(\mathbf{u}, \mathbf{W})$.

If we want to show that $\mathbf{W}$ fills Ω for every minimizer $(\mathbf{u}, \mathbf{W})$, it is reasonable to require some strict monotonicity.

Lemma 12.2 *Assume that for each $\mathbf{W} \in \mathcal{W}(\Omega)$ that does not fill Ω, we can find $\mathbf{W}' \in \mathcal{W}(\Omega)$ such that $\mathbf{W} \preceq \mathbf{W}'$, $\mathbf{W}'$ fills Ω, and $F(\mathbf{W}') < F(\mathbf{W})$. Then $\mathbf{W}$ fills Ω whenever $(\mathbf{u}, \mathbf{W})$ is a minimizer for J in $\mathcal{F}$.*

Again the sufficient condition is satisfied as soon as F is a decreasing function of some variable W_i, i.e., if $F(\mathbf{W}') < F(\mathbf{W})$ whenever $\mathbf{W}' \in \mathcal{W}(\Omega)$, $W_i \subset W_i'$, $|W_i' \setminus W_i| > 0$, and $W_j' = W_j$ for $j \neq i$. This lemma also is obvious: if $(\mathbf{u}, \mathbf{W})$ is a minimizer and $\mathbf{W}'$ does not fill Ω, the hypothesis gives $\mathbf{W}' \in \mathcal{W}(\Omega)$ such that $(\mathbf{u}, \mathbf{W}') \in \mathcal{F}$ (as before) and $J(\mathbf{u}, \mathbf{W}') = J(\mathbf{u}, \mathbf{W}) + F(\mathbf{W}') - F(\mathbf{W}) < J(\mathbf{u}, \mathbf{W})$.

Notice that both lemmas are atypical in the world of Alt, Caffarelli, and Friedman free boundaries, because when F is given by (1.7), they tend to require $q_i(x) \geq 0$, or even $q_i(x) > 0$, for some i (that may depend on x, but even so).

In the context of eigenfunctions, it can make sense to assume that F is a non increasing function of the volumes, so as to get a partition of Ω by the W_i. We still can add a convexity assumption on F to try to get regularity properties on the W_i, except those which have the minimal volume. See Section 15.

Next we want to state sufficient conditions for $\mathbf{u}$ to be nonzero almost everywhere on Ω. Of course, if we want this to happen, we need $\mathbf{W}$ to fill Ω, and also each u_i to be nonzero almost everywhere on the corresponding W_i. If our usual assumptions are satisfied, $\mathbf{u}$ is continuous, and since $u_i = 0$ almost everywhere on $\mathbb{R}^n \setminus W_i$, we also get that $u_i = 0$ on $\mathbb{R}^n \setminus W_i$, and even on its closure. So the best that we can do is to make sure that $u_i(x) \neq 0$ a.e. on the interior of W_i, and then we will also need to show that the interiors of the W_i fill Ω. A simple way to make sure that

$$u_i(x) > 0 \text{ on the interior of } W_i \tag{12.1}$$

(of course, if the definition of $\mathcal{F}$ allows positive functions u_i) is to require that $g_i(x) > 0$ almost everywhere on W_i. Let us not check this for the moment because we shall prove a more general result later; the general idea is that we first check that $u_i \geq 0$ on W_i, then use the fact that $J(\mathbf{u}, \mathbf{W})$ does not decrease when we replace u_i by $u_i + t\varphi$, where φ is a bump function supported on a small ball contained in the interior of W_i. But the argument seems to require a bootstrap, that we shall do later.

Of course we could also assume that $g_i < 0$ a.e. on W_i, to get that $u_i < 0$, but if $g_i = 0$ on W_i, we shall get $u_i = 0$. We may also get $u_i = 0$ on large pieces of W_i if we allow g to change signs and vanish on a small disk, or if our definition of $\mathcal{F}$ only allows nonnegative functions u_i, and we take g very negative somewhere. [Recall that we can always pick u_i in advance, and then choose $g_i = -2\Delta u_i$ and $f_i = 0$ to make a counterexample.] So the assumption that $g_i > 0$ on W_i is reasonable. But remember that we shall also need to make sure that the interiors of the W_i almost cover Ω.

We are ready for a first statement. For this one, we require F to depend only on the volumes of the W_i. That is, we suppose that there is a function $\widetilde{F} : [0, |\Omega|]^N \to \mathbb{R}$ such that

$$F(W_1, \ldots, W_N) = \widetilde{F}(|W_1|, \ldots, |W_N|) \text{ for } (W_1, \ldots, W_N) \in \mathcal{W}(\Omega). \tag{12.2}$$

Proposition 12.3 *Assume that Ω is open, that the f_i and g_i are bounded, that F is given by (12.2), that $(\mathbf{u}, \mathbf{W})$ is a minimizer for J in $\mathcal{F}$, and that $\mathbf{W}$ fills Ω. Further assume that*

$$g_i(x) > 0 \quad \textit{for } 1 \leq i \leq N \textit{ and almost every } x \in \Omega, \tag{12.3}$$

and that the definition of $\mathcal{F}$ allows all the functions u_i to take positive values. Then $\mathbf{u}(x) \neq 0$ almost everywhere on Ω.

Recall that we say that $\mathbf{W}$ fills Ω when $\left|\Omega \setminus \bigcup_{i=1}^N W_i\right| = 0$. Of course this is needed if we want to have $\mathbf{u}(x) \neq 0$ a.e. on Ω (because $u_i = 0$ a.e. on $\mathbb{R}^n \setminus W_i$. If the hypothesis of Lemma 12.1 holds, we do not need to assume that $\mathbf{W}$ fills Ω, because Lemma 12.1 provides $\mathbf{W}'$ such that $(\mathbf{u}, \mathbf{W}')$ is a minimizer and $\mathbf{W}'$ fills Ω; we apply Proposition 12.3 to $(\mathbf{u}, \mathbf{W}')$ and get the desired conclusion. Of course we then get that $\mathbf{W}$ fills Ω anyway, as a consequence of the proposition.

We can see the proposition as a very weak regularity result on the free boundary set $\Omega \setminus \big\{\mathbf{u}(x) \neq 0\big\}$, since it says that this set is Lebesgue negligible.

We do not need to require any regularity on $\widetilde{F}$, because our proof will only use competitors for which the volumes of the W_i do not change.

The statement allows some u_i to be valued in $\mathbb{R}$, while other ones are valued in $\mathbb{R}_+$. We could also work with the assumption that $\varepsilon_j g_j > 0$, with a sign that depends on j, provided that we exclude again the ridiculous case when we require that $\varepsilon_j u_i \leq 0$ (because, if $\varepsilon_j g_j \geq 0$, we can be sure that $u_i = 0$ in that case).

We are happy with (12.2) for our initial setup with eigenfunctions, because we intended to use such an F anyway. When F is given by (1.7), (12.2) requires all the q_i to be constant, and this may be a little too much to ask. So we state a second result, which will be proved together with Proposition 12.2, and which is a little more flexible in this respect.

This time we select one index i (and for convenience we will pick $i = 1$) and require something like a negative half derivative of F in the direction of that variable. More precisely, we shall assume that there exist $\varepsilon > 0$, that may even depend on the minimizer $(\mathbf{u}, \mathbf{W})$, such that $F(\mathbf{W}') \leq F(W)$ for every choice of $\mathbf{W}' = (W_1, \ldots, W_N)$ such that

$$W_i' \subset W_i \text{ and } |W_i \setminus W_i'| \leq \varepsilon \text{ for } 2 \leq i \leq N, \tag{12.4}$$

and

$$W_1' = W_1 \bigcup \Big(\bigcup_{i \geq 2} (W_i \setminus W_i')\Big). \tag{12.5}$$

That is, we transfer small pieces of the W_i, $i \geq 2$, into W_1; notice that this gives $\mathbf{W}' \in \mathcal{W}(\Omega)$.

When $F(\mathbf{W}) = \sum_i \int_{W_i} q_i$ as in (1.7), this property holds for all ε as soon as $q_1 \leq \min(q_2, \dots q_N)$ everywhere. When F is given by (12.2), it holds as soon as

$$\widetilde{F}(V_1 + t_2 + \dots t_N, V_2 - t_2, \dots, V_N - t_N) \leq \widetilde{F}(V_1, \dots, V_N) \tag{12.6}$$

for some $\varepsilon > 0$ and all choices of $t_2, \dots t_N \in [0, \varepsilon]$ such that $0 \leq t_i \leq V_i$ for $2 \leq i \leq N$.

We add another requirement, that the reader probably implicitly assumed already, that F is insensitive to zero sets, in the sense that $F(\mathbf{W}) = F(\mathbf{W}')$ when the W_i coincide with the W_i' almost everywhere, i.e. when $|W_i \Delta W_i'| = 0$. Notice that this is contained in our continuity assumptions.

Proposition 12.4 *Assume that Ω is open, that the f_i and g_i are bounded, that $(\mathbf{u}, \mathbf{W})$ is a minimizer for J in $\mathcal{F}$, and that $\mathbf{W}$ fills Ω. Further assume that F is insensitive to zero sets and we can find $\varepsilon > 0$ such that $F(\mathbf{W}') \leq F(W)$ whenever $\mathbf{W}'$ satisfies (12.4) and (12.5), and that*

$$g_1(x) > 0 \ \text{ for almost every } x \in \Omega, \tag{12.7}$$

and that the definition of $\mathcal{F}$ allows all the functions u_i to take positive values. Then $\mathbf{u}(x) \neq 0$ almost everywhere on Ω.

The same sort of comments as for Proposition 12.3 apply here. In particular, if the hypothesis of Lemma 12.1 holds, we don't need to assume that $\mathbf{W}$ fills Ω. The advantage of picking $i = 1$ first is that we just need to check that $g_1 > 0$.

We shall prove Propositions 12.3 and 12.4 at the same time, and the idea will be to add a small bump function to one of the u_i near a density point of $\{\mathbf{u} = 0\}$. But some surgery will be needed, so we shall first prove a lemma that applies to any minimizer (regardless of our assumptions on F), and says that the energy of $\mathbf{u}$ decays rather fast near such a point.

Lemma 12.5 *Assume that the f_i and g_i are bounded, and that $(\mathbf{u}, \mathbf{W})$ is a minimizer for J in $\mathcal{F}$. Then let $x_0 \in \Omega$ be a Lebesgue density point of the set $Z = \big\{x \in \mathbb{R}^n \,;\, \mathbf{u}(x) = 0\big\}$. Then*

$$\lim_{r \to 0} r^{-n-2} \int_{B(x_0,r)} |\nabla u|^2 = 0. \tag{12.8}$$

Proof. Let $(\mathbf{u}, \mathbf{W})$ and x_0 be as in the statement. Without loss of generality, we assume that $x_0 = 0$. The general idea is that by Poincaré, $\mathbf{u}$ should stay very small near the origin, and even smaller if $\fint_{B(x_0,r)} |\nabla u|^2$ is small; then the M-term of the functional should only play a small role, and in turn there is no reason for the energy to be large to compensate.

In practice we shall repeatedly test $(\mathbf{u}, \mathbf{W})$ against the cut-off competitor of Section 6, and use this to shows that $\fint_{B(x_0,r)} |\nabla u|^2$ decays rapidly. We shall use the quantities

$$\theta_i(r) = r^{-n} \big|\big\{x \in B(0,r) \,;\, u_i(x) \neq 0\big\}\big| \tag{12.9}$$

and

$$\theta(r) = \sum_i \theta_i(r) = r^{-n}\big|\big\{x \in B(0,r)\,;\, \mathbf{u}(x) \neq 0\big\}\big|; \tag{12.10}$$

notice that our assumption that 0 is a Lebesgue density point of Z exactly means that $\lim_{r\to 0} \theta(r) = 0$. We shall restrict our attention to radii r so small that

$$\theta(r) \leq \eta^n, \tag{12.11}$$

where the small number η will be chosen soon.

Fix such an r, and apply the analogue of (4.7) (with $p = 2$) to the ball $B(0,r)$ (we observed before that the proof of (4.7) that we gave on spheres also work on balls). We get that

$$\begin{aligned}\int_{B(0,r)} |u_i|^2 &\leq C\big|\big\{x \in B(0,r)\,;\, u_i(x) \neq 0\big\}\big|^{\frac{2}{n}} \int_{B(0,r)} |\nabla u_i|^2 \\ &= Cr^2\theta_i(r)^{\frac{2}{n}} \int_{B(0,r)} |\nabla u_i|^2 \leq Cr^2\eta^2 \int_{B(0,r)} |\nabla u_i|^2.\end{aligned} \tag{12.12}$$

Now consider the cut-off competitor $(\mathbf{u}^*, \mathbf{W})$ described at the beginning of Section 6. We take $I = [1, N]$ (i.e, multiply all the u_i by $\varphi(|x|)$, as in (6.2)) and $a = 1/2$. Notice that we do not even need $B(0,r)$ to be contained in Ω for this one, because $u_i^* = 0$ whenever $u_i = 0$. We now estimate the terms that we get from (6.5)-(6.7). Let $\tau > 0$ be small, to be chosen soon; then (6.5), with $p = +\infty$, yields

$$\begin{aligned}\int_{B(0,r)} |\nabla u_i^*|^2 &\leq (1+\tau)\int_{B(0,r)\setminus B(0,ar)} |\nabla u_i|^2 + 4(1-a)^{-2}(1+\tau^{-1})r^{-2}\int_{B(0,r)\setminus B(0,ar)} |u_i|^2 \\ &\leq (1+\tau)\int_{B(0,r)\setminus B(0,r/2)} |\nabla u_i|^2 + C\tau^{-1}\eta^2 \int_{B(0,r)} |\nabla u_i|^2 \\ &\leq \int_{B(0,r)\setminus B(0,r/2)} |\nabla u_i|^2 + \Big(\tau + C\tau^{-1}\eta^2\Big)\int_{B(0,r)} |\nabla u_i|^2\end{aligned} \tag{12.13}$$

or equivalently

$$\int_{B(0,r)} |\nabla u_i^*|^2 - \int_{B(0,r)} |\nabla u_i|^2 \leq -\int_{B(0,r/2)} |\nabla u_i|^2 + \Big(\tau + C\tau^{-1}\eta^2\Big)\int_{B(0,r)} |\nabla u_i|^2. \tag{12.14}$$

For the two M-terms, (6.6) yields

$$\begin{aligned}\Big|\int_\Omega (u_i^*)^2 f_i - \int_\Omega u_i^2 f_i\Big| &\leq \int_B |u_i^2 f_i| \leq ||f_i||_\infty \int_{B(0,r)} |u_i|^2 \\ &\leq Cr^2\eta^2 \int_{B(0,r)} |\nabla u_i|^2\end{aligned} \tag{12.15}$$

by (12.12), and (6.7) gives

$$\begin{aligned} \Big|\int_{\Omega} u_i^* g_i - \int_{\Omega} u_i g_i\Big| &\leq \int_{B(0,r)} |u_i g_i| \leq ||g_i||_\infty \int_{B(0,r)} |u_i| \leq C r^{n/2} \Big\{ \int_{B(0,r)} |u_i|^2 \Big\}^{1/2} \\ &\leq C r \eta\, r^{n/2} \Big\{ \int_{B(0,r)} |\nabla u_i|^2 \Big\}^{1/2}. \end{aligned} \tag{12.16}$$

Set $E(r) = \int_{B(0,r)} |\nabla \mathbf{u}|^2$ as usual; we sum (12.15) and (12.16) over i and get that

$$|M(\mathbf{u}^*) - M(\mathbf{u})| \leq C r^2 \eta^2 E(r) + C r \eta\, r^{n/2} E(r)^{1/2} =: \pi(r), \tag{12.17}$$

where the last part is the definition of $\pi(r)$. There is no difference in the volume terms, because we did not change $\mathbf{W}$, so the fact that $(\mathbf{u}, \mathbf{W})$ minimizes J yields

$$\begin{aligned} 0 &\leq J(\mathbf{u}^*, \mathbf{W}^*) - J(\mathbf{u}, \mathbf{W}) = \int_{B(0,r)} |\nabla \mathbf{u}^*|^2 - \int_{B(0,r)} |\nabla \mathbf{u}|^2 + M(\mathbf{u}^*) - M(\mathbf{u}) \\ &\leq \int_{B(0,r)} |\nabla \mathbf{u}^*|^2 - \int_{B(0,r)} |\nabla \mathbf{u}|^2 + \pi(r) \\ &\leq -\int_{B(0,r/2)} |\nabla \mathbf{u}|^2 + \Big(\tau + C\tau^{-1}\eta^2\Big) \int_{B(0,r)} |\nabla \mathbf{u}|^2 + \pi(r) \end{aligned} \tag{12.18}$$

by (12.17) and (12.14) (summed over i). That is,

$$\begin{aligned} E(r/2) &\leq \Big(\tau + C\tau^{-1}\eta^2\Big) E(r) + \pi(r) \\ &\leq \Big(\tau + C\tau^{-1}\eta^2\Big) E(r) + C r \eta\, r^{n/2} E(r)^{1/2} \end{aligned} \tag{12.19}$$

if $r \leq 1$, say, so that the first term of (12.17) is smaller.

We now choose η so small, depending on τ that will be chosen soon, that (12.19) implies that

$$E(r/2) \leq 2\tau E(r) + \tau r^{1+\frac{n}{2}} E(r)^{1/2}. \tag{12.20}$$

Let us rewrite this in terms of $e(r) = r^{-n-2} E(r)$; we get that

$$\begin{aligned} e(r/2) &= 2^{n+2} r^{-n-2} E(r/2) \\ &= 2^{n+2} r^{-n-2} \big[2\tau E(r) + \tau r^{1+\frac{n}{2}} E(r)^{1/2} \big] \\ &= 2^{n+2} r^{-n-2} \big[2\tau r^{n+2} e(r) + \tau r^{1+\frac{n}{2}} r^{\frac{n}{2}+1} e(r)^{1/2} \big] \\ &= 2^{n+3} \tau e(r) + 2^{n+2} \tau e(r)^{1/2} \end{aligned} \tag{12.21}$$

Thus, if we set $e_k = e(2^{-k} r)$, we get the induction relation $e_{k+1} \leq 2^{n+3} \tau e_k + 2^{n+2} \tau e_k^{1/2}$. If τ is so small enough, then $2^{n+3}\tau < 1/4$, and $e_{k+1} \leq e_k/4 + 2^{n+2} \tau e_k^{1/2}$. If $e_k \geq 2^{2n+8}\tau^2$, then $2^{n+2}\tau e_k^{1/2} \leq e_k/4$ and so $e_{k+1} \leq e_k/2$. Otherwise, $e_{k+1} \leq e_k/4 + 2^{n+2}\tau e_k^{1/2} \leq 2^{2n+6}\tau^2 + 2^{2n+6}\tau \leq 2^{2n+7}\tau$. It is then easy to see that $e_k \leq 2^{2n+7}\tau$ for k large. In the present situation, we can choose τ as small as we want, and then we get that

$$\lim_{r\to 0} r^{-n-2}E(r) = \lim_{r\to 0} e(r) = 0; \tag{12.22}$$

this completes our proof of Lemma 12.5. ∎

We are now ready to prove our two propositions. We are given a minimizer $(\mathbf{u}, \mathbf{W})$ of J, and we want to prove that $\mathbf{u} \neq 0$ almost everywhere on Ω, so we proceed by contradiction, and assume that $\{x \in \Omega\,;\, \mathbf{u}(x) = 0\}$ has positive measure. Then we can find a Lebesgue density point x_0 in that set, and we can even choose it so that for $1 \leq i \leq N$,

$$\lim_{r\to 0} \fint_{B(x_0,r)} |g_i(x) - g_i(x_0)|dx = 0, \tag{12.23}$$

because this Lebesgue density property for g_i holds for almost every x_0 (see [M]), and

$$g_i(x_0) > 0 \tag{12.24}$$

for all i if we prove proposition 12.3, or for $i = 1$ only if we prove proposition 12.4. Without loss of generality, we may assume that $x_0 = 0$ (just to save notation).

Now we want to try a slightly different competitor, with the promised bump function. Let $r > 0$ be given, and denote by $\mathbf{u}^*$ the cut-off competitor that we used in Lemma 12.5. We want to use the fact that $\mathbf{u}^* = 0$ in $B(0, r/2)$ to modify $\mathbf{u}^*$ again in $B(0, r/2)$ (and in particular add a small bump function to some u_i).

Let ψ be a smooth, nonnegative bump function, with compact support in $B(0, 1/3)$ and $\int \psi = 1$. For the sake of Proposition 12.3, we choose the support of ψ a little smaller, so that

$$\big|\{x \in B(0, 1/2)\,;\, \psi(x) \neq 0\}\big| \leq \frac{1}{N}|B(0, 1/2)|. \tag{12.25}$$

We first define a new function $\mathbf{u}^\sharp$. We first select an index i. In the case of Proposition 12.3, choose i such that

$$|W_i \cap B(0, r/2)| \geq \frac{1}{N}|B(0, r/2)|. \tag{12.26}$$

For Proposition 12.4, choose $i = 1$. Then define $\mathbf{u}^\sharp$ by

$$\begin{aligned} u_i^\sharp(x) =&\ \ u_i^*(x) + cr^2\psi(x/r) && \text{for } x \in \mathbb{R}^n \\ u_j^\sharp(x) =&\ \ u_j^*(x) && \text{for } x \in \mathbb{R}^n \text{ and } j \neq i, \end{aligned} \tag{12.27}$$

where the small constant c will be chosen soon.

We also need to define sets $W_j^\sharp$. For both propositions, we keep

$$W_j^\sharp \setminus B(0,r/2) = W_j^* \setminus B(0,r/2) = W_j \setminus B(0,r/2) \tag{12.28}$$

for all j. For Proposition 12.3, we want to keep the same volumes, so we choose the $W_j^\sharp \cap B(0,r/2)$ so that they are disjoint, that $|W_j^\sharp \cap B(0,r/2)| = |W_j \cap B(0,r/2)|$ for all j, and that $W_i^\sharp \cap B(0,r/2)$ contains $\{x \in B(0,r/2)\,;\, \psi(x/r) \neq 0\}$. This is possible, precisely by (12.26) and because we chose the support of ψ small enough in (12.25).

For Proposition 12.4, the most efficient is to take $W_1^\sharp \cap B(0,r/2) = B(0,r/2)$, and $W_j^\sharp \cap B(0,r/2) = \emptyset$ for $j > 1$.

It is easy to see that the $W_i^\sharp$ are disjoint, that $u^\sharp \in W^{1,2}(\mathbb{R}^n)$, and that $u^\sharp = 0$ almost everywhere on $\mathbb{R}^n \setminus W_i$. If r is small enough, $B(0,r) \subset \Omega$, the $W_i^\sharp$ are contained in Ω, and $(\mathbf{u}^\sharp, \mathbf{W}^\sharp) \in \mathcal{F}$; this was the reason why we required Ω to be open.

Now we estimate the functional, starting with the volume term. For Proposition 12.3, we did not change the volumes $|W_i|$, so by (12.2) $F(\mathbf{W}^\sharp) = F(\mathbf{W})$. For Proposition 12.4, we took some pieces of the W_j, $j \geq 2$ and threw them inside W_1. In fact, we also threw the set $B(0,r/2) \setminus \cup_{i=1}^N W_i$ in W_1, but that since $\mathbf{W}$ fills Ω, this set has vanishing measure. Thus $\mathbf{W}^\sharp$ satisfies (12.4) and (12.5), modulo this set of measure zero that does not matter because we assumed that F is insensitive to zero sets, and if r is so small that $|B(0,r/2)| \leq \varepsilon$. Thus we get that

$$F(\mathbf{W}^\sharp) \leq F(\mathbf{W}) \tag{12.29}$$

in this case too.

We also modified the M-term a little, because we added $cr^2\psi(x/r)$ to u_i. But

$$\begin{aligned}
M(\mathbf{u}^\sharp) - M(\mathbf{u}^*) &= \int_{B(0,r/2)} [(u_i^\sharp)^2 f_i - u_i^\sharp g_i] = c^2 r^4 \int_{B(0,r/2)} \psi(x/r)^2 f_i - cr^2 \int_{B(0,r/2)} \psi(x/r) g_i \\
&\leq Cc^2 r^{4+n} - cr^2 \int_{B(0,r/2)} \psi(x/r) g_i \\
&\leq Cc^2 r^{4+n} - cr^2 g_i(0) \int_{B(0,r/2)} \psi(x/r) + cr^2 \int_{B(0,r/2)} ||\psi||_\infty |g_i(x) - g_i(0)| \\
&= Cc^2 r^{4+n} - cr^2 g_i(0) + Ccr^{2+n} ||\psi||_\infty \fint_{B(0,r/2)} |g_i(x) - g_i(0)|
\end{aligned} \tag{12.30}$$

by (1.4), because $\mathbf{u}^* = 0$ in $B(0,r/2)$, and by (12.27). For the energy term,

$$\int |\nabla \mathbf{u}^\sharp|^2 - \int |\nabla \mathbf{u}^*|^2 = c^2 r^4 \int_{B(0,r/2)} |\nabla \psi(\cdot/r)|^2 = c^2 r^{2+n} \int_{B(0,1/2)} |\nabla \psi|^2. \tag{12.31}$$

Hence, by (1.5) and because $F(\mathbf{W}^\sharp) \leq F(\mathbf{W})$,

$$\begin{aligned}
J(\mathbf{u}^\sharp, \mathbf{W}^\sharp) - J(\mathbf{u}^*, \mathbf{W}^*) &\leq M(\mathbf{u}^\sharp) - M(\mathbf{u}^*) + \int |\nabla \mathbf{u}^\sharp|^2 - \int |\nabla \mathbf{u}^*|^2 \\
&\leq -cr^{2+n} g_i(0) + \pi_1(r),
\end{aligned} \tag{12.32}$$

where

$$\pi_1(r) = Cc^2 r^{4+n} + Ccr^{2+n}||\psi||_\infty \fint_{B(0,r/2)} |g_i(x) - g_i(0)| + Cc^2 r^{2+n} \int_{B(0,1/2)} |\nabla\psi|^2. \tag{12.33}$$

Recall from (12.18) and (12.17) that if r is so small that (12.11) holds,

$$\begin{aligned} J(\mathbf{u}^*, \mathbf{W}^*) - J(\mathbf{u}, \mathbf{W}) &\leq -\int_{B(0,r/2)} |\nabla \mathbf{u}|^2 + \Big(\tau + C\tau^{-1}\eta^2\Big) \int_{B(0,r)} |\nabla \mathbf{u}|^2 + \pi(r) \\ &\leq \Big(\tau + C\tau^{-1}\eta^2\Big) \int_{B(0,r)} |\nabla \mathbf{u}|^2 + \pi(r) \\ &= \Big(\tau + C\tau^{-1}\eta^2\Big) E(r) + \pi(r) \qquad (12.34) \\ &\leq \Big(\tau + C\tau^{-1}\eta^2\Big) E(r) + Cr\eta\, r^{n/2} E(r)^{1/2} \end{aligned}$$

(we also use the fact that $r \leq 1$ to control the first half of $\pi(r)$). We no longer need to optimize too much, so let us choose $\tau = 1$ and even $\eta = 1$, and deduce from this that

$$J(\mathbf{u}^*, \mathbf{W}^*) - J(\mathbf{u}, \mathbf{W}) \leq CE(r) + Crr^{n/2}E(r)^{1/2} =: \pi_2(r), \tag{12.35}$$

and where the last part is a definition of $\pi_2(r)$. But $(\mathbf{u}, \mathbf{W})$ is a minimizer for J, so $J(\mathbf{u}, \mathbf{W}) \leq J(\mathbf{u}^\sharp, \mathbf{W}^\sharp)$, which means that

$$cr^{2+n} g_i(0) \leq \pi_1(r) + \pi_2(r), \tag{12.36}$$

by (12.32) and (12.35). Let us now check that we can choose $c > 0$ so small that (12.36) fails for r small; this contradiction will prove that our initial assumption that $\big|\big\{x \in \Omega\,;\, \mathbf{u}(x) = 0\big\}\big| > 0$ was false, and the proposition will follow.

The first term of $\pi_1(r)$ is $Cc^2 r^{4+n} = o(r^{2+n})$. The second term is $Ccr^{2+n} \fint_{B(0,r/2)} |g_i(x) - g_i(0)| = o(r^{2+n})$, by (12.23). The last term, $c^2 r^{2+n} \int_{B(0,1/2)} |\nabla\psi|^2$, is smaller than $cr^{2+n} g_i(0)/2$ if c is small enough. It does not matter that c depends on $g_i(0)$ or our choice of ψ. For $\pi_2(r)$, we observe that since $x_0 = 0$ lies in the open set Ω and is a density point for $\big\{x \in \Omega\,;\, \mathbf{u}(x) = 0\big\}$, we can apply Lemma 12.5; we get that $E(r) = o(r^{n+2})$; then $\pi_2(r) = o(r^{n+2})$ too, which is much smaller than $cr^{2+n} g_i(0)/2$. So (12.36) fails for r small, and Propositions 12.3 and 12.4 follow. ∎

Notice that we have no margin in our last estimates, i.e., the decay exponent in Lemma 12.5 is just enough for our purposes. This probably means that since the amount of J that we can save by adding a small jump function is of higher order, our boundedness assumption on the f_i and g_i are about right.

13 Sufficient conditions for minimizers to be nontrivial

In this section, we check that if the volume functional is defined by

$$F(\mathbf{W}) = \sum_{i=1}^{N} a|W_i| + b|W_i|^{1+\alpha} \tag{13.1}$$

for some $\alpha > 2/n$ and suitable constants a and b, depending on Ω, the f_i, and the g_i, then the minimizers of our functional J are not trivial, in the sense that at least one function u_i is nonzero, and $|W_i| < |\Omega|$ for all i.

For this we need some assumptions on Ω, the f_i, and the g_i, which we shall not try to optimize. Let us assume that (3.1) and (3.2) hold. That is, $|\Omega| < +\infty$, the f_i are bounded and nonnegative, and the g_i lie in L^2. With these assumptions and F as in (13.1), Theorem 3.1 says that J admits minimizers.

Our first result says that if b is large enough and $(\mathbf{u}, \mathbf{W})$ is a minimizer for J, the W_i cannot be too large.

Lemma 13.1 *Suppose Ω, the f_i, and the g_i satisfy (3.1) and (3.2). For each choice of $\alpha \geq 0$ and $\eta > 0$, we can find $b_0 > 0$, that depends only on n, α, $|\Omega|$, the $||f_i||_\infty$, and the $||g_i||_2$, so that if F is given by (13.1) for some $a \geq 0$ and $b \geq b_0$, and $(\mathbf{u}, \mathbf{W})$ is a minimizer for J, then*

$$(13.2) \qquad |W_i| \leq \eta \quad \text{for } 1 \leq i \leq N.$$

Proof. Let us first estimate the first two terms of $J(\mathbf{u}, \mathbf{W})$. Let $(\mathbf{u}_0, \mathbf{W}_0)$ denote the trivial competitor for which $u_i = 0$ and $W_i = \emptyset$ for $1 \leq i \leq N$. Then

$$(13.3) \qquad E(\mathbf{u}) + M(\mathbf{u}) \leq E(\mathbf{u}) + M(\mathbf{u}) + F(\mathbf{W}) = J(\mathbf{u}, \mathbf{W}) \leq J(\mathbf{u}_0, \mathbf{W}_0) = 0$$

by (1.3)-(1.5) and because $F(\mathbf{W}) \geq 0$. Since $f_i \geq 0$ by (3.2), (1.4) yields

$$(13.4) \qquad E(\mathbf{u}) \leq -M(\mathbf{u}) \leq \sum_{i=1}^{N} \int u_i(x) g(x) dx \leq \sum_{i=1}^{N} ||g_i||_2 ||u_i||_2.$$

By Lemma 3.2, $||u_i||_2^2 \leq C|\Omega|^{2/n} \int |\nabla u_i|^2 \leq C|\Omega|^{2/n} E(\mathbf{u})$, hence

$$(13.5) \qquad E(\mathbf{u}) \leq C|\Omega|^{1/n} E(\mathbf{u})^{1/2} \sum_{i=1}^{N} ||g_i||_2,$$

and so $E(\mathbf{u}) \leq C'$, where C' depends on the data as above. We return to (13.3) and notice that for $1 \leq i \leq N$,

$$(13.6) \quad b|W_i|^{1+\alpha} \leq F(\mathbf{W}) \leq -M(\mathbf{u}) \leq \sum_{i=1}^{N} ||g_i||_2 ||u_i||_2 \leq C|\Omega|^{1/n} E(\mathbf{u})^{1/2} \sum_{i=1}^{N} ||g_i||_2 \leq C'',$$

where C'' depends only on n, $|\Omega|$, the $||f_i||_\infty$, and the $||g_i||_2$. The conclusion (13.2) follows easily (for $b \geq b_0$ and if b_0 is large enough). ∎

The next result says that if at least one of the g_i is nontrivial and a in (13.1) is small enough, the minimizers for J are nontrivial.

Lemma 13.2 *Suppose Ω, the f_i, and the g_i satisfy (3.1) and (3.2). Suppose in addition that Ω is open (and non empty) and that $g_i \neq 0$ for some i. Then for each choice of parameters $\alpha > 2/n$ and $b > 0$, we can find $a_0 > 0$ such that if F is given by (13.1) for some $a \in [0, a_0]$ and $(\mathbf{u}, \mathbf{W})$ is a minimizer for J, then $\mathbf{u} \neq 0$.*

Proof. Here a_0 will also depend on g_i in a more complicated way that its norm, through the choice of a small ball where an average of g_i is not too small.

For the proof we may assume that $i = 1$. Let x_0 be a point of Ω such that $g_1(x_0) \neq 0$ and x_0 is a Lebesgue point for g_1, in the sense that

$$\lim_{r \to 0} \frac{1}{r^n} \int_{B(x_0,r)} |g_1(x) - g_1(x_0)| dx = 0; \tag{13.7}$$

such a point x_0 exists because (13.7) holds almost everywhere and $g_1(x_0) \neq 0$ on a set of positive measure. As in Proposition 12.3, we required Ω to be open so that small bump functions near x_0 yield competitors for J.

Let φ be a smooth radial bump function, supported in $B(0,1)$ and such that $\int \varphi = 1$, and for $r > 0$, set

$$\varphi_r(x) = \beta r^2 \varphi((x - x_0)/r), \tag{13.8}$$

where the small constant β will be chosen near the end of the proof. If $r <$ dist $(x_0, \mathbb{R}^n \setminus \Omega)$, we can use φ_r to define an admissible pair $(\mathbf{u}, \mathbf{W}) \in \mathcal{F}(\Omega)$. That is, we take $u_1 = \varphi_r$, $W_1 = B(x_0, r)$, and for $i > 1$ we take $u_i = 0$ and $W_i = \emptyset$; it is then clear that $(\mathbf{u}, \mathbf{W})$ satisfies the requirements of Definition 1.1.

We just need to prove that $J(\mathbf{u}, \mathbf{W}) < 0$ (if a_0, β, and r are chosen correctly), because then $\mathbf{u} = 0$ cannot yield a minimizer, no matter which choice of $\mathbf{W}$ we associate to it. So let us evaluate all the terms in $J(\mathbf{u}, \mathbf{W})$. We start with the energy

$$E(\mathbf{u}) = \int |\nabla \varphi_t|^2 = \beta^2 r^2 ||\nabla \varphi||_2^2 \leq C \beta^2 r^{n+2}, \tag{13.9}$$

where we do not need to worry about the dependence of C on φ. Let us also record that

$$F(\mathbf{W}) = a|B(x_0, r)| + b|B(x_0, r)|^{1+\alpha} \leq C a r^n + C b r^{n(1+\alpha)} \tag{13.10}$$

The first part of $M(\mathbf{u})$ is

$$\sum_i \int u_i^2 f_i = \int \varphi_r f_1 \leq ||f_1||_\infty ||\varphi_r||_2^2 = ||f_1||_\infty \beta^2 r^4 r^n ||\varphi||_2^2 \leq C \beta^2 r^{n+4}. \tag{13.11}$$

The remaining part of $M(\mathbf{u})$ is

$$\begin{aligned} -\sum_i \int u_i g_i &= -\int \varphi_r g_1 \leq -g_1(x_0) \int \varphi_r + \int \varphi_r(x) |g(x) - g(x_0)| dx \\ &\leq -\beta r^{n+2} g_1(x_0) + \beta r^2 ||\varphi||_\infty \int_{B(x_0,r)} |g(x) - g(x_0)| dx. \end{aligned} \tag{13.12}$$

We shall take β small, with the same sign as $g_1(x_0)$, and we want the negative term $-\beta r^{n+2} g_1(x_0)$ in (13.12) to dominate all the other ones. Let us now choose our parameters r, β, and a_0 so that this is the case. For the second term $\beta r^2 ||\varphi||_\infty \int_{B(x_0,r)} |g(x) - g(x_0)| dx$ of (13.12), this happens as soon as r is small enough, because of (13.7). For $C\beta^2 r^{n+4}$ in (13.11), this is also true as soon as r is small enough (because we'll take $|\beta| \leq 1$). The term $Cbr^{n(1+\alpha)}$ in (13.10) can be treated the same way, because $\alpha > 2/n$.

At this stage we choose r so small that $B(x_0, r) \subset \Omega$ and the terms mentionned above are smaller than $\beta r^{n+2} g_1(x_0)/10$. Then we choose β small, so that $E(\mathbf{u}) < \beta r^{n+2} g_1(x_0)/10$, and a_0 so small (hence depending on our choice of r) that $Ca_0 r^n < \beta r^{n+2} g_1(x_0)/10$. Now (13.9)-(13.12) imply that $J(\mathbf{u}, \mathbf{W}) < 0$ as soon as $0 \leq a \leq a_0$, and the lemma follows. ∎

14 A bound on the number of components

This short section answers a natural question on the implementation of our functional: even if we choose to allow a very large number N of regions, will the functional naturally limit the number of indices i such that $u_i \neq 0$ somewhere?

We shall check that under reasonable assumptions on F, there is a lower bound on the volume of W_i when $u_i(x) \neq 0$ somewhere. If $|\Omega|$ is assumed to be bounded, this will give the desired bound on the number of nontrivial components W_i.

Let us state our main assumption for the index $i = 1$. We assume that there exist an exponent $1 \leq p < \frac{n+2}{n}$ and a constant $\lambda \geq 0$ such that there exist disjoint subsets $A_1, A_2, \ldots, A_N$ of W_1 such that

$$F(A_1, W_2 \cup A_2, \ldots, W_N \cup A_N) \leq F(W_1, \ldots, W_N) - \lambda |W_1|^p. \tag{14.1}$$

Thus we have the right to take away a part of W_1, dispatch some of it among the other components, and this will make the volume form somewhat smaller. A simple special case of this is when

$$F(\emptyset, W_2, \ldots, W_N) \leq F(W_1, \ldots, W_N) - \lambda |W_1|^p, \tag{14.2}$$

where $\lambda |W_1|^p$ is a minimum price that we had to pay for the volume $|W_1|$. Even more specifically, (14.1) holds if

$$F(W_1, \ldots, W_N) = \sum_{i=1}^{N} a_i |W_i| + b_i |W_i|^2, \tag{14.3}$$

with $a_i \geq \lambda > 0$ and $b_i \geq 0$.

If we do not assume anything, i.e., if volume is too cheap, the functional may decide to have a tiny components W_1 (even if this is not very useful), win something on the M-part of the functional, pay less in the energy term if u_1 is small enough (the homogeneity of $\int |\nabla u_1|^2$ is higher than for $\int u_1 g_1$), and essentially not pay for it in the volume term.

We shall only need (14.1) when $\mathbf{W}$ comes from the minimizer $(\mathbf{u}, \mathbf{W})$ (and then we will get bounds on $|W_1|$ that depend on p, λ, and n), but in general we do not expect to know $(\mathbf{u}, \mathbf{W})$ in advance, so we may need to require (14.1) for all $\mathbf{W} \in \mathcal{W}(\Omega)$.

Our condition (14.1) is a simple form of the main nondegeneracy assumption that will be introduced in Section 15.

Proposition 14.1 *Let $(\mathbf{u}, \mathbf{W})$ be a minimizer for J, and suppose that $|W_1| > 0$ and that (14.1) holds for some choice of $p \in [1, \frac{n+2}{n})$ and $\lambda > 0$. Also suppose that f_1 and g_1 are bounded. Then*

$$|W_1| \geq C^{-1}(||g_1||_\infty^{\frac{2n}{n+2-np}} + ||f_1||_\infty^{n/2})^{-1}, \tag{14.4}$$

with a constant C that depends only on n, p, and λ.

Proof. The reader should not worry about the case when $f_1 = g_1 = 0$; the proof below will just show that it does not happen. We construct a simple competitor that will be compared to $(\mathbf{u}, \mathbf{W})$. Set $\mathbf{u}^* = (0, u_2, \ldots, u_N)$ and $\mathbf{W}^* = (A_1, W_2 \cup A_2, \ldots, W_N \cup A_N)$, where the A_i are the same as in (14.1). It is easy to see that $(\mathbf{u}^*, \mathbf{W}^*)$ is an acceptable pair, i.e., lies in the class $\mathcal{F}(\Omega)$ of Definition 1.1; simply notice that the A_i are contained in Ω because they are contained in W_1. Thus $J(\mathbf{u}, \mathbf{W}) \leq J(\mathbf{u}^*, \mathbf{W}^*)$, and when we remove the identical parts of $E(\mathbf{u}) + M(\mathbf{u})$ and $E(\mathbf{u}^*) + M(\mathbf{u}^*)$, we are left with

$$\int_{W_1} |\nabla u_1|^2 + u_1^2 f_1 - u_1 g_1 + F(\mathbf{W}) \leq F(\mathbf{W}^*) \leq F(\mathbf{W}) - \lambda |W_1|^p, \tag{14.5}$$

by (1.4) and (14.1). Set $E = \int_{W_1} |\nabla u_1|^2$ and $M = \big|\int_{W_1} u_1^2 f_1 - u_1 g_1\big|$, and then apply the Poincaré inequality from Lemma 3.2; we get that

$$\begin{aligned} M \leq ||f_1||_\infty \int_{W_1} u_1^2 + ||g_1||_\infty \int_{W_1} |u_1| &\leq C||f_1||_\infty |W_1|^{2/n} E + C||g_1||_\infty |W_1|^{1/2} ||u_1||_2 \\ &\leq C||f_1||_\infty |W_1|^{2/n} E + C||g_1||_\infty |W_1|^{\frac{n+2}{2n}} E^{1/2} \end{aligned} \tag{14.6}$$

and (14.5) yields

$$E + \lambda |W_1|^p \leq M \leq C||f_1||_\infty |W_1|^{2/n} E + C||g_1||_\infty |W_1|^{\frac{n+2}{2n}} E^{1/2}. \tag{14.7}$$

If $C||f_1||_\infty |W_1|^{2/n} \geq 1/2$, we are happy because (14.4) holds. Otherwise we simplify (14.7) and get that

$$E + 2\lambda |W_1|^p \leq 2C||g_1||_\infty |W_1|^{\frac{n+2}{2n}} E^{1/2}. \tag{14.8}$$

Set $\alpha = C||g_1||_\infty |W_1|^{\frac{n+2}{2n}}$; then $E - 2\alpha E^{1/2} = (E^{1/2} - \alpha)^2 - \alpha^2 \geq -\alpha^2$, so (14.8) implies that $2\lambda |W_1|^p \leq 2\alpha E^{1/2} - E \leq \alpha^2$, and hence $2\lambda \leq |W_1|^{-p} \alpha^2 \leq C^2 ||g_1||_\infty^2 |W_1|^{\frac{n+2}{n} - p}$. Then (14.4) holds, and the proposition follows. ∎

In fact the proof of Proposition 14.1 shows that the domain W_1 is not too thin, in the sense that its Poincaré constant is fairly large. That is, denote by $V(W_1)$ the smallest constant such that

$$\int_{W_1} |u|^2 \leq V(W_1)^{2/n} \int_{W_1} |\nabla u|^2 \tag{14.9}$$

for every function $u \in W^{1,2}(\mathbb{R}^n)$ such that $u(x) = 0$ almost every $x \in \mathbb{R}^n \setminus W_1$. We make $V(W_1)$ scale like a volume to simplify the computations below. Then we have the same bounds as above for $V(W_1)$.

Corollary 14.2 *Let* $(\mathbf{u}, \mathbf{W})$ *and* W_1 *be as in Proposition 14.1; then*

$$V(W_1) \geq C^{-1}(||g_1||_\infty^{\frac{2n}{n+2-np}} + ||f_1||_\infty^{n/2})^{-1}, \tag{14.10}$$

with a constant C *that depends only on* n, p, *and* λ.

Proof. Observe that

$$V(W_1) \leq C|W_1|, \tag{14.11}$$

for instance by Lemma 3.2; thus (14.10) is better than (14.4). But let us follow the argument above. The proof of (14.6) yields

$$M \leq ||f_1||_\infty V(W_1)^{2/n} E + C||g_1||_\infty |W_1|^{1/2} V(W_1)^{1/n} E^{1/2}. \tag{14.12}$$

Then (14.7) becomes

$$E + \lambda |W_1|^p \leq M \leq C||f_1||_\infty V(W_1)^{2/n} E + C||g_1||_\infty |W_1|^{1/2} V(W_1)^{1/n} E^{1/2} \tag{14.13}$$

If $C||f_1||_\infty V(W_1)^{2/n} \geq 1/2$, we are happy as before, and otherwise we are left with

$$E + 2\lambda |W_1|^p \leq 2C||g_1||_\infty |W_1|^{1/2} V(W_1)^{1/n} E^{1/2}. \tag{14.14}$$

Then we set $\alpha = C||g_1||_\infty |W_1|^{1/2} V(W_1)^{1/n}$ and the computation above yields

$$2\lambda |W_1|^p \leq 2\alpha E^{1/2} - E \leq \alpha^2 = C^2 ||g_1||_\infty^2 |W_1| \, V(W_1)^{2/n}. \tag{14.15}$$

We assumed that $p \geq 1$, because it did not disturb and now we can say that

$$C^2 ||g_1||_\infty^2 V(W_1)^{2/n} \geq 2\lambda |W_1|^{p-1} \geq C^{-1} \lambda V(W_1)^{p-1} \tag{14.16}$$

and conclude as before. ∎

For the initial goal of the section, it is important to notice that our constant C does not depend on N. So, if $|\Omega| < +\infty$ the f_i and the g_i are bounded, and the analogue of (14.1) holds for all i (all this with constants that do not depend on N), we get a bound on the number of indices i such that $|W_i| > 0$.

Remark 14.3 *In Section 15 we will get more precise lower bounds on the measure of $W_i \cap B$ when B is s small ball centered on the boundary of $\Omega_1 = \{u_1(x) > 0\}$, but the more complicated proofs will make the constants depend on N. Thus it seems that we cannot use them to get easily the result of this section. On the contrary, because of the result of this section, we can apply the results of Section 15 to an equivalent minimizer with N bounded, and get that the constants of Section 15 do not depend on N.*

15 The main non degeneracy condition; good domains

In this section we return to one of the main schemes of the study of free boundaries and give a a sufficient condition for the positive part of the function u_i associated to a minimizer $(\mathbf{u}, \mathbf{W})$ to behave like the distance to the boundary of $\Omega_i = \{x \in \Omega\,;\, u_i(x) > 0\}$.

This is a condition on F, which essentially says that we can remove any small part A of W_i, distribute some of it among the other W_j, and win in F an amount which is proportional to $|A|$, and then all sorts of useful non degeneracy properties for u_i and Ω_i follow. See Theorems 15.1, 15.2, 15.3, and 15.4 below.

But let us first describe this condition. Without loss of generality, we shall restrict our attention to $i = 1$. We say that $i = 1$ is a good index, or (with a small abuse of notation) that W_1 is a good domain, when there exist $\lambda > 0$ and $\varepsilon > 0$ such that, for each measurable set $A \subset W_1$, with $0 < |A| \leq \varepsilon$, we can find disjoint subsets $A_j \subset A$, $2 \leq j \leq N$, such that

$$F(W_1 \setminus A, W_2 \cup A_2, \ldots, W_N \cup A_N) \leq F(\mathbf{W}) - \lambda |A|. \tag{15.1}$$

Again, this means that if we have a small set $A \subset W_1$, and we can somehow dispense with it (typically, because it is not very useful for making $E(\mathbf{u}) + M(\mathbf{u})$ small, we can then give some of it to the other regions W_j, $j \geq 2$, throw out the rest, and we will win something substantial in the F-term of the functional. We will see that for good regions of a minimizer, due to the fact that the whole set W_1 is really needed, we have some additional regularity properties of u near the free boundary $\partial\big(\{u_1 \neq 0\}\big)$).

Here are some simple sufficient conditions for W_1 to be a good region. First assume that $F(\mathbf{W})$ is a function of $\mathbf{V} = (|W_1|, \ldots, |W_N|)$, i.e., that $F(W_1, \ldots, W_N) = \widetilde{F}(|W_1|, \ldots, |W_N|)$ for some function $\widetilde{F} : [0, |\Omega|]^N \to \mathbb{R}$, as in (12.3). If $\widetilde{F}$ is differentiable at the point $\mathbf{V}$ (coming from the minimizer $(\mathbf{u}, \mathbf{W})$); then (15.1) holds (for some choice of λ and ε) as soon as

$$\frac{\partial \widetilde{F}}{\partial V_1}(\mathbf{V}) > \inf\Big(0, \frac{\partial \widetilde{F}}{\partial V_2}(\mathbf{V}), \ldots, \frac{\partial \widetilde{F}}{\partial V_N}(\mathbf{V})\Big). \tag{15.2}$$

Even more specifically, if $F((W_1, \ldots, W_N) = \sum_{1 \leq i \leq N} F_i(|W_i|)$, and each F_i is differentiable at $|W_i|$, then (15.1) and (15.2) hold as soon as

$$F_1'(|W_1|) > 0 \ \text{ or } \ F_1'(|W_1|) > F_j'(|W_j|) \ \text{ for some } j > 1. \tag{15.3}$$

In the more familiar context of Alt, Caffarelli, and Friedman where $F(\mathbf{W}) = \sum_i \int_{W_i} q_i$ as in (1.7), (15.1) holds as soon as

$$q_1(x) \geq \lambda + \min(0, q_2(x), \ldots, q_N(x)) \ \text{ almost everywhere on } \Omega \tag{15.4}$$

(and often the q_i are nonnegative and the condition becomes $q_1 \geq \lambda$). Notice that if $q_1 \geq \lambda$ everywhere, we do not need to compare q_1 with the other q_i.

We state two similar nondegeneracy results now, which we distinguish because they have slightly different assumptions, then prove them, and then state and prove other ones.

Theorem 15.1 *Let $(\mathbf{u}, \mathbf{W})$ is a minimizer in $\mathcal{F}$ of the functional J, suppose that the data f_1 and g_1 are bounded, and that (15.1) holds for some choice of $\lambda > 0$ and $\varepsilon > 0$. Also let $x \in \mathbb{R}^n$ and $r > 0$ be such that $\mathbf{u}$ is continuous on $B(x,r)$ and*

(15.5) $\quad r \leq \min(1, \varepsilon^{1/n})$ *and $B(x, r/2)$ meets the boundary of* $\Omega_1 = \big\{x \in \Omega\,;\, u_1(x) > 0\big\}$.

Then

$$\fint_{B(x,r)} |u_{1,+}|^2 \geq c_1 r^2, \tag{15.6}$$

where we set $u_{1,+} = \max(0, u_1)$, and

$$|\Omega_1 \cap B(x,r)| \geq c_2 r^n, \tag{15.7}$$

with constants $c_1 > 0$ and $c_2 > 0$ that depend only on n, λ, $||f_1||_\infty$, $||g_1||_\infty$, and an upper bound for $\fint_{B(x,r)} |\nabla u_{1,+}|^2$.

The rest of the paper is full of sufficient conditions for $\mathbf{u}$ to be continuous (and anyway we just ask this so that we can talk about the open set Ω_1), and ways to estimate $\fint_{B(x,r)} |\nabla u_1|^2$; we state the theorem like this to stress the small amount of information that we will use.

Theorem 15.2 *Let $(\mathbf{u}, \mathbf{W})$ is a minimizer in $\mathcal{F}$ of the functional J, suppose that f_1 and g_1 are bounded and that (15.1) holds for some choice of $\lambda > 0$ and $\varepsilon > 0$. Let $x \in \mathbb{R}^n$ and $r > 0$ be such that $\mathbf{u}$ is continuous on $B(x,r)$, that (15.5) holds, and that for some $C_0 > 0$, either*

$$u_1 \text{ is } C_0\text{-Lipschitz on } B(0,r), \tag{15.8}$$

or

$$|B(0,r) \setminus \Omega_1| \geq C_0^{-1} r^n, \tag{15.9}$$

or both. Then

$$\fint_{B(x,r)} |\nabla u_{1,+}|^2 \geq c_3. \tag{15.10}$$

The constant c_3 depends only on n, λ, $||f_1||_\infty$, $||g_1||_\infty$, and C_0.

These two results are standard in the context of Alt, Caffarelli, and Friedman; our proof is slightly different because we try to rely more on the cut-off competitors, but it is not surprising. Even the assumption (15.8) will not cost us much in practice, because we are ready to assume that Ω is bounded and $C^{1+\alpha}$ and apply Theorem 11.1, and also because the conclusions of Theorem 15.2 are easier to use when we know that $\mathbf{u}$ is Lipschitz. In addition, (15.9) would automatically hold if $x \in \partial\Omega$ and r is small, under very weak assumptions on Ω. Other than that, we shall not use the geometry of Ω in this section.

We shall prove the two theorems at the same time. The main ingredient will be a comparison with the cut-off competitor, which we shall use repeatedly with sometimes slightly different estimates.

We shall start the proof with any ball $B(y, r)$, which may be different from the ball of the theorem, and we shall only assume that

$$r \le 1 \ \text{ and } |B(y, r/2)| \le \varepsilon, \tag{15.11}$$

where ε still comes from (15.1).

For simplicity, we shall assume that $y = 0$. We choose the following the cut-off competitor: we select the first index $i = 1$, take $a = 1/2$, define φ as in (6.1), and denote by $\mathbf{u}^*$ the function defined by $u_1^*(x) = \varphi(|x|)u_1(x)$ for $x \in \Omega_1$, $u_1^*(x) = u_1(x)$ for $x \in \mathbb{R}^n \setminus \Omega_1$, and $u_j^* = u_j$ for $j \ge 2$. Another equivalent definition of u_1^* is $u_1^*(x) = \min(u_1(x), \varphi(|x|)u_1(x))$ (because $0 \le \varphi \le 1$), from which the fact that $u_1^* \in W^{1,2}(\mathbb{R}^n)$ is more obvious.

To define $\mathbf{W}^*$, we use the fact that $u_1 = 0$ on $\Omega_1 \cap B(0, r/2)$ to change the W_i to our advantage. Since (15.11) holds, we can set $A = \Omega_1 \cap W_1 \cap B(0, r/2)$ and choose $A_2, \dots A_N$ so that (15.1) holds (we intersect with W_1 because formally Ω_1 is only almost everywhere contained in W_1). Then we set $\mathbf{W}^* = (W_1 \setminus A, W_2 \cup A_2, \dots, W_N \cup A_N)$, and (15.1) says that

$$F(\mathbf{W}^*) \le F(\mathbf{W}) - \lambda|A| = F(\mathbf{W}) - \lambda|\Omega_1 \cap B(0, r/2)|. \tag{15.12}$$

As in Section 6, it is easy to see that $(\mathbf{u}^*, \mathbf{W}^*) \in \mathcal{F}$, the integrals $\int |\nabla u_j|^2$, $j \ge 2$, stay the same, and for $j = 1$, $\int_{\mathbb{R}^n \setminus \Omega_1} |\nabla u_1|^2$ stays the same, while the analogue of (6.5) for $u_{1,+}$, with $a = 1/2$ says that

$$\begin{aligned} \int_{\Omega_1 \cap B(0,r)} |\nabla u_1^*|^2 &= \int_{B(0,r)} |\nabla u_{1,+}^*|^2 \\ &\le (1+\tau) \int_{B(0,r) \setminus B(0,r/2)} |\nabla u_{1,+}|^2 + 16(1+\tau^{-1})\, r^{-2} \int_{B(0,r) \setminus B(0,r/2)} |u_{1,+}|^2, \end{aligned} \tag{15.13}$$

where we can choose $\tau > 0$ as we like. That is,

$$\begin{aligned} \int_{B(0,r)} |\nabla u_1^*|^2 - \int_{B(0,r)} |\nabla u_1|^2 \ &\le \ -\int_{B(0,r/2)} |\nabla u_{1,+}|^2 + \tau \int_{B(0,r) \setminus B(0,r/2)} |\nabla u_{1,+}|^2 \\ &\quad + 16(1+\tau^{-1})\, r^{-2} \int_{B(0,r) \setminus B(0,r/2)} |u_{1,+}|^2. \end{aligned} \tag{15.14}$$

Then we estimate the M-terms. Notice that only $u_{1,+}$ changes, so

$$\begin{aligned}
|M(\mathbf{u}^*) - M(\mathbf{u})| &= \Big| \int [(u_1^*)^2 f_1 - u_1^2 f_1 - u_1^* g_1 + u_1 g_1] \Big| \\
&\leq ||f_1||_\infty \int_{B(0,r)} u_{1,+}^2 + ||g_1||_\infty \int_{B(0,r)} u_{1,+} \\
&\leq C \int_{B(0,r)} u_{1,+}^2 + C r^{n/2} \Big\{ \int_{B(0,r)} u_{1,+}^2 \Big\}^{1/2} \qquad (15.15)
\end{aligned}$$

because $0 \leq |u_1^*| \leq |u_1|$ everywhere and by Cauchy-Schwarz. Since

$$\begin{aligned}
0 &\leq J(\mathbf{u}^*, \mathbf{W}^*) - J(\mathbf{u}, \mathbf{W}) \\
&= \int_{B(0,r)} |\nabla u_1^*|^2 - \int_{B(0,r)} |\nabla u_1|^2 + M(\mathbf{u}^*) - M(\mathbf{u}) - F(\mathbf{W}^*) - F(\mathbf{W}) \qquad (15.16)
\end{aligned}$$

by minimality, we deduce from (15.12), (15.14), and (15.15) that

$$\int_{B(0,r/2)} |\nabla u_{1,+}|^2 + \lambda |\Omega_1 \cap B(0,r/2)| \leq \alpha(r), \qquad (15.17)$$

where

$$\begin{aligned}
\alpha(r) &= \tau \int_{B(0,r)\setminus B(0,r/2)} |\nabla u_{1,+}|^2 + 16(1+\tau^{-1}) r^{-2} \int_{B(0,r)\setminus B(0,r/2)} u_{1,+}^2 + |M(\mathbf{u}^*) - M(\mathbf{u})| \\
&\leq \tau \int_{B(0,r)} |\nabla u_{1,+}|^2 + [16(1+\tau^{-1}) + Cr^2] \int_{B(0,r)} r^{-2} u_{1,+}^2 + C r^{n/2} \Big\{ \int_{B(0,r)} u_{1,+}^2 \Big\}^{1/2} \\
&\leq \tau \int_{B(0,r)} |\nabla u_{1,+}|^2 + C\tau^{-1} \int_{B(0,r)} r^{-2} u_{1,+}^2 + C r r^{n/2} \Big\{ \int_{B(0,r)} r^{-2} u_{1,+}^2 \Big\}^{1/2} \qquad (15.18)
\end{aligned}$$

because we shall take $\tau \leq 1/2$, and since $r \leq 1$. It will be easier to use this when (15.9) holds, because then the analogue for the ball $B(0,r)$ of the Poincaré estimate (4.6) yields

$$\int_{B(0,r)} r^{-2} u_{1,+}^2 \leq C C_0 \int_{B(0,r)} |\nabla u_{1,+}|^2 \qquad (15.19)$$

and (15.18) implies that

$$\alpha(r) \leq (\tau + C C_0 \tau^{-1}) \int_{B(0,r)} |\nabla u_{1,+}|^2 + C C_0^{1/2} r^{\frac{n}{2}+1} \Big\{ \int_{B(0,r)} |\nabla u_{1,+}|^2 \Big\}^{1/2} \qquad (15.20)$$

when (15.9) holds. Similarly, set

$$\theta(r) = r^{-n} |\Omega_1 \cap B(0,r)| = r^{-n} \big| \{ x \in B(0,r)\,;\, u_1(x) > 0 \} \big| \qquad (15.21)$$

almost as in (12.9); when $\theta(r)$ is small, we can rather use the analogue of (4.7) for the ball $B(0,r)$, which yields

$$\int_{B(0,r)} r^{-2} u_{1,+}^2 \leq C\theta(r)^{2/n} \int_{B(0,r)} |\nabla u_{1,+}|^2. \tag{15.22}$$

In this case, it is also our interest to revise our application of Cauchy-Schwarz in (15.15), because we can say that

$$\begin{aligned} ||g_1||_\infty \int_{B(0,r)} u_{1,+} &\leq C|\Omega_1 \cap B(0,r)|^{1/2} \Big\{ \int_{B(0,r)} u_{1,+}^2 \Big\}^{1/2} \\ &\leq C\theta(r)^{\frac{1}{2}+\frac{1}{2n}}\, r^{\frac{n}{2}+1} \Big\{ \int_{B(0,r)} |\nabla u_{1,+}|^2 \Big\}^{1/2}. \end{aligned} \tag{15.23}$$

Thus we can win an extra $\theta(r)^{1/2}$ in our estimate, and deduce from the proof of (15.20) that

$$\alpha(r) \leq \big(\tau + C\tau^{-1}\theta(r)^{2/n}\big) \int_{B(0,r)} |\nabla u_{1,+}|^2 + C\theta(r)^{\frac{1}{2}+\frac{1}{2n}}\, r^{\frac{n}{2}+1} \Big\{ \int_{B(0,r)} |\nabla u_{1,+}|^2 \Big\}^{1/2}. \tag{15.24}$$

To be honest, we don't really need this extra power if we are willing to take r small, but we shall use it to show that we don't need the extra power of r that we also get in the last term of (15.24), and which we could use to make things small. Set

$$e(r) = r^{-n} \int_{B(0,r)} |\nabla u_{1,+}|^2 \tag{15.25}$$

to simplify our discussion, and let us show that there exists a constant $c_4 > 0$ (that depends only on n, $||f_1||_\infty$, and $||g_1||_\infty$) such that

$$e(r/2) + \theta(r/2) \leq \frac{1}{2}\big(e(r) + \theta(r)\big) \ \text{ if } \theta(r) \leq c_4. \tag{15.26}$$

Indeed, we can use (15.24), and then

$$\begin{aligned} e(r/2) + \theta(r/2) &= 2^n r^{-n} \Big\{ \int_{B(0,r/2)} |\nabla u_{1,+}|^2 + |B(0,r) \cap \Omega_1| \Big\} \leq 2^n r^{-n} \lambda^{-1} \alpha(r) \\ &\leq C\lambda^{-1}(\tau + C\tau^{-1}\theta(r)^{2/n}) e(r) + C\lambda^{-1}\theta(r)^{\frac{1}{2}+\frac{1}{2n}} r e(r)^{1/2} \end{aligned} \tag{15.27}$$

by (15.17), because we can safely assume that $\lambda \leq 1$, and by (15.24). We drop the extra power of r, use the assumption that $\theta(r) \leq c_4$, and get that

$$e(r/2) + \theta(r/2) \leq C\lambda^{-1}\tau e(r) + C\lambda^{-1}\tau^{-1} c_4^{2/n} e(r) + C\lambda^{-1} c_4^{1/2n} (\theta(r) e(r))^{1/2}. \tag{15.28}$$

We choose τ so small (depending on λ) that $C\lambda^{-1}\tau \leq 1/8$, and choose c_4, depending on τ and λ, so small that (15.28) yields

$$e(r/2) + \theta(r/2) \leq \frac{1}{4} e(r) + \frac{1}{4}(\theta(r) e(r))^{1/2} \leq \frac{1}{4} e(r) + \frac{1}{4}(e(r) + \theta(r)), \tag{15.29}$$

(because $ab \leq a^2 + b^2$), as needed for (15.26). Observe now that

$$u_{1,+}(0) = 0 \ \text{ if } \theta(r) \leq c_4 \text{ for } r \text{ small enough}, \tag{15.30}$$

because iterations of (15.26) imply that $\lim_{k\to+\infty} \theta(2^{-k}r) = 0$, which is impossible if $u_1(0) > 0$ (recall that u is continuous).

We may now return to the proof of our two theorems. Let $B(x,r)$ be as in any of the two statements, and first assume that $r \leq \min(1, \varepsilon^{1/n})$ but (15.6) fails. We want to show that $u_1(y) \leq 0$ for every $y \in B(x, r/2)$, and as before we may assume for simplicity that $y = 0$. We want to apply the estimates above to the ball $B(y, r_1)$, with $r_1 = (2\sqrt{n})^{-1}r$. We choose this strange formula for r_1 just to make sure that $|B(y, r_1/2)| \leq \varepsilon$ since $r^n \leq \varepsilon$ (as in our assumptions). This way (15.11) holds for r_1, we can use the estimates above, and we get that

$$e(r_1/2) + \theta(r_1/2) \leq 2^n r_1^{-n} \lambda^{-1} \alpha(r_1) \tag{15.31}$$

by the first part of (15.27). Then, by (15.18),

$$\begin{aligned} r_1^{-n}\alpha(r_1) &\leq C\tau \fint_{B(0,r_1)} |\nabla u_{1,+}|^2 + C\tau^{-1} \fint_{B(0,r_1)} r^{-2}u_{1,+}^2 + Cr\Big\{ \fint_{B(0,r_1)} r^{-2}u_{1,+}^2 \Big\}^{1/2} \\ &\leq C\tau \fint_{B(x,r)} |\nabla u_{1,+}|^2 + C\tau^{-1} \fint_{B(x,r)} r^{-2}u_{1,+}^2 + Cr\Big\{ \fint_{B(x,r)} r^{-2}u_{1,+}^2 \Big\}^{1/2} \end{aligned} \tag{15.32}$$

Set $\Lambda = \fint_{B(x,r)} |\nabla u_{1,+}|^2$ (recall that we are allowed to choose c_1 depending on Λ), and use the fact that (15.6) fails. This gives

$$r_1^{-n}\alpha(r_1) \leq C\tau\Lambda + C\tau^{-1}c_1 + Crc_1^{1/2} \tag{15.33}$$

and, by (15.31), $e(r_1/2) + \theta(r_1/2) \leq C\lambda^{-1}[\tau\Lambda + \tau^{-1}c_1 + rc_1^{1/2}]$. If we choose $\tau > 0$ small enough, then c_1 even smaller, this implies that $e(r_1/2) + \theta(r_1/2) \leq c_4$. Then we can apply (15.26) to the radius $r_1/2$, and get that $e(r_1/4) + \theta(r_1/4) \leq c_4/2$. We iterate the argument and find that $e(2^{-k}r_1) + \theta(2^{-k}r_1) \leq 2^{-k+1}c_4$ for $k \geq 1$, hence $\lim_{\rho\to 0} \theta(\rho) = 0$. And now (15.30) says that $u_1(y) \leq 0$ (recall that we assumed that $y = 0$ for simplicity).

Thus we proved that if $r \leq \min(1, \varepsilon^{1/n})$ but (15.6) fails, $u_1 \leq 0$ on $B(x, r/2)$. This is impossible under our assumption (15.5), so (15.6) holds.

We now prove (15.7) similarly. Suppose that $r \leq \min(1, \varepsilon^{1/n})$ but (15.7) fails, pick any $y \in B(x, r/2)$, and assume by translation invariance that $y = 0$. Let $k \geq 0$ be an integer, that will be chosen soon in terms of $\Lambda = \fint_{B(x,r)} |\nabla u_{1,+}|^2$. Set $r_1 = (2\sqrt{n})^{-1}r$ as above, and observe that for $0 \leq j \leq k$,

$$\theta(2^{-j}r_1) = 2^{nj}r_1^{-n}|\Omega_1 \cap B(0, 2^{-j}r_1)| \leq C2^{nj}r^{-n}|\Omega_1 \cap B(x,r)| \leq C2^{nk}c_2 < c_4 \tag{15.34}$$

by (15.21), because (15.7) fails, and if c_2 is chosen small enough (depending on k).

Thus we can apply (15.26) to the radii $2^{-j}r_1$, $0 \le j \le k$, and we get that

$$\theta(2^{-k-1}r_1) + e(2^{-k-1}r_1) \le 2^{-k-1}(\theta(r_1) + e(r_1)) \le 2^{-k-1}(c_4 + C\Lambda) < c_4, \tag{15.35}$$

if k is chosen large enough. Starting from that point, we can use (15.26) to prove by induction that $\theta(2^{-j}r_1) + e(2^{-j}r_1) \le 2^{-j}(c_4 + C\Lambda) < c_4$ for $j \ge k+1$, as we did below (15.33). Again (15.30) says that $u_1(y) \le 0$, and we can conclude as above. This concludes our proof of Theorem 15.1. ∎

Now we switch to Theorem 15.2, and start under the assumption (15.9). Suppose that $r \le \min(1, \varepsilon^{1/n})$ and (15.9) holds, but that (15.10) fails, and pick any $y \in B(x, r/2)$. As usual, assume that $y = 0$ for simplicity. Choose r_1 as above; because of (15.9), we can use (15.19), and hence

$$\begin{aligned}
\lambda\theta(r_1/2) &= 2^n \lambda r_1^{-n}|\Omega_1 \cap B(0, r_1/2)| \le 2^n r_1^{-n}\alpha(r_1) \\
&\le C\tau \fint_{B(0,r_1)} |\nabla u_{1,+}|^2 + C\tau^{-1}\fint_{B(0,r_1)} r^{-2}u_{1,+}^2 + Cr\Big\{\fint_{B(0,r_1)} r^{-2}u_{1,+}^2\Big\}^{1/2} \\
&\le C\tau \fint_{B(x,r)} |\nabla u_{1,+}|^2 + C\tau^{-1}\fint_{B(x,r)} r^{-2}u_{1,+}^2 + Cr\Big\{\fint_{B(x,r)} r^{-2}u_{1,+}^2\Big\}^{1/2} \\
&\le C[\tau + \tau^{-1}C_0]\fint_{B(x,r)} |\nabla u_{1,+}|^2 + CrC_0^{1/2}\Big\{\fint_{B(x,r)} |\nabla u_{1,+}|^2\Big\}^{1/2} \\
&\le C[\tau + \tau^{-1}C_0]c_3 + CrC_0^{1/2}c_3^{1/2}
\end{aligned} \tag{15.36}$$

by (15.17), (15.18), (15.19), and because (15.10) fails. In addition, $e(r_1/2) \le C \fint_{B(x,r)} |\nabla u_{1,+}|^2 \le Cc_3 < c_4/2$ by the definition (15.25) and because (15.10) fails.

We take $\tau = 1/2$, then choose c_3 very small and get that $\theta(r_1/2) + e(r_1/2) \le c_4$. Then we can apply (15.26) iteratively, as we did twice before, get that $u_1(y) \le 0$, and conclude as before. Again we managed not to use the extra r in the last term.

We are left with the case when we only assume that u_1 is C_0-Lipschitz, as in (15.8). Let $B(x,r)$ satisfy the assumptions of the theorem, and use (15.5) to pick $y \in B(x, r/2)$, with $u_1(y) = 0$. Then let $\eta > 0$ be small, to be chosen soon, and notice that $|u_1(z)| \le C_0\eta r$ for $z \in B(y, \eta r)$. Set $m = \fint_{B(x,r)} u_{1,+}$; then by Poincaré (see (4.2)),

$$\begin{aligned}
m &= \fint_{B(y,\eta r)} \big[u_{1,+}(z) - (u_{1,+}(z) - m)\big] \le \fint_{B(y,\eta r)} \big[|u_{1,+}(z)| + |u_{1,+}(z) - m|\big] \\
&\le C_0\eta r + \fint_{B(y,\eta r)} |u_{1,+}(z) - m| \le C_0\eta r + \eta^{-n}\fint_{B(x,r)} |u_{1,+}(z) - m| \\
&\le C_0\eta r + C\eta^{-n} r\fint_{B(x,r)} |\nabla u_{1,+}| \le C_0\eta r + C\eta^{-n} r\Big\{\fint_{B(x,r)} |\nabla u_{1,+}|^2\Big\}^{1/2} \\
&\le C_0\eta r + C\eta^{-n}c_3^{1/2} r
\end{aligned} \tag{15.37}$$

if (15.10) fails. Then, by Poincaré again,

$$
\begin{aligned}
\fint_{B(x,r)} |u_{1,+}|^2 &\leq 2m^2 + 2\fint_{B(x,r)} |u_{1,+} - m|^2 \leq 2m^2 + Cr^2 \fint_{B(x,r)} |\nabla u_{1,+}|^2 \\
&\leq 2m^2 + Cr^2 c_3 \leq CC_0^2\eta^2 r^2 + C\eta^{-2n} c_3 r^2. \qquad (15.38)
\end{aligned}
$$

We shall choose $c_3 \leq 1$ soon; then let c_1 be as in Theorem 15.1, with the bound $\fint_{B(x,r)} |\nabla u_{1,+}|^2 \leq 1$. If we choose η, and then c_3 small enough (depending on this c_1), we deduce from (15.38) that $\fint_{B(x,r)} |u_{1,+}|^2 \leq c_1 r^2$, i.e., that (15.6) fails. This is impossible, by Theorem 15.1, and this contradiction completes our proof of Theorem 15.2. ∎

For the next nondegeneracy result, we assume that (15.1) holds and u_1 is continuous, and compare $u_1(x)$ with the distance

$$
\delta(x) = \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega_1), \text{ where again } \Omega_1 = \big\{x \in \Omega\,;\, u_1(x) > 0\big\}. \qquad (15.39)
$$

Theorem 15.3 *Let $(\mathbf{u}, \mathbf{W})$ is a minimizer in $\mathcal{F}$ of the functional J, suppose that u_1 is continuous, that f_1 and g_1 are bounded, and that (15.1) holds for some choice of $\lambda > 0$ and $\varepsilon > 0$. For each $C_0 \geq 1$, there is a constant $c_5 > 0$, that depends only on n, λ, $||f_1||_\infty$, $||g_1||_\infty$, and C_0, such that*

$$
u_1(x) \geq c_5 \min(\delta(x), \varepsilon^{1/n}, 1) \qquad (15.40)
$$

as soon as $x \in \Omega_1$ and u_1 is C_0-Lipschitz on $B(x, \delta(x)/2)$.

Our Lipschitz assumption looks complicated because it depends on $\delta(x)$; of course the simplest is to assume that $\mathbf{u}$ is globally Lipschitz, for instance because of Theorem 11.1, but we should also be able to manage without smoothness assumption on Ω. Indeed, $u_1 > 0$ on $B = B(x, \delta(x))$, so B is almost everywhere contained in Ω (because $u_1(x) = 0$ on $\mathbb{R}^n \setminus \Omega$), which means that Ω is equivalent to a set that contains B. Then we may use Lemma 10.6 to get local Lipschitz bounds on $B(x, \delta(x)/2)$, which we can then use to get (15.40).

Notice also that if u_1 is C-Lipschitz in $B(x, \delta(x))$, we immediately get the opposite inequality $|u_1(x)| \leq C\delta(x)$.

Proof. The general idea will be that if a positive harmonic function is very small near the center of the ball, it must also be small on average on the whole ball; if $u_1(x)$ is too small, we shall try to approximate u_1 with such a harmonic function, show that the average of u_1^2 on a ball is small, and use a cut-off competitor to conclude.

Let $x \in \Omega_1$ be as in the statement; as usual, we can assume that $x = 0$ to simplify the notation. Let $r > 0$ be small, to be chosen later. For the moment, let us just assume that

$$
r < \delta(x)/2. \qquad (15.41)
$$

This way, we know that $u_1 > 0$ on $B(x,r)$, hence $B(x,r) \subset \Omega$ (modulo replacing Ω with an equivalent domain, as above) and u_1 is C_0-Lipschitz near $\overline{B}(x,r)$. Notice that the restriction of u_1 to $S_r = \partial B(0,r)$ is Lipschitz, so we can define its harmonic extension u_1^* to $B(0,r)$ more easily than in Section 6. Also set $u_1^*(x) = u_1(x)$ for $x \in \mathbb{R}^n \setminus B(0,r)$, take $u_j^* = 0$ on $B(0,r)$ for $j \geq 2$, keep $\mathbf{u}$ as it was on $\mathbb{R}^n \setminus B(0,r)$, and set $\mathbf{W}^* = \mathbf{W}$. Thus $(\mathbf{u}^*, \mathbf{W}^*)$ is a simpler variant of the harmonic competitor of Section 6, we get that $(\mathbf{u}^*, \mathbf{W}^*) \in \mathcal{F}$ at once (because $u_1^* \in W^{1,2}(\mathbb{R}^n)$), and we shall now see what the comparison yields. For the energy part, the usual computation yields $\int_{B(0,r)} |\nabla u_1^*|^2 \leq \int_{B(0,r)} |\nabla u_1|^2$ (because u_1^* minimizes the energy with the given boundary data; see (6.13)), and even

$$\int_{B(0,r)} |\nabla u_1|^2 - \int_{B(0,r)} |\nabla u_1^*|^2 = \int_{B(0,r)} |\nabla (u_1 - u_1^*)|^2 \tag{15.42}$$

because $u_1^* + t(u_1 - u_1^*)$, $t \in \mathbb{R}$, is a competitor for u_1^*, and by Pythagorus (see the proof of (10.32)). There is no difference in the F-terms, and

$$\begin{aligned} |M(\mathbf{u}^*) - M(\mathbf{u})| &= \Big| \int_{B(0,r)} [(u_1^*)^2 f_1 - u_1^2 f_1 - u_1^* g_1 + u_1 g_1] \Big| \\ &\leq \Big(||(u_1^* + u_1) f_1||_{L^\infty(B(0,r))} + ||g_1||_\infty \Big) \int_{B(0,r)} |u_1^* - u_1|. \end{aligned} \tag{15.43}$$

If $u_1(0) \geq 1$, we are happy because (15.40) holds with $c_5 = 1$, so we may assume that $u_1(0) \leq 1$; then $|u_1(x)| \leq 1 + C_0 r$ on $B(0,r)$, the same thing holds for u_1^* by the maximum principle, and hence

$$||(u_1^* + u_1) f_1||_{L^\infty(B(0,r))} \leq 2(1 + C_0 r) ||f_1||_\infty \leq C \tag{15.44}$$

if we forget to write the dependence on C_0 and demand that $r \leq 1$. Thus

$$|M(\mathbf{u}^*) - M(\mathbf{u})| \leq C \int_{B(0,r)} |u_1^* - u_1| \leq C r^{n/2} \Big\{ \int_{B(0,r)} |u_1^* - u_1|^2 \Big\}^{1/2}. \tag{15.45}$$

Next we claim that since $u_1^* = u_1$ on S_r,

$$\int_{B(0,r)} |u_1^* - u_1|^2 \leq C r^2 \int_{B(0,r)} |\nabla (u_1^* - u_1)|^2 \tag{15.46}$$

because $u_1^* - u_1 \in W^{1,2}(\mathbb{R}^n)$ (see near (4.18)) and by Lemma 3.2, or, if the reader prefers, because Lemma 4.2 says that $\big| \fint_{B(0,r)} (u_1^* - u_1) \big| \leq C r \fint_{B(0,r)} |\nabla (u_1^* - u_1)|$, and then by the usual Poincaré inequality (4.2). In addition,

$$\begin{aligned} \int_{B(0,r)} |\nabla (u_1 - u_1^*)|^2 &= \int_{B(0,r)} |\nabla u_1|^2 - \int_{B(0,r)} |\nabla u_1^*|^2 \\ &= J(\mathbf{u}, \mathbf{W}) - M(\mathbf{u}) - J(\mathbf{u}^*, \mathbf{W}^*) + M(\mathbf{u}^*) \\ &\leq |M(\mathbf{u}^*) - M(\mathbf{u})| \leq C r^{n/2} \Big\{ \int_{B(0,r)} |u_1^* - u_1|^2 \Big\}^{1/2} \\ &\leq C r^{\frac{n}{2}+1} \Big\{ \int_{B(0,r)} |\nabla (u_1^* - u_1)|^2 \Big\}^{1/2} \end{aligned} \tag{15.47}$$

by (15.42), because $F(\mathbf{W}^*) = F(\mathbf{W})$ and $(\mathbf{u}, \mathbf{W})$ is a minimizer, and by (15.45) and (15.46). We simplify and get that

$$\int_{B(0,r)} |\nabla(u_1 - u_1^*)|^2 \leq C r^{n+2}. \tag{15.48}$$

Let $\eta > 0$ be small, to be chosen soon. Then

$$u(z) \leq u(0) + C_0 \eta r \quad \text{for } z \in B(0, \eta r), \tag{15.49}$$

hence

$$\begin{aligned}
|u_1^*(0)| &= \Big| \fint_{B(0,\eta r)} u_1^* \Big| \leq u(0) + C_0 \eta r + \fint_{B(0,\eta r)} |u_1^* - u_1| \\
&\leq u(0) + C_0 \eta r + \eta^{-n} \fint_{B(0,r)} |u_1^* - u_1| \\
&\leq u(0) + C_0 \eta r + \eta^{-n} \Big\{ \fint_{B(0,r)} |u_1^* - u_1|^2 \Big\}^{1/2} \qquad (15.50) \\
&\leq u(0) + C_0 \eta r + C \eta^{-n} r \Big\{ \fint_{B(0,r)} |\nabla(u_1^* - u_1)|^2 \Big\}^{1/2} \leq u(0) + C_0 \eta r + C \eta^{-n} r^2
\end{aligned}$$

because u_1^* is harmonic, and by (15.46) and (15.48). In addition, $u_1^* = u_1$ on S_r and by (15.41) $u_1 > 0$ on $\overline{B}(x,r) \subset \Omega_1$, so $u_1^* > 0$ on S_r and on $B(0,r)$ (by the maximum principle). So

$$\fint_{B(0,r)} |u_1^*| = \fint_{B(0,r)} u_1^* = u_1^*(0) \tag{15.51}$$

and, by simple estimates on harmonic functions,

$$\fint_{B(0,r/2)} |\nabla u_1^*|^2 \leq C r^{-2} \Big\{ \fint_{B(0,r)} |u_1^*| \Big\}^2 = C r^{-2} u_1^*(0)^2. \tag{15.52}$$

Then

$$\fint_{B(0,r/2)} |\nabla u_1|^2 \leq 2 \fint_{B(0,r/2)} |\nabla u_1^*|^2 + 2 \fint_{B(0,r/2)} |\nabla(u_1 - u_1^*)|^2 \leq C r^{-2} u_1^*(0)^2 + C r^2 \tag{15.53}$$

by (15.48). Set $m = \fint_{B(0,r/2)} u_1$; By Poincaré,

$$\fint_{B(0,r/2)} |u_1 - m|^2 \leq C r^2 \fint_{B(0,r/2)} |\nabla u_1|^2 \leq C u_1^*(0)^2 + C r^4. \tag{15.54}$$

Let $\eta_1 > 0$ be another small number, to be chosen later, notice that $u_1(z) \leq u_1(0) + C_0 \eta_1 r$ for $z \in B(0, \eta_1 r)$, and use this and Poincaré to estimate m:

$$\begin{aligned}
m^2 &= \fint_{B(0,\eta_1 r)} m^2 \leq 2 \fint_{B(0,\eta_1 r)} u_1^2 + 2 \fint_{B(0,\eta_1 r)} |u_1 - m|^2 \\
&\leq 2(u_1(0) + C_0 \eta_1 r)^2 + 2^{1-n} \eta_1^{-n} \fint_{B(0,r/2)} |u_1 - m|^2 \\
&\leq 2(u_1(0) + C_0 \eta_1 r)^2 + C \eta_1^{-n} u_1^*(0)^2 + C \eta_1^{-n} r^4 \qquad (15.55)
\end{aligned}$$

by (15.54). Finally,

$$\begin{aligned}
\fint_{B(0,r/2)} u_1^2 &\leq 2m^2 + 2\fint_{B(0,r/2)} |u_1 - m|^2 \leq 2m^2 + Cu_1^*(0)^2 + Cr^4 \\
&\leq 4(u_1(0) + C_0\eta_1 r)^2 + C\eta_1^{-n} u_1^*(0)^2 + C\eta_1^{-n} r^4 \\
&\leq Cu_1(0)^2 + C\eta_1^2 r^2 + C\eta_1^{-n}\big(u_1(0) + \eta r + \eta^{-n} r^2\big)^2 + C\eta_1^{-n} r^4 \\
&\leq C\eta_1^{-n} u_1(0)^2 + C\eta_1^2 r^2 + C\eta_1^{-n}\eta^2 r^2 + \eta_1^{-n}\eta^{-2n} r^4 =: r^2\alpha(r),
\end{aligned} \tag{15.56}$$

by (15.54), (15.55), and (15.50) and where the last identity is a definition of $\alpha(r)$.

We now have enough information to compare $(\mathbf{u}, \mathbf{W})$ with the cut-off competitor associated to $B(0, r/2)$ and the constant $a = 1/2$. That is, we want to replace u_1 with a cut-off function u_1^*, defined by $u_1^*(x) = u_1(x)\varphi(|x|)$ as in Section 6 or in the beginning of this section. We do not need to touch the u_j, $j \geq 2$, because they all vanish on $B(0, r/4)$. But we want to take advantage of (15.1) to modify $\mathbf{W}$. Let us assume, in addition to (15.41) and the fact that $r \leq 1$, that r is so small that

$$|B(0, r/4)| \leq \varepsilon \tag{15.57}$$

Observe that $B(0,r) \subset \Omega_1 \subset W_1$ (modulo a null set), because $u_1 > 0$ there; we apply (15.1) to $A = B(0, r/4) \cap W_1$, and get disjoint sets $A_j \subset A$, $j \geq 2$. Then set $\mathbf{W}^* = (W_1 \setminus A, W_2 \cup A_1, \dots, W_N \cup A_N)$; it is easy to see that $(\mathbf{u}^*, \mathbf{W}^*) \in \mathcal{F}$, and (15.1) says that

$$F(\mathbf{W}^*) \leq F(\mathbf{W}) - \lambda|A| = F(\mathbf{W}) - \lambda|B(0, r/4)|. \tag{15.58}$$

From (6.5) with $\tau = 1$ we deduce that

$$\begin{aligned}
\int_{B(0,r/2)} |\nabla u_1^*|^2 &\leq 2\int_{B(0,r/2)} |\nabla u_1|^2 + Cr^{-2}\int_{B(0,r/2)} |u_1|^2 \\
&\leq Cr^{n-2}|u_1^*(0)|^2 + Cr^{n+2} + Cr^n\alpha(r) =: r^n\beta(r)
\end{aligned} \tag{15.59}$$

by (15.53) and (15.56), and where the last identity is a definition of $\beta(r)$. Concerning $M(\mathbf{u}^*) - M(\mathbf{u})$, even though we are now looking at a different competitor, we can still use the estimates (15.43)-(15.45). We get that

$$\begin{aligned}
|M(\mathbf{u}^*) - M(\mathbf{u})| &\leq C\int_{B(0,r/2)} |u_1^* - u_1| \leq Cr^{n/2}\Big\{\int_{B(0,r/2)} |u_1^* - u_1|^2\Big\}^{1/2} \\
&\leq Cr^{\frac{n}{2}+1}\Big\{\int_{B(0,r/2)} |\nabla(u_1^* - u_1)|^2\Big\}^{1/2} \\
&\leq Cr^{\frac{n}{2}+1}\Big\{\int_{B(0,r/2)} |\nabla u_1^*|^2 + |\nabla u_1)|^2\Big\}^{1/2} \leq Cr^{n+1}\beta(r)^{1/2}
\end{aligned} \tag{15.60}$$

by (15.45), Cauchy-Schwarz, Poincaré and the fact that $u_1^* = u_1$ on $\partial B(0, r/2)$ (as in (15.46)-(15.47)), and (15.59). Now

$$\begin{aligned}
\lambda |B(0, r/4)| &\leq F(\mathbf{W}) - F(\mathbf{W}^*) \\
&= J(\mathbf{u}, \mathbf{W}) - J(\mathbf{u}^*, \mathbf{W}^*) + [M(\mathbf{u}^*) - M(\mathbf{u})] + \int_{B(0,r/2)} |\nabla u_1^*|^2 - |\nabla u_1|^2 \\
&\leq C r^{n+1} \beta(r)^{1/2} + r^n \beta(r) \qquad (15.61)
\end{aligned}$$

by (15.58), (1.5), because $(\mathbf{u}, \mathbf{W})$ is a minimizer, and by (15.59) and (15.60). The game now consists in checking out the right-hand side of (15.61), and showing that all the terms that we get, except those coming from $u_1(0)$, are much smaller than λr^n. For this we shall put the additional constraint that $r \leq r_0$, where r_0 depends on n, λ, $||f_1||_\infty$, $||g_1||_\infty$, and C_0.

We first look at $\alpha(r)$ in (15.56). If we chose η_1 small enough, then η small, and then r_0 small enough, we get that $\alpha(r) \leq C\eta_1^{-n} r^{-2} u(0)^2 + o(1)$, where $o(1)$ is as small as we want. The rest of $\beta(r)$ is

$$\beta_1(r) = r^{-2} u_1^*(0)^2 + Cr^2 \leq Cr^{-2} u(0)^2 + C\eta^2 + C\eta^{-2n} r^2 + Cr^2 \qquad (15.62)$$

(see (15.59) and (15.50)), which is of the same type. Then (15.61) says that

$$\begin{aligned}
\lambda &\leq C r^{-n} \lambda |B(0, r/4)| \leq C r \beta(r)^{1/2} + C\beta(r) \\
&\leq C r \eta_1^{-n/2} r^{-1} u(0) + C\eta_1^{-n} r^{-2} u(0)^2 + o(1) \qquad (15.63)
\end{aligned}$$

which, if we choose our constants so that $o(1) \leq \lambda/2$, yields $r^{-1}u(0) \geq c$ for some $c > 0$. Our constraints on r (namely (15.41), (15.57), $r \leq 1$, and $r \leq r_0$) allow us to chose $r \geq C^{-1} \min(\delta(0), \varepsilon^{1/n}, 1)$, and we obtain (15.40) (recall that we took $x = 0$). Theorem 15.3 follows. ∎

The last result of this section concerns the size of the complement of a good region where $u_1 \geq 0$.

Theorem 15.4 *Let $(\mathbf{u}, \mathbf{W})$ be a minimizer in $\mathcal{F}$ of the functional J, suppose that the f_i and the g_i in the data are bounded, that F is Lipschitz (i.e., (10.2) holds), and that (15.1) holds for some choice of $\lambda > 0$ and $\varepsilon > 0$. For each $C_0 \geq 1$, we can find $r_0 > 0$ and $c_6 > 0$, such that if $x \in \mathbb{R}^n$ and $0 \leq r \leq r_0$ are such that $B(x, r) \subset \Omega$,*

$$\mathbf{u} \text{ is } C_0\text{-Lipschitz on } B(x, r), \qquad (15.64)$$

and $B(x, r/2)$ meets the boundary of $\Omega_1 = \{x \in \Omega\,;\, u_1(x) > 0\}$, then

$$\big|\{x \in B(x, r)\,;\, u_1(x) \leq 0\}\big| \geq c_6 r^n. \qquad (15.65)$$

The constant $c_6 > 0$ depends only on n, the $||f_i||_\infty$, the $||g_i||_\infty$, the Lipschitz constant for F (in (10.2)) λ, and C_0, and r_0 depends on these constants, plus ε.

The main case is probably when $u_1 \geq 0$. Then (15.65) implies that $|B(x,r) \setminus W_1| \geq c_6 r^n$, because the remaining set $\{x \in W_1\,;\, u_1(x) \leq 0\} = \{x \in W_1\,;\, u_1(x) = 0\}$ is negligible, because otherwise we could just use (15.1) to send some of it to the other W_i and some of it to the trash, and make a profit.

We decided also to include the case when u_1 is real-valued, and then (15.65) also counts the part of W_1 where $u_1 < 0$.

We did not mention the case when $B(x,r)$ is centered near $\partial\Omega$, because even though the result is still true in that case (if $\partial\Omega$ is reasonably smooth), this is for the stupid reason that $B(x,r) \setminus \Omega$ is already large enough. So we shall restrict to $B(x,r) \subset \Omega$. In many cases, we will be able to deduce (15.64) from Theorem 10.1, applied to $B(x,2r)$ if it is contained in Ω; but then C_0 will depend on $\int_{B(x,2r)} |\nabla \mathbf{u}|^2$.

Proof. The proof will be based on the following (admittedly vague) idea. We want to show that u_1 is close to a harmonic function v_1 in some ball centered on $\partial\Omega_1$; if (15.65) fails, u_1 and v_1 should almost be positive. Theorem 15.1 says that u_1, hence also v_1, is reasonably large on average, so the mean value property for v_1 says that it should also be large near the center of the ball. But the closeby u_1 vanishes on that center (a contradiction). The harmonic function v_1 will be the same one as in the harmonic competitor.

Let the pair (x,r) be as in the statement, and suppose in addition that (15.65) fails. By assumption, we can find $y \in B(x,r/2) \cap \partial\Omega_1$, and without lost of generality, we may assume that $y = 0$. We first want to select a radius $\rho \in (r/4, r/2)$, with the following properties. Since we want to use the harmonic competitor (with main function u_1), we first demand that the restriction of $\mathbf{u}$ to S_ρ lie in $W^{1,2}(S_\rho)$, with a derivative that can be computed from the restriction of $\nabla \mathbf{u}$ to S_ρ, and with the usual estimates

$$\int_{S_\rho} |\nabla_t u_i|^2 \leq 20 N r^{-1} \int_{B(x,r)} |\nabla u_i|^2 \tag{15.66}$$

for $1 \leq i \leq N$ that we can easily obtain by Chebyshev. In the present case, we know that $\mathbf{u}$ is Lipschitz, so we don't need to be as prudent as usual about restrictions. We also require that

$$u_i(z) = 0 \ \ \sigma\text{-almost everywhere on } S_\rho \setminus W_i \tag{15.67}$$

(as in (6.16)), which is true for almost every ρ, and that

$$\sigma\big(\{z \in S_\rho\,;\, u_1(z) \leq 0\}\big) \leq 20 c_6 r^{n-1}. \tag{15.68}$$

We can also get this last condition, by Chebyshev and because we assume (15.65) to fail.

Define the harmonic competitor $(\mathbf{u}^*, \mathbf{W}^*)$ as we did near (6.11) (but with the radius ρ); notice in particular that the requirement (6.9) holds because $B(0,\rho) \subset B(x,r) \subset \Omega$. This time, we shall need to take $a \in [1/2, 1)$ close to 1, to be chosen later. Notice that $\mathbf{u}_1^*$ is given in terms of a harmonic function v_1 (see (6.14) and above), and a good part of the estimates that follow is aimed at showing that $v_1 - u_1$ is quite small.

Let us first estimate $F(\mathbf{W}^*)-F(\mathbf{W})$. Recall the definition (6.17)-(6.18) of $\mathbf{W}^*$. For $i \geq 2$, $|W_i \setminus W_i^*| \leq |W_i| \leq c_6 r^n$ because $u_1 = 0$ on W_i. Also, $|W_i^* \setminus W_i| \leq |W_i^*| \leq B(0,\rho)\setminus B(0,a\rho) \leq C(1-a)r^n$. Next, $|W_1 \setminus W_1^*| \leq |B(0,\rho) \setminus W_1^*| \leq |B(0,\rho) \setminus B(0,a\rho)| \leq C(1-a)r^n$. Finally, $|W_1^* \setminus W_1| \leq |B(0,\rho) \setminus W_1| \leq |B(0,\rho) \setminus \Omega_1| \leq c_6 r^n$. Thus by (10.2)

$$|F(\mathbf{W}^*) - F(\mathbf{W})| \leq C \sum_{i=1}^{N} |W_i \Delta W_i^*| \leq C(c_6 + (1-a))r^n. \tag{15.69}$$

Notice that $u_1(y) = 0$ because $y \in \partial\Omega_1$, and all the other u_i also vanish somewhere in $B(y,\rho)$ (in fact, anywhere on $B(y,\rho) \cap \Omega_1$), because (15.65) fails. Since $\mathbf{u}$ is assumed to be C_0-Lipschitz on $B(x,r)$, we get that $|\mathbf{u}| \leq Cr$ on $B(0,\rho)$, and (by the maximum principle) the same thing holds for $\mathbf{u}^*$. [We shall not keep track of the dependence of our constants on C_0, or the other constants mentioned in the statement.] Then

$$|M(\mathbf{u}^*) - M(\mathbf{u})| \leq \sum_{i=1}^{N} \int_{B(0,\rho)} \left[||f_i||_\infty |u^2 - (u^*)^2| + ||g_i||_\infty |u - u^*| \right] \leq Cr^{n+1}. \tag{15.70}$$

For the energy part, we start with $i \geq 2$. By (6.19),

$$\int_{B(0,\rho)} |\nabla u_i^*|^2 \leq (1-a)ra^{2-n} \int_{S_\rho} |\nabla_t u_i|^2 + 4(1-a)^{-1} r a^{-n} \int_{S_\rho} |\rho^{-1} u_i|^2. \tag{15.71}$$

We continue as usual, but at this point we could also use the fact that $\mathbf{u}$ is Lipschitz on $\overline{B}(0,\rho)$ to get the same conclusion. We want to use (4.7) with $E = S_\rho \setminus \Omega_1$, so we first observe that if $z \in S_\rho \setminus E = S_\rho \cap \Omega_1$, then $z \in W_1$ almost surely (by (15.67) for $i = 1$), so $z \in S_\rho \setminus W_i$, and σ-almost surely $u_i(z) = 0$ (by (15.67) for i). So $u_i(z) = 0$ almost everywhere on $S_\rho \setminus E$. Also, $u_1(z) \leq 0$ on E, so $\sigma(E) \leq 20 c_6 r^{n-1}$ by (15.68), and

$$\int_{S_\rho} |\rho^{-1} u_i|^2 \leq C\rho^{-2} \sigma(E)^{\frac{2}{n-1}} \int_{S_\rho} |\nabla_t u_i|^2 \leq C c_6^{\frac{2}{n-1}} \int_{S_\rho} |\nabla_t u_i|^2 \tag{15.72}$$

by (4.7) and

$$\begin{aligned}
\int_{B(0,\rho)} |\nabla u_i^*|^2 &\leq C[(1-a) + (1-a)^{-1} c_6^{\frac{2}{n-1}}]\, r \int_{S_\rho} |\nabla_t u_i|^2 \\
&\leq C[(1-a) + (1-a)^{-1} c_6^{\frac{2}{n-1}}] \int_{B(x,r)} |\nabla u_i|^2 \\
&\leq C[(1-a) + (1-a)^{-1} c_6^{\frac{2}{n-1}}] r^n
\end{aligned} \tag{15.73}$$

by (15.71) and (15.66), and brutally because $\mathbf{u}$ is Lipschitz in $\overline{B}(0,\rho)$.

We have the estimate (6.20) for the exterior part of $\int |\nabla u_1^*|^2$, namely,

$$\begin{aligned}
\int_{B(0,\rho)\setminus B(0,a\rho)} |\nabla u_1^*|^2 &\leq (1-a)\rho a^{2-n} \int_{S_\rho} |\nabla_t u_1|^2 \\
&\leq C(1-a) \int_{B(x,r)} |\nabla u_1|^2 \leq C(1-a) r^n
\end{aligned} \tag{15.74}$$

(by (15.66) again). The last term is

$$\begin{aligned}
\int_{B(0,a\rho)} |\nabla u_1^*|^2 &= a^{n-2} \int_{B(0,\rho)} |\nabla v_1|^2 \\
&= a^{n-2} \inf\Big\{ \int_{B(0,\rho)} |\nabla v|^2 \,;\, v \in W^{1,2}(B(0,\rho)) \text{ and } v = u_1 \text{ on } S_\rho \Big\}, \\
&\leq a^{n-2} \int_{B(0,\rho)} |\nabla u_1|^2 \qquad (15.75)
\end{aligned}$$

by (6.21). Set $\Delta = \int_{B(0,\rho)} |\nabla u_1|^2 - \int_{B(0,\rho)} |\nabla v_1|^2$; then by the minimizing property in (15.75), the fact that $v_1 + t(u_1 - v_1)$ is a competitor for v_1 for all t, and the usual Pythagorus argument (see for instance the proof of (10.32)),

$$\Delta = \int_{B(0,\rho)} |\nabla(u_1 - v_1)|^2. \tag{15.76}$$

But also, by the first line of (15.75) and because $a \leq 1$,

$$\begin{aligned}
a^{n-2}\Delta &= a^{n-2} \int_{B(0,\rho)} |\nabla u_1|^2 - \int_{B(0,a\rho)} |\nabla u_1^*|^2 \\
&\leq \int_{B(0,\rho)} |\nabla u_1|^2 - \int_{B(0,a\rho)} |\nabla u_1^*|^2 \qquad (15.77) \\
&= \int_{B(0,\rho)} |\nabla u_1|^2 - \int_{B(0,\rho)} |\nabla u_1^*|^2 + \int_{B(0,\rho)\setminus B(0,a\rho)} |\nabla u_1^*|^2
\end{aligned}$$

We now add the contribution of the other components. Notice that $\int_{B(0,\rho)} |\nabla u_1|^2 \leq \int_{B(0,\rho)} |\nabla \mathbf{u}|^2$, and $\int_{B(0,\rho)} |\nabla u_1^*|^2 = \int_{B(0,\rho)} |\nabla \mathbf{u}^*|^2 - \sum_{i\geq 2} \int_{B(0,\rho)} |\nabla u_i^*|^2$. We replace and get that

$$a^{n-2}\Delta \leq \int_{B(0,\rho)} |\nabla \mathbf{u}|^2 - \int_{B(0,\rho)} |\nabla \mathbf{u}^*|^2 + \sum_{i\geq 2} \int_{B(0,\rho)} |\nabla u_i^*|^2 + \int_{B(0,\rho)\setminus B(0,a\rho)} |\nabla u_1^*|^2 \tag{15.78}$$

Let us use the minimality of $(\mathbf{u}, \mathbf{W})$. Since $\mathbf{u} = \mathbf{u}^*$ on $\mathbb{R}^n \setminus B(0,\rho)$,

$$\begin{aligned}
\int_{B(0,\rho)} |\nabla \mathbf{u}|^2 - \int_{B(0,\rho)} |\nabla \mathbf{u}^*|^2 &= J(\mathbf{u}) - M(\mathbf{u}) - J(\mathbf{u}^*) + M(\mathbf{u}^*) \\
&\leq M(\mathbf{u}^*) - M(\mathbf{u}) \leq C r^{n+1} \qquad (15.79)
\end{aligned}$$

by (1.5), the minimality of $(\mathbf{u}, \mathbf{W})$, and (15.70). Then

$$\begin{aligned}
\Delta &\leq 2a^{n-2}\Delta \leq C r^{n+1} + \sum_{i\geq 2} \int_{B(0,\rho)} |\nabla u_i^*|^2 + \int_{B(0,\rho)\setminus B(0,a\rho)} |\nabla u_1^*|^2 \\
&\leq C r^{n+1} + C[(1-a) + (1-a)^{-1} c_6^{\frac{2}{n-1}}] r^n + C(1-a) r^n \qquad (15.80) \\
&\leq C[(1-a) + (1-a)^{-1} c_6^{\frac{2}{n-1}} + r] r^n
\end{aligned}$$

because we shall take a close to 1, and by (15.78), (15.79) (15.73) and (15.74).

We can now use this to show that u_1 is close to the harmonic function v_1. Notice that $v_1 - u_1$ lies in $W^{1,2}(\mathbb{R}^n)$ and vanishes on $\mathbb{R}^n \setminus B(0,\rho)$ so Lemma 3.2 (or if the reader prefers, Lemma 4.2 and the standard Poincaré inequality), implies that

$$\int_{B(0,\rho)} |u_1 - v_1|^2 \leq C\rho^2 \int_{B(0,\rho)} |\nabla(u_1 - v_1)|^2 \leq Cr^2\Delta, \tag{15.81}$$

by (15.76).

Next we want to apply Theorem 15.1 to $B(0,\rho)$; the assumption (15.5) is satisfied, because $0 = y \in \partial\Omega_1$ and if $r_0 \leq \inf(1, \varepsilon^{1/n})$ (recall that $2\rho \leq r \leq r_0$). So (15.6) says that $\fint_{B(0,\rho)} u_{1,+}^2 \geq c_1\rho^2 \geq c_1 r^2/16$, where c_1 also depends on C_0 through an upper bound for $\fint_{B(0,\rho)} |\nabla u_{1,+}|^2$. Since in addition $|u_1(z)| \leq Cr$ for $z \in B(0,\rho)$, because $u_1(y) = 0$ (see below (15.69)), we get that

$$\frac{c_1 r^2}{16} \leq \fint_{B(0,\rho)} u_{1,+}^2 \leq Cr \fint_{B(0,\rho)} u_{1,+} \tag{15.82}$$

If (15.65) fails, we also get that

$$\fint_{B(0,\rho)} u_{1,-} \leq Cr\rho^{-n} \big|\{x \in B(x,r)\,;\, u_1(x) \leq 0\}\big| \leq Cc_6 r \tag{15.83}$$

because $|u_1(z)| \leq Cr$ for $z \in B(0,\rho)$. If c_6 is small enough compared to c_1, we deduce from (15.82) and (15.83) that

$$\fint_{B(0,\rho)} u_1 \geq C^{-1}c_1 r. \tag{15.84}$$

Next set $m = \fint_{B(0,\rho} v_1$; we deduce from (15.81) that

$$\begin{aligned} \Big|m - \fint_{B(0,\rho)} u_1\Big| &= \Big|\fint_{B(0,\rho)} (v_1 - u_1)\Big| \leq \fint_{B(0,\rho)} |v_1 - u_1| \\ &\leq \Big\{\fint_{B(0,\rho)} |v_1 - u_1|^2\Big\}^{1/2} \leq Cr(r^{-n}\Delta)^{1/2}. \end{aligned} \tag{15.85}$$

Let $\eta > 0$ be small, to be chosen soon, and observe that $|u_1(z)| \leq C\eta\rho$ for $z \in B(0,\eta\rho)$ (because $u_1(0) = 0$ and $\mathbf{u}$ is C_0-Lipschitz in $B(0,\rho)$). But $\fint_{B(0,\eta\rho)} v_1 = m$ because v_1 is harmonic; then

$$\begin{aligned} |m| &= \Big|\fint_{B(0,\eta\rho)} v_1\Big| \leq C\eta\rho + \Big|\fint_{B(0,\eta\rho)} (v_1 - u_1)\Big| \leq C\eta r + \fint_{B(0,\eta\rho)} |v_1 - u_1| \\ &\leq C\eta r + \eta^{-n} \fint_{B(0,\rho)} |v_1 - u_1| \leq C\eta r + Cr\eta^{-n}(r^{-n}\Delta)^{1/2} \end{aligned} \tag{15.86}$$

by the end of (15.85). Now (15.84)-(15.86) yield

$$\begin{aligned}
C^{-1}c_1 &\leq r^{-1}\fint_{B(0,\rho)} u_1 \leq r^{-1}\Big|m - \fint_{B(0,\rho)} u_1\Big| + r^{-1}|m| \\
&\leq C(r^{-n}\Delta)^{1/2} + C\eta + C\eta^{-n}(r^{-n}\Delta)^{1/2} \\
&\leq C\eta + C\eta^{-n}[(1-a) + (1-a)^{-1}c_6^{\frac{2}{n-1}} + r]^{1/2} \qquad (15.87)
\end{aligned}$$

by (15.80). If we choose η, then $1-a$, then c_6 and r_0 small enough (also recall that we need to take $r_0 \leq \inf(1, \varepsilon^{1/n})$ to apply Theorem 15.1), we obtain the desired contradiction, which proves (15.65). Theorem 15.4 follows. ∎

16 The boundary of a good region is rectifiable

Our assumptions for this section will be roughly the same as for Section 15. We shall assume that

$$\text{the } f_i \text{ and the } g_i,\ 1 \leq i \leq N, \text{ are bounded} \qquad (16.1)$$

and restrict our attention on an open ball B_0 such that

$$u_1 \text{ is } C_0\text{-Lipschitz on } B_0 \qquad (16.2)$$

for some $C_0 \geq 0$. As usual, previous sections give sufficient conditions for this to happen. We shall also assume the Lipschitz condition (10.2), and that W_1 is a good region for F, in the sense that we can find $\varepsilon > 0$ and $\lambda > 0$ such that the nondegeneracy condition (15.1) holds. And we start to study the regularity in B_0 of the boundary of the open set

$$\Omega_1 = \big\{x \in \Omega\,;\, u_1(x) > 0\big\}. \qquad (16.3)$$

Maybe we should point out that Ω_1 is the good free boundary to study, as opposed to the larger set $\big\{\mathbf{u} = 0\big\}$. Suppose that $u_1 \geq 0$ to simplify the discussion. On the other side of Ω_1, there may be other components, possibly not all good, and/or the black zone $\Omega \setminus \cup_i W_i$, and these may be much less regular. Even when $n = 2$, $N = 2$, the u_i are required to be positive, and $(\mathbf{u}, \mathbf{W})$ is a local minimizer of the most standard Alt-Caffarelli functional with $q_1 q_2 = 1$ (or take $q_2 = 2$ if you are afraid of a potential degeneracy), it can happen that Ω_1 and Ω_2 are smooth regions, with a black zone $\Omega \setminus (\Omega_1 \cup \Omega_2)$ in the middle, with lots of thin parts, cusps, and islands.

In this section we want to show that $\partial\Omega_1$ is a locally Ahlfors regular and (even uniformly) rectifiable set. We shall also get a reproducing formula for $\Delta u_{1,+}$, which will be used later, once we have a better description of the blow-up limits of $\mathbf{u}$. See Proposition 16.2.

For all this, we shall mostly follow the initial arguments of [AC]; this will be not be too hard to do because, as soon as we have the nondegeneracy results of Section 15, the other components u_i $i \geq 2$, only play a small role in the estimates. Set

$$v = u_{1,+} = \max(0, u_1) = u_1 \mathbb{1}_{\Omega_1}. \qquad (16.4)$$

The hero of this section is μ, the restriction of the distribution Δv to $\partial\Omega_1$ (see a correct definition below). We shall prove that μ is in fact a (positive!) locally Ahlfors-regular measure, and this will help us prove that Ω_1 is locally a Caccioppoli set (a set with finite perimeter), with a reduced boundary almost equal to $\partial\Omega_1$. The local Ahlfors-regularity and rectifiability of $\partial\Omega$, and the representation formulas of Proposition 16.2 will then easily follow.

Recall from (9.6) that in Ω_1, $v = u_1$ satisfies the equation $\Delta v = f_1 v - \frac{1}{2} g_1$; the official definition of our hero μ is the distribution

$$\mu = \Delta v - [f_1 v - \frac{1}{2} g_1]\, \mathbb{1}_{\Omega_1}. \tag{16.5}$$

Proposition 16.1 *Let* $(\mathbf{u}, \mathbf{W})$*,* W_1 *and* B_0 *satisfy the assumptions above (up to (16.3)), and let* μ *be the distribution defined by (16.5). Then* μ *is, in* B_0*, a locally Ahlfors-regular positive measure whose support is* $\partial\Omega_1$*. More precisely, there are constants* $C_1 \geq 1$ *and* $r_0 \leq 1$ *such that*

$$C_1^{-1} r^{n-1} \leq \mu(B(x,r)) \leq C_1 r^{n-1} \tag{16.6}$$

for $x \in \partial\Omega_1$ *and* $0 < r \leq r_0$ *such that* $B(x, 2r) \subset B_0$*. The constant* C_1 *depends on* n*, the* L^∞ *bounds in (16.1), the Lipschitz bound in (16.2), the Lipschitz constant in (10.2), and the constant* λ *in that shows up in (15.1). The radius* r_0 *depends on these constants, plus the* $\varepsilon > 0$ *from (15.1).*

Proof. We start with the positivity. By (9.6), $\Delta v = f_1 v - \frac{1}{2} g_1 \geq -C$ in $\Omega_1 \cap B_0$, where C is a large constant that we don't even want to compute. Then Remark 1.4 in [CJK] says that $\Delta v \geq -C$. That is, $\Delta v + C$ is a positive distribution, and this implies that it is also a positive measure. Then Δv and μ are measures too. We want to check that these three measures have the same restriction to $\partial\Omega_1 \cap B_0$. Let us first check that

$$|\partial\Omega_1 \cap B_0| = 0. \tag{16.7}$$

Suppose not. Then we can find a point $x_0 \in \partial\Omega_1 \cap B_0$, which is a Lebesgue density point of $\partial\Omega_1 \cap B_0$. We apply Theorem 15.1 to this point, with a very small radius r, and find out that $|\Omega_1 \cap B(x,r)| \geq c_2 r^n$ by (15.7). When r is small, this is not compatible with the definition of a Lebesgue point for $\partial\Omega_1$; hence (16.7) holds.

It easily follows from (16.7) that the restrictions of our three measures to $\Omega_1 \cap B_0$ are the same; this shows that μ is positive, like $\Delta v + C$, and we also get from this that in B_0,

$$\mu \text{ is the restriction of } \Delta v \text{ to } \partial\Omega_1, \tag{16.8}$$

just because the definition (16.5) makes it vanish on $\partial\Omega_1$, and the three measure vanish on the open set $\mathbb{R}^n \setminus \overline{\Omega}_1$, where $v = 0$.

Next we want to check that μ is locally Ahlfors-regular of dimension $n-1$. We start with the upper estimate. Let $B(x,r)$ be as in the statement, and let φ be a nonnegative smooth function such that $\varphi(y) = 1$ for $y \in B(x,r)$, $|\nabla\varphi| \leq Cr^{-1}$ everywhere, and φ is compactly supported in $B(x,2r)$. Then

$$\begin{aligned} \mu(B(x,r)) &\leq \int \varphi d\mu = \langle \Delta v, \varphi \rangle - \int_{\Omega_1} [f_1 v - \frac{1}{2} g_1]\varphi \\ &\leq \langle \Delta v, \varphi \rangle + C||\varphi||_\infty |B(x,2r)| \leq |\langle \nabla v, \nabla \varphi \rangle| + Cr^n \\ &\leq C||\nabla v||_\infty ||\nabla\varphi||_{L^\infty(B_0)} r^n + Cr^n \leq Cr^{n-1} \end{aligned} \tag{16.9}$$

because μ is positive, by definition of μ and of a distribution derivative (and because we know that v is locally Lipschitz), and because $|\nabla\varphi| \leq Cr^{-1}$, $|\nabla v| \leq C_0$, and $r \leq r_0 \leq 1$. This was shockingly easy, but positivity helped a great deal.

For the lower bound, we shall need to choose our test function more precisely and use results of Section 15. Let us first choose an intermediate radius $\rho \in (0,r)$. If r is small enough (depending on ε in the condition (15.1)), Theorem 15.1 says that $\fint_{B(x,r)} u_{1,+}^2 \geq c_1 r^2$, where $u_{1,+} = \max(0, u_1) = v$ and for some $c_1 > 0$ that depends only on the constants cited in the statement of Lemma 16.1. In particular the needed bound on $\fint_{B(x,r)} |\nabla u_1|^2$ follows at once from our Lipschitz bound (16.2). By (4.3) we can choose $\rho \in (0,r)$ such that

$$\int_{\partial B(x,\rho)} v^2 \geq r^{-1} \int_{B(x,r)} v^2 \geq C^{-1} r^{n+1}. \tag{16.10}$$

But $x \in \partial\Omega_1$, so $u_1(x) = 0$, $v(x) = 0$, and so $v \leq Cr$ on $B(x,r)$, by (16.2). So $\fint_{\partial B(x,\rho)} v^2 \leq Cr \fint_{\partial B(x,\rho)} v$ and (16.10) yields

$$\fint_{\partial B(x,\rho)} v \geq C^{-1} r. \tag{16.11}$$

Notice that this forces $\rho \geq C^{-1} r$, because $\fint_{\partial B(x,\rho)} v \leq C_0 \rho$ (again because $v(x) = 0$ and by (16.2)).

Let $\tau > 0$ be small (so that $\tau r < \rho$ in particular), and define a test function φ by

$$\varphi(y) = f(|y-x|), \tag{16.12}$$

where $f : [0, +\infty) \to [0,1]$ is defined as follows. We set $f(t) = 1$ for $0 \leq t \leq \tau r$, $f(t) = 0$ for $t \geq \rho$, and, in the remaining region where $\tau r \leq t \leq \rho$,

$$f(t) = \frac{\tau^{n-2}}{1-\tau^{n-2}} \frac{\rho^{n-2} - t^{n-2}}{t^{n-2}} \quad \text{when } n \geq 2 \tag{16.13}$$

and

$$f(t) = (\log(1/\tau))^{-1} \log(\rho/t) \quad \text{when } n = 2. \tag{16.14}$$

[Here is a typical place when we don't really want to consider $n = 1$.] The point of this choice is that f is continuous and $f'(t) = -a_n(\tau)\rho^{n-2}t^{1-n}$ for $\tau r < t < r$, where $a_n(\tau)$ is a positive constant that we don't need to compute. We would like to use the fact that

$$\begin{aligned}
\langle \Delta v, \varphi \rangle &= -\langle \nabla v, \nabla \varphi \rangle = -\int_{B(x,\rho)\setminus B(x,\tau r)} f'(|y-x|)\frac{\partial v}{\partial r}(y)dy \\
&= a_n(\tau)\rho^{n-2}\int_{B(x,\rho)\setminus B(x,\tau r)} |y-x|^{1-n}\frac{\partial v}{\partial r}(y)dy,
\end{aligned} \tag{16.15}$$

but φ is not a real test function, so we have to do something about it. There is no problem with the other identities, because v is Lipschitz; in particular the second line just comes from our specific choice of φ. Rather than doing an approximating now, let us keep it for later. In the mean time, let us still compute right-hand side of (16.15). Set $g(t) = \fint_{\partial B_(x,t)} v$ for $\tau r \leq t \leq r$, and observe that g is Lipschitz, with

$$g'(t) = \frac{\partial}{\partial t}\Big(\fint_{S_1} v(x+t\theta)d\sigma(\theta)\Big) = \fint_{S_1}\frac{\partial v}{\partial r}(x+t\theta)d\sigma(\theta) = \sigma(S_1)^{-1}t^{1-n}\int_{\partial B(x,t)}\frac{\partial v}{\partial r}, \tag{16.16}$$

so that

$$\begin{aligned}
g(\rho) - g(\tau r) &= \int_{t=\tau r}^{\rho} g'(t)dt = \sigma(S_1)^{-1}\int_{t=\tau r}^{\rho} t^{1-n}\int_{\partial B(x,t)}\frac{\partial v}{\partial r} \\
&= \sigma(S_1)^{-1}\int_{B(x,\rho)\setminus B(x,\tau r)} |y-x|^{1-n}\frac{\partial v}{\partial r} = -\sigma(S_1)^{-1}a_n(\tau)^{-1}\rho^{2-n}\langle \nabla v, \nabla\varphi\rangle
\end{aligned} \tag{16.17}$$

by the correct part of (16.15). Recall that $g(\rho) \geq C^{-1}r$ by (16.11), and that, since v is C_0-Lipschitz, $g(\tau r) \leq C_0\tau r$. We choose τ so small that this implies that $g(\rho) - g(\tau r) \geq (2C)^{-1}r$. Then (16.17) yields

$$-\langle \nabla v, \nabla\varphi\rangle = \sigma(S_1)a_n(\tau)\rho^{n-2}[g(\rho) - g(\tau r)] \geq C(\tau)^{-1}r^{n-1}, \tag{16.18}$$

because $\rho \geq r/C$.

Now we approximate φ. For $\eta > 0$ small, pick a function φ_η, such that $\varphi_\eta(y) = f_\eta(|y-x|)$, where f_η is smooth, coincides with f except on the interval $(\tau r, \tau+\eta)$, and yet $|f'_\eta| \leq C(\tau, r)$ for some constant $C(\tau, r)$ that does not depend on η. Then

$$\begin{aligned}
\big|\langle \nabla v, \nabla\varphi\rangle - \langle \nabla v, \nabla\varphi_\eta\rangle\big| &\leq \int_{B(x,\tau r+\eta)\setminus B(x,\tau)} |\nabla v|\,|\nabla\varphi - \nabla\varphi_\eta| \\
&\leq C||\nabla\varphi - \nabla\varphi_\eta||_\infty |B(x,\tau r+\eta)\setminus B(x,\tau)| \leq C'(\tau, r)\eta
\end{aligned} \tag{16.19}$$

because $\nabla\varphi_t = \nabla\varphi$ most of the time. So, if η is small enough (depending on τ and r), (16.18) yields

$$-\langle \nabla v, \nabla\varphi_\eta\rangle \geq (2C(\tau))^{-1}r^{n-1}. \tag{16.20}$$

But now φ_η is smooth, so $\langle \Delta v, \varphi_\eta \rangle = -\langle \nabla v, \nabla \varphi_\eta \rangle$ by definitions, and

$$\begin{aligned} \mu(B(x,r)) &\geq \int \varphi_\eta d\mu = \langle \Delta v, \varphi_\eta \rangle - \int [f_1 v - \frac{1}{2} g_1] \, \mathbb{1}_{\Omega_1} \varphi \\ &\geq (2C(\tau))^{-1} r^{n-1} - C r^n \geq (4C(\tau))^{-1} r^{n-1} \end{aligned} \tag{16.21}$$

because μ is a positive measure and (we can very easily arrange that) $0 \leq \varphi_\eta \leq \mathbb{1}_{B(x,\rho)} \leq \mathbb{1}_{B(x,r)}$, then by (16.5), (16.18), and if r_0 is small enough. This completes our proof of (16.6) and Proposition 16.1. ∎

Proposition 16.1 has a few consequences that we record now. Let $B_1 \subset B_0$ be a strictly smaller open ball. Since (16.6) holds for every ball of small enough radius centered on $\partial\Omega_1 \cap B_1$, an easy covering argument shows that on B_1, μ is equivalent to the restriction of $\mathcal{H}^{n-1}$ to $\partial\Omega_1 \cap B_1$, in the sense that

$$C^{-1}\mathcal{H}^{n-1}(E \cap \partial\Omega_1) \leq \mu(E) \leq C\mathcal{H}^{n-1}(E \cap \partial\Omega_1) \text{ for every Borel set } E \subset B_1. \tag{16.22}$$

See for instance Lemma 18.11 on page 109 of [D]. In particular, $\mathcal{H}^{n-1}(\partial\Omega_1 \cap B_1) < +\infty$ (by (16.6)), and it is classical that in such a case, Ω_1 is a set of finite perimeter in B_1, with $Per(\Omega_1; B_1) \leq \mathcal{H}^{n-1}(\partial\Omega_1 \cap B_1) < +\infty$. See for instance [Gi]. This means that the restriction to B_1 of the vector-valued distribution $\nabla \mathbb{1}_{\Omega_1}$ is in fact a (vector valued) measure, whose total variation, often denoted by $Per(\Omega_1; B_1)$, is finite. More precisely, we may write

$$\nabla \mathbb{1}_{\Omega_1} = n\nu \tag{16.23}$$

in B_1, where ν is the (total) variation of $\nabla \mathbb{1}_{\Omega_1}$ (a positive measure), and n is a measurable function, with values in the set of unit vectors in $\mathbb{R}^n$ (the inwards unit vector, which is defined ν-almost everywhere). It is also known (see [Gi]) that inside B_1, ν is in fact the restriction of $\mathcal{H}^{n-1}$ to the reduced boundary of Ω_1, which is usually denoted by $\partial^*\Omega_1$, and is in general a Borel set that is strictly contained in $\partial\Omega_1$. For instance, corners of an otherwise smooth domain would lie in $\partial\Omega_1 \setminus \partial^*\Omega_1$. Let us check that in fact

$$\mathcal{H}^{n-1}(B_0 \cap \partial\Omega_1 \setminus \partial^*\Omega_1) = 0 \text{ and } \nu = \mathcal{H}^{n-1}_{|\partial\Omega_1}. \tag{16.24}$$

The second affirmation will follow at once from the first one (since ν is the restriction of $\mathcal{H}^{n-1}$ to $\partial^*\Omega_1$), and for the first one let us first prove that for $B_1 \subset B_0$ as above there exist constants $C > 0$ and $r_1 > 0$ such that

$$\nu(B(x,r)) \geq C^{-1} r^{n-1} \tag{16.25}$$

for $x \in \partial\Omega_1 \cap B_1$ and $0 < r \leq r_1$. Indeed, for such $B(x,r)$ (and if r_1 is small enough, in particular so that $B(x,2r) \subset B_0$), (15.7) says that $|\Omega_1 \cap B(x,r)| \geq c_2 r^n$, and (15.65) says that $|B(x,r) \setminus \Omega_1| = \big|\{x \in B(x,r)\,;\, u_1(x) \leq 0\}\big| \geq c_6 r^n$. By the isoperimetric inequality in $B(x,r)$, we deduce from this that $\nu(B(x,r)) = Per(\Omega_1; B(x,r)) \geq C^{-1} r^{n-1}$, as announced. Since $\nu(B(x,r)) = H^{n-1}(\partial^*\Omega_1 \cap B(x,r)) \leq \mathcal{H}^{n-1}(\partial\Omega_1 \cap B(x,r)) \leq C r^{n-1}$ by (16.6), we see that ν also is locally Ahlfors-regular, and hence equivalent to $\mathcal{H}^{n-1}_{|\partial\Omega_1}$. In particular, $\mathcal{H}^{n-1}(\partial\Omega_1 \setminus \partial^*\Omega_1) \leq C\nu(\partial\Omega_1 \setminus \partial^*\Omega_1) = 0$ (because ν is supported on the Borel set $\partial^*\Omega_1$), which proves (16.24).

Notice that since the reduced boundary of a set with locally finite perimeter is always rectifiable, (16.24) implies that

$$\partial\Omega_1 \cap B_0 \text{ is rectifiable.} \tag{16.26}$$

In fact, we can even prove that $\partial\Omega_1$ is uniformly rectifiable in B_1, with big pieces of Lipschitz graphs. See Proposition 16.3 below. But let us continue with the same program as in [AC] and discuss a representation formula for μ. We know from Proposition 16.1 and its consequence (16.22) that μ is absolutely continuous with respect to $\mathcal{H}^{n-1}_{|\partial\Omega_1}$, which is therefore locally finite. A differentiation result for measures in $\mathbb{R}^n$ allow us to write

$$\mu = h\mathcal{H}^{n-1}_{|\partial\Omega_1} = h\nu \tag{16.27}$$

on B_0, where the density h can be computed by differentiation, i.e.,

$$h(x) = \lim_{r\to 0} \frac{\mu(B(x,r))}{\mathcal{H}^{n-1}(\partial\Omega_1 \cap B(x,r))} = \lim_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} \tag{16.28}$$

for $\mathcal{H}^{n-1}$-almost every $x \in \partial\Omega_1 \cap B_0$. See for instance [M] for this and the next discussion. Now $\partial\Omega_1 \cap B_0$ is rectifiable, so $\lim_{r\to 0} r^{1-n}\mathcal{H}^{n-1}(\partial\Omega_1 \cap B(x,r)) = \omega_{n-1}$ for $\mathcal{H}^{n-1}$-almost every $x \in \partial\Omega_1 \cap B_0$, where $\omega_{n-1} = \mathcal{H}^{n-1}(\mathbb{R}^{n-1} \cap B(0,1))$ denotes the $\mathcal{H}^{n-1}$-measure of the unit ball in $\mathbb{R}^{n-1}$. We could also say that $\lim_{r\to 0} r^{1-n}\nu(B(x,r)) = \omega_{n-1}$ everywhere on $\partial^*\Omega_1$, by definition of the reduced boundary, and get the same conclusion. Anyway, we deduce from (16.28) that for $\mathcal{H}^{n-1}$-almost every $x \in \partial\Omega_1 \cap B_0$,

$$h(x) = \frac{1}{\omega_{n-1}} \lim_{r\to 0} r^{1-n}\mu(B(x,r)). \tag{16.29}$$

Also notice that in (16.5), the contribution of $[f_1 v - \frac{1}{2}g_1]\,\mathbb{1}_{\Omega_1}$ to small balls $B(x,r)$ is negligible compared to r^{n-1}, so (16.29) also says that

$$h(x) = \frac{1}{\omega_{n-1}} \lim_{r\to 0} r^{1-n}\langle \Delta u_{1,+}, \mathbb{1}_{B(x,r)}\rangle, \tag{16.30}$$

where the right-hand side makes sense because $\Delta u_{1,+}$ is a locally finite measure. Let us summarize this and a little bit of the previous results.

Proposition 16.2 *Let $(\mathbf{u}, \mathbf{W})$ be a minimizer for J, assume that the f_i and g_i are bounded (as in (16.1)), that F is Lipschitz (i.e., (10.2) holds), and that W_1 is a good region (i.e. (15.1) holds). Also assume that u_1 is Lipschitz on some open ball B_0. Then, inside B_0, the Laplacian of $\mathbf{u}_{1,+} = \max(u_1, 0)$ can be decomposed as $\Delta u_{1,+} = \mu + [f_1 v - \frac{1}{2}g_1]\,\mathbb{1}_{\Omega_1}$, where μ is a locally Ahlfors-regular measure supported on $\partial\Omega_1$, and in addition $\mu = h\mathcal{H}^{n-1}_{|\partial\Omega_1}$, with a density h that can be computed by (16.28), (16.29), or (16.30).*

The reader may wonder why we find it so interesting to have an expression of $\Delta u_{1,+}$ in terms of its integrals on small balls. In some cases, for instance when we have a good control on the blow-up limits of $(\mathbf{u}, \mathbf{W})$, it will be possible to estimate $\langle \Delta u_{1,+}, \mathbb{1}_{B(x,r)} \rangle$ and compute it. See Section 22.

Notice that in very smooth cases, $\Delta u_{1,+}$ is expected to be the jump of the normal derivative of $u_{1,+}$ along $\partial\Omega_1$, which plays some role in the first variation of our functional J.

Proposition 16.3 *Let $(\mathbf{u}, \mathbf{W})$, W_1 and B_0 satisfy the assumptions of Proposition 16.2. Then $\partial\Omega_1$ is locally uniformly rectifiable in B_0, with big pieces of Lipschitz graphs. This means that there are constants $C_2 \geq 1$ and $r_1 \leq 1$ such that, for each $x \in \partial\Omega_1$ and $0 < r \leq r_1$ such that $B(x, 2r) \subset B_0$, we can find a C_2-Lipschitz graph Γ such that*

$$\mathcal{H}^{n-1}(\Gamma \cap \partial\Omega_1 \cap B(x,r)) \geq C_2^{-1} r^{n-1}. \tag{16.31}$$

The constant C_2 depends on n, the L^∞ bounds in (16.1), the Lipschitz bound in (16.2), the Lipschitz constant in (10.2), and the constant λ in that shows up in (15.1). The radius r_1 depends on these constants, plus the $\varepsilon > 0$ from (15.1).

Of course Proposition 16.3 is a natural complement to Proposition 16.2; we would usually not talk about uniform rectifiability unless $\partial\Omega_1$ is locally Ahlfors-regular. A C_2-Lipschitz graph is just the graph, in some set of orthonormal coordinates of $\mathbb{R}^n$, of a real valued Lipschitz function with a Lipschitz norm at most C_2. We refer to [DS] for more information on uniform rectifiability.

Of course Proposition 16.3 is stronger that (16.26), even though this is not completely obvious from the definition. The proof below does not use regularity properties of sets of finite perimeters, so it could seen as an alternative route to (16.26), but in fact it relies on heavy machinery too. We shall deduce Proposition 16.3 from the following lemma, which says that $\partial\Omega_1$ satisfies the so called Condition B locally in B_0.

Lemma 16.4 *Keep the assumptions of Propositions 16.2 and 16.3. Then there are positive constants c_7, c_8, and $r_2 \leq 1$ such that, for each $x \in \partial\Omega_1$ and $0 < r \leq r_2$ such that $B(x, 2r) \subset B_0$, we can points y_+ and y_- in $B(x,r)$ such that $B(y_+, c_7 r) \subset B(x,r) \cap \Omega_1$ and $B(y_-, c_8 r), \subset B(x,r) \setminus \Omega_1$. The constants c_7 and c_8 depend on the same parameters as announced for C_2 above, and r_2 may also depend on ε.*

Proposition 16.3 is a direct consequence of Lemma 16.4, because any Ahlfors-regular set E that satisfies Condition B contains big pieces of Lipschitz graphs. The condition was introduced by Semmes [Se], who proved that it implies the L^2-boundedness of many singular integral operators on E. The existence of big pieces of Lipschitz graphs were proved slightly later; see [Daa] or [DJ]. Here there is a minor additional detail, which is that the references above concern unbounded Ahlfors-regular sets, and we want a local version of these results. This is not a serious issue; one check that both proofs go through (and we recommend the shorter second one).

So we just need to prove Lemma 16.4. Let $B(x,r)$ be as in the statement; if r_1 is small enough, we can apply Theorem 15.1 to $B(x,r/2)$, and (15.6) says that $\fint_{B(x,r/2)} |u_{1,+}|^2 \geq c_1 r^2$. So we can find $y_+ \in B(x,r/2)$ such that $u_{1,+}(y_+)^2 \geq c_1 r^2/4$. But $u_{1,+}$ is C_0-lipschitz on $B(x,r)$, so $u_{1,+}(z) > 0$ if $z \in B(x,r)$ is such that $|z-y_+| < C_0^{-1}\sqrt{c_1}r/2$; that is, $B(y_+, c_7 r) \subset \Omega_1$ as soon as $c_7 < \frac{1}{2}\min(\sqrt{c_1}C_0^{-1}, 1)$.

We do not have such a good estimate on the other component $\mathbb{R}^n \setminus \overline{\Omega}_1$, but at least Proposition 15.4 says that for $B(x,r)$ as above (hence with r small enough),

$$\big|\big\{z \in B(x,r)\,;\, u_1(z) \leq 0\big\}\big| \geq c_6 r^n. \tag{16.32}$$

It turns out that this is enough: the existence of the points y_- (again for r small enough, and with a constant c_8 which is much smaller than c_7) follows from (16.32) and the existence of the points y_+ by a fairly simple porosity argument. Here is the idea. Suppose that for some pair (x,r) we cannot find y_-; we want to show that the set $H = B(x,r/2) \setminus \Omega_1$ is porous, at least at the scales $\rho \in [c_8 r, r/2]$. Indeed, the fact that we cannot find y_- implies that $\operatorname{dist}(y, \partial\Omega_1) \leq c_8 r$ for each $y \in H$, so we we can choose $z = z(y) \in \partial\Omega_1$ such that $|z-y| \leq c_8 r$, and for each radius $\rho \in [c_8 r, r/2]$, we apply the known existence of points y_+ to the pairs (y,ρ), and get a point $y_+(y,\rho)$ such that $B(y_+(y,\rho), c_7\rho) \subset B(z,\rho) \cap \Omega_1 \subset B(y, 2\rho) \setminus H$. This is what we mean by porous at the scales $\rho \in [c_8 r, r/2]$. It is known that porous sets (at all scales) have vanishing measure, and are even of Hausdorff dimension smaller than n; the proof of this fact also shows that $|H| \leq \eta(c_7, c_8) r^n$, with a function η which for each fixed value of c_7 (the porosity constant) tends to 0 when c_8 tends to 0; then we take c_8 small and contradict (16.32). We shall not include the details of the argument here, because they are the same as in Proposition 10.3 of [DT], for instance, which even concerns a similar situation, and uses fairly close notation. ∎

17 Limits of minimizers

The main point of this section is not to give a general theory of limits, but mostly to allow a description of the blow-up limits of a given minimizer at a point, and give a little more information on the convergence to the blow-up limits. It will be convenient to work with the following notion of local minimizers.

We are given an open set $\mathcal{O}$, where we will work (and an open ball of $\mathbb{R}^n$ is our main example), and a measurable set $\Omega \subset \mathcal{O}$, and we define $\mathcal{F} = \mathcal{F}(\mathcal{O}, \Omega)$ to be the set of pairs $(\mathbf{u}, \mathbf{W})$ such that $\mathbf{u} = (u_1, \dots u_N)$ is a N-uple of functions $u_i \in W^{1,2}_{loc}(\mathcal{O})$, and $\mathbf{W} = (W_1, \dots W_N)$ is a N-uple of disjoint measurable subsets of Ω such that $u_i = 0$ almost everywhere on $\mathcal{O} \setminus W_i$. When we say that $u_i \in W^{1,2}_{loc}(\mathcal{O})$, we just mean that $u_i \in W^{1,2}(B)$ for every ball B such that $\overline{B} \subset \mathcal{O}$.

We are also given a function F, defined on the set $\mathcal{W}(\Omega)$ of N-uples of disjoint measurable subsets of Ω, and measurable functions f_i and g_i, $1 \leq i \leq N$, defined on $\mathcal{O}$ (but only the values on Ω matter). We shall assume that these functions are bounded; lesser assumptions, in particular on the g_i (and then on the $g_{i,k}$ below) would be enough, but usually we shall need these strong ones to check the other assumptions of the Theorem 17.1 anyway.

We say that a pair $(\mathbf{u}^*, \mathbf{W}^*)$ is a competitor for $(\mathbf{u}, \mathbf{W}) \in \mathcal{F}$ in $\mathcal{O}$, relative to Ω, when $(\mathbf{u}^*, \mathbf{W}^*) \in \mathcal{F}$ there is a compact set $K \subset \mathcal{O}$ such that $(\mathbf{u}^*, \mathbf{W}^*)$ coincides with $(\mathbf{u}, \mathbf{W})$ in $\mathcal{O} \setminus K$. Then we say that $(\mathbf{u}, \mathbf{W}) \in \mathcal{F}$ is a local minimizer for J in $\mathcal{O}$, relative to Ω, when

$$(17.1)\quad \int_K |\nabla \mathbf{u}|^2 + \sum_i \int_K [u_i^2 f_i - u_i g_i] + F(\mathbf{W}) \leq \int_K |\nabla \mathbf{u}^*|^2 + \sum_i \int_K [(u_i^*)^2 f_i - u_i^* g_i] + F(\mathbf{W}^*)$$

whenever $(\mathbf{u}^*, \mathbf{W}^*)$ is a competitor for $(\mathbf{u}, \mathbf{W}) \in \mathcal{F}$ in $\mathcal{O}$ and $K \subset \mathcal{O}$ is a compact set such that $(\mathbf{u}^*, \mathbf{W}^*)$ coincides with $(\mathbf{u}, \mathbf{W})$ in $\mathcal{O} \setminus K$. We would have liked to say that $J(\mathbf{u}, \mathbf{W}) \leq J(\mathbf{u}^*, \mathbf{W}^*)$, but these numbers may be infinite, so (17.1) is a good substitute for this.

Local minimizers can be thought of as a substitute of minimizers under a Dirichlet constraint at the boundary of $\mathcal{O}$ (in a strong way, since competitors have to coincide with $(\mathbf{u}, \mathbf{W})$ in a neighborhood of $\partial\mathcal{O}$), where the Dirichlet data is not given in advance, and just comes from the pair $(\mathbf{u}, \mathbf{W})$ itself.

Our definition is not perfect for large sets $\mathcal{O}$, because we decided to use functionals F that may not be local, so we are essentially forced to assume that F is defined globally on $\mathcal{W}(\Omega)$, and finite. For functions F defined by (1.7), we should restrict F to K and rewrite (17.1) accordingly.

Even when $\mathcal{O}$ bounded, the definition allows the possibility that $\int_\Omega |\nabla \mathbf{u}|^2 = +\infty$, and then we should use (17.1). But most often $J(\mathbf{u}, \mathbf{W}) < +\infty$ and we can use the simpler form $J(\mathbf{u}, \mathbf{W}) \leq J(\mathbf{u}^*, \mathbf{W}^*)$.

Notice that the regularity results that we proved so far also hold, locally inside $\mathcal{O}$, for local minimizers for J in $\mathcal{O}$, and with essentially the same proof. Let us say more specifically what we mean by this in the case of Theorem 11.1. We claim that if Ω is smooth, F is Lipschitz, the f_i and the g_i satisfy (10.1) and (10.2), and $(\mathbf{u}, \mathbf{W})$ is a local minimizer for J in $\mathcal{O}$, then for each ball $B(x, r)$ such that $B(x, 3r) \subset \mathcal{O}$, the restriction of $\mathbf{u}$ to $B(x, r)$ is Lipschitz, with bounds that depend only on n, N, the regularity constants for Ω, L^∞ bounds for the f_i and the g_i, r, and initial bounds $\int_{B(x,2r)} |\nabla \mathbf{u}|^2$ and $||\mathbf{u}||_{L^\infty(B(x,2r))}$. The proof just consists in following the proof of Theorem 11.1. Probably, we could even dispense with this last bound on $||\mathbf{u}||_{L^\infty(B(x,r))}$, but this is not the point of the remark.

Let us now set the notation for the next result. We are given an open set $\mathcal{O}$ (for instance, an open ball), a sequence $\{\Omega_k\}_{k \geq 0}$ of measurable subsets of $\mathcal{O}$, sequences $\{f_{i,k}\}$ and $\{g_{i,k}\}$ of bounded functions on $\mathcal{O}$, with $f_{i,k} \geq 0$ on $\mathcal{O}$, and even a sequence of functions F_k defined on the corresponding $\mathcal{W}(\Omega_k)$. We shall assume that there is a constant C_0 such that

$$(17.2)\qquad ||f_{i,k}||_\infty \leq C_0 \ \text{ and } \ ||g_{i,k}||_\infty \leq C_0 \text{ for every } k,$$

and that there exist weak limits f_i and g_i on $\mathcal{O}$, by which we mean that

$$(17.3)\qquad \lim_{k \to +\infty} \int f_{i,k}\varphi = \int f_i \varphi \ \text{ and } \ \lim_{k \to +\infty} \int g_{i,k}\varphi = \int g_i \varphi$$

for every continuous function φ with compact support in $\mathcal{O}$.

We shall also make our life simpler, as in Section 3, and assume that F_k has a simple form for which it will be easy to take limits. We shall start with the case when F_k is coming from a function of the volumes, i.e., when

$$F_k(W_1, \dots, W_N) = \widetilde{F}_k(|W_1|, \dots, |W_N|) \tag{17.4}$$

for some function $F_k : [0, |\mathcal{O}|]^N \to \mathbb{R}$. We shall assume that $|\mathcal{O}| < +\infty$, and that

$$\text{each } \widetilde{F}_k : [0, |\mathcal{O}|]^n \to \mathbb{R} \text{ is continuous, and the } \widetilde{F}_k \text{ converge uniformly to a limit } \widetilde{F}. \tag{17.5}$$

We required $\widetilde{F}_k$ to be defined on $[0, |\mathcal{O}|]^N$ when $[0, |\Omega_k|]^N$ would have been enough, so that the $\widetilde{F}_k$ have a common domain of definition. It costs us very little, because functions $\widetilde{F}_k$ on $[0, |\Omega_k|]^N$ would be easy to extend.

These functions F_k fit with the original motivation of the paper, but we shall also give a statement for functions F_k defined by (1.7), as in the standard setting of Alt, Caffarelli, and Friedman, are possible. See Corollary 17.5 at the end of the section.

For the domains Ω_k, we assume the existence of a measurable set Ω such that

$$\lim_{k\to+\infty} \mathbb{1}_{\Omega_k} = \mathbb{1}_\Omega \text{ in } L^1(\mathcal{O}), \tag{17.6}$$

and the following weak regularity property of Ω, which will be used to approximate Sobolev functions by compactly supported ones. We suppose that for each compact set $K \subset \mathcal{O}$, there exist $r_K > 0$ and $c_K > 0$ such that

$$|B(x, r) \setminus \Omega| \geq c_K r^n \text{ for } x \in K \cap \partial\Omega \text{ and } 0 < r \leq r_K. \tag{17.7}$$

In addition, (17.6) is a little too weak to prevent something that we don't want: the Ω_k may have islands of $\mathcal{O} \setminus \Omega_k$ inside them, with very small masses so that (17.6) does not see it, but which become dense in Ω. If this happens, it could be that the $\mathbf{u}_k$ converge to 0 because they need to vanish on $\mathcal{O} \setminus \Omega_k$, but Ω is a nice ball for which some bump function will do better. So we also assume that for each compact set $K \subset \mathcal{O}$,

$$\lim_{k\to+\infty} \delta(K, k) = 0, \quad \text{where } \delta(K, k) = \sup\big\{ \operatorname{dist}(x, \mathcal{O} \setminus \Omega) \,;\, x \in K \setminus \Omega_k \big\}. \tag{17.8}$$

Since the numbers $\delta(K, k)$ are sensitive to adding small useless pieces to $\partial\Omega_k$, we should probably replace $\mathcal{O} \setminus \Omega$ by the smaller set $Z(k) = \big\{x \in \mathcal{O} \,;\, |\mathcal{O} \cap B(x, r) \setminus \Omega| > 0 \text{ for } r > 0\big\}$ before we check (17.8). It is easy to see that $|(\mathcal{O} \setminus \Omega_k) \setminus Z(k)| = 0$, so Ω_k and $\mathcal{O} \setminus Z(k)$ are equivalent for our functional. We do not need to take this precaution for Ω, because it is already included in (17.7). Finally, if the Ω_k satisfy (17.7) uniformly, then (17.8) follows from (17.6); see the second part of the proof of Lemma 19.1 below.

These three assumptions sound weak, but remember that we shall need to assume more regularity on the Ω_k if we want to make sure that the $\mathbf{u}_k$ are uniformly Lipschitz, as in (17.10).

In addition to the data, we are also given a sequence of pairs $(\mathbf{u}_k, \mathbf{W}_k) \in \mathcal{F}(\mathcal{O}, \Omega_k)$, and we assume that

$$(17.9) \qquad (\mathbf{u}_k, \mathbf{W}_k) \text{ is a local minimizer for } J_k \text{ in } \mathcal{O}, \text{ relative to } \Omega_k,$$

where J_k is the analogue of J, but defined with the data Ω_k, $f_{i,k}$, $g_{i,k}$, and F_k. We assume that for each compact ball $B \subset \mathcal{O}$, there is a constant $C(B)$ such that

$$(17.10) \qquad \mathbf{u}_k \text{ is } C(B)\text{-Lipschitz in } B,$$

and also that there is a function $\mathbf{u} \in W^{1,2}(\mathcal{O})$ such that

$$(17.11) \qquad \mathbf{u}(x) = \lim_{k \to +\infty} \mathbf{u}_k(x) \ \text{ for } x \in \mathcal{O}.$$

In practice, we shall obtain (17.10) by an application of Theorem 11.1, which means that we will have stronger assumptions on the Ω_k, and also explains why we don't try to give weaker assumptions on the g_i, for instance. The existence of a subsequence for which the weak limits in (17.3) and the limit in (17.11) will then be rather easy to get. But again we do not try to give optimal assumptions here.

Theorem 17.1 *Assume all the conditions above. Then we can find* $\mathbf{W} \in \mathcal{W}(\Omega_k)$ *such that* $(\mathbf{u}, \mathbf{W}) \in \mathcal{F}(\mathcal{O}, \Omega)$ *and* $(\mathbf{u}, \mathbf{W})$ *is a local minimizer for* J *in* $\mathcal{O}$*, relative to* Ω*. In addition,*

$$(17.12) \qquad \lim_{k \to +\infty} \mathbf{u}_k = \mathbf{u} \textit{ in } W^{1,2}(B) \textit{ for every ball such that } \overline{B} \subset \mathcal{O}.$$

Proof. The first thing that we need to do is define sets W_i, so that $(\mathbf{u}, \mathbf{W}) \in \mathcal{F}(\mathcal{O}, \Omega)$, and we shall proceed as in Section 3. Because of (17.6), we can find a subsequence $\{k_j\}$ for which

$$(17.13) \qquad \lim_{j \to +\infty} \mathbb{1}_{\Omega_{k_j}}(x) = \mathbb{1}_{\Omega}(x)$$

for almost every $x \in \mathcal{O}$.

Denote by Z the bad set of points $x \in \mathcal{O}$ such that (17.13) fails, or there exists $k \geq 0$ such that $x \in W_{i,k}$ but $u_{i,k}(x) \neq 0$; then $|Z| = 0$ by definitions. Next set

$$(17.14) \qquad W'_{i,k} = \big\{ x \in \Omega_k \setminus Z \,;\, u_{i,k}(x) \neq 0 \big\}$$

for each $k \geq 0$, and

$$(17.15) \qquad W'_i = \big\{ x \in \Omega \setminus Z \,;\, u_i(x) \neq 0 \big\}.$$

Also set $\mathbf{W}' = (W'_1, \ldots, W'_N)$; we want to show that $(\mathbf{u}, \mathbf{W}') \in \mathcal{F}(\mathcal{O}, \Omega)$, and then we will obtain the desired $\mathbf{W}$ by adding an extra piece to some W'_i, when needed. Let us first check that

$$(17.16) \qquad \text{the } W'_i,\ 1 \leq i \leq N, \text{ are disjoint.}$$

If $x \in W'_{i,k}$, then $x \in W_{i,k}$ (by definition of Z); hence the $W'_{i,k}$ are disjoint. Next, if $x \in W'_i$, then $x \in \Omega_{k_j}$ for j large (because (17.13) holds), and $u_{i,k_j}(x) \neq 0$ (by (17.11)), so $x \in W'_{i,k_j}$ for j large. Hence (17.16) holds.

In addition, we just proved that $\mathbb{1}_{W'_i} \leq \liminf_{j\to+\infty} \mathbb{1}_{W'_{i,k_j}}$ everywhere, and by Fatou

$$|W'_i| = \int \mathbb{1}_{W'_i} \leq \liminf_{j\to+\infty} \int \mathbb{1}_{W'_{i,k_j}} = \liminf_{j\to+\infty} |W'_{i,k_j}| \leq \liminf_{j\to+\infty} |W_{i,k_j}|. \tag{17.17}$$

We also need to know that $u_i(x) = 0$ almost everywhere on $\mathcal{O} \setminus W'_i$, and indeed if $u_i(x) \neq 0$ but $x \notin W'_i \cup Z$, then for j large, $u_{i,k_j}(x) \neq 0$, hence $x \in W_{k_j}$ (because $x \notin Z$), so $x \in \Omega_{k_j}$, and then (by (17.13)) $x \in \Omega$, which contradicts the definition (17.15). Notice also that $\mathbf{u} \in W^{1,2}_{loc}(\mathcal{O})$, because $\mathbf{u}$ is locally Lipschitz by (17.13); thus we proved that $(\mathbf{u}, \mathbf{W}') \in \mathcal{F}(\mathcal{O}, \Omega)$.

Observe that although our precise definition of $\mathbf{W}'$ depends on the subsequence $\{k_j\}$, this dependence is only through the set Z; since $|Z| = 0$, different subsequences would yield slightly different, but equivalent sets W'_i.

We may not be happy with the W'_i because there may be a way to increase some of their volumes and make $F(\mathbf{W})$ smaller, so we will replace the W'_i with possibly larger ones. We could try to make $F(\mathbf{W})$ as large as possible (given the natural constraints), but in fact making sure that the volumes of the $W^{i,k}$ go to the limit along a subsequence will be enough for our purposes. Here is the place where we shall use the special form of the F_k.

Let us choose a new subsequence $\{k_j\}$, which we even extract from the previous one, so that

$$l_i = \lim_{j\to+\infty} |W_{i,k_j}| \text{ exists for each } i; \tag{17.18}$$

recall that we assumed that $|\mathcal{O}| < +\infty$. With this new information, (17.17) just says that $|W'_i| \leq l_i$.

Recall that for each k, the $W_{i,k}$ are disjoint and contained in Ω_k (because $(\mathbf{u}_k, \mathbf{W}_k) \in \mathcal{F}(\mathcal{O}, \Omega_k)$); then $\sum_i |W_{i,k_j}| \leq |\Omega_{k_j}|$ for each j and

$$\sum_i |W'_i| \leq \sum_i l_i = \lim_{j\to+\infty} \sum_i |W_{i,k_j}| \leq \liminf_{j\to+\infty} |\Omega_{k_j}| = |\Omega|, \tag{17.19}$$

by (17.6). We claim that we can chose disjoint measurable sets W_i, $1 \leq i \leq N$, so that

$$W'_i \subset W_i \subset \Omega \text{ and } |W_i| = l_i \text{ for } 1 \leq i \leq N. \tag{17.20}$$

Indeed, the W'_i are disjoint and contained in Ω (see (17.15) and (17.16)), so we just need to cut a part of $\Omega \setminus (\cup_i W'_i)$ into pieces, and add them to the W'_i as needed. We could do this with the l_i replaced by any numbers such that $l_i \geq |W'_i|$ and $\sum_i l_i \leq |\Omega|$, but the present choice will be enough.

This is how we define $\mathbf{W} = (W_1, \dots, W_N)$. Notice that

$$(\mathbf{u}, \mathbf{W}) \subset \mathcal{F}(\mathcal{O}, \Omega) \tag{17.21}$$

by construction, and our next task is to show that it is a local minimizer. That is, we are given a competitor $(\mathbf{u}^*, \mathbf{W}^*) \subset \mathcal{F}(\mathcal{O}, \Omega)$ for $(\mathbf{u}, \mathbf{W})$ in $\mathcal{O}$, and we want to prove that (17.1) holds. As usual, the idea is to modify $(\mathbf{u}^*, \mathbf{W}^*)$ into a competitor $(\mathbf{u}_k^*, \mathbf{W}_k^*)$ for $(\mathbf{u}_k, \mathbf{W}_k)$, k large, and use the minimality of $(\mathbf{u}_k, \mathbf{W}_k)$ to get some estimates.

We denote by K_0 a compact subset of $\mathcal{O}$ such that $(\mathbf{u}^*, \mathbf{W}^*)$ coincides with $(\mathbf{u}, \mathbf{W})$ on $\mathcal{O} \setminus K_0$ (as in the definition of competitors). Our construction will depend on four small positive constants, ε_0, ε, δ, and η, that eventually will all tend to 0. Our first action is to replace K_0 with a larger compact set K, with $K_0 \subset K \subset \mathcal{O}$, and so large that

$$|\mathcal{O} \setminus K| \leq \varepsilon_0. \tag{17.22}$$

This is possible, because $|\mathcal{O}| < +\infty$ and by the regularity of the Lebesgue measure, and this will be helpful when we control the volume terms of the functional, because whatever happens in $\mathcal{O} \setminus K$ will not change this term much.

In the estimates that follow, we shall not mark the dependence of the various constants on ε_0 and K, but we will be more careful about ε, δ, and η. We intend to choose ε_0 and K first, then ε, then δ, then η, and our estimates will typically hold as soon as k is large enough, depending on all these constants.

Set $K^\varepsilon = \big\{x \in \mathbb{R}^n\,;\ \mathrm{dist}\,(x, K) \leq \varepsilon\big\}$; we restrict to $\varepsilon > 0$ so small that K^ε is a compact subset of $\mathcal{O}$, and we decide to take

$$\mathbf{u}_k^* = \mathbf{u}_k \text{ and } \mathbf{W}_k^* = \mathbf{W}_k \text{ on } \mathcal{O} \setminus K^\varepsilon. \tag{17.23}$$

In the intermediate region $K^\varepsilon \setminus K$, we want to do two things. First, we want to use a smooth cut-off function φ_ε such that

$$\begin{array}{rll} \varphi_\varepsilon(x) & = 1 & \text{for } x \in K^{\varepsilon/2} \\ \varphi_\varepsilon(x) & = 0 & \text{for } x \in \mathbb{R}^n \setminus K^{2\varepsilon/3} \\ 0 \leq \varphi_\varepsilon(x) & \leq 1 & \text{for } x \in K^\varepsilon \setminus K^{\varepsilon/2} \\ |\nabla \varphi_\varepsilon(x)| & \leq 10\varepsilon^{-1} & \text{everywhere} \end{array} \tag{17.24}$$

to interpolate between $\mathbf{u}_k$ and $\mathbf{u}$. But also, we want to replace $\mathbf{u}$ with a slightly smaller function with coordinates $v_i = h \circ u_i$, where h is a smooth function such that

$$\begin{array}{rcll} h(t) & = & t & \text{for } |t| \geq 2\delta \\ h(t) & = & 0 & \text{for } |t| \leq \delta \\ 0 \leq h'(t) & \leq & 3 & \text{for } |t| \leq 2\delta. \end{array} \tag{17.25}$$

We should observe now that

$$\lim_{k \to +\infty} ||\mathbf{u} - \mathbf{u}_k||_{L^\infty(K^\varepsilon)} = 0, \tag{17.26}$$

i.e., the $\mathbf{u}_k$ converge to $\mathbf{u}$ uniformly on K^ε; indeed, we can cover K^ε by a finite number of compact balls $B \subset \mathcal{O}$, and use (17.10) to get the uniform convergence in each B. Because of (17.26), we can decide to restrict to integers k such that

$$||\mathbf{u} - \mathbf{u}_k||_{L^\infty(K^\varepsilon)} < \delta. \tag{17.27}$$

Then we set

$$u_{k,i}^*(x) = \varphi_\varepsilon(x) h(u_i(x)) + (1-\varphi_\varepsilon(x)) u_{k,i}(x) \text{ for } x \in K^\varepsilon \setminus K \text{ and } 1 \leq i \leq N. \tag{17.28}$$

Let us abuse notation slightly, and rewrite this as

$$\mathbf{u}_k^* = \varphi_\varepsilon\, h \circ \mathbf{u} + (1-\varphi_\varepsilon)\mathbf{u}_k \text{ on } x \in K^\varepsilon \setminus K, \tag{17.29}$$

where we now write $h \circ \mathbf{u}$ for the function whose coordinates are the $h \circ u_i$. Observe that if $x \in K^\varepsilon \setminus K$ is such that $h(u_i(x)) > 0$ for some i, then $|u_i(x)| \geq \delta$, hence $u_{k,i} \neq 0$ (by (17.26)) and, almost surely, $x \in W_{k,i}$. So we may take

$$\mathbf{W}_k^* \cap (K^\varepsilon \setminus K) = \mathbf{W}_k \cap (K^\varepsilon \setminus K), \tag{17.30}$$

and we still get that $\mathbf{u}_{k,i}^*(x) = 0$ almost everywhere on $(K^\varepsilon \setminus K) \setminus W_{k,i}^*$. Let us record the fact that

$$\mathbf{u}_k^* = h \circ \mathbf{u} = h \circ \mathbf{u}^* \text{ on } K^{\varepsilon/2} \setminus K \tag{17.31}$$

because $\varphi_\varepsilon = 1$ there and by definition of $K \supset K_0$.

Next we want to define $\mathbf{u}_k^*$ on K. We would have liked to take $\mathbf{u}_k^* = h \circ \mathbf{u}^*$ (that is, $u_{k,i}^* = h \circ u_i^*$ for $1 \leq i \leq N$), but it could be that $h \circ u^* \neq 0$ somewhere on $\mathcal{O} \setminus \Omega_k$, and this is not allowed. [Recall that we only get from the definitions that $\mathbf{u}^* = 0$ on $\mathcal{O} \setminus \Omega$.] So we will have to kill $h \circ u_i^*$ near $\mathcal{O} \setminus \Omega_k$ with another cut-off function. Let $\eta \in (0, \varepsilon/100)$ be our third small number, and let ψ be a smooth function such that

$$\begin{array}{rcll} \psi(x) & = & 0 & \text{when } \operatorname{dist}(x, \mathcal{O} \setminus \Omega) \leq \eta \\ \psi(x) & = & 1 & \text{when } \operatorname{dist}(x, \mathcal{O} \setminus \Omega) \geq 2\eta \\ 0 \leq \psi(x) & \leq & 1 & \text{when } \eta \leq \operatorname{dist}(x, \mathcal{O} \setminus \Omega) \leq 2\eta \\ |\nabla \psi(x)| & \leq & 2\eta^{-1} & \text{everywhere.} \end{array} \tag{17.32}$$

We intend to take

$$\mathbf{u}_k^*(x) = \psi(x)\, h \circ \mathbf{u}^*(x) \text{ for } x \in K^{\varepsilon/2}, \tag{17.33}$$

with an overlap of domains that will be useful to prove that $W^{1,2}_{loc}(\mathcal{O})$, but then we shall need to check that the two definitions coincide on $K^{\varepsilon/2} \setminus K$. We know that $\mathbf{u}^* = \mathbf{u}$ on that set, by definition of K, so just need to check that $\psi(x)\, h \circ \mathbf{u}(x) = h \circ \mathbf{u}(x)$ on $K^{\varepsilon/2} \setminus K$. When $\operatorname{dist}(x, \mathcal{O} \setminus \Omega) \geq 2\eta$, this is clear because $\psi(x) = 1$. Set

$$H = \left\{ x \in K^{\varepsilon/2} \,;\ \operatorname{dist}(x, \mathcal{O} \setminus \Omega) \leq 2\eta \right\} \tag{17.34}$$

If we show that

$$|\mathbf{u}(x)| \leq \delta \text{ and } h \circ \mathbf{u}(x) = 0 \text{ for } x \in H, \tag{17.35}$$

the second part will give the desired result.

The second part follows from the first one, because $h(t) = 0$ when $|t| \leq \delta$. For the first part, let us back up a little and do a construction that depends only on ε (we just want to avoid any confusion about what our constants depend on). Let Y be a maximal subset of $K^{3\varepsilon/4} \setminus \Omega$ whose points lie at mutual distances larger than $\varepsilon/100$. For each $y \in Y$, the ball $D_y = \overline{B}(y, \varepsilon/10)$ is contained in K^ε, so (17.10) gives a constant C_y such that the $\mathbf{u}_k$, and then $\mathbf{u}$ too, are C_y-Lipschitz on D_y. Denote by C_ε the largest of these numbers; the notation is fair because we can compute C_ε as soon as ε is chosen, and it will not depend on δ and η.

Now let $x \in H$ be given. First assume that $B(x,\eta) \subset \mathcal{O} \setminus \Omega$; then $\mathbf{u} = 0$ almost everywhere on $B(x,\eta)$. Since $x \in K^{3\varepsilon/4} \setminus \Omega$ we can find $y \in Y$ such that $|y - x| \leq \varepsilon/100$, so $B(x,\eta) \subset D_y$ (recall that $\eta \leq \varepsilon/100$), $\mathbf{u}$ is Lipschitz near x, and $\mathbf{u}(x) = 0$. Now suppose that $B(x,\eta)$ meets Ω; since $\operatorname{dist}(x, \mathcal{O} \setminus \Omega) \leq 2\eta$, we can find $z \in \partial\Omega \cap B(x, 3\eta)$. By the weak regularity condition (17.7), applied with a sufficiently small radius r, we can find a set of positive measure A such that $A \subset B(z,\eta) \setminus \Omega$, and hence $\mathbf{u} = 0$ almost everywhere on A. Also, $z \in K^{3\varepsilon/4} \setminus \Omega$ because $\eta \leq \varepsilon/100$, so we can find $y \in Y$ such that $|y - z| \leq \varepsilon/100$, and then A and x both lie in D_y where $\mathbf{u}$ is C_ε-Lipschitz. Pick $w \in A$, with $\mathbf{u}(w) = 0$; then $\mathbf{u}(x) \leq C_\varepsilon |x - w| \leq 4C\varepsilon\eta < \delta$ if η is small enough, depending on δ.

This proves the first part of (17.35); as we saw before, the second part and then the fact that the two definitions (17.33) and (17.31) coincide on $K^\varepsilon \setminus K$ follow.

We now have a complete definition of $\mathbf{u}_k^*$ on $\mathcal{O}$, and let us check that

$$\mathbf{u}_k^* \in W^{1,2}_{loc}(\mathcal{O}). \tag{17.36}$$

By definitions, it is enough to show that for each $x \in \mathcal{O}$ there is a small ball B_x centered at x such that $\mathbf{u}_k^* \in W^{1,2}(B_x)$; the verification would involve covering any open ball B such that $\overline{B} \subset \mathcal{O}$ by a finite number of balls B_x, and then using a partition of the function 1 to compute $\langle \nabla u_k^*, \varphi \rangle = -\int u_k^* \nabla\varphi$ locally in the small balls.

When $x \in \mathcal{O} \setminus K^{2\varepsilon/3}$, we choose $B_x \subset\subset \mathcal{O}$ so that $B_x \subset \mathcal{O} \setminus K^{2\varepsilon/3}$, observe that $\mathbf{u}_k^* = \mathbf{u}^*$ on B_x, either by (17.23) or by (17.29) and the fact that $\varphi_\varepsilon = 0$ on B_x, and just use our assumption that $(\mathbf{u}^*, \mathbf{W}^*) \in \mathcal{F}(\mathcal{O}, \Omega)$ to get that $\mathbf{u}_k^* \in W^{1,2}(B_x)$. When $x \in K^{2\varepsilon/3} \setminus K$, we choose $B_x \subset K^\varepsilon \setminus K$, notice that (17.29) is valid on B_x, observe that the first piece $\varphi_\varepsilon(\cdot) h \circ u(\cdot)$ is even Lipschitz on B_x (because of (17.10) and (17.11)), and get that $\mathbf{u}_k^* \in W^{1,2}(B_x)$ too. When $x \in K$, we use (17.33), observe that $\mathbf{u}^* \in W^{1,2}(B_x)$ by assumption, and then that the composition with the smooth function h with a bounded derivative and then the multiplication with ψ, preserve this (see for instance [Z]). So (17.36) holds.

Next we complete the definition of $\mathbf{W}_k^*$ by taking

$$W_{k,i}^* \cap K = \Omega_k \cap W_i^* \cap K \tag{17.37}$$

and we want to check that

$$(\mathbf{u}_k^*, \mathbf{W}_k^*) \text{ is a competitor for } (u_k, \mathbf{W}_k) \text{ in } \mathcal{O}, \text{ relative to } \Omega_k. \tag{17.38}$$

We already know that $(\mathbf{u}_k^*, \mathbf{W}_k^*)$ coincides with $(u_k, \mathbf{W}_k)$ on $\mathcal{O} \setminus K^\varepsilon$, so we just need to check that $(\mathbf{u}_k^*, \mathbf{W}_k^*) \in \mathcal{F}(\mathcal{O}, \Omega_k)$. The $W_{k,i}^*$ are disjoint: on $\mathcal{O} \setminus K$, this is because the $W_{k,i}$ are disjoint (see (17.23) and (17.30)), and on K we use the fact that the W_i^* are disjoint. Also, $W_{k,i}^* \subset \Omega_k$ (on $\mathcal{O} \setminus K$, use the fact that this is true with the $W_{k,i}$, and on K we forced it in (17.37)). So $\mathbf{W}_k^* \in \mathcal{W}(\Omega_k)$. Since we know that $\mathbf{u}_k^* \in W_{loc}^{1,2}(\mathcal{O})$, we just need to check that

$$\mathbf{u}_{k,i}^*(x) = 0 \text{ almost everywhere on } \mathcal{O} \setminus W_{i,k}^*. \tag{17.39}$$

In $\mathcal{O} \setminus K^\varepsilon$, this comes from (17.23) (because $(\mathbf{u}_k, \mathbf{W}_k) \in \mathcal{F}(\mathcal{O}, \Omega_k)$). We checked this on $K^\varepsilon \setminus K$, just below (17.30); so we are left with $x \in K \setminus W_{i,k}^*$.

If $x \in K \setminus W_i^*$, then almost surely $u_i^*(x) = 0$ (because $(\mathbf{u}^*, \mathbf{W}^*) \in \mathcal{F}(\mathcal{O}, \Omega)$), hence $u_{i,k}(x) = 0$ by (17.33). Otherwise, (17.37) says that $x \in K \cap W_i^* \setminus W_{i,k}^* = K \cap W_i^* \setminus \Omega_k \subset K \cap \Omega \setminus \Omega_k$. By (17.8), $\operatorname{dist}(x, \mathcal{O} \setminus \Omega) \leq \delta(K,k)$, which is less than η if k is large enough (depending on K and η). By (17.32), $\psi(x) = 0$, and by (17.33) $\mathbf{u}_{k,i}^*(x) = 0$. This proves (17.39), and (17.38) follows.

We shall now start comparing our various function; we start with an estimate of $\mathbf{u}_k^* - \mathbf{u}^*$ on K^ε.

Lemma 17.2 *We have that*

$$\int_{K^\varepsilon} |\nabla(\mathbf{u}_k^* - \mathbf{u}^*)|^2 = o(1), \tag{17.40}$$

where by convention $o(1)$ is a number that can be made as small as we want (once ε_0 and K are chosen) by choosing ε, then δ, then η small enough, and taking k large enough.

Proof. We write $\int_{K^\varepsilon} |\nabla(\mathbf{u}_k^* - \mathbf{u}^*)|^2 = A_1 + A_2$, with

$$A_1 = \int_{K^\varepsilon \setminus K} |\nabla(\mathbf{u}_k^* - \mathbf{u}^*)|^2 \text{ and } A_2 = \int_K |\nabla(\mathbf{u}_k^* - \mathbf{u}^*)|^2. \tag{17.41}$$

We start with A_1. Recall from (17.29) that on $K^\varepsilon \setminus K$,

$$\mathbf{u}_k^* - \mathbf{u}^* = \varphi_\varepsilon\, h \circ \mathbf{u} + (1 - \varphi_\varepsilon)\mathbf{u}_k - \mathbf{u}^* = \varphi_\varepsilon[h \circ \mathbf{u} - \mathbf{u}] + (1 - \varphi_\varepsilon)[\mathbf{u}_k - \mathbf{u}] \tag{17.42}$$

because $\mathbf{u}^* = \mathbf{u}$ on $\mathcal{O} \setminus K$ by definition of K. This naturally gives a decomposition of $\nabla(\mathbf{u}_k^* - \mathbf{u}^*)$, and then A_1, into four pieces. We start with

$$A_{1,1} = \int_{K^\varepsilon \setminus K} |\nabla\varphi_\varepsilon|^2 |h \circ \mathbf{u} - \mathbf{u}|^2 \leq \int_{K^\varepsilon \setminus K} (100\varepsilon^{-2})(9\delta^2) = o(1) \tag{17.43}$$

by (17.24) and (17.25), and because we can choose δ small, depending on K and ε. For the next piece, cover K by a finite number of open balls B_i such that $\overline{B}_i \subset \mathcal{O}$, and then apply (17.10) and (17.11); we get that the $\mathbf{u}_k$, and then also $\mathbf{u}$, are Lipschitz on the B_i, with uniform bounds, and so there is a constant C_K such that

$$|\nabla\mathbf{u}_k|^2 + |\nabla\mathbf{u}|^2 \leq C_K \text{ on } K^\varepsilon. \tag{17.44}$$

Our next term is

$$A_{1,2} = \int_{K^\varepsilon \setminus K} \varphi_\varepsilon^2 |\nabla[h \circ \mathbf{u} - \mathbf{u}]|^2 \leq 9 \int_{K^\varepsilon \setminus K} |\nabla \mathbf{u}|^2 \leq 9 C_K |K^\varepsilon \setminus K| = o(1) \tag{17.45}$$

by the chain rule and (17.25), and because the monotone intersection of the $K^\varepsilon \setminus K$, when ε tend to 0, is empty. We continue with

$$A_{1,3} = \int_{K^\varepsilon \setminus K} |\nabla \varphi_\varepsilon|^2 |\mathbf{u}_k - \mathbf{u}|^2 \leq 100 \varepsilon^{-2} \int_{K^\varepsilon \setminus K} |\mathbf{u}_k - \mathbf{u}|^2 \leq 100 \varepsilon^{-2} \delta |K^\varepsilon \setminus K| = o(1) \tag{17.46}$$

by (17.24) and because $||\mathbf{u} - \mathbf{u}_k||_{L^\infty(K^\varepsilon)} < \delta$ by (17.27). Finally,

$$A_{1,4} = \int_{K^\varepsilon \setminus K} (1 - \varphi_\varepsilon)^2 |\nabla (\mathbf{u}_k - \mathbf{u})|^2 \leq 2 \int_{K^\varepsilon \setminus K} |\nabla \mathbf{u}_k|^2 + |\nabla \mathbf{u}|^2 \leq 2 C_K |K^\varepsilon \setminus K| = o(1) \tag{17.47}$$

by (17.44) and as in (17.45).

We are thus left with A_2. On the set K, (17.33) says that

$$\mathbf{u}_k^* - \mathbf{u}^* = \psi\, h \circ \mathbf{u}^* - \mathbf{u}^* = (\psi - 1) h \circ \mathbf{u}^* + [h \circ \mathbf{u}^* - \mathbf{u}^*], \tag{17.48}$$

which gives a natural decomposition of A_2 into three terms. The most interesting one is

$$A_{2,1} = \int_K |\nabla \psi|^2 |h \circ \mathbf{u}^*|^2. \tag{17.49}$$

Notice that $\mathbf{u}^*$ and $h \circ \mathbf{u}^*$ vanish almost everywhere on $\mathcal{O} \setminus \Omega$, because $(\mathbf{u}^*, \mathbf{W}^*) \in \mathcal{F}(\mathcal{O}, \Omega)$, so we only need to integrate on $K \cap \Omega$. In addition, (17.32) says that $\nabla \psi = 0$ unless $\eta \leq \operatorname{dist}(x, \mathcal{O} \setminus \Omega) \leq 2\eta$, so we just need to integrate on

$$H_\eta = \big\{ x \in K \cap \Omega\, ;\ \operatorname{dist}(x, \mathcal{O} \setminus \Omega) \leq 2\eta \big\}. \tag{17.50}$$

Cover H_η by balls B_j of radius 4η centered on H_η and such the B_j have bounded overlap. Obviously

$$\begin{aligned} A_{2,1} &\leq \int_{H_\eta} |\nabla \psi|^2 |h \circ \mathbf{u}^*|^2 \leq 4 \eta^{-2} \int_{H_\eta} |h \circ \mathbf{u}^*|^2 \\ &\leq 4 \eta^{-2} \sum_i \int_{B_j} |h \circ \mathbf{u}^*|^2 \leq 36 \eta^{-2} \sum_i \int_{B_j} |\mathbf{u}^*|^2 \end{aligned} \tag{17.51}$$

by (17.32) and (17.25), and now we shall use the weak regularity assumption (17.7) to estimate $|\mathbf{u}^*|^2$. For each j, $B_j = B(x_j, 4\eta)$ for some $x_j \in H_\eta$. By definition of H_η, $x \in \Omega$ but $\operatorname{dist}(x_j, \mathcal{O} \setminus \Omega) \leq 2\eta$, so we can find $y_j \in \partial\Omega$ such that $|y_j - x_j| \leq 2\eta$. Notice that $y_j \in K^\varepsilon \subset\subset \Omega$ because $x \in K$ and $\eta \leq \varepsilon/100$. If η is small enough (depending on K^ε) we can apply (17.7) to $B(y_j, \eta)$ and get that $|B(y_j, \eta) \setminus \Omega| \geq c_\varepsilon \eta^n$, where c_ε depends on ε through K^ε.

Notice that this is the only place where we seriously use (17.7) (the previous time, we just needed to say that $\mathbf{u}(x) = 0$ on $\partial\Omega$, in a place where $\mathbf{u}$ was in fact Lipschitz).

Anyway, return to $A_{2,1}$ and the analogue of (4.6) to $\mathbf{u}^*$ on B_j, with $E = B_j \cap \Omega$. We get that

$$(17.52) \qquad \int_{B_j} |\mathbf{u}^*|^2 = \int_E |\mathbf{u}^*|^2 \leq C(4\eta)^2 \frac{|B_j|}{|B_j \cap E|} \int_{B_j} |\nabla \mathbf{u}^*|^2 \leq C\eta^2 c_\varepsilon^{-1} \int_{B_j} |\nabla \mathbf{u}^*|^2.$$

We now return to (17.51), use the fact that the B_j have bounded overlap, and get that

$$(17.53) \qquad A_{2,1} \leq C\eta^{-2} \sum_i \int_{B_j} |\mathbf{u}^*|^2 \leq C c_\varepsilon^{-1} \sum_i \int_{B_j} |\nabla \mathbf{u}^*|^2 \leq C c_\varepsilon^{-1} \int_{\cup_j B_j} |\nabla \mathbf{u}^*|^2.$$

Recall that each B_j contains a point $y_j \in \partial\Omega$, and is centered on K; thus $\cup_j B_j \subset Z(\eta)$, with

$$(17.54) \qquad Z(\eta) = \big\{x \in K^{\varepsilon/2}\,;\ \mathrm{dist}\,(x, \partial\Omega) \leq 4\eta\big\}$$

In addition, we claim that $\nabla \mathbf{u}^* = 0$ almost-everywhere on $K^{\varepsilon/2} \cap \partial\Omega$. Indeed, the Rademacher-Calderón theorem says that for almost every $x \in K^{\varepsilon/2} \cap \partial\Omega$, $\mathbf{u}^*$ is differentiable at x, with a differential that coincides with the distributional derivative. It is easy to compute that $\nabla \mathbf{u}^*(x) = 0$ for such an x, because by (17.7) every small ball centered at x contains a fixed proportion of points of $\mathcal{O} \setminus \Omega$, where $\mathbf{u}^* = 0$ almost everywhere because $(\mathbf{u}^*, \mathbf{W}^*) \in \mathcal{F}(\mathcal{O}, \Omega)$. Because of this,

$$(17.55) \qquad A_{2,1} \leq C c_\varepsilon^{-1} \int_{\cup_j B_j} |\nabla \mathbf{u}^*|^2 \leq C c_\varepsilon^{-1} \int_{Z(\eta) \setminus \partial\Omega} |\nabla \mathbf{u}^*|^2 = o(1)$$

because $|\nabla \mathbf{u}^*|^2$ is integrable near $K^{\varepsilon/2}$ and when ε is fixed and η decreases to 0, the set $Z(\eta) \setminus \partial\Omega$ decreases to the empty set.

The next term is

$$(17.56) \qquad A_{2,2} = \int_K (1-\psi)^2 |\nabla(h \circ \mathbf{u}^*)|^2 \leq 9 \int_K (1-\psi)^2 |\nabla \mathbf{u}^*|^2$$

by the chain rule and (17.25). By (17.32), we integrate on the set $\big\{x \in K\,;\ \mathrm{dist}\,(x, \mathcal{O} \setminus \Omega) \leq 2\eta\big\}$, but $\nabla \mathbf{u}^*$ almost everywhere on $\mathcal{O} \setminus \Omega$, because $\mathbf{u}^* = 0$ there, and also almost everywhere on $\mathcal{O} \cap \partial\Omega$, by the same argument as for $A_{2,1}$. Thus we may integrate on the smaller set $\big\{x \in K\,;\, 0 < \mathrm{dist}\,(x, \partial\Omega) \leq 2\eta\big\}$, and $A_{2,2} = o(1)$ by the proof of (17.55). We are left with

$$(17.57) \qquad A_{2,3} = \int_K |\nabla[h \circ \mathbf{u}^* - \mathbf{u}^*]|^2.$$

By (17.25), we only integrate on the set $\big\{x \in K\,;\, |\mathbf{u}^*(x)| \leq 2N\delta\big\}$ (we added N to account for the different coordinates of $\mathbf{u}^*$), and we can even remove the set where $\mathbf{u}^*(x) = 0$, because $\nabla[h \circ \mathbf{u}^* - \mathbf{u}^*] = 0$ almost everywhere on that set. We are left with

$$(17.58) \quad A_{2,3} = \int_{\big\{x \in K\,;\, 0 < |\mathbf{u}^*(x)| \leq N\delta\big\}} |\nabla[h \circ \mathbf{u}^* - \mathbf{u}^*]|^2 \leq 9 \int_{\big\{x \in K\,;\, 0 < |\mathbf{u}^*(x)| \leq N\delta\big\}} |\nabla \mathbf{u}^*|^2 = o(1)$$

by (17.25) and because the intersection, when δ tends to 0, of the domains of integration is empty. This completes our proof of Lemma 17.2. ∎

Our next estimate is on the difference $\mathbf{u}_k^* - \mathbf{u}^*$ itself. We claim that

$$\int_{K^\varepsilon} |\mathbf{u}_k^* - \mathbf{u}^*|^2 = o(1). \tag{17.59}$$

We want the be a little less brutal that we have been so far, with respect to estimates on K^ε. Choose a new constant ε_1, that depends on K but not on ε, δ, or η, so small that $K^{\varepsilon_1} \subset \mathcal{O}$, and observe that $\mathbf{u}$ is bounded on K^{ε_1} (cover K^{ε_1} by a finite collection of balls B such that $2B \subset \mathcal{O}$, and apply (17.10) and (17.11). Then $\mathbf{u}$ is also bounded on K^ε for $\varepsilon \leq \varepsilon_1$, with a bound that does not depend on ε, and by (17.26) we also get that $|\mathbf{u}_k| \leq C$ for k large, with a bound that does not depend on ε, δ, and η. The fact that how large k may depend on these constants does not matter.

We start our proof of (17.59) with the contribution of $K^\varepsilon \setminus K$. On this set, $\mathbf{u}^* = \mathbf{u}$ by definition of K, and $\mathbf{u}_k^*$ is given by (17.29), so it is bounded as well. Thus

$$\int_{K^\varepsilon \setminus K} |\mathbf{u}_k^* - \mathbf{u}^*|^2 \leq C|K^\varepsilon \setminus K| = o(1) \tag{17.60}$$

and we are left with

$$\int_K |\mathbf{u}_k^* - \mathbf{u}^*|^2 \leq 2\int_K \big|(\psi - 1) h \circ \mathbf{u}^*\big|^2 + 2\int_K \big|h \circ \mathbf{u}^* - \mathbf{u}^*\big|^2 \tag{17.61}$$

by (17.33). The second integral is easily estimated, since $|h \circ \mathbf{u}^* - \mathbf{u}^*| \leq C\delta$ everywhere, $|K| < +\infty$, and we can take δ small. For the first one, we observe that it is enough to integrate on Ω (because $h \circ \mathbf{u}^* = \mathbf{u}^* = 0$ on $\mathcal{O} \setminus \Omega$), and even on $H_\eta = \big\{x \in K \cap \Omega\,;\ \mathrm{dist}\,(x, \mathcal{O} \setminus \Omega) \leq 2\eta\big\}$, because otherwise $\psi(x) = 1$ by (17.32). This is the same set as in (17.50), so the second parts of (17.51) and of (17.53), and then (17.55), show that

$$\begin{aligned}
\int_K \big|(\psi - 1) h \circ \mathbf{u}^*\big|^2 &\leq \int_{H_\eta} \big|h \circ \mathbf{u}^*\big|^2 \leq C \sum_i \int_{B_j} |\mathbf{u}^*|^2 \leq C c_\varepsilon^{-1} \eta^2 \int_{\cup_j B_j} |\nabla \mathbf{u}^*|^2 \\
&\leq C c_\varepsilon^{-1} \eta^2 \int_{Z(\eta) \setminus \partial\Omega} |\nabla \mathbf{u}^*|^2 = o(1);
\end{aligned} \tag{17.62}$$

either for the same reason as in (17.55), or because $Z(\eta) \subset K^{\varepsilon/2}$ and we can use the extra η^2 to make the right-hand side small. This completes our proof of the claim (17.59).

We are now ready to take limits. Let us assume that ε is chosen so that the boundary of K^ε has vanishing Lebesgue measure. Since the boundaries $\partial K^\varepsilon = \big\{x\,;\ \mathrm{dist}\,(x, K) = \varepsilon\big\}$, $\varepsilon > 0$, are all disjoint, this is the case for almost every ε. Recall that by (17.10) and (17.11), the $\mathbf{u}_k$ converge to $\mathbf{u}$ uniformly on K^ε. By the lowersemicontinuity of the homogeneous $W^{1,2}$ norm on the interior of K^ε,

$$\int_{K^\varepsilon} |\nabla u_i|^2 = \int_{K^\varepsilon \setminus \partial K^\varepsilon} |\nabla u_i|^2 \leq \liminf_{k \to +\infty} \int_{K^\varepsilon} |\nabla u_{i,k}|^2 \tag{17.63}$$

for each i (recall that we can evaluate these norm by duality with compactly supported smooth functions, and then use the uniform convergence to control integrals like $\int_K u_{i,k}\partial_j\varphi$).

Let us also look at the convergence of the M-terms. Write

$$\Big|\int_{K^\varepsilon} u_i g_i - u_{i,k} g_{i,k}\Big| \leq \Big|\int_{K^\varepsilon} u_i(g_i - g_{i,k})\Big| + \Big|\int_{K^\varepsilon} (u_i - u_{i,k}) g_{i,k}\Big|, \tag{17.64}$$

observe the second term tends to 0 because $u_{i,k}$ tends to u_i uniformly on K and $||g_{i,k}||_\infty \leq C_0$, and the first term tends to 0 by (17.3) and because $||u_i - u_{i,k}||_\infty \leq 2C_0$ by (17.2) and $u_i \mathbb{1}_{K^\varepsilon}$ can be approximated in $L^1(\mathcal{O})$ by continuous compactly supported function φ. Thus

$$\lim_{k\to+\infty} \Big|\int_{K^\varepsilon} u_i g_i - u_{i,k} g_{i,k}\Big| = 0. \tag{17.65}$$

The same argument works for the $\int u_i^2 f_i$ and yields

$$\lim_{k\to+\infty} \Big|\int_{K^\varepsilon} u_i^2 f_i - u_{i,k}^2 f_{i,k}\Big| = 0. \tag{17.66}$$

Set

$$J_-(\mathbf{u}) = \int_{K^\varepsilon} \Big[|\nabla \mathbf{u}|^2 + \sum_i u_i^2 f_i - \sum_i u_i g_i\Big] \tag{17.67}$$

(we leave the volume terms for later); we just checked that

$$J_-(\mathbf{u}) \leq \liminf_{k\to+\infty} J_{k,-}(\mathbf{u}_k), \tag{17.68}$$

where

$$J_{k,-}(\mathbf{u}_k) = \int_{K^\varepsilon} \Big[|\nabla \mathbf{u}_k|^2 + \sum_i u_{i,k}^2 f_{i,k} - \sum_i u_{i,k} g_{i,k}\Big]. \tag{17.69}$$

Next, by the minimizing property of $(\mathbf{u}_k, \mathbf{W}_k)$, (17.38), and (17.23)

$$J_{k,-}(\mathbf{u}_k) \leq J_{k,-}(\mathbf{u}_k^*) - F_k(\mathbf{W}_k) + F_k(\mathbf{W}_k^*). \tag{17.70}$$

We may soon replace $\mathbf{u}_k^*$ with $\mathbf{u}^*$, because

$$\begin{aligned} |J_{k,-}(\mathbf{u}_k^*) - J_{k,-}(\mathbf{u}^*)| &\leq \int_{K^\varepsilon} |\nabla(\mathbf{u}_k^* - \mathbf{u}^*)|^2 + C\int_{K^\varepsilon} \sum_i \big[|u_{k,i}^* - u_i^*| + |(u_{k,i}^*)^2 - (u_i^*)^2|\big] \\ &\leq o(1) + C\int_{K^\varepsilon} |\mathbf{u}_k^* - \mathbf{u}^*|(1 + |\mathbf{u}^*| + |\mathbf{u}_k^* - \mathbf{u}^*|) = o(1) \end{aligned} \tag{17.71}$$

by Lemma 17.2, because $|(u_{k,i}^*)^2 - (u_i^*)^2| = |u_{k,i}^* - u_i^*|\,|u_{k,i}^* + u_i^*|$, and by (17.59) and Cauchy-Schwarz. In turn,

$$J_{k,-}(\mathbf{u}^*) - J_-(\mathbf{u}^*) = \int_{K^\varepsilon} \sum_i \Big[(u_i^*)^2[f_i - f_{i,k}] - u_i^*[g_i - g_{i,k}]\Big] = o(1) \tag{17.72}$$

because only the M-term changes, by the weak limit assumption (17.3), and because $\mathbb{1}_{K^\varepsilon}\mathbf{u}^*$ and $\mathbb{1}_{K^\varepsilon}(\mathbf{u}^*)^2$ can be approximated in $L^1(\mathcal{O})$ by continuous functions with compact support in $\mathcal{O}$ (see the proof of (17.65)).

By (17.68), (17.70), (17.71), and (17.72),

$$J_-(\mathbf{u}) \leq J_-(\mathbf{u}^*) - F_k(\mathbf{W}_k) + F_k(\mathbf{W}_k^*) + o(1), \tag{17.73}$$

and we now need to worry about the F-terms. We want to use the continuity of F, so let us estimate some symmetric differences between sets; for $1 \leq i \leq N$,

$$\begin{aligned} |W_{k,i}^* \Delta W_i^*| &\leq |(\mathbf{W}_{k,i}^* \cap K)\Delta(W_i^* \cap K)| + |\mathcal{O} \setminus K| \\ &= |(\Omega_k \cap W_i^* \cap K)\Delta(W_i^* \cap K)| + |O \setminus K| \\ &\leq |\Omega \setminus \Omega_k| + \varepsilon_0 = o(1) + \varepsilon_0 = o'(1) \end{aligned} \tag{17.74}$$

by (17.37), because $W_i^* \subset \Omega$, by (17.22) and (17.6), and with the convention that $o'(1)$ is a number that can be made as small as we want, by choosing ε_0 small, then choosing K and the other constants, and finally restricting to k large enough.

We are ready to use the special form of F_k in (17.4) for the first time. Recall also that by (17.5), the $\widetilde{F}_k$ converge uniformly to the continuous function $\widetilde{F}$, so that

$$\begin{aligned} F_k(\mathbf{W}_k^*) &= \widetilde{F}_k(|W_{1,k,}^*|, \dots, |W_{N,k}^*|) = \widetilde{F}(|W_{1,k}^*|, \dots, |W_{N,k}^*|) + o(1) \\ &= \widetilde{F}(|W_1^*|, \dots, |W_N^*|) + o'(1) = F(\mathbf{W}^*) + o'(1) \end{aligned} \tag{17.75}$$

by (17.74). We shall restrict our attention to $k = k_j$, where $\{k_j\}$ is the subsequence that we chose to get (17.18) and to define $\mathbf{W}$. We get that

$$\begin{aligned} F_{k_j}(\mathbf{W}_{k_j}) &= \widetilde{F}_{k_j}(|W_{1,k_j}|, \dots, |W_{N,k_j}|) = \widetilde{F}(|W_{1,k_j}|, \dots, |W_{N,k_j}|) + o(1) \\ &= \widetilde{F}(l_1, \dots, l_N) + o(1) = \widetilde{F}(|W_1|, \dots, |W_N|) + o(1) = F(\mathbf{W}) + o(1) \end{aligned} \tag{17.76}$$

by (17.18) and (17.20). Thus (17.73), restricted to the subsequence $\{k_j\}$, yields

$$J_-(\mathbf{u}) \leq J_-(\mathbf{u}^*) - F(\mathbf{W}) + F(\mathbf{W}^*) + o'(1). \tag{17.77}$$

We may now let ε_0, ε, δ, η tend to 0 in the prescribed order, let k_j tend to $+\infty$, and get that $J_-(\mathbf{u}) \leq J_-(\mathbf{u}^*) - F(\mathbf{W}) + F(\mathbf{W}^*)$. This is the same thing as (17.1), with the compact set K^ε (recall the definition (17.67) of J_-, and that the pairs $(\mathbf{u}, \mathbf{W})$ and $(\mathbf{u}^*, \mathbf{W}^*)$ coincide outside of K by definition of K).

We just completed the proof of minimality for our pair $(\mathbf{u}, \mathbf{W})$, but we still need to check the strong limit property in (17.12). Let B be a ball in $\mathcal{O}$, with $\overline{B} \subset \mathcal{O}$, and observe that for each i,

$$\int_{\overline{B}} |\nabla u_i|^2 = \int_B |\nabla u_i|^2 \leq \liminf_{k\to+\infty} \int_B |\nabla u_{i,k}|^2 \tag{17.78}$$

because $\mathbf{u}$ is Lipschitz near $\overline{B}$ and by the lower semicontinuity of $\int_B |\nabla u_i|^2$ (see the proof of (17.63)). Suppose that for some i,

$$\int_{\overline{B}} |\nabla u_i|^2 - \liminf_{k\to+\infty} \int_B |\nabla u_{i,k}|^2 = \alpha > 0, \tag{17.79}$$

and run all the proof above with $(\mathbf{u}^*, \mathbf{W}^*) = (\mathbf{u}, \mathbf{W})$, and $K \supset \overline{B}$. We can improve (17.63) and write instead

$$\begin{aligned}
\int_{K^\varepsilon} |\nabla u_i|^2 &= \int_{K^\varepsilon\setminus\partial K^\varepsilon} |\nabla u_i|^2 = \int_{K^\varepsilon\setminus(\partial K^\varepsilon\cup\overline{B})} |\nabla u_i|^2 + \int_{\overline{B}} |\nabla u_i|^2 \\
&\leq \liminf_{k\to+\infty} \int_{K^\varepsilon\setminus(\partial K^\varepsilon\cup\overline{B})} |\nabla u_{i,k}|^2 + \liminf_{k\to+\infty} \int_B |\nabla u_{i,k}|^2 - \alpha \\
&\leq \liminf_{k\to+\infty} \int_{K^\varepsilon} |\nabla u_{i,k}|^2 - \alpha,
\end{aligned} \tag{17.80}$$

where we have applied our lower semicontinuity estimate on the open set $K^\varepsilon \setminus (\partial K^\varepsilon \cup \overline{B})$.

We follow the rest of the proof, and eventually get an improved version of (17.77), that says that

$$J_-(\mathbf{u}) \leq J_-(\mathbf{u}^*) - \alpha - F(\mathbf{W}) + F(\mathbf{W}^*) + o'(1) = J_-(\mathbf{u}) - \alpha + o'(1) \tag{17.81}$$

(because $(\mathbf{u}^*, \mathbf{W}^*) = (\mathbf{u}, \mathbf{W})$). This is impossible, so $\int_B |\nabla u_i|^2 = \liminf_{k\to+\infty} \int_B |\nabla u_{i,k}|^2$. We already know that $\nabla u_{i,k}$ converges weakly to ∇u_i in $L^2(B)$, and the fact that the norms converge now imply that the convergence is strong. This proves (17.12), and completes our proof of Theorem 17.1. ∎

The following remark is probably more amusing than useful.

Remark 17.3 *In Theorem 17.1, if the volumes $|W_{k,i}|$ do not have a limit for each $i \in [1, N]$, then there are more than one minimizers, associated to the same function $\mathbf{u}$, but with different N-uples $\mathbf{W}$. The values of F for all these $\mathbf{W}$ is the same.*

Let us be more specific. To each k, we associate the N-uple $\mathbf{V}_k = (|W_{1,k}|, \dots, |W_{N,k}|)$ of volumes. If the sequence $\{\mathbf{V}_k\}$ does not have a limit in $[0, |\mathcal{O}|]$, then for each point of accumulation $\mathbf{V}_\infty$ of this sequence, we constructed a minimizer $(\mathbf{u}, \mathbf{W})$ such that $|W_i| = V_{i,\infty}$ for $1 \leq i \leq N$. If $\mathbf{W}^\sharp$ is another N-uple of $\mathcal{W}(\Omega)$ for which $(\mathbf{u}, \mathbf{W}^\sharp)$ is a local minimizer, we can use W to construct competitors for $(\mathbf{u}, \mathbf{W}^\sharp)$ and show that $F(\mathbf{W}) \leq F(\mathbf{W}^\sharp)$: we just keep $\mathbf{W}$ in some very large compact set K, replace $\mathbf{W}$ by $\mathbf{W}^\sharp$ on $\mathcal{O} \setminus K$, keep the same $\mathbf{u}$, notice that this still gives a pair in $\mathcal{F}(\mathcal{O}, \Omega)$, and then let K tend to $\mathcal{O}$. The same argument shows that $F(\mathbf{W}^\sharp) \leq F(\mathbf{W})$ too.

Also recall that we defined sets W_i', $1 \leq i \leq N$. We first chose a sequence $\{k_j\}$ so that the $|W_{i,k_j}|$ converge to a limit l_i, and given a point of accumulation $\mathbf{V}_\infty$ as above, we could easily choose the sequence so that $l_i = V_{i,\infty}$. Then we constructed a pair $(\mathbf{u}, \mathbf{W}') \in \mathcal{F}(\mathcal{O}, \Omega)$, such that $|W_i'| \leq l_i$. Finally we chose $\mathbf{W}$ above so that $W_i \supset W_i'$ and $|W_i| = l_i$, and this was useful to prove that $(\mathbf{u}, \mathbf{W})$ is a local minimizer. but we could have tried other choices, with different measures.

Let $\mathcal{V}$ denote the set of N-uples $\mathbf{V} = (v_1, \ldots, v_N)$ such that $|W_i'| \leq v_i$ for $1 \leq i \leq N$ and $\sum_i v_i \leq |\Omega|$. As was explained below (17.19), for each $\mathbf{V} \in \mathcal{V}$ we can find $\mathbf{W}^\sharp \in \mathcal{W}(\Omega)$ such that $W_i^\sharp = v_i$ and $W_i' \subset W_i^\sharp$ for $1 \leq i \leq N$; then $(\mathbf{u}, \mathbf{W}^\sharp) \in \mathcal{F}(\mathcal{O}, \Omega)$, and the local minimality of $(\mathbf{u}, \mathbf{W})$ shows that $F(\mathbf{W}) \leq F(\mathbf{W}^\sharp)$ (just use $\mathbf{W}$ to produce local competitors for $(\mathbf{u}, \mathbf{W}^\sharp)$, as above. In other words, we get that

$$F(\mathbf{W}) = \widetilde{F}(\mathbf{V}_\infty) \leq \widetilde{F}(\mathbf{V}) \ \text{ for } \mathbf{V} \in \mathcal{V}. \tag{17.82}$$

But recall that the class $\mathcal{V}$ depends on the point of accumulation $\mathbf{V}_\infty$ through the the W_i' and the subsequence $\{k_j\}$.

Remark 17.4 *The conclusion of Theorem 17.1 still holds, with the same proof, if instead of assuming that $(\mathbf{u}_k, \mathbf{W}_k)$ is a local minimizer for J_k in $\mathcal{O}$, as in (17.9), we only assume that this is asymptotically true on compact sets, i.e., that for each compact set $K \subset \mathcal{O}$, there is a sequence $\{\alpha_k\}$ that tends to 0, such that*

$$J_k(\mathbf{u}_k, \mathbf{W}_k) \leq J_k(\mathbf{u}_k^*, \mathbf{W}_k^*) + \alpha_k \tag{17.83}$$

for every competitor $(\mathbf{u}_k^*, \mathbf{W}_k^*)$ for $(\mathbf{u}_k, \mathbf{W}_k)$ in $\mathcal{O}$, relative to Ω_k, which coincides with $(\mathbf{u}_k, \mathbf{W}_k)$ on $\mathcal{O} \setminus K$.

Proof. Indeed nothing changes until (17.70), when we apply the minimality of $(\mathbf{u}_k, \mathbf{W}_k)$. What we get instead is

$$J_{k,-}(\mathbf{u}_k) \leq J_{k,-}(\mathbf{u}_k^*) - F_k(\mathbf{W}_k) + F_k(\mathbf{W}_k^*) + \alpha_k, \tag{17.84}$$

where the sequence $\{\alpha_k\}$ is associated to the compact set K^ε (see (17.23)). Then we continue the argument as above, with an extra term α_k that does not disturb because it tends to 0. ■

We promised to say a few words about the analogue of Theorem 17.1 in the context of Alt, Caffarelli, and Friedman. Suppose, instead of (17.4), that the F_k are given by

$$F_k(W_1, \ldots, W_N) = \sum_{i=1}^{N} \int_{W_i} q_{i,k}(x) dx, \tag{17.85}$$

where the $q_{i,k}$ are nonnegative measurable functions. We shall assume that the $q_{i,k}$ are locally bounded, with uniform estimates. That is, for each compact set $K \subset \mathcal{O}$, we assume that there is a constant $C(K)$ such that

$$0 \leq q_{i,k}(x) \leq C(K) \ \text{ for } k \geq 0,\ 1 \leq i \leq N, \text{ and } x \in K. \tag{17.86}$$

We also suppose that the $q_{i,k}$ converge weakly to locally bounded functions q_i, by which we mean that for each compact set $K \subset \mathcal{O}$ and each measurable set $E \subset K$,

$$\int_E q_i(x)dx = \lim_{k\to+\infty} \int_E q_{i,k}(x)dx. \tag{17.87}$$

We could have chosen compact sets E, or integrated the $q_{i,k}$ and q_i against continuous function with compact support, without changing the notion (because of our L^∞ bounds). We just picked the definition that we shall use.

Of course in many cases, $|\mathcal{O}|$ will be finite, the functions $q_{i,k}$ will be uniformly bounded (regardless of compact support), and the functional F_k defined above will take finite values. In the other cases, (17.85) and its analogue for the q_i may be infinite, but we can define local minimizers as we did before, except that we replace (17.1) by the simpler condition

$$\sum_i \int_{K\cap W_i} |\nabla u_i|^2 + u_i^2 f_i - u_i g_i + q_i \leq \sum_i \int_{K\cap W_i^*} |\nabla u_i^*|^2 + (u_i^*)^2 f_i - u_i^* g_i + q_i. \tag{17.88}$$

That is, we say that $(\mathbf{u}, \mathbf{W})$ is a local minimizer for J in $\mathcal{O}$ if $(\mathbf{u}, \mathbf{W}) \in \mathcal{F}(\mathcal{O}, \Omega)$ and (17.88) holds for every compact set $K \subset \mathcal{O}$ and every pair $(\mathbf{u}^*, \mathbf{W}^*) \in \mathcal{F}(\mathcal{O}, \Omega)$ that coincides with $(\mathbf{u}, \mathbf{W})$ in $\mathcal{O} \setminus K$. We define local minimizers for the J_k similarly, with Ω_k, the $f_{i,k}$, $g_{i,k}$, and $q_{i,k}$.

Corollary 17.5 *Assume that the hypotheses of Theorem 17.1 are satisfied, except that we replace (17.4) and (17.5) with (17.85), (17.86), and (17.87), and we modify the definition of local minimizers as above. Then we can find $\mathbf{W}$ such that $(\mathbf{u}, \mathbf{W}) \in \mathcal{F}(\mathcal{O}, \Omega)$ and $(\mathbf{u}, \mathbf{W})$ is a local minimizer for J in $\mathcal{O}$. In addition, (17.12) holds.*

Our proof will really use the fact that the $q_{i,k}$ are nonnegative (otherwise, we get the problem that we may not extract a sequence for which the $\mathbb{1}_{W_{i,k}}$ converge).

We shall no longer need the assumption that $|\mathcal{O}| < +\infty$. This is an advantage of choosing a local functional F, and the improvement is not a real one, because we can easily reduce to the case when $\mathcal{O}$ is bounded by observing that $(\mathbf{u}, \mathbf{W})$ is a local minimizer for J in $\mathcal{O}$ if and only if its restriction to $\mathcal{O} \cap B(0, R)$ is a local minimizer for J in $\mathcal{O} \cap B(0, R)$ for every $R > 0$. We could not do this trick with our non local functional F (but we could try to use Remark 17.4 in some cases).

Proof. The proof is even simpler. We are happy to take $\mathbf{W} = (W_1', \ldots, W_N')$, where the W_i' are defined near (17.15), because taking larger sets may only make $F(\mathbf{W})$ larger anyway.

Then we repeat the same proof as before, except that we need to rewrite (17.70) and modify the limiting argument (17.75)-(17.76). Instead of (17.70), the minimality of $(\mathbf{u}_k, \mathbf{W}_k)$ is now expressed by

$$J_{k,-}(\mathbf{u}_k) \leq J_{k,-}(\mathbf{u}_k^*) - \sum_i \int_{K^\varepsilon \cap W_{i,k}} q_{i,k}(x)dx + \sum_i \int_{K^\varepsilon \cap W_{i,k}^*} q_{i,k}(x)dx \tag{17.89}$$

(compare with (17.88)). That is, we replace $-F_k(\mathbf{W}_k)+F_k(\mathbf{W}_k^*)$ with the expression

$$-\sum_i \int_{K^\varepsilon\cap W_{i,k}} q_{i,k}(x)dx + \sum_i \int_{K^\varepsilon\cap W_{i,k}^*} q_{i,k}(x)dx \tag{17.90}$$

in (17.70), and then in (17.73).

Instead of (17.75), we restrict to K^ε and say that

$$\begin{aligned}\sum_{i=1}^N \int_{K^\varepsilon\cap W_{i,k}^*} q_{i,k}(x)dx &= \sum_{i=1}^N \int_{K^\varepsilon\cap W_i^*} q_{i,k}(x)dx + o'(1)\\ &= \sum_{i=1}^N \int_{K^\varepsilon\cap W_i^*} q_i(x)dx + o'(1)\end{aligned} \tag{17.91}$$

by (17.74) and (17.86), and then (17.87). For the analogue of (17.76), we use the subsequence $\{k_j\}$ that defines $W_i = W_i'$ and first say that

$$\sum_{i=1}^N \int_{K^\varepsilon\cap W_{i,k_j}} q_{i,k}(x)dx \geq \sum_{i=1}^N \int_{K^\varepsilon\cap W_{i,k_j}\cap W_i} q_{i,k}(x)dx \tag{17.92}$$

because $q_{i,k}(x)\geq 0$. Next observe that

$$\mathbb{1}_{W_i} = \mathbb{1}_{W_i'} \leq \liminf_{j\to+\infty} \mathbb{1}_{W_{i,k_j}'} \leq \liminf_{j\to+\infty} \mathbb{1}_{W_{i,k_j}} \tag{17.93}$$

by the remark above (17.17) and because $W_{i,k_j}' \subset W_{i,k_j}$ (see below (17.16)). So $\mathbb{1}_{W_i} = \lim_{j\to+\infty} \mathbb{1}_{W_{i,k_j}\cap W_i}$ and the right-hand side of (17.92) is

$$\begin{aligned}\sum_{i=1}^N \int_{K^\varepsilon\cap W_{i,k_j}\cap W_i} q_{i,k}(x)dx &= \sum_{i=1}^N \int_{K^\varepsilon\cap W_i} q_{i,k}(x)dx + o(1)\\ &= \sum_{i=1}^N \int_{K^\varepsilon\cap W_i} q_i(x)dx + o(1)\end{aligned} \tag{17.94}$$

by (17.87). We insert (17.91), (17.92), and (17.94) in the expression (17.90) as we did for (17.75) and (17.76), and we get the following replacement (17.77):

$$J_-(\mathbf{u}) \leq J_-(\mathbf{u}^*) - \sum_i \int_{K^\varepsilon\cap W_i} q_i + \sum_i \int_{K^\varepsilon\cap W_i^*} q_i + o'(1). \tag{17.95}$$

This is then enough to conclude as before. ∎

18 Blow-up Limits are minimizers

We shall now apply the results of the previous section to describe blow-up limits of minimizers of our main functional. In this section, we show that they are themselves local minimizers of a simpler functional; in later sections, we shall use previous work in the context of [AC] and [ACF] to describe the blow-up limits more precisely, and then obtain some information on the initial minimizers themselves.

In this section we start from a minimizer $(\mathbf{u}, \mathbf{W})$ for the usual functional J (see Section 1), and choose an origin $x_0 \in \mathbb{R}^n$ where we do the blow up. Here we shall just take $x_0 = 0$ to save some notation. We give ourselves a sequence $\{r_k\}$, with

$$\lim_{k \to +\infty} r_k = 0 \tag{18.1}$$

and consider the pairs $(\mathbf{u}_k, \mathbf{W}_k)$ given by

$$\mathbf{u}_k(x) = r_k^{-1}\mathbf{u}(r_k x) \ \text{ and } \ \mathbf{W}_k = r_k^{-1}\mathbf{W} \tag{18.2}$$

(i.e., $W_{i,k} = r_k^{-1} W_i$ for $1 \leq i \leq N$). Notice the normalization of $\mathbf{u}_k$, which keeps the Lipschitz constants intact.

Let us describe our main assumptions now, that will allow us to apply the results of the previous section. First of all, we assume that

$$\mathbf{u}(0) = 0. \tag{18.3}$$

This is needed because we will require the $u_{k,i}$ to have pointwise limits $u_{\infty,i}$, in particular at the origin, and this will not hurt because otherwise (under weak assumptions that make $\mathbf{u}$ Hölder continuous) only one component W_i is present near 0, and there is no free boundary to study.

We also assume that there is a ball $B(0, \rho_0)$ such that for $1 \leq i \leq N$

$$f_i \geq 0 \text{ on } B(0, \rho_0), \quad \text{and} \quad ||f_i||_{L^\infty(B(0,\rho_0))} + ||g_i||_{L^\infty(B(0,\rho_0))} < +\infty \tag{18.4}$$

and

$$\mathbf{u} \text{ is Lipschitz in } B(0, \rho_0). \tag{18.5}$$

It is just as simple here not to say how we know that (18.5) holds, but recall that we could obtain it by applying Theorem 10.1 (if 0 is an interior point of Ω) or 11.1 (if $0 \in \partial\Omega$).

Next let $R > 0$ be given, and assume that there is a Lipschitz function $\mathbf{u}_\infty$ on $B(0, R)$ such that

$$\mathbf{u}_\infty(x) = \lim_{k \to +\infty} \mathbf{u}_k(x) \ \text{ for } x \in B(0, R). \tag{18.6}$$

Notice that if $\mathbf{u}$ is C_0-Lipschitz on $B(0, \rho_0)$, then $\mathbf{u}_k$ is C_0-Lipschitz on $B(0, R)$ as soon as $r_k \leq R^{-1}\rho_0$. Since $\mathbf{u}(0) = 0$, it is easy to extract subsequences for which (18.6) holds for any given R, or even all integers R at the same time. Because of this, we shall often be able to assume that $\mathbf{u}_\infty$ is defined on the whole $\mathbb{R}^n$ and that $\mathbf{u}_\infty(x) = \lim_{k\to+\infty} \mathbf{u}_k(x)$ for every $x \in \mathbb{R}^n$.

Since $\mathbf{u}_k$ is C_0-Lipschitz on $B(0,R)$ for k large, $\mathbf{u}$ also is Lipschitz, and the convergence in (18.6) is automatically uniform.

In some cases the origin lies on $\partial\Omega$, and then we also need to worry about a limit for the sets

$$\Omega_k = r_k^{-1}\Omega. \tag{18.7}$$

We shall essentially keep the mild assumptions of the previous section. Let us assume that there is a measurable set $\Omega_\infty = \Omega_{\infty,R} \subset B(0,R)$ such that

$$\mathbb{1}_{\Omega_\infty} = \lim_{k\to+\infty} \mathbb{1}_{B(0,R)\cap\Omega_k} = \lim_{k\to+\infty} \mathbb{1}_{B(0,R)\cap r_k^{-1}\Omega} \text{ in } L^1(B(0,R)), \tag{18.8}$$

as in (17.6), that for $0 < T < R$, there exist small constants $r_T > 0$ and $c_T > 0$ such that

$$|B(x,r) \setminus \Omega_\infty| \geq c_T r^n \text{ for } x \in B(0,T) \cap \partial\Omega_\infty \text{ and } 0 < r \leq r_T, \tag{18.9}$$

as in (17.7), and that for $0 < T < R$

$$\lim_{k\to+\infty} \delta(k,T) = 0, \text{ where } \delta(k,T) = \sup\big\{ \operatorname{dist}(x, B(0,R)\setminus\Omega_\infty)\,;\, x \in B(0,T)\setminus\Omega_k \big\}, \tag{18.10}$$

as in (17.8) and to avoid tiny little islands of $B(0,R) \setminus \Omega_k$ in the middle of Ω_k that (18.8) would not detect. Take $\delta(k,T) = 0$ when $B(0,T) \setminus \Omega_k = \emptyset$.

If Ω is very general, we cannot guarantee that we can find sequences for which such an Ω_∞ exists, but fairly weak regularity properties of Ω will ensure all this. We give in Lemma 19.1 below a sufficient condition for this to happen, where we ask $\partial\Omega$ to be porous near 0 (so that we can find a subsequence and Ω_∞ such that (18.8) holds), and Ω to satisfy (18.9) near 0 (so that (18.10) holds for Ω_∞).

In the mean time, notice that if Ω has a C^1 boundary near the origin, we don't even need to extract a subsequence and the limit is a half space. Notice also that the special case when 0 is an interior point of Ω is still included here; the set $\Omega_{\infty,R} = B(0,R)$ satisfies the conditions above because $\Omega_k \supset B(0,r)$ for k large.

Next we describe our assumptions on the functional F. We assume that there exist real constants λ_i, $1 \leq i \leq N$, and a function $\varepsilon(r)$ defined for r small and such that $\lim_{r\to0} \varepsilon(r) = 0$, such that

$$\Big| F(\mathbf{W}) - F(\mathbf{W}') - \sum_{i=1}^{N} \lambda_i \big[|W_i \cap B(0,r)| - |W_i' \cap B(0,r)| \big] \Big| \leq r^n \varepsilon(r) \tag{18.11}$$

for every N-uple $\mathbf{W}' = (W_1', \dots, W_N') \in \mathcal{W}(\Omega)$ such that $W_i \setminus B(0,r) = W_i' \setminus B(0,r)$ for $0 \leq i \leq N$. Recall that $\mathbf{W}$ is the N-uple that comes from our minimizer $(\mathbf{u}, \mathbf{W})$.

Thus, very near 0, we require F to look a lot like the function F defined by (1.7), with constant functions $q_i = \lambda_i$. Slightly surprisingly, we shall be able to content ourselves with the error $r^n\varepsilon(r)$, rather than $\varepsilon(r)|\mathbf{W}\Delta\mathbf{W}'|$ (which would vaguely correspond to requiring a derivative of the special form suggested by (18.11) in some directions.

Let us give two examples where this condition is satisfied. Naturally the first one is when F is given by the formula (1.7), with functions $q_i(x)$ that are continuous at $x = 0$. In fact, it is even enough to take the $q_i(x)$ locally integrable, and to assume that 0 is a Lebesgue point for each q_i, in the sense that

$$\lim_{r\to 0} \fint_{B(0,r)} |q_i(x) - q_i(0)| = 0. \tag{18.12}$$

Indeed, if $\mathbf{W}' = (W_1', \dots, W_N') \in \mathcal{W}(\Omega)$ such that $W_i \setminus B(0,r) = W_i' \setminus B(0,r)$ for $0 \leq i \leq N$,

$$\begin{aligned} F(\mathbf{W}) - F(\mathbf{W}') &= \sum_{i=1}^N \int_{B(x,r)} [\mathbb{1}_{W_i}(x) - \mathbb{1}_{W_i'}(x)] q_i(x) dx \\ &= \sum_{i=1}^N q_i(0) \big[|W_i \cap B(0,r)| - |W_i' \cap B(0,r)|\big] + E, \end{aligned} \tag{18.13}$$

with

$$\begin{aligned} |E| &= \Big|\sum_{i=1}^N \int_{B(x,r)} [\mathbb{1}_{W_i}(x) - \mathbb{1}_{W_i'}(x)](q_i(x) - q_i(0)) dx\Big| \\ &\leq \sum_{i=1}^N \int_{B(x,r)} |q_i(x) - q_i(0)| dx = o(r^n). \end{aligned} \tag{18.14}$$

Thus, (18.11) holds, with $\lambda_i = q_i(0)$. Notice that we do not require the q_i to be nonnegative.

Our second example is when F is a function of the volumes, i.e., $F(\mathbf{Z}) = \widetilde{F}(|Z_1|, \dots, |Z_N|)$ for $\mathbf{Z} \in \mathcal{W}(\Omega)$, with a function $\widetilde{F}$ of N variables which is differentiable at $(|W_1|, \dots, |W_N|)$. That is, assume that near $(|W_1|, \dots, |W_N|)$,

$$\widetilde{F}(v_1, \dots, v_N) - \widetilde{F}(|W_1|, \dots, |W_N|) = \sum_{i=1}^N \lambda_i (v_i - |W_i|) + o\big(\sum_i |v_i - |W_i||\big); \tag{18.15}$$

then (18.11) holds, with the same numbers λ_i, because $\sum_i |v_i - |W_i|| \leq Cr^n$ when $\mathbf{Z} = \mathbf{W}$ outside of $B(0,r)$.

So our last assumption (18.11) is reasonably mild. Associated to Ω_∞ and the λ_i, there is a functional J_∞, defined on $\mathcal{F}(B(0,R), \Omega_\infty)$ by

$$J_\infty(\mathbf{v}, \mathbf{Z}) = \sum_{i=1}^N \int_{B(0,R)} |\nabla v_i|^2 + \sum_{i=1}^N \lambda_i |Z_i|. \tag{18.16}$$

Theorem 18.1 *Let $(\mathbf{u}, \mathbf{W}) \in \mathcal{F}$ be a minimizer for the functional J of Section 1, and let $\{r_k\}$ be a sequence such that $\lim_{k\to+\infty} r_k = 0$. Assume that (18.3)-(18.5) hold for some $R > 0$ and some Lipschitz function $\mathbf{u}_\infty$ on $B(0,R)$, that (18.8)-(18.10) hold for some $\Omega_\infty \subset B(0,R)$, and that F satisfies (18.11). Then we can find $\mathbf{W}_\infty$ such that $(\mathbf{u}_\infty, \mathbf{W}_\infty) \in \mathcal{F}(B(0,R), \Omega_\infty)$ and $(\mathbf{u}_\infty, \mathbf{W}_\infty)$ is a local minimizer for J_∞ in $B(0,R)$. We also have that*

$$\lim_{k\to+\infty} \mathbf{u}_k = \mathbf{u}_\infty \text{ in } W^{1,2}(B(0,R)). \tag{18.17}$$

See near (17.1) for the definition of $\mathcal{F}(B(0,R), \Omega_\infty)$ and local minimizers for J_∞; in the present case, J_∞ has no M-term and $\int_{B(0,R)} |\nabla U_\infty| < +\infty$, so (17.1) can be rewritten as

$$J_\infty(\mathbf{u}_\infty, \mathbf{W}_\infty) \leq J_\infty(\mathbf{v}, \mathbf{Z}), \tag{18.18}$$

and the local minimality of $(\mathbf{u}_\infty, \mathbf{W}_\infty)$ means that (18.18) holds for every pair $(\mathbf{v}, \mathbf{Z}) \in \mathcal{F}(B(0,R), \Omega_\infty)$ that coincides with $(\mathbf{u}_\infty, \mathbf{W}_\infty)$ on some $B(0,R) \setminus B(0,r)$, $r < 1$.

See Corollary 18.3 for the more natural rephrasing of this theorem with $R = +\infty$.

Proof. Naturally we want to deduce this from Theorem 17.1 and Remark 17.4. Ironically, we cannot use Corollary 17.5, even when F is given by (1.7), because we assumed that the q_i are nonnegative there. We first compute the functional J_k of which the pair $(\mathbf{u}_k, \mathbf{W}_k)$ defined in (18.2) is a minimizer.

Lemma 18.2 *If $(\mathbf{u}, \mathbf{W}) \in \mathcal{F}$ is a minimizer for J in the class $\mathcal{F}(\Omega)$ of Definition 1.1, then the pair $(\mathbf{u}_k, \mathbf{W}_k)$ is a minimizer for J_k in $\mathcal{F}(\Omega_k)$, where $\Omega_k = r_k^{-1}\Omega$ and J_k is defined like J in (1.5), except that we use the functions $f_{i,k}$ and $g_{i,k}$ given by*

$$f_{i,k}(x) = r_k^2 f_i(r_k x) \quad \text{and} \quad g_{i,k}(x) = r_k g_i(r_k x) \tag{18.19}$$

and the functional F_k defined by

$$F_k(H_1, \ldots, H_N) = r_k^{-n}\, F(r_k H_1, \ldots, r_k H_N) =: r_k^{-n}\, F(r_k \mathbf{H}). \tag{18.20}$$

Proof. Notice the extra powers in (18.19), which come from the scaling and the fact that we base our normalization of J_k on the energy. For a general pair $(\mathbf{u}^*, \mathbf{W}^*)$, define $\mathbf{u}_k^*$ and $\mathbf{W}_k^*$ as we did for $\mathbf{u}_k$ and $\mathbf{W}_k$ (in (18.2)). It is clear that $(\mathbf{u}^*, \mathbf{W}^*) \in \mathcal{F}(\Omega)$ if and only if $(\mathbf{u}_k^*, \mathbf{W}_k^*)$ lies in $\mathcal{F}(\Omega_k)$, and that $F_k(\mathbf{W}_k^*)$ is well defined by (18.20) (because $r_k\mathbf{H} \in \mathcal{W}(\Omega)$). Let us show that

$$J_k(\mathbf{u}_k^*, \mathbf{W}_k^*) = r_k^{-n} J(\mathbf{u}^*, \mathbf{W}^*); \tag{18.21}$$

the conclusion will easily follow. For the last term, this is just a consequence of the definition, since $F_k(\mathbf{W}_k^*) = r_k^{-n} F(\mathbf{W}^*)$ by (18.20). For the energy, we just change variables and get that

$$\begin{aligned} E(\mathbf{u}_k^*) &= \int |\nabla u_k^*|^2 = \int |\nabla[r_k^{-1}\mathbf{u}^*(r_k\,\cdot)]|^2 \\ &= \int |(\nabla \mathbf{u}^*)(r_k x)|^2 dx = r_k^{-n} \int |\nabla \mathbf{u}^*|^2 = r_k^{-n} E(\mathbf{u}^*). \end{aligned} \tag{18.22}$$

The M-term of the functional is computed similarly:

$$\begin{aligned} M_k(\mathbf{u}_k^*) &=: \sum_{i=1}^N \int [u_{i,k}^*(x)^2 f_{i,k}(x) - u_{i,k}^*(x) g_{i,k}(x)] dx \\ &= \sum_{i=1}^N \int [u_i^*(r_k x)^2 f_i(r_k x) - u_i^*(r_k x) g_i(r_k x)] dx = r_k^{-n} M(\mathbf{u}^*) \end{aligned} \tag{18.23}$$

because $u_{i,k}^*(x) = r_k^{-1} u_i(r_k x)$ by (18.2), and by (18.19). Hence (18.21) holds and the lemma follows. ∎

We return to the proof of Theorem 18.1. Let $R > 0$ be as in the statement. We want to apply Theorem 17.1 and Remark 17.4 to the restriction to $\mathcal{O} = B(0,R)$ of our sequence of minimizers $(\mathbf{u}_k, \mathbf{W}_k)$. We use the domains $\Omega_k \cap \mathcal{O}$ as our measurable subsets of $\mathcal{O}$, and just the fixed J_∞ as our variable functional; of course the $(\mathbf{u}_k, \mathbf{W}_k)$ will only be approximate local minimizers for J_∞, and this is why we shall use Remark 17.4. Then (17.2) and (17.3) are trivial (we take $f_i = f_{i_k} = g_i = g_{i,k} = 0$), (17.4) and (17.5) hold with $\widetilde{F}_k(v_1, \dots, v_N) = \sum_i \lambda_i v_i$ and $\widetilde{F} = \widetilde{F}_k$, (17.6) is just a translation of (18.8), (17.7) is the same as (18.9), and (17.8) follows from (18.10).

We will need to replace (17.9) by (17.83) but the uniform Lipschitz bound in (17.10) holds by (18.5), and (17.11) follows from (18.6). This completes our list of easy verifications; as soon as we check (17.83), Remark 17.4 will say that the conclusion of Theorem 17.1 holds for the limit function $\mathbf{u}_\infty$, and Theorem 18.1 will follow. Notice that the conclusion (17.12) looks a little weaker than (18.17), but here $\mathbf{u}_\infty$ is Lipschitz on $B(0,R)$ (by (18.5) and (18.6)), so $\lim_{k\to+\infty} \mathbf{u}_k = \mathbf{u}_\infty$ in $W^{1,2}(B(0,R))$ as soon as this happens in $B(0,T)$ for every $T < R$.

So we check (17.83), which means that we take a competitor $(\mathbf{v}, \mathbf{H})$ (previously called $(\mathbf{u}_k^*, \mathbf{W}_k^*)$) for $(\mathbf{u}_k, \mathbf{W}_k)$ in $\mathcal{O} = B(0,R)$, relative to $\Omega_k \cap \mathcal{O}$, and we want to show that

$$J_\infty(\mathbf{u}_k, \mathbf{W}_k) \leq J_\infty(\mathbf{v}, \mathbf{H}) + \alpha_k, \tag{18.24}$$

for some α_k that tends to 0; we shall not need the dependence on the compact set K here.

Notice that the pair $(\mathbf{u}_k, \mathbf{W}_k)$ was actually defined on the larger set $\mathbb{R}^n$, but $(\mathbf{v}, \mathbf{H}) \in \mathcal{F}(B(0,R), \Omega_k)$ is only defined on $B(0,R)$. This is easy to fix: set $\mathbf{v}(x) = \mathbf{u}_k(x)$ on $\mathbb{R}^n \setminus B(0,R)$ and define $\mathbf{H}'$ by $H_i' = H_i \cup (W_{i,k} \setminus B(0,R)$ for $1 \leq i \leq N$. It is easy to see that $(\mathbf{v}, \mathbf{H}') \in \mathcal{F}(\Omega_k) = \mathcal{F}(\mathbb{R}^n, \Omega_k)$; in particular, the fact that $\mathbf{v} \in W^{1,2}(\mathbb{R}^n)$ is trivial, because $\mathbf{v}$ coincides with $\mathbf{u}_k$ in $B(0,R) \setminus B(0,T)$ for some $T < R$, which gives enough room to glue $v \in W^{1,2}_{loc}(B(0,R)$ with $\mathbf{u} \in W^{1,2}(\mathbb{R}^n)$. Notice also that $(\mathbf{v}, \mathbf{H}')$ is a competitor for $(\mathbf{u}_k, \mathbf{W}_k)$, which implies that

$$J_k(\mathbf{u}_k, \mathbf{W}_k) \leq J_k(\mathbf{v}, \mathbf{H}'). \tag{18.25}$$

Recall from Lemma 18.2 that

$$J_k(\mathbf{v}, \mathbf{H}') = \int_{\Omega_k} |\nabla \mathbf{v}|^2 + M_k(\mathbf{v}) + r_k^{-n} F(r_k \mathbf{H}), \tag{18.26}$$

where

$$M_k(\mathbf{v}) = \sum_{i=1}^N \int_{\Omega_k} [v_i(x)^2 f_{i,k}(x) - v_i(x) g_{i,k}(x)] dx \tag{18.27}$$

Let us remove the large constant

$$A_k = \int_{\Omega_k \setminus B(0,R)} |\nabla \mathbf{u}_k|^2 + \int_{\Omega_k \setminus B(0,R)} [u_{i,k}(x)^2 f_{i,k}(x) - u_{i,k}(x) g_{i,k}(x)] dx + r_k^{-n} F(\mathbf{W}) \tag{18.28}$$

from this; we get that

$$J_k(\mathbf{v}, \mathbf{H}') - A_k = \int_{B(0,R)} |\nabla \mathbf{v}|^2 + M_k'(\mathbf{v}) + r_k^{-n} \big(F(r_k \mathbf{H}) - F(\mathbf{W})\big), \tag{18.29}$$

where

$$M_k'(\mathbf{v}) = \sum_{i=1}^N \int_{B(0,R)} [v_i(x)^2 f_{i,k}(x) - v_i(x) g_{i,k}(x)] dx. \tag{18.30}$$

Also, the special case when $(\mathbf{v}, \mathbf{H}') = (\mathbf{u}_k, \mathbf{W}_k)$ yields

$$J_k(\mathbf{u}_k, \mathbf{W}_k) - A_k = \int_{B(0,R)} |\nabla \mathbf{u}_k|^2 + M_k'(\mathbf{u}_k) \tag{18.31}$$

because $r_k \mathbf{W}_k = \mathbf{W}$. We now estimate various terms. First observe that for $x \in B(0,R)$,

$$|u_k(x)| = r_k^{-1} |\mathbf{u}(r_k x)| = r_k^{-1} |\mathbf{u}(r_k x) - \mathbf{u}(0)| \leq C \tag{18.32}$$

by (18.2), (18.3), and (18.5); similarly, $\mathbf{u}_k$ is C-Lipschitz on $B(0,R)$, so we easily get that

$$|M_k'(\mathbf{u}_k)| \leq C |B(0,R)| (||f_{i,k}||_\infty + ||g_{i,k}||_\infty) \leq C r_k \tag{18.33}$$

by (18.19) and (18.4), and with a constant C that does not depend on k.

Notice that we may assume that $J_\infty(\mathbf{v}, \mathbf{H}) \leq J_\infty(\mathbf{u}_k, \mathbf{W}_k)$, because otherwise (18.24) is satisfied for any $\alpha_k \geq 0$. This yields

$$\int_{B(0,R)} |\nabla \mathbf{v}|^2 \leq J_\infty(\mathbf{v}, \mathbf{H}) + C \leq J_\infty(\mathbf{u}_k, \mathbf{W}_k) + C \int_{B(0,R)} |\nabla \mathbf{u}_k|^2 + 2C \leq C' \tag{18.34}$$

where our constants C and C' depend on the λ_i and R, but again not on k, and we used again the fact that $\mathbf{u}_k$ is C-Lipschitz. Recall also that the Sobolev function $\mathbf{v} - \mathbf{u}_k$ vanishes outside of $B(0,R)$, so the Poincaré inequality (3.7) yields

$$\int_{B(0,R} |\mathbf{v} - \mathbf{u}_k|^2 \leq C R^2 \int_{B(0,R} |\nabla (\mathbf{v} - u_k|^2 \leq C \tag{18.35}$$

hence $\int_{B(0,R} |\mathbf{v}|^2 \leq C$, by (18.32). Thus

$$|M_k'(\mathbf{v})| \leq (||f_{i,k}||_\infty + ||g_{i,k}||_\infty) \int_{B(0,R} (|\mathbf{v}| + |\mathbf{v}|^2) \leq C r_k. \tag{18.36}$$

Finally notice that $\mathbf{H}'$ coincides with $\mathbf{W}_k$ outside of $B(0,R)$, which implies that $r_k\mathbf{H}' = r_k\mathbf{W}_k = \mathbf{W}$ outside of $B(0, r_k R)$. So we can apply (18.11), and we get that

$$r_k^{-n}\big(F(r_k\mathbf{H}') - F(\mathbf{W})\big) = r_k^{-n}\Big\{\sum_i \lambda_i\big[|r_k H_i' \cap B(0,r_kR)| - |W_i \cap B(0,r_kR)|\big] + error\Big\} \tag{18.37}$$

with $|error| \leq (r_kR)^n \varepsilon(r_kR)$. Since $W_i \cap B(0,r_kR) = r_k(W_{i,k} \cap B(0,R))$ and $r_kH_i' \cap B(0,r_kR) = r_k(H_i' \cap B(0,R)) = r_kH_i$ by definition of $\mathbf{H}'$, (18.37) is the same as

$$r_k^{-n}\big(F(r_k\mathbf{H}') - F(\mathbf{W})\big) = \sum_i \lambda_i\big[|H_i| - |W_{i,k} \cap B(0,R)|\big] + r_k^{-n} error. \tag{18.38}$$

We may now put things together:

$$\begin{aligned} J_\infty(\mathbf{u}_k, \mathbf{W}_k) - J_\infty(\mathbf{v}, \mathbf{H}) &= \int_{B(0,R)} |\nabla \mathbf{u}_k|^2 - |\nabla \mathbf{v}|^2 - \sum_{i=1}^N \lambda_i\big[|H_i| - |W_{i,k} \cap B(0,R)|\big] \\ &= \int_{B(0,R)} |\nabla \mathbf{u}_k|^2 - |\nabla \mathbf{v}|^2 - r_k^{-n}\big(F(r_k\mathbf{H}') - F(\mathbf{W})\big) + r_k^{-n} error \\ &= J_k(\mathbf{u}_k, \mathbf{W}_k) - J_k(\mathbf{v}, \mathbf{H}') - M_k'(\mathbf{u}_k) + M_k'(\mathbf{v}) + r_k^{-n} error \\ &\leq -M_k'(\mathbf{u}_k) + M_k'(\mathbf{v}) + r_k^{-n} error \leq C r_k + \varepsilon(r_kR) \end{aligned} \tag{18.39}$$

by (18.16), (18.29), (18.31), (18.25), (18.33), and (18.36). This proves (18.24), and we know that Theorem 18.1 follows. ∎

Usually it does not hurt to treat all the balls $B(0,R)$ at the same time, because extracting a sequence from an original sequence $\{r_k\}$ so that (18.6) holds for every $R > 0$ is just as easy, if we know that $\mathbf{u}$ is Lipschitz near 0 (as in (18.5)), as getting it for a single R. Here is the corresponding statement.

Corollary 18.3 *Let $(\mathbf{u}, \mathbf{W}) \in \mathcal{F}$ and the sequence $\{r_k\}$ satisfy the assumptions of Theorem 18.1 for each integer $R > 0$. Then the $\mathbf{u}_k$ converge uniformly on compact subsets of $\mathbb{R}^n$ to a Lipschitz function $\mathbf{u}_\infty : \mathbb{R}^n \to \mathbb{R}^N$, there is a measurable set $\Omega_\infty \subset \mathbb{R}^n$ such that (18.8) holds for every $R > 0$, and we can find disjoint sets $W_{i,\infty} \subset \Omega_\infty$ with the following property. Set $\mathbf{W}_\infty = (W_{1,\infty}, \ldots, W_{N,\infty})$; then $(\mathbf{u}_\infty, \mathbf{W}_\infty)$ lies in $\mathcal{F}(\mathbb{R}^n, \Omega_\infty)$ and is a local minimizer for J_∞ in $\mathbb{R}^n$. In addition, (18.17) holds for all $R > 0$.*

See the beginning of Section 17 for the definition $\mathcal{F}(\mathbb{R}^n, \Omega_\infty)$. The functional is still defined as in (18.16), except that we would now integrate on $\mathbb{R}^n$; so, when we say that

$(\mathbf{u}_\infty, \mathbf{W}_\infty)$ is a local minimizer for J_∞ in $\mathbb{R}^n$, we mean as in (17.1), or rather (17.88) that if $(\mathbf{v}, \mathbf{H}) \in \mathcal{F}(\mathbb{R}^n, \Omega_\infty)$ coincides with $(\mathbf{u}_\infty, \mathbf{W}_\infty)$ outside of a ball $B(0,R)$, then

$$(18.40) \qquad \int_{B(0,R)} |\nabla u_{i,\infty}|^2 + \sum_{i=1}^N \lambda_i |W_{i,\infty} \cap B(0,R)| \leq \int_{B(0,R)} |\nabla v_i|^2 + \sum_{i=1}^N \lambda_i |H_i \cap B(0,R)|.$$

We did not require directly the existence of a limit $\mathbf{u}_\infty$ that works for all R, but it is easy to see that if the $\mathbf{u}_k$ converge in $B(0,R)$ for every integer R, the limit does not depend on R. This is our function $\mathbf{u}_\infty$, and the fact that the convergence is uniform on compact subsets of $\mathbb{R}^n$ follows easily from the fact that for each R, the $\mathbf{u}_k$, k large enough, are uniformly Lipschitz by (18.5).

Proof. Compared to Theorem 18.1, there is just a small amount of new information here. We already discussed the fact that the limit of the $\mathbf{u}_k(x)$ does not depend on the radius R, and that this limit is uniform on compact sets. We also need to say that, modulo sets of measure 0, the limit set Ω_∞ on (18.8) does not depend on R either. It could be that if we are too clumsy with the gluing of the various Ω_∞, the condition (18.9) does not hold any more (on the whole $\mathbb{R}^n$). We do not need this to prove the corollary, and we could probably easily fix this problem if it became an issue for some other question.

Finally, we also need to show that we can find a fixed $\mathbf{W}_\infty$ that works for all R. One way we can do this is to modify slightly our choice of $\mathbf{W}$ (starting a little above (17.18)). Indeed the proof of (17.17) also yields that for $A_R = B(0,R+1) \setminus B(0,R)$, $R \in \mathbb{N}$,

$$(18.41) \qquad |W_i' \cap A_R| \leq \liminf_{j \to +\infty} |W_{i,k_j} \cap A_R|.$$

Then we extract a new subsequence $\{k_j\}$, so that for each integer R, the limits $l_{i,R} = \lim_{j\to+\infty} |W_{i,k_j} \cap A_R|$ exist (as in (17.17)), we still have an analogue of (17.20) in each annulus, and this allows us to complete the W_i', independently on each annulus, so that we have a stronger form of (17.20) with $|W_i \cap A_R| = l_{i,R}$. With this construction, the restriction of $\mathbf{W}$ to a given ball does not depend on R.

We could also use the fact that J_∞ has a special form to replace directly $\mathbf{W}'$ with an optimal choice, where we keep $W_i = W_i'$ for all i, except perhaps for one i_0 for which λ_{i_0} is the smallest. If $\lambda_{i_0} \geq 0$ we still keep $W_{i_0} = W_{i_0}'$, but otherwise we take $W_{i_0} = W_{i_0}' \cap \big[\Omega_\infty \setminus \big(\cup_{i \neq i_0} W_i'\big)\big]$. This candidate works at least as well as our old one, and it is easy to see that it does not depend on R (if we always choose the same i_0). ∎

19 Blow-up limits with 2 phases

Our interest in the blow-up limits of our minimizers $(\mathbf{u}, \mathbf{W})$ comes from the fact that they should be simpler to study, and their description should still provide useful information on $(\mathbf{u}, \mathbf{W})$. We also intend to use the fact that since our blow-up limits are given by a more standard Alt, Caffarelli, and Friedman formula, they were studied intensively and we can use some of the results.

In this Section we keep the same assumptions as in the previous one, add a minor regularity property for Ω to make sure that we can apply Corollary 18.3, and study the blow-up limits of $\mathbf{u}$ when we can find two different choices of pairs (i, ε), where $i \in [1, N]$ and ε is a sign, such that the functional $\Phi(r)$ associated in Section 9 to the two functions $(\varepsilon u_i)_+$ has a nonzero limit when r tends to 0.

We will see that in this case, all the blow-up limits $\mathbf{u}_\infty$ of $\mathbf{u}$ at the origin are composed of just two non-trivial affine functions defined on the two components of the complement of some hyperplane H, and which vanish on H; the other components of $\mathbf{u}_\infty$ are null.

It will follow that 0 lies in the interior of Ω, and that the other components of $\mathbf{u}$ are small near the origin. See Corollary 19.4.

We will also see that in this case the natural free boundaries associated to $\mathbf{u}$ stay quite close to hyperplanes in the small balls $B(0, r)$. See Corollary 19.5. This is not such an impressive regularity result, but we can get it without nondegeneracy assumptions like (15.1), or the size of the $q_i(0)$ when F is given by (1.7) with functions q_i that are nearly continuous at 0 (as in (18.12)).

The main assumptions for this section are almost the same as for Corollary 18.3. We consider a minimizer $(\mathbf{u}, \mathbf{W})$ for J and a sequence $\{r_k\}$ that tend to 0, we suppose that the f_i are nonnegative and bounded and the g_i are bounded (as in (18.4)), that $\mathbf{u}(0) = 0$, $\mathbf{u}$ is Lipschitz near 0, and the $\mathbf{u}_k$ defined by (18.2) converge pointwise (or uniformly on compact sets, this is the same) to a function $\mathbf{u}_\infty$, as in (18.3), (18.5), and (18.6), that we assume for all $R > 0$.

We also systematically assume that the volume functional F satisfies the regularity condition (18.11).

When we want information for a specific blow-up limit of $(\mathbf{u}, \mathbf{W})$ at the origin, we shall assume that that for each $R > 0$, there is a limit $\Omega_\infty = \Omega_{\infty,R}$, as in (18.8) and (18.10), and that satisfies the weak regularity assumption (18.9). Then Corollary 18.3 will give a limiting domain Ω_∞ (that does not depend on R, but this is not a surprise), and a N-uple $\mathbf{W}_\infty$ such that $(\mathbf{u}_\infty, \mathbf{W}_\infty)$ is a local minimizer for the functional J_∞ in $\mathbb{R}^n$, relative to Ω_∞ (i.e., in the class $\mathcal{F}(\mathbb{R}^n, \Omega_\infty)$). We recall what this means near (18.40).

But for Corollary 19.4 we will need to know that for each sequence $\{r_k\}$ that tends to 0, we can find a subsequence that satisfies the assumptions above. There is no problem with the convergence of $\mathbf{u}_k$, since $\mathbf{u}$ is Lipschitz near 0, but if Ω is too ugly near the origin, it may be hard to find the $\Omega_{\infty,R}$. For instance, the functions $\mathbf{1}_{\Omega_k}$ may converge weakly to the constant 1/2. Let us give a condition which will prevent that.

Lemma 19.1 *Suppose that there is a radius $\rho > 0$ and a constant $\tau > 0$ such that, for $x \in \partial\Omega \cap B(0, \rho)$ and $0 < r \leq \rho$,*

$$|B(x, r) \setminus \Omega| \geq \tau r^n \tag{19.1}$$

and we can find $y \in B(x, r)$ such that

$$\text{dist}\,(y, \partial\Omega) \geq \tau r. \tag{19.2}$$

Then for each sequence $\{r_k\}$ that tends to 0, we can find a subsequence and a measurable set Ω_∞ for which (18.8)-(18.10) hold for each $R > 0$.

It is amusing that we do not need to know on which side of $\partial\Omega$ the point y lies. Of course if we assume that $\operatorname{dist}(y, \Omega) \geq \tau r$, we get (19.1) for free.

We included the lemma mostly for fun, and the the reader that would not be convinced can skip and assume that Ω is a Lipschitz (or even C^1) domain near the origin; then the existence of a good subsequence is really easy.

Proof. First we take a subsequence (which as usual we still denote by $\{r_k\}$) for which the boundaries $\partial\Omega_k$ of converge to a closed set Z locally in $\mathbb{R}^n$, for the Hausdorff distance. This just means that for each $R > 0$, the numbers d_R tend to 0, where

$$d_R = \sup\big\{\operatorname{dist}(z, \partial\Omega_k)\,;\, z \in Z\big\} + \sup\big\{\operatorname{dist}(w, Z)\,;\, w \in \partial\Omega_k\big\}, \tag{19.3}$$

and the existence of such a subsequence comes from the standard compactness property of the Hausdorff distance.

Then we show that Z is porous. Let $z \in Z$ and $r > 0$ be given. Choose R such that $B(z, r) \subset B(0, R)$, and then k so large that $d_{2R} \leq \tau r/10$ and $r_k R < \rho$. Pick $w \in \partial\Omega_k$ such that $|w - z| \leq \tau t/10$, and apply our second assumption to $x = r_k w \in \partial\Omega \cap B(0, \rho)$ and the radius $r_k r < \rho$; this gives a point $y \in B(x, r_k r)$ such that $\operatorname{dist}(y, \partial\Omega) \geq \tau r_k r$. Then $y' = r_k^{-1} y$ lies in $B(w, r) \subset B(z, 2r)$ and $\operatorname{dist}(y', \partial\Omega_k) \geq \tau r$. Since $d_{2R} \leq \tau r/10$, we get that $\operatorname{dist}(y', Z) \geq \tau r/2$. Thus every ball $B(z, 2r)$ centered on Z contains a ball of radius $\tau r/2$ that does not meet Z, our definition of porous.

It follows that $|Z| = 0$ (if Z is porous, it cannot have a Lebesgue point of density). We now need to say which part of $\mathbb{R}^n \setminus Z$ lies in Ω_∞ and which part lies in $\mathbb{R}^n \setminus \Omega_\infty$, and for this we shall extract a subsequence again. Let $\{y_j\}$, $j \geq 0$, be a dense sequence in $\mathbb{R}^n \setminus Z$, and cover $\mathbb{R}^n \setminus Z$ by the balls $B_j = B(y_j, \operatorname{dist}(y_j, Z)/2)$. For each $j \geq 0$, we define a sequence $\{m_{j,k}\}$, $k \geq 0$. Set $m_{j,k} = 1$ when $B_j \subset \Omega_k$, $m_{j,k} = -1$ when $B_j \subset \mathbb{R}^n \setminus \Omega_k$, and $m_{j,k} = 0$ otherwise. Since $\operatorname{dist}(B_j, Z) > 0$, we get that $B_j \cap \partial\Omega_k$ for k large, so $m_{j,k} \neq 0$ for k large. We extract our new subsequence so that $\{m_{j,k}\}$ has a limit l_j for each j (which is therefore either 1 or -1). Then we set $\Omega_\infty = \bigcup_{j \geq 0; l_j = 1} B_j$ and $\Omega^\sharp = \bigcup_{j \geq 0; l_j = -1} B_j$. Let us check that

$$\mathbb{R}^n \setminus Z \text{ is the disjoint union of } \Omega_\infty \text{ and } \Omega^\sharp. \tag{19.4}$$

Both sets are contained in $\mathbb{R}^n \setminus Z$ because $B_j \subset \mathbb{R}^n \setminus Z$ for $j \geq 0$. If $x \in \mathbb{R}^n \setminus Z$, then $x \in B_j$ for some j, and then x lies in the corresponding set. But if $x \in B_j \cap B_i$, then for k large, B_i and B_j are contained in Ω_k or $\mathbb{R}^n \setminus \Omega_k$, and this has to be the same set for both balls (the set that contains x). So $l_i = l_j$, and our two sets are disjoint. So (19.4) holds.

Now we have a candidate Ω_∞, and we just need to check (18.8)-(18.10). We start with Let $R > 0$ and $\varepsilon > 0$ be given, and choose a compact set $K \subset B(0, R) \setminus Z$ such that $|K| \geq |B(0, R) \setminus Z| - \varepsilon = |B(0, R)| - \varepsilon$. Cover K by a finite number of balls B_j, $j \in J$. Notice that for k large, B_j is either contained in Ω_k (if $l_j = 1$) or in $\mathbb{R}^n \setminus \Omega_k$ (if $l_j = -1$). Now for each $x \in K$, choose $j \in J$ such that $x \in B_j$; if $l_j = 1$, then $x \in \Omega_k \cap \Omega_\infty$. If $l_j = -1$, then $x \in (\mathbb{R}^n \setminus \Omega_k) \cap \Omega^\sharp \subset (\mathbb{R}^n \setminus \Omega_k) \cap (\mathbb{R}^n \setminus \Omega_\infty)$ (by (19.4)). That is $K \cap \Omega_\infty = K \cap \Omega_k$, and hence $||\mathbb{1}_{\Omega_\infty} - \mathbb{1}_{\Omega_k}||_{L^1(B(0,R))} \leq |B(0,R) \setminus K| \leq \varepsilon$. This holds for k large; (18.8) follows.

Next we want to deduce from (18.8) and (19.1) that

$$|B(x,r) \setminus \Omega_\infty| \geq \tau 2^{-n} r^n \text{ for } x \in \partial\Omega_\infty \text{ and } r > 0; \tag{19.5}$$

obviously (18.9) will follow (we even get some additional scale invariance).

Let $x \in \partial\Omega_\infty$ and $r > 0$ be given. Observe that $x \in Z$, because otherwise x would lie in one of the open balls B_i, which cannot meet $\partial\Omega_\infty$ because they are contained in Ω_∞ or in $\Omega^\sharp \subset \mathbb{R}^n \setminus \Omega_\infty$. For k large, we can find $w \in \partial\Omega_k$ such that $|w - x| \leq r/2$. Then (19.1), applied to the ball $B(r_k w, r_k r/2)$ (which is contained in $B(0,\rho)$ for k large), says that

$$|B(x,r) \setminus \Omega_k| \geq |B(w,r/2) \setminus \Omega_k| = r_k^{-n} |B(r_k w, r_k r/2) \setminus \Omega| \geq \tau 2^{-n} r^n. \tag{19.6}$$

But (18.8) says that $|B(x,r) \setminus \Omega_\infty| = \lim_{k\to+\infty} |B(x,r) \setminus \Omega_k|$ so we take a limit in (19.6) and get that $|B(x,r) \cap \setminus\Omega_\infty| \geq \tau 2^{-n} r^n$; (19.5) and (18.9) follow.

For (18.10) we fix $0 < T < R$ and $\varepsilon > 0$, and show that for k large, $\delta(k,T) := \sup\big\{ \operatorname{dist}(x, B(0,R) \setminus \Omega_\infty)\, ;\, x \in B(0,T) \setminus \Omega_k \big\}, \leq \varepsilon$. We may safely assume that $\varepsilon < R - T$. Pick $x \in B(0,T) \setminus \Omega_k$, and first assume that $\operatorname{dist}(x, \Omega_k) \leq \varepsilon/2$. Then $\operatorname{dist}(x, \partial\Omega_k) \leq \varepsilon/2$ too, and if k is large enough (depending on ε and R), we also get that $\operatorname{dist}(x, \partial\Omega_k) \leq 2\varepsilon/3$, because Z is the limit of the $\partial\Omega_k$. By (19.5) (applied to a small ball centered on $Z \cap B(x,\varepsilon)$, Ω_∞ meets $B(x,\varepsilon)$, and $\operatorname{dist}(x, B(0,R) \setminus \Omega_\infty) < \varepsilon$, because $x \in B(0,T)$ and $\varepsilon < R - T$.

When $\operatorname{dist}(x, \Omega_k) \geq \varepsilon/2$, we just say that

$$\begin{aligned} |\Omega_\infty \cap B(x,\varepsilon/2)| &\geq |\Omega_k \cap B(x,\varepsilon/2)| - ||\mathbb{1}_{\Omega_\infty} - \mathbb{1}_{\Omega_k}||_{L^1(B(0,R))} \\ &= |B(x,\varepsilon/2)| - ||\mathbb{1}_{\Omega_\infty} - \mathbb{1}_{\Omega_k}||_{L^1(B(0,R))} > 0 \end{aligned} \tag{19.7}$$

if k is small enough (again depending on ε and R), then $B(x,\varepsilon/2)$ meets Ω_∞ and we can conclude as above. So $\delta(k,T) \leq \varepsilon$, and (18.10) follows.

Notice that our proof of (18.8) only uses the second condition (19.2), and that our proof of (18.9) and (18.10) only uses the first one and (18.8). ∎

Return to the general case (without the assumptions of the lemma).

When $\mathbf{u}_\infty$ is the pointwise limit of a sequence $\{\mathbf{u}_k\}$ as above (and this includes the existence, for $R > 0$, of a limit Ω_∞ such that (18.8)-(18.10) hold) we shall say that it is a <u>regular blow-up limit of</u> $\mathbf{u}$ (at the origin, associated to the sequence $\{r_k\}$); by a slight abuse of notation (due to the fact that $\mathbf{W}_\infty$ is not really determined by the sequence $\{r_k\}$), we also say that $(\mathbf{u}_\infty, \mathbf{W}_\infty)$ is a <u>regular blow-up limit</u> of $(\mathbf{u}, \mathbf{W})$. The reader should not worry, we shall not use this terminology too much; we just want to insist on the fact that Lemma 19.2 below, for instance, only works for regular blow-up limits.

Our first result says that under the (other) assumptions above, the functionals Φ introduced in Section 9 are monotone, and their normalized versions for the $\mathbf{u}_k$ go to the limit, so that their analogues for $\mathbf{u}_\infty$ are constant on $(0, +\infty)$, with the value $L = \lim_{r\to 0} \Phi(r)$. We shall then study more carefully the case when $L > 0$, which is somewhat easier.

Let us use again, and even expand slightly, the ugly notation of Section 9. The point is that in the discussion below, the main objects will not be the functions u_i themselves, but their positive or negative part. Denote by I the set of pairs $\varphi = (i, \varepsilon)$, where $i \in [1, N]$ as usual, and $\varepsilon \in \{-1, +1\}$ is a sign. For $\varphi \in I$, we define the function v_φ by

$$v_\varphi(x) = [\varepsilon u_i(x)]_+ = \max(0, \varepsilon u_i(x)) \in [0, +\infty) \text{ for } x \in \mathbb{R}^n. \tag{19.8}$$

We shall often refer φ as a phase, and to v_φ as a phase of $\mathbf{u}$. Hopefully, the reader will not be too disturbed by this additional notion, but otherwise the trick that we used for Lemma 10.2 can be applied here, to reduce to the situation where all the u_i are required to be nonnegative, and the notion of phase is useless (i.e., we could take $I = [1, N]$). Incidentally, the reader may also want to remove from I the phases $\varphi = (i, \varepsilon)$ for which we demanded that $\varepsilon u_i \leq 0$ in the definition of $\mathcal{F}(\Omega)$ (the class of admissible pairs). These phases will not really disturb, but they are useless because $v_\varphi = 0$.

We want to take blow-up limits, so we set

$$v_{\varphi,k}(x) = r_k^{-1} v_\varphi(r_k x) = [\varepsilon u_{i,k}(x)]_+ \tag{19.9}$$

for $k \geq 0$ and $x \in \mathbb{R}^n$ and, corresponding to $k = +\infty$,

$$v_{\varphi,\infty} = [\varepsilon u_{i,\infty}]_+. \tag{19.10}$$

We shall now denote by Φ^0_φ the function that we defined in (9.4) in terms of v_φ, with $x_0 = 0$. That is, we set

$$\Phi^0_\varphi(r) = \frac{1}{r^2} \int_{B(0,r)} \frac{|\nabla v_\varphi|^2}{|x|^{n-2}}\, dx \text{ for } r > 0. \tag{19.11}$$

Notice that by (18.5) (our Lipschitz condition),

$$\Phi^0_\varphi(r) \leq \frac{1}{r^2} \int_{B(0,r)} \frac{C}{|x|^{n-2}}\, dx \leq C' \text{ for } 0 < r < \rho_0. \tag{19.12}$$

Also define the analogue of Φ^0_φ for $k \geq 0$ and $k = \infty$, by

$$\Phi_{\varphi,k}(r) = \frac{1}{r^2} \int_{B(0,r)} \frac{|\nabla v_{\varphi,k}|^2}{|x|^{n-2}}\, dx. \tag{19.13}$$

Lemma 19.2 *For each $r > 0$,*

$$\Phi_{\varphi,\infty}(r) = \lim_{k \to +\infty} \Phi_{\varphi,k}(r) = \lim_{k \to +\infty} \Phi^0_\varphi(r_k r). \tag{19.14}$$

Proof. Fix $r > 0$ and for each $\varepsilon > 0$, cut an integral as

$$\begin{aligned} r^2|\Phi_{\varphi,k}(r) - \Phi_{\varphi,\infty}(r)| &= \int_{B(0,r)} \frac{\big||\nabla v_{\varphi,k}|^2 - |\nabla v_{\varphi,\infty}|^2\big|}{|x|^{n-2}}\, dx \\ &\leq \int_{x\in B(0,\eta r)} \frac{|\nabla v_{\varphi,k}|^2 + |\nabla v_{\varphi,\infty}|^2}{|x|^{n-2}} + (\eta r)^{2-n} \int_{B(0,r)\setminus B(0,\eta r)} \big||\nabla v_{\varphi,k}|^2 - |\nabla v_{\varphi,\infty}|^2\big| \\ &\leq C \int_{x\in B(0,\eta r)} \frac{1}{|x|^{n-2}} + (\eta r)^{2-n} \int_{B(0,r)\setminus B(0,\eta r)} \big||\nabla v_{\varphi,k}|^2 - |\nabla v_{\varphi,\infty}|^2\big| \end{aligned} \tag{19.15}$$

because our functions are uniformly Lipschitz. The first term can be made as small as we want, by choosing η small, and then the second term also, because (18.17) holds for every $R > 0$, and says that $\nabla v_{\varphi,k}$ converges to $\nabla v_{\varphi,\infty}$ in $L^2(B(0,r))$. This is where we use the fact that $\mathbf{u}_\infty$ is a regular blow-up limit. This proves the first part of (19.14).

For the second part we just compute. By (19.9), (18.2), and (9.3), $v_{\varphi,k}(x) = r_k^{-1} v_\varphi(r_k x)$, so $\nabla v_{\varphi,k}(x) = \nabla v_\varphi(r_k x)$ and hence (setting $y = r_k x$ in the integral)

$$\Phi_{\varphi,k}(r) = \frac{1}{r^2}\int_{B(0,r)} \frac{|\nabla v_{\varphi,k}(x)|^2}{|x|^{n-2}}\, dx = \frac{1}{r^2}\int_{B(0,r_k r)} \frac{|\nabla v_\varphi(y)|^2}{|r_k^{-1}y|^{n-2}}\, r_k^{-n} dy = \Phi^0_\varphi(r_k r) \tag{19.16}$$

for $k \geq 0$ and $r > 0$. Lemma 19.2 follows. ∎

We now come to the nearly monotone functional of Section 9. Pick two different indices $\varphi_1, \varphi_2 \in I$, and consider the product of the corresponding functions Φ_φ. Our function Φ from (9.4) is now called

$$\Phi^0_{\varphi_1,\varphi_2} = \Phi^0_{\varphi_1}\Phi^0_{\varphi_2} \tag{19.17}$$

and we also set, for $k \geq 0$ and $k = \infty$

$$\Phi_{\varphi_1,\varphi_2,k} = \Phi_{\varphi_1,k}\Phi_{\varphi_2,k}. \tag{19.18}$$

It was observed in the proof of Theorem 9.1 that the functions v_{φ_1} and v_{φ_2} satisfy the assumptions of Theorem 1.3 in [CJK], and this was even the main ingredient in the proof. Now we also know that $\mathbf{u}(0) = 0$ (by (18.3)), and since $\mathbf{u}$ is Lipschitz near the origin (by (18.5)), the additional size assumption in Theorem 1.6 of [CJK] is satisfied. Then that theorem says that the limit

$$L(\varphi_1, \varphi_2) = \lim_{\rho\to 0} \Phi^0_{\varphi_1,\varphi_2}(\rho) = \lim_{\rho\to 0} \Phi^0_{\varphi_1}(\rho)\Phi^0_{\varphi_2}(\rho) \text{ exists.} \tag{19.19}$$

It follows from Lemma 19.2 that the function $\Phi_{\varphi_1,\varphi_2,\infty}$ of (19.18) is constant, with

$$\Phi_{\varphi_1,\varphi_2,\infty}(r) = L(\varphi_1, \varphi_2) \text{ for } r > 0. \tag{19.20}$$

In this section we shall concentrate on the case when the limit $L(\varphi_1, \varphi_2)$ is positive for some choice of phases $\varphi_1 \neq \varphi_2$. This will be made easier, because the following theorem gives a very good description of $\mathbf{u}_\infty$ when this happens. For this theorem we forget a little minimizers and blow-up limits, and concentrate on pairs of harmonic functions with essentially disjoint supports.

Theorem 19.3 *Let v_1 and v_2 be two nonnegative Lipschitz functions on $\mathbb{R}^n$ such that $v_1 v_2 = 0$ everywhere, each v_j is harmonic on the open set $\mathcal{O}_j = \{x \in \mathbb{R}^n\,;\, v_j(x) > 0\}$, and there is a constant $L > 0$ such that*

$$\Phi_1(r)\Phi_2(r) = L \quad \text{for } r > 0, \tag{19.21}$$

where we set

$$\Phi_j(r) = \frac{1}{r^2}\int_{B(0,r)} \frac{|\nabla v_j|^2}{|x|^{n-2}}. \tag{19.22}$$

Then there is a unit vector $e \in \mathbb{R}^n$, and two positive constants a_1 and a_2, such that

$$v_1(x) = a_1 \max(0, \langle x, e\rangle) \text{ and } v_2(x) = a_2 \max(0, \langle x, -e\rangle) \text{ for } x \in \mathbb{R}^n. \tag{19.23}$$

Of course we shall use this when the two v_j are phases of our blow-up limit $\mathbf{u}_\infty$; they are harmonic because they minimize $\int |\nabla v_i|^2$ locally; see the proof of (9.6) with $f_i = g_i = 0$.

Proof. The fact that $\Phi_1(r)\Phi_2(r)$ is nondecreasing when v_1 and v_2 are nonnegative Lipschitz functions such that $v_1 v_2 = 0$ and v_i is harmonic on $\mathcal{O}_i$ was proved in [ACF], and we shall follow their proof (Lemma 5.1 in [ACF]), see where equality occurs in the argument, and try to conclude from there. So we need to recall how the proof of [ACF] goes.

We start with some notation. Set $\rho(x) = |x|$ for $x \in \mathbb{R}^n$, and

$$A_j(r) = r^2\Phi_j(r) = \int_{B(0,r)} \frac{|\nabla v_j(x)|^2}{|x|^{n-2}}\,dx = \int_{B(0,r)} \rho^{2-n}|\nabla v_j|^2\,dx \tag{19.24}$$

for $j = 1, 2$. Set $B_r = B(0,r)$ and $S_r = \partial B(0,r)$, and denote by σ the surface measure on S_r.

Notice that $S_r \cap \mathcal{O}_j$ is not empty, because otherwise $v_j = 0$ on S_r, hence also on B_r (by the maximum principle), and then $\Phi_1(r)\Phi_2(r) = 0$ for r small; we excluded this in (19.21). Now let $r^2\alpha_j(r) \in (0, +\infty]$ be the square root of the Sobolev constant on $S_r \cap \mathcal{O}_j$, i.e., the smallest constant such that

$$\int_{S_r\cap\mathcal{O}_j} |u|^2 d\sigma \leq r^2\alpha_j(r)\int_{S_r\cap\mathcal{O}_j} |\nabla_t u|^2 d\sigma \tag{19.25}$$

for a function $u \in W^{1,2}(S_r)$ with compact support in $S_r \cap \mathcal{O}_j$, and where ∇_t denotes the gradient on the sphere. This is the same thing as (5.4) in [ACF], except that we intend to keep the dependence on r apparent, and we normalize $\alpha_j(r)$ so that it is dimensionless. Notice that $\alpha_j(r) < +\infty$ because the other domain meets S_r, so there is a nontrivial Sobolev inequality. Naturally we want to apply (19.25) to the restriction of v_j to S_r, which in our case even lies in $W^{1,2}(S_r)$ for every r because it is Lipschitz. Of course it is not compactly supported in $S_r \cap \mathcal{O}_j$, but this is easy to fix, because v_j is easy to approximate by such functions (for instance, try $u_\varepsilon = [v_j - \varepsilon]_+$), so we get (19.25) for v_j as well.

The proof will use the fact that

$$\begin{aligned} 2\int_{B_r} \rho^{2-n}|\nabla v_j|^2 &\leq 2r^{2-n}\int_{S_r} v_j \frac{\partial v_j}{d\rho} + (n-2)r^{1-n}\int_{S_r} v_j^2 \\ &= r^{3-n}\int_{S_r}\Big[2\frac{v_j}{r}\frac{\partial v_j}{d\rho} + (n-2)\frac{v_j^2}{r^2}\Big] \end{aligned} \tag{19.26}$$

(see (5.2) in [ACF]). The reader may be surprised that this is only an inequality, but this is because its proof uses the fact that $\Delta v_i \geq 0$ (as a distribution). Also, we only get it for almost every r because some limit of integrals on thin annuli near S_r is taken.

So we consider $\Phi_1(r)\Phi_2(r) = r^{-4}A_1(r)A_2(r)$ and differentiate it. Here we do not even care that $\Phi_1\Phi_2$ is the integral of its derivative; we know that the derivative is zero, and we just need to compute it at almost every r. We get that

$$0 = -4r^{-5}A_1(r)A_2(r) + r^{-4}A_1'(r)A_2(r) + r^{-4}A_1(r)A_2'(r) \tag{19.27}$$

with,

$$A_j'(r) = \int_{S_r} \rho^{2-n}|\nabla v_j|^2 = r^{2-n}\int_{S_r}|\nabla v_j|^2 \tag{19.28}$$

by (4.3). We divide by $r^{-5}A_1(r)A_2(r) = r^{-1}L \neq 0$ and get that

$$\frac{rA_1'(r)}{A_1(r)} + \frac{rA_2'(r)}{A_2(r)} = 4. \tag{19.29}$$

Then we evaluate each $\frac{rA_j'(r)}{A_j(r)}$. We choose numbers $\beta_j(r) \in (0,1)$ such that

$$\frac{1-\beta_j(r)^2}{\alpha_j(r)} = (n-2)\frac{\beta_j(r)}{\sqrt{\alpha_j(r)}} \tag{19.30}$$

(see (5.4) in [ACF]; there is only one solution $\beta_j(r) \in (0,1)$, which is even computed seven lines later in [ACF], but this is not the point yet). We write that

$$\frac{2\beta_j(r)}{\sqrt{\alpha_j(r)}}\int_{S_r}\frac{v_j}{r}\frac{\partial v_j}{d\rho} \leq \frac{2\beta_j(r)}{\sqrt{\alpha_j(r)}}\Big\{\int_{S_r}\frac{v_j^2}{r^2}\Big\}^{1/2}\Big\{\int_{S_r}\Big|\frac{\partial v_j}{d\rho}\Big|^2\Big\}^{1/2} \tag{19.31}$$

by Cauchy-Schwarz, then use (19.25) to get that

$$\frac{\beta_j(r)}{\sqrt{\alpha_j(r)}}\Big\{\int_{S_r}\frac{v_j^2}{r^2}\Big\}^{1/2} \leq \Big\{\int_{S_r}\beta_j(r)^2|\nabla_t v_j|^2\Big\}^{1/2}, \tag{19.32}$$

and then use the fact that $2AB = A^2 + B^2$, with

$$A = \Big\{\int_{S_r}\Big|\frac{\partial v_j}{d\rho}\Big|^2\Big\}^{1/2} \text{ and } B = \Big\{\int_{S_r}\beta_j(r)^2|\nabla_t v_j|^2\Big\}^{1/2}, \tag{19.33}$$

to get that

$$(19.34) \qquad \frac{2\beta_j(r)}{\sqrt{\alpha_j(r)}} \int_{S_r} \frac{v_j}{r} \frac{\partial v_j}{d\rho} \leq 2AB \leq A^2 + B^2 = \int_{S_r} \Big|\frac{\partial v_j}{d\rho}\Big|^2 + \beta_j(r)^2 |\nabla_t v_j|^2.$$

We use (19.25) a second time, to say that

$$(19.35) \qquad \frac{1-\beta_j(r)^2}{\alpha_j(r)} \int_{S_r} \frac{v_j^2}{r^2} \leq \int_{S_r} (1-\beta_j(r)^2)|\nabla_t v_j|^2,$$

and then we add this to (19.34) to get that

$$(19.36) \qquad \frac{2\beta_j(r)}{\sqrt{\alpha_j(r)}} \int_{S_r} \frac{v_j}{r} \frac{\partial v_j}{d\rho} + \frac{1-\beta_j(r)^2}{\alpha_j(r)} \int_{S_r} \frac{v_j^2}{r^2} \leq \int_{S_r} |\nabla v_j|^2 = r^{n-2} A_j'(r)$$

by (19.29). Because of (19.30), the left-hand side is equal to

$$(19.37) \qquad \frac{\beta_j(r)}{\sqrt{\alpha_j(r)}} \int_{S_r} \Big[2\frac{v_j}{r}\frac{\partial v_j}{d\rho} + (n-2)\frac{v_j^2}{r^2}\Big] \geq 2\frac{\beta_j(r)}{\sqrt{\alpha_j(r)}} r^{n-3} A_j(r),$$

where the last part comes from(19.26). Thus

$$(19.38) \qquad 2\frac{\beta_j(r)}{\sqrt{\alpha_j(r)}} A_j(r) \leq r A_j'(r).$$

Hence

$$(19.39) \qquad \frac{rA_1'(r)}{A_1(r)} + \frac{rA_2'(r)}{A_2(r)} \geq 2\frac{\beta_1(r)}{\sqrt{\alpha_j(r)}} + 2\frac{\beta_2(r)}{\sqrt{\alpha_j(r)}}.$$

Then we compute the constants, and find out that

$$(19.40) \qquad \frac{\beta_1(r)}{\sqrt{\alpha_j(r)}} = \gamma_j(r),$$

where the numbers $\gamma_j(r)$ are defined by

$$(19.41) \qquad \gamma_j(r)(\gamma_j(r) + n - 2) = \frac{1}{\alpha_j(r)}\,;$$

see (5.8) and (5.9) in [ACF], and observe again that these numbers are dimensionless. Now the situation is that for all choices of disjoint domains $\mathcal{O}_1$ and $\mathcal{O}_2$ that intersect the sphere, the numbers $\gamma_j(r)$ are such that

$$(19.42) \qquad \gamma_1(r) + \gamma_2(r) \geq 2,$$

from which [ACF] deduces that $\frac{rA_1'(r)}{rA_1(r)}+\frac{A_2'(r)}{A_2(r)} \geq 4$ and (returning to variable r and integrating) that $\Phi_1\Phi_2$ is nondecreasing. This is roughly how one proceeds in [ACF].

In the situation of Theorem 19.3, we know that $\frac{A_1'(r)}{A_1(r)} + \frac{A_2'(r)}{A_2(r)} = 4$ (by (19.40)), so all the inequalities above are in fact identities (the quantities in play are all positive), and we need to derive information from that.

The fastest route would use an unpublished paper of W. Beckner, C. Kenig, and J. Pipher [BKP] which says that when $\gamma_1(r)+\gamma_2(r) = 2$, the two domains $\mathcal{O}_j \cap S_t$ are complementary hemispheres. This helps greatly, but let us see what we can get easily.

Because we have equality in (19.31) (Cauchy-Schwarz), we get that the two functions $\frac{v_j}{r}$ and $\frac{\partial v_j}{\partial \rho}$ are proportional. That is, there is a constant $c_j(r)$ such that

$$\frac{\partial v_j}{\partial \rho} = c_j(r)\frac{v_j}{r} \ \text{ on } \mathcal{O}_j \cap S_r. \tag{19.43}$$

We don't need to say almost everywhere, because both functions are smooth on $\mathcal{O}_j$, and we know that $0 < c_j(r) < +\infty$ because otherwise the left-hand side of (19.31) would not be positive like the right-hand side. We can even compute $c_j(r)$; indeed (19.32) is an equality, and $A = B$ in (19.33) (because (19.34) is an equality), so

$$\frac{\beta_j(r)}{\sqrt{\alpha_j(r)}}\Big\{ \int_{S_r} \frac{v_j^2}{r^2}\Big\}^{1/2} = B = A = \Big\{ \int_{S_r} \big|\frac{\partial v_j}{d\rho}\big|^2\Big\}^{1/2} \tag{19.44}$$

and hence $c_j(r) = \frac{\beta_j(r)}{\sqrt{\alpha_j(r)}} = \gamma_j(r)$ (by (19.40)). Notice that (19.43) holds for almost every $r > 0$; let us use this to check that

$$\mathcal{O}_j \text{ is a cone.} \tag{19.45}$$

Pick $x \in \mathcal{O}_j$, write $x = r_0\xi$ for some $\xi \in S_1$, and consider $h(r) = \log(v_j(r\xi))$; we know that h is defined and locally Lipschitz as long as $r\xi \in \mathcal{O}_j$, and (19.43) yields $h'(r) = r^{-1}c_j(r) = r^{-1}\gamma_j(r)$ almost everywhere. Since $\gamma_j(r) \leq 2$, we get that $h(r) \geq h(r_0)| - 2|\log(r/r_0)|$ as long as $r\xi \in \mathcal{O}_j$. This gives a lower bound for $v_j(r\xi)$ and proves that in fact $r\xi \in \mathcal{O}_j$ for all $r > 0$; (19.45) follows.

By (19.45), the numbers $\alpha_j(r)$, and then $\beta_j(r)$ and $\gamma_j(r)$, do not depend on r. Then we can solve the differential equation (19.43), and we get that $v_j(tx) = t^{\gamma_j}v_j(x)$ for $x \in \mathcal{O}_j$ and $t > 0$. Since v_j is Lipschitz near the origin and $\nabla_t v_j \neq 0$ somewhere, $\gamma_j \leq 1$. Since $\gamma_1 + \gamma_2 = 2$, $\gamma_1 = \gamma_2 = 1$.

Now we really need some information on Sobolev constants, and we shall use results from [BZ] (that are in fact a little posterior to [ACF]). Fix a pole in $z_0 \in S_1$, and denote by Γ_j the spherical cap centered at z_0 (meaning, the intersection $S_1 \cap B$, where B is a ball centered at z_0) such that $\sigma(\Gamma_j) = \sigma(\mathcal{O}_j \cap S_1)$. Denote by α_j^* and γ_j^* the constants associated to Γ_j as above. Then Theorem 5.1 on page 175 of [BZ], with $A(t) = t^2$, says the following things.

First, if $u \in W^{1,2}(S_1)$ and u^* denotes its symmetric rearrangement, then $u^* \in W^{1,2}(S_1)$ and

$$\int_{S_r} |\nabla_t u^*|^2 \leq \int_{S_r} |\nabla_t u|^2. \tag{19.46}$$

Notice that if u is compactly supported in $\mathcal{O}_j \cap S_1$, then u^* is compactly supported in Γ_j; since $\int |u^*|^2 = \int |u|^2$, the definition (19.25) says that $\alpha_j^* \leq \alpha_j$, and then $\gamma_j^* \geq \gamma_j = 1$ by (19.41).

It is easy to check that α_j^* is a (strictly) decreasing function of (the volume of) Γ_j, hence γ_j^* is increasing. Also, $\gamma_j^* = 1$ when Γ_j is a hemisphere (see later), hence $\sigma(\Gamma_j) \geq \sigma(S_1)/2$. But $\sigma(\Gamma_j) = \sigma(\mathcal{O}_j \cap S_1)$ and the $\mathcal{O}_j$ are disjoint, so $\sigma(\Gamma_j) = \sigma(\mathcal{O}_j \cap S_1) = 1$ for both j, $\gamma_j = \gamma_j^* = 1$, and $\alpha_j = \alpha_j^*$.

Notice that $\int_{S_1} |v_j|^2 = \alpha_j \int_{S_1} |\nabla_t v_j|^2$ because (19.35) is an equality for almost every r and v_1 is homogeneous of degree 1, hence by (19.46) and (19.25) for Γ_j,

$$\int_{S_r} |\nabla_t u^*|^2 \leq \int_{S_r} |\nabla_t u|^2 = \alpha_j^{-1} \int_{S_1} |v_j|^2 = \alpha_j^{-1} \int_{S_1} |v_j^*|^2 \leq \alpha_j^{-1} \alpha_j^* \int_{S_r} |\nabla_t u^*|^2; \tag{19.47}$$

we know that $\alpha_j = \alpha_j^*$, so (19.25) is an equality for v_j. So v_j^* is a Sobolev minimizer in a half sphere. It is well known that then v_j^* is a first eigenfunction for the Laplacian on the sphere, and (because we can use a symmetry argument to reduce to the sphere) that it is the restriction to S_1 of an affine function. This is also how one computes that $\gamma_j^* = 1$.

Return to Theorem 5.1 of [BZ]; its most important part is that its says that since (19.46) is an equality for v_j, v_j is of the form $v_j^* \circ R$, where R is a rotation. There are just two assumptions to check: first, that $A(t)$ is increasing (this is trivial because $A(t) = t^2$), and also that

$$\left| \left\{ x \in S_1 \,;\, v_1^*(x) > 0 \text{ and } \nabla_t v_j^* = 0 \right\} \right| = 0, \tag{19.48}$$

which holds because v_j^* comes from an affine function. So Brothers and Ziemer's theorem applies and says that both v_j are also equal to affine functions on the sphere; we easily deduce the representation formula (19.23) for the v_j from this, because they are homogeneous of degree 1, and Theorem 19.3 follows. ∎

Let us now return to our minimizer $(\mathbf{u}, \mathbf{W})$ and its regular blow-up limits. As was said above, if $\mathbf{u}_\infty$ is as in the beginning of the section and we can find different indices $\varphi_1, \varphi_2 \in I$ such that $L(\varphi_1, \varphi_2) > 0$ in (19.20), the corresponding functions $v_j = v_{\varphi_j,\infty}$ satisfy the hypotheses of Theorem 19.3 (because they are harmonic, see (9.6)), and so they are described by (19.23).

Because of this, all the other functions $v_{\varphi,\infty}$, $\varphi \neq \varphi_1, \varphi_2$ are null because, by definition of the class $\mathcal{F}(\mathbb{R}^n, \Omega_\infty)$, they vanish almost everywhere on the sets where $v_{\varphi_j,\infty} > 0$.

But Corollary 18.3 also says that $(\mathbf{u}_\infty, \mathbf{W}_\infty)$ is a minimizer for the functional J_∞, which means that (18.40) holds for all competitors of $(\mathbf{u}_\infty, \mathbf{W}_\infty)$ in a ball $B(0, R)$. We could simplify this and write it in terms of the single function $v = v_1 - v_2$ and usual Alt, Caffarelli, and Friedman minimizers in $\mathbb{R}^n$, but let us not do this for the moment. Write $\varphi_1 = (i_1, \varepsilon_1)$ and $\varphi_2 = (i_2, \varepsilon_2)$. We claim that because of this additional minimality property (where we may move the free boundary),

$$a_1^2 - a_2^2 = \lambda_{i_1} - \lambda_{i_2}. \tag{19.49}$$

This can be proved easily, with a computations of first variation where we modify $v_1 - v_2$. The proof is classical, but we shall do it (later) for the convenience of the reader. See near (22.10) We can also check whether making $v_1 = v_2 = 0$ near the separating hyperplane would make things better, and the first variation computation for this gives the additional constraints that

$$a_1^2 \geq \lambda_{i_1} - \min(0, \lambda_1, \dots, \lambda_N) \text{ and } a_2^2 \geq \lambda_{i_2} - \min(0, \lambda_2, \dots, \lambda_N). \tag{19.50}$$

See near (22.12), and notice that most often all the λ_i are nonnegative, so we get the simpler relations $a_j^2 \geq \lambda_{i_j}$.

There is another relation, which is more a matter of definitions:

$$\begin{aligned} L(\varphi_1, \varphi_2) &= \Phi_{\varphi_1,\varphi_2,\infty}(1) = \Phi_{\varphi_2,\infty}(1)\Phi_{\varphi_1,\infty}(1) \\ &= \Big\{\int_{B(0,1)} \frac{|\nabla v_{\varphi_1,\infty}(x)|^2}{|x|^{n-2}}\Big\}\Big\{\int_{B(0,1)} \frac{|\nabla v_{\varphi_2,\infty}(x)|^2}{|x|^{n-2}}\Big\} \\ &= \frac{a_1 a_2}{4}\Big\{\int_{B(0,1)} |x|^{2-n}\Big\}^2 = \frac{a_1 a_2}{16}\sigma(S_1)^2 = \frac{n^2 a_1 a_2}{4}|B(0,1)|^2 \end{aligned} \tag{19.51}$$

by (19.20), (19.18), (19.16), and (19.23). So we can always compute a_1 and a_2, if we know $L(\varphi_1, \varphi_2)$ and the λ_i.

Let us gather a few easy consequences of this discussion on the minimizer $(\mathbf{u}, \mathbf{W})$ itself.

In the next statement, a blow-up limit of $\mathbf{u}$ is any function $\mathbf{u}_\infty$ such that $\mathbf{u}_\infty(x) = \lim_{k\to+\infty} r_k^{-1}\mathbf{u}(r_k x)$ for $x \in \mathbb{R}^n$, for some sequence $\{r_k\}$ that tends to 0. Since $\mathbf{u}$ is assumed to be Lipschitz near 0, such limits exist and the convergence is uniform on compact sets of $\mathbb{R}^n$. We won't need to say that $\mathbf{u}_\infty$ is a regular blow-up limit, because this will follow from the assumptions of Lemma 19.1.

Corollary 19.4 *Let* $(\mathbf{u}, \mathbf{W})$ *be a minimizer for* J*. Suppose that* $\mathbf{u}$ *satisfies* (18.3) *and* (18.5) *that the* f_i *and* g_i *satisfy* (18.4)*, that the domain* Ω *satisfies the two assumptions of Lemma 19.1, and that* F *satisfies* (18.11)*. Suppose in addition that there are phases* $\varphi_1 = (i_1, \varepsilon_1) \in I$ *and* $\varphi_2 = (i_2, \varepsilon_2) \in I \setminus \{\varphi_1\}$ *such that*

$$L(\varphi_1, \varphi_2) = \lim_{\rho\to 0} \Phi^0_{\varphi_1}(\rho)\Phi^0_{\varphi_2}(\rho) = \lim_{\rho\to 0}\Big\{\frac{1}{\rho^4}\int_{B(0,\rho)} \frac{|\nabla v_{\varphi_1}|^2}{|x|^{n-2}}\int_{B(0,\rho)} \frac{|\nabla v_{\varphi_2}|^2}{|x|^{n-2}}\Big\} \neq 0 \tag{19.52}$$

(see (19.19) *and* (19.11)*). Then*

(i) For each blow-up limit $\mathbf{u}_\infty$ *of* $\mathbf{u}$ *at* 0*, the functions* $v_j = v_{\varphi_j,\infty} = [\varepsilon_j u_{\varphi_j,\infty}]_+$ *take the form given by* (19.23)*, with constants* $a_j > 0$ *such that* (19.49)*,* (19.50)*, and* (19.51) *hold, and all the other components* $v_{\varphi,\infty}$*,* $\varphi \in I \setminus \{\varphi_1, \varphi_2\}$*, of* $\mathbf{u}_\infty$ *are null.*
(ii) For each $\varphi \in I \setminus \{\varphi_1, \varphi_2\}$*,*

$$\lim_{\rho\to 0} \Phi^0_\varphi(\rho) := \lim_{\rho\to 0} \frac{1}{\rho^2} \int_{B(0,\rho)} \frac{|\nabla v_\varphi|^2}{|x|^{n-2}} = 0. \tag{19.53}$$

(iii) the origin is an interior point of Ω*.*

In (i), the coefficients a_j in the description (19.23) do not depend on the blow-up sequence $\{r_k\}$ because they can be computed in terms of $L(\varphi_1, \varphi_2)$ and the λ_i, but in principle the unit vector e does. Unless we prove a better regularity result for $\mathbf{u}$ near 0.

Also, (iii) only says that $0 \in \mathrm{int}(\Omega)$ under our assumptions, which includes some weak regularity assumptions on Ω. Probably (19.52) can happen when Ω has an inward cusp at 0, with a free boundary that continues in Ω in the direction of the cusp.

Proof. We included the assumptions of Lemma 19.1 to make sure that when $\mathbf{u}_\infty$ is a blow-up limit associated to any sequence $\{r_k\}$ that tends to 0, the assumptions of the beginning of this section (we were missing the conditions on Ω_∞) are satisfied for some subsequence. Then (i) is a consequence of the discussion above.

Suppose that (19.53) fails for some $\varphi \in I \setminus \{\varphi_1, \varphi_2\}$, and choose a sequence $\{r_k\}$ such that

$$\lim_{k\to+\infty} \Phi^0_\varphi(r_k) = \limsup_{k\to\infty} \Phi^0_\varphi(\rho) > 0. \tag{19.54}$$

Then use Lemma 19.1 to replace $\{r_k\}$ with a subsequence that satisfies the assumptions at the beginning of this section. Of course $\mathbf{u}_\infty$ stays the same for this subsequence, and (i) gives two nontrivial components $v_{\varphi_j,\infty}$ of $\mathbf{u}_\infty$. Notice that

$$\begin{aligned} \liminf_{k\to+\infty} \Phi^0_{\varphi_1}(r_k) &\geq \frac{\liminf_{k\to\infty}\left[\Phi^0_{\varphi_1}(r_k)\Phi^0_{\varphi_2}(r_k)\right]}{\limsup_{k\to\infty} \Phi^0_{\varphi_2}(r_k)} \\ &= \frac{L(\varphi_1,\varphi_2)}{\limsup_{k\to\infty} \Phi^0_{\varphi_2}(r_k)} \geq C^{-1} L(\varphi_1,\varphi_2) > 0 \end{aligned} \tag{19.55}$$

by (19.52) and (19.12), so

$$\begin{aligned} L(\varphi,\varphi_1) &= \lim_{\rho\to 0} \Phi^0_\varphi(\rho)\Phi^0_{\varphi_1}(\rho) = \lim_{k\to+\infty} \Phi^0_\varphi(r_k)\Phi^0_{\varphi_1}(r_k) \\ &\geq C^{-1} L(\varphi_1,\varphi_2) \limsup_{k\to\infty} \Phi^0_\varphi(\rho) > 0 \end{aligned} \tag{19.56}$$

by (19.19), (19.54), and (19.55). Then the component $v_{\varphi,\infty}$ of $\mathbf{u}_\infty$ is also given by a formula like (19.23), which is impossible because the other two don't leave any room for its support. This proves (ii).

Finally suppose that 0 is an interior point of $\partial\Omega$. Then Lemma 19.1 gives a limiting domain Ω_∞ such that in particular $|B(x,r) \setminus \Omega_\infty| > 0$ for every ball $B(x,r)$ centered on $\partial\Omega_\infty$; see (19.5). But $\mathbf{u}_\infty = 0$ almost everywhere on $\mathbb{R}^n \setminus \Omega_\infty$, by definition of $\mathcal{F}(\mathbb{R}^n, \Omega_\infty)$ (see the lines above Lemma 19.1, and the first lines of Section 17. This contradicts that description (19.23) of the $v_{\varphi_j,\infty}$, proves (iii), and completes the proof of Corollary 19.4. ∎

We can find sufficient conditions for the condition (19.52) to hold. The main one will use good domains and the nondegeneracy condition of Section 15, and we shall discuss it in Section 21. Let us just say now that if

$$\liminf_{\rho\to 0} \Phi^0_{\varphi_j}(\rho) > 0 \tag{19.57}$$

for $j = 1, 2$ (and $\mathbf{u}$ is Lipschitz near 0), then (19.52) holds because we already knew from (19.19) that the limit of the product exits. In general, if we merely assume that

$$\limsup_{\rho\to 0} \Phi^0_{\varphi_j}(\rho) > 0, \tag{19.58}$$

this should not be enough to conclude. A priori it can happen that v_{φ_1} is dormant at some scales (i.e., ∇v_{φ_1} is very small), and revives at smaller scales, but never at the same time as for the other phase φ_2; then $L(\varphi_1, \varphi_2)$ may be null, and we won't find a blow-up sequence that would show both phases.

The following proposition can be seen as a weak regularity result for the free boundaries associated to phases that satisfy the conditions of Corollary 19.4. Set

$$\Omega_\varphi = \big\{x \in \mathbb{R}^n\,;\, v_\varphi(x) > 0\big\} = \big\{x \in \mathbb{R}^n\,;\, \varepsilon u_i(x) > 0\big\} \tag{19.59}$$

for $\varphi = (i, \varepsilon) \in I$. Then let φ_1 and $\varphi_2 \in I$ be as in Corollary 19.4; we want to measure the flatness of $\partial\Omega_{\varphi_1} \cup \partial\Omega_{\varphi_2}$ in small balls $B(0,r)$.

Proposition 19.5 *Let $(\mathbf{u}, \mathbf{W})$ and the pairs φ_1 and $\varphi_2 \in I$ satisfy the hypotheses of Corollary 19.4. Then there exists numbers $\beta(r) \in [0,1]$ such that*

$$\lim_{r\to 0} \beta(r) = 0 \tag{19.60}$$

and, for $r > 0$, a unit vector $e = e(r)$ such that

$$v_{\varphi_1}(x) > 0 \ \text{ for } x \in B(0,r) \text{ such that } \langle x, e(r)\rangle \geq \beta(r) r, \tag{19.61}$$

$$v_{\varphi_2}(x) > 0 \ \text{ for } x \in B(0,r) \text{ such that } \langle x, e(r)\rangle \leq -\beta(r) r, \tag{19.62}$$

$$(\partial\Omega_{\varphi_1} \cup \partial\Omega_{\varphi_2}) \cap B(0,r) \subset \big\{x \in B(x,r)\,;\, |\langle x, e(r)\rangle| \leq \beta(r) r\big\}, \tag{19.63}$$

and

$$\Omega_\varphi \cap B(0,r) \subset \big\{x \in B(x,r)\,;\, |\langle x, e(r)\rangle| \leq \beta(r) r\big\} \ \text{ for } \varphi \in I \setminus \{\varphi_1, \varphi_1\}. \tag{19.64}$$

Thus all the action is on the thin band where $|\langle x, e(r)\rangle| \leq \beta(r)r$. There may be a lot of action though, because if volume for the other W_i is cheap, lots of them may find it convenient to squat some of that band. Even when we assume that all the regions are good, there may be a part of the band that lies on $\Omega \setminus \cup_i W_i$.

As for the blow-up limits, we do not know whether the direction $e(r)$ really depends on r, but by lack of a better regularity theorem (which may be hard with the present degree of generality), we have to assume that it does.

Proof. Let us first observe that $\Omega_{\varphi_1} \cap \Omega_{\varphi_2} = \emptyset$, by definition if $\mathcal{F}$ if $i_1 \neq i_2$ and by definition of v_{φ_i} otherwise. So (19.63) is an immediate consequence of (19.61) and (19.62). Similarly, Ω_φ does not meet the Ω_{φ_j} and (19.64) follows from (19.61) and (19.62).

Notice also that (19.61) and (19.62) are trivial when $\beta(r) = 1$, so the point is to show that we may make it tend to 0.

We shall prove the proposition by contradiction and compactness. Suppose that we cannot find the $\beta(r)$ and $e(r)$ as above; then we can find a positive number $\alpha > 0$, and a sequence $\{r_k\}$ that tends to 0, with the following property. For each k, there is no choice of a unit vector e such that (19.61) and (19.62) hold with $r = r_k$ and $\beta(r) = \alpha$.

Extract from $\{r_k\}$ a subsequence for which the assumptions of Corollary 18.3 are satisfied; we have seen at the beginning of the proof of Corollary 19.4 that Lemma 19.1 allow us to do this. Then we get a blow-up limit $\mathbf{u}_\infty$, associated to our subsequence, which admits the description of (19.23). This gives a unit vector e, which we may try in (19.61) and (19.62). By definition of $\{r_k\}$, this does not work well, so we get a point $x_k \in B(0, r_k)$ such that

$$\langle x_k, e(r)\rangle \geq \alpha r_k \ \text{ but } v_{\varphi_1}(x_k) = 0, \tag{19.65}$$

or (a point $x_k \in B(0, r)$ such that)

$$\langle x_k, e(r)\rangle \leq -\alpha r_k \ \text{ but } v_{\varphi_2}(x_k) = 0. \tag{19.66}$$

We extract a new subsequence so that we get (19.65) for all k, or we get (19.65) for all k, and in addition $y_k = r_k^{-1} x_k$ has a limit $y_\infty \in \overline{B}(0, 1)$.

Suppose for instance that (19.65) holds for all k. Then $v_{\varphi_1,k}(y_k) = r_k^{-1} v_{\varphi_1}(r_k y_k) r_k^{-1} v_{\varphi_1}(x_k) = 0$ by (19.9), and

$$|v_{\varphi_1,\infty}(y_\infty)| \leq |v_{\varphi_1,\infty}(y_\infty) - v_{\varphi_1,k}(y_\infty)| + |v_{\varphi_1,k}(y_\infty) - v_{\varphi_1,k}(y_k)|. \tag{19.67}$$

The first term tends to 0 because $v_{\varphi_1,\infty}$ is the limit of the $v_{\varphi_1,k}$ in $\mathbb{R}^n$, and the second term because the $v_{\varphi_1,k}$ are uniformly Lipschitz and y_k tends to y_∞. So $v_{\varphi_1,\infty}(y_\infty) = 0$. But $\langle y_k, e(r)\rangle = r_k^{-1}\langle x_k, e(r)\rangle \geq \alpha$ by (19.65), so $\langle y_\infty, e(r)\rangle \geq \alpha$, which contradicts (19.23) (recall that we set $v_j = v_{\varphi_j,\infty}$ there). The case when (19.66) holds for all k would be treated the same way, and Proposition 19.5 follows from the contradiction. ∎

20 Blow-up limits with one phase

In this section we keep the same general assumptions as in Section 19 and study the blow-up limits of $\mathbf{u}$ when all the limits $L(\varphi_1, \varphi_2)$ of (19.19) are null, but, say

$$\limsup_{r\to 0} \Phi^0_{1,1}(r) := \limsup_{r\to 0} \frac{1}{r^2}\int_{B(0,r)} \frac{|\nabla u_{1,+}|^2}{|x|^{n-2}}\, dx > 0. \tag{20.1}$$

In fact, we shall rapidly assume that F is Lipschitz and $i = 1$ is a good index, i.e., that

$$\text{(10.2) holds and there exists } \lambda > 0 \text{ and } \varepsilon > 0 \text{ such that (15.1) holds.} \tag{20.2}$$

The two go together because we want to apply Proposition 16.1, for instance. At this stage, this is not so much to assume, because we shall see that if (20.1) holds and 0 is an interior point of Ω, then

$$\lambda_1 > \min(0, \lambda_2, \dots, \lambda_N), \tag{20.3}$$

where the λ_i are as in our assumption (18.11). In this case, (20.2) just amounts to requiring a little bit more regularity on F than we do in (18.11).

Then, if 0 is an interior point of Ω, or at least Ω looks enough like cones near 0, we shall be able to show that some blow-up limits $\mathbf{u}_\infty$ of $\mathbf{u}_1$ at the origin are nontrivial, one-phase minimizers of the standard Alt, Caffarelli, and Friedman functional (in $\mathbb{R}^n$ or in a cone), that are also homogeneous of degree 1. See Theorem 20.2 below. When $n \leq 3$ and the cone is $\mathbb{R}^n$, it was proved in [CJK2] that the first coordinate of such functions u_∞ is of the form $u(x) = a(x)_+ = \max(a(x), 0)$, where a is affine, so this will give a good description of the corresponding blow-up limits of $\mathbf{u}$ at 0 when $n \leq 3$ and 0 is an interior point of Ω. See Corollary 20.3.

The main ingredient for this section is a functional introduced by G. S. Weiss [We], its monotonicity properties, and what happens when it is constant.

We shall try to add assumptions as they are used. The initial assumptions for this section are, as for most of Section 20, that

$$\mathbf{u} \text{ satisfies (18.3) and (18.5), the } f_i \text{ and } g_i \text{ satisfy (18.4), } F \text{ satisfies (18.11),} \tag{20.4}$$

$$\text{the domain } \Omega \text{ satisfies the two assumptions of Lemma 19.1,} \tag{20.5}$$

but for the main result we shall assume that 0 is an interior point of Ω, in which case the issue does not arise. We also assume that

$$L(\varphi_1, \varphi_2) := \lim_{\rho\to 0} \Phi^0_{\varphi_1}(\rho)\Phi^0_{\varphi_2}(\rho) = 0 \;\text{ for } \varphi_1 \in I \text{ and } \varphi_1 \in I \setminus \{\varphi_1\} \tag{20.6}$$

(see the definitions (19.19) and (19.11)), and (20.1) or (20.2).

Let us define the Weiss functional associated to $u_{1,+}$. Set $v = u_{1,+} = \max(0, u_1)$ and $\Omega_1 = \big\{x \in \mathbb{R}^n\,;\, u_1(x) > 0\big\}$ to save notation, and then define Ψ by

$$\begin{aligned}
\Psi(r) &= r^{-n}\int_{B(0,r)} |\nabla v|^2 + r^{-n}\lambda_1|\Omega_1 \cap B(0,r)| - \fint_{0\le t\le r} t^{1-n}\int_{\partial B(0,t)} \Big|\frac{\partial v}{d\rho}\Big|^2 d\sigma dt \\
&= r^{-n}\int_{B(0,r)} |\nabla v|^2 + r^{-n}\lambda_1|\Omega_1 \cap B(0,r)| - \frac{1}{r}\int_{B(0,r)} |x|^{1-n}\Big|\frac{\partial v}{d\rho}(x)\Big|^2 dx
\end{aligned}\tag{20.7}$$

for $0 < r < \rho_0$ (where ρ_0 comes from our Lipschitz assumption (18.5)). Here λ_1 comes from (18.11) and $\frac{\partial u}{d\rho}$ denotes the radial derivative of u (also written $\langle \nabla u, \nu\rangle$ in [We]), and there is no convergence problem for the integrals, since $\mathbf{u}$ is Lipschitz near 0. This is the same function as p on page 319 of [We], with $Q(0) = \lambda_1$.

Next let $\mathbf{u}_\infty$ be any blow-up limit of $\mathbf{u}$ at 0. That is, assume that there is a sequence $\{r_k\}$ that tends to 0, such that $\mathbf{u}_\infty(x) = \lim_{k\to+\infty} r_k^{-1}\mathbf{u}(r_k x)$ on $\mathbb{R}^n$. As usual, set

$$v_k(x) = r_k^{-1}v(r_k x) = r_k^{-1}[u_1(r_k x)]_+ \ \text{ for } x \in \mathbb{R}^n,\tag{20.8}$$

$$v_\infty(x) = [\mathbf{u}_{\infty,1}(x)]_+ = \lim_{k\to+\infty} r_k^{-1}v(r_k x)\tag{20.9}$$

for $x \in \mathbb{R}^n$, and

$$\Omega_{1,k} = \big\{x \in \mathbb{R}^n\,;\, v_k(x) > 0\big\} \ \text{ for } k \ge 0 \text{ and } k = +\infty.\tag{20.10}$$

We shall soon use the limit Weiss function Ψ_∞ defined on $(0, +\infty)$ by

$$\Psi_\infty(r) = r^{-n}\int_{B(0,r)} |\nabla v_\infty|^2 + r^{-n}\lambda_1|\Omega_{1,\infty} \cap B(0,r)| - \frac{1}{r}\int_{B(0,r)} |x|^{1-n}\Big|\frac{\partial v_\infty}{d\rho}(x)\Big|^2 dx\tag{20.11}$$

but let us first talk about the minimizing properties of $\mathbf{u}_\infty$.

Because of Lemma 19.1, we can replace $\{r_k\}$ with some subsequence for which (18.8)-(18.10) hold, and the we can apply Corollary 18.3. Thus there is a domain Ω_∞ and a N-uple $\mathbf{W}_\infty$ such that

$$(\mathbf{u}_\infty, \mathbf{W}_\infty) \text{ is a local minimizer for } J_\infty \text{ in } \mathcal{F}(\mathbb{R}^n, \Omega_\infty);\tag{20.12}$$

see near (18.40) for the definitions. We claim that because of (20.6)

$$u_{1,\infty} \ge 0 \ \text{ and } u_{i,\infty} = 0 \text{ for } i \ge 2\tag{20.13}$$

if we chose $\{r_k\}$ such that

$$\liminf_{k\to+\infty} \Phi^0_{1,1}(r_k) > 0.\tag{20.14}$$

Of course we can find a sequence like this, by (20.1). And indeed, suppose that (20.14) holds. Set $\varphi_1 = (1,1)$ and let $\varphi \in I \setminus \{\varphi_1\}$ be given. By (20.14), $\liminf_{k\to+\infty} \Phi^0_{1,1}(r_k R) > 0$ for each

$R > 1$ as well (just because $\Phi^0_{1,1}(r_kR) \geq R^{-2}\Phi^0_{1,1}(r_k)$ by (19.11)); by (20.6), $\Phi^0_{\varphi_1}(r_kR)\Phi^0_{\varphi}(r_kR)$ tends to 0, so $\Phi^0_{\varphi}(r_kR)$ tends to 0, and then Lemma 19.2 says that

$$\Phi_{\varphi,\infty}(R) = \lim_{k\to+\infty} \Phi^0_{\varphi}(r_kR) = 0. \tag{20.15}$$

Then $\nabla v_\varphi = 0$ almost everywhere, and (20.13) follows.

Next we check that (20.3) holds if 0 is an interior point of Ω. Suppose not. Choose a sequence $\{r_k\}$ such that (20.14) holds, and then use Lemma 19.1 to get a subsequence such that Corollary 18.3 applies, so that we get a local minimizer $(\mathbf{u}_\infty, \mathbf{W}_\infty)$ as in (20.12). Notice that here $\Omega_\infty = \mathbb{R}^n$. Set $\varphi = (1,1)$, and notice that $\Phi_{\varphi,\infty}(1) = \lim_{k\to+\infty} \Phi^0_{\varphi}(r_k) > 0$ by Lemma 19.2, so $u_{1,\infty} > 0$ somewhere.

We build a competitor for $(\mathbf{u}_\infty, \mathbf{W}_\infty)$. Pick a ball $B = B(0,r)$ such that $u_{1,\infty} > 0$ somewhere on ∂B, then denote by u_1^* the harmonic extension to B of the restriction of $u_{1,\infty}$ to $\partial B(0,r)$. Keep $u_1^* = u_{1,\infty}$ on $\mathbb{R}^n \setminus B$, and also set $u_i^* = 0$ for $i \geq 2$. Also set $W_1^* = W_{1,\infty} \cup B$ and $W_i^* = W_{i,\infty} \cup B$ for $i \geq 2$. It is easy to see that $(\mathbf{u}^*, \mathbf{W}^*) \in \mathcal{F}(\mathbb{R}^n, \Omega_\infty)$ (because $\Omega_\infty = \mathbb{R}^n$ and by (20.13)). So (18.40) holds, even with $R = r$. Since $\sum_i \lambda_i |W_1^* \cap B| = \lambda_1 |B| \leq \sum_i \lambda_i |W_{i,\infty} \cap B|$ because (20.3) fails, we do not lose anything on the volume term, and (18.40) yields $\int_B |\nabla u_{1,\infty}|^2 \leq \int_B |\nabla u_1^*|^2$. But u_1^* is the only minimizer of $\int_B |\nabla u|^2$ with the given boundary values on ∂B, so $u_{1,\infty} = u_1^*$ on B. This is not possible, because $\mathbf{u}(0) = 0$ and $u_1^*(0) > 0$ (recall that $u_{1,\infty} \geq 0$ by (20.13), and that it is positive somewhere on ∂B).

When 0 lies on the boundary, it is harder to say much; Ω_1 may fill the entire region Ω, be harmonic (or satisfy the equation (9.4)) there, and naturally vanish at the boundary because this was our initial constraint. In this case, (20.1) seems to reflect more on the shape of the boundary than on the λ_i. But even in this case we shall decide to assume (20.2).

We return to the Weiss functional of (20.7) and (20.11) and prove that it goes to the limit.

Lemma 20.1 *Let $\mathbf{u}_\infty$ be a blow-up limit of $\mathbf{u}$ at 0, associated to the sequence $\{r_k\}$. Suppose, in addition to (20.4) and (20.5), that (20.2) holds. Then, maybe after replacing $\{r_k\}$ by some subsequence,*

$$\Psi_\infty(r) = \lim_{k\to+\infty} \Psi(r_kr) \quad \textit{for } r > 0. \tag{20.16}$$

Proof. Because of Lemma 19.1, we can replace $\{r_k\}$ with some subsequence for which (18.8)-(18.10) hold, and then we can apply Corollary 18.3. We get that there is a domain Ω_∞ and a N-uple $\mathbf{W}_\infty$ such that (20.12) holds, but this is not what we care about here, we just need to know that for $r > 0$,

$$\nabla v_\infty = \lim_{k\to+\infty} \nabla v_k \quad \text{in } L^2(B(0,r)), \tag{20.17}$$

as in (18.17). Then

$$r^{-n}\int_{B(0,r)}|\nabla v_\infty|^2 = \lim_{k\to+\infty} r^{-n}\int_{B(0,r)}|\nabla v_k(x)|^2dx = \lim_{k\to+\infty} r^{-n}\int_{B(0,r)}|\nabla v(r_kx)|^2dx$$
$$= \lim_{k\to+\infty}\int_{B(0,r_kr)}|\nabla v(y)|^2dy \tag{20.18}$$

by (20.8) and a change of variable, and similarly, for each small $\varepsilon > 0$,

$$\frac{1}{r}\int_{B(0,r)\setminus B(0,\varepsilon)}|x|^{1-n}\Big|\frac{\partial v_\infty}{d\rho}(x)\Big|^2dx = \lim_{k\to+\infty}\frac{1}{r}\int_{B(0,r)\setminus B(0,\varepsilon)}|x|^{1-n}\Big|\frac{\partial v_k}{d\rho}(x)\Big|^2dx$$
$$= \lim_{k\to+\infty}\frac{1}{r}\int_{B(0,r)\setminus B(0,\varepsilon)}|x|^{1-n}\Big|\frac{\partial v}{d\rho}(r_kx)\Big|^2dx$$
$$= \lim_{k\to+\infty}\frac{1}{r_kr}\int_{B(0,r_kr)\setminus B(0,r_k\varepsilon)}|x|^{1-n}\Big|\frac{\partial v}{d\rho}(x)\Big|^2dx. \tag{20.19}$$

The small missing pieces are estimated with the Lipschitz norms, i.e.,

$$\frac{1}{r}\int_{B(0,\varepsilon)}|x|^{1-n}\Big|\frac{\partial v_\infty}{d\rho}(x)\Big|^2dx \leq \frac{C}{r}\int_{B(0,\varepsilon)}|x|^{1-n} \leq \frac{C\varepsilon}{r}, \tag{20.20}$$

and

$$\frac{1}{r_kr}\int_{B(0,r_k\varepsilon)}|x|^{1-n}\Big|\frac{\partial v}{d\rho}(x)\Big|^2dx \leq \frac{C}{r_kr}\int_{B(0,r_k\varepsilon)}|x|^{1-n} \leq \frac{C\varepsilon}{r}. \tag{20.21}$$

Thus the last term of the functional Ψ goes to the limit too, and the lemma will follow as soon as we prove that

$$r^{-n}\lambda_1|\Omega_{1,\infty}\cap B(0,r)| = \lim_{k\to+\infty}(r_kr)^{-n}\lambda_1|\Omega_1\cap B(0,r_kr)|. \tag{20.22}$$

This is where the nondegeneracy assumption (20.2) will be useful. Set $\Omega_{1,k} = r_k^{-1}\Omega_1 = \big\{x\,;\, v_k > 0\big\}$ and simplify (20.22); we just need to check that

$$|\Omega_{1,\infty}\cap B(0,r)| = \lim_{k\to+\infty}|\Omega_{1,k}\cap B(0,r)|. \tag{20.23}$$

If $x\in\Omega_{1,\infty}\cap B(0,r)$, then $v_\infty(x) > 0$, hence $v_k(x) > 0$ for k large, and $x\in\Omega_{1,k}\cap B(0,r)$ for k large. So $\mathbb{1}_{\Omega_{1,\infty}\cap B(0,r)} \leq \liminf_{k\to+\infty}\mathbb{1}_{\Omega_{1,\infty}\cap B(0,r)}$ and by Fatou

$$|\Omega_{1,\infty}\cap B(0,r)| \leq \liminf_{k\to+\infty}|\Omega_{1,k}\cap B(0,r)|. \tag{20.24}$$

Next set $\varepsilon_k = ||v_\infty - v_k||_{L^\infty(B(0,r))}$ and

$$\mathcal{O}_k = \big\{x\in B(0,r)\,;\, 0 < v_k(x) \leq 2\varepsilon_k\big\}. \tag{20.25}$$

By definition, $\Omega_{1,k} \cap B(0,r) \subset \Omega_{1,\infty} \cup \mathcal{O}_k$; if we prove that

$$\lim_{k\to+\infty} |\mathcal{O}_k| = 0, \tag{20.26}$$

we will get that

$$\limsup_{k\to+\infty} |\Omega_{1,k} \cap B(0,r)| \leq |\Omega_{1,\infty} \cap B(0,r)| + \limsup_{k\to+\infty} |\mathcal{O}_k| = |\Omega_{1,\infty} \cap B(0,r)|, \tag{20.27}$$

and (20.23) and the lemma will follow.

Let us first use Theorem 15.3 to show that for k large,

$$\mathcal{O}_k \subset A_k := \big\{y \in B(0,r)\,;\ \mathrm{dist}\,(y, \partial\Omega_{1,k}) \leq C\varepsilon_k\big\}. \tag{20.28}$$

Pick any $x \in \Omega_1 \cap B(0,\rho_0/2)$, where ρ_0 is still such that $\mathbf{u}$ is Lipschitz on $B(0,\rho_0)$. Set $\delta(x) = \mathrm{dist}\,(x, \mathbb{R}^n \setminus \Omega_1)$ as in (15.39). Since $\mathbf{u}(0) = 0$, we get that $\delta(x) < \rho_0/2$, hence $\mathbf{u}$ is Lipschitz on $B(x, \delta(x)/2)$. The other assumptions of Theorem 15.3 follow from (20.2) and (20.4), so (15.40) holds. That is,

$$u_1(x) \geq c_5 \min(\delta(x), \varepsilon^{1/n}, 1) \ \text{ for } x \in B(0,\rho_0/2), \tag{20.29}$$

where ε is a constant that comes from (15.1). Set $y = r_k^{-1}x$; this yields that for $y \in \Omega_{1,k} \cap B(0, r_k^{-1}\rho_0/2)$,

$$v_k(y) = r_k^{-1} v(r_k y) = r_k^{-1} u_1(r_k y) = r_k^{-1} u_1(x) \geq c_5 r_k^{-1} \min(\delta(r_k y), \varepsilon^{1/n}, 1) \tag{20.30}$$

(by (20.8) and because $u_1(r_k y)$). If k is large enough, this holds for $y \in \Omega_{1,k} \cap B(0,r)$, and in addition $\delta(r_k y) = \mathrm{dist}\,(r_k y, \mathbb{R}^n \setminus \Omega_1) \leq |r_k y| < \min(\varepsilon^{1/n}, 1)$. That is,

$$v_k(y) \geq c_5 r_k^{-1}\, \mathrm{dist}\,(r_k y, \mathbb{R}^n \setminus \Omega_1) = c_5\, \mathrm{dist}\,(y, \mathbb{R}^n \setminus \Omega_{1,k}) \tag{20.31}$$

for $y \in \Omega_{1,k} \cap B(0,r)$; (20.28) follows because $y \in \Omega_{k,1}$ and $v_k(y) \leq 2\varepsilon_k$ when $y \in \mathcal{O}_k$.

Now we shall use the uniform local Ahlfors regularity of the $\partial\Omega_k$ to estimate $|A_k|$. Let us apply Proposition 16.1 to some the ball $B_0 = B(0,r_0)$, where r_0 is chosen small so that the assumptions of the proposition are satisfied. Notice that we did not forget to add (10.2) in (20.2). We get a measure μ such that the local Ahlfors regularity condition (16.6) is satisfied for all balls B centered on $\partial\Omega_1$ such that $2B \subset B_0$. Set $\mu_k(A) = r_k^{1-n}\mu(r_k A)$ for Borel sets A (to preserve the homogeneity); then by (16.6)

$$C_1^{-1}\rho^{n-1} \leq \mu_k(B(y,\rho)) = r_k^{1-n}\mu(B(r_k y, r_k\rho)) \leq C_1 \rho^{n-1} \tag{20.32}$$

for $y \in \partial\Omega_{1,k} = r_k^{-1}\partial\Omega_1$ and $\rho > 0$ such that $B(y,2\rho) \subset B(0, r_k^{-1}r_0)$. For k large enough, this includes all the balls $B(y,\rho)$ such that $y \in \partial\Omega_k \cap B(0,2r)$ and $0 \leq \rho \leq 3r$. We shall gladly restrict to such k.

For each $t \in (0, r)$, denote by $Y_{k,t}$ a subset of $\partial\Omega_k \cap B(0, 2r)$, whose points lie at distances at least t from each other, and which is maximal with this property. The number of points of $Y_{k,t}$ is easy to estimate, as

$$
\begin{aligned}
\sharp Y_{k,t} &\leq C \sum_{y \in Y_{k,t}} t^{-d} \mu_k(B(y,t)) \leq C t^{-d} \mu_k\big(\bigcup_{y \in Y_{k,t}} B(y,t)\big) \\
&\leq C t^{-d} \mu_k(B(0,3r)) \leq C t^{1-n} r^{n-1} \qquad (20.33)
\end{aligned}
$$

by (20.32) and because the balls have bounded overlap. We apply this with $t = C\varepsilon_k$, where C is as in (20.28), notice that the balls $B(y, 2t)$, $y \in Y_{k,t}$, cover A_k, and deduce from (20.28) that

$$
|\mathcal{O}_k| \leq |A_k| \leq \sum_{y \in Y_{k,t}} |B(y, 2t)| \leq C t^n \sharp Y_{k,t} \leq C t r^{n-1} = C \varepsilon_k r^{n-1} \qquad (20.34)
$$

for k large; Since ε_k tends to 0 because the v_k converge to v_∞ uniformly in $B(0, r)$, (20.26), then (20.23) and Lemma 20.1, follow. ∎

The next stage is to apply the monotonicity argument. The wiser thing to do would probably be to restrict to the case when 0 is an interior point of Ω, but we shall try to include boundary points for some time. We shall assume that

(20.35) each limit set Ω_∞ that can be obtained from Ω by applying Lemma 19.1 to a sequence $\{r_k\}$ that tend to 0 is equal a.e. to an open cone centered at 0;

this is not very explicit because we did not really choose our notion of convergence for the domains, but fairly weak conditions of approximation of Ω by cones at 0 would imply this.

Theorem 20.2 *Assume that $0 \in \partial\Omega_1 = \partial\big\{x \in \mathbb{R}^n \,;\, u_1(x) > 0\big\}$, and that (20.2), (20.4), (20.5), (20.6), and (20.35) hold. Then there is a sequence $\{r_k\}$ that tend to 0 such that the blow-up limit $\mathbf{u}_\infty$ defined by $\{r_k\}$ exists, is non trivial, gives a minimizer of J_∞ as in (20.12), and is homogeneous of degree 1.*

By "exists", we would just mean that the $r^{-k}\mathbf{u}(r_k x)$ have a limit, but (20.12) asks for more anyway (a domain Ω_∞ and a partner $\mathbf{W}_\infty$). Since all the components other than $(u_{1,\infty})_+$ vanish by (20.13), by nontrivial we just mean that $u_{1,\infty}(x) > 0$ somewhere.

Note that when $0 \in \partial\Omega$, the assumptions of Theorem 20.2 put a nontrivial restriction on the shape of Ω near 0, because the limit cone Ω_∞ needs to be large enough to host a subdomain $\Omega_{1,\infty}$ and a nontrivial positive harmonic function on $\Omega_{1,\infty}$ that vanishes on the boundary and is homogeneous of degree 1. For instance, if $\partial\Omega$ is flat at 0, Ω_∞ is a half space, and there is just enough room to put such a harmonic function on Ω_∞. In this case, we could prove that $(u_{1,\infty})_+$ coincides with an affine function on Ω_∞. The point is that the Poincaré constant for $\Omega_{1,\infty} \cap \partial B(0,1)$ is at least as small as for a half sphere, which gives a control on the first eigenvalue of the Laplacian on $\Omega_{1,\infty} \cap \partial B(0,1)$, and then on the existence of homogeneous harmonic functions on $\Omega_{1,\infty}$; we would get that the Poincaré constant is the same as for the half sphere, then that $\Omega_{1,\infty} \cap \partial B(0,1)$ is a half sphere, $\Omega_{1,\infty}$ is a half space, and $(u_{1,\infty})_+$ is affine on $\Omega_{1,\infty} = \Omega_\infty$.

Proof. Let us first check that (20.1) follows from (20.2). We claim that (15.1) and (20.4) even imply that

$$\liminf_{r\to 0} \Phi^0_{1,1}(r) = \liminf_{r\to 0} \frac{1}{r^2}\int_{B(0,r)} \frac{|\nabla u_{1,+}|^2}{|x|^{n-2}}\,dx > 0, \tag{20.36}$$

where the first part comes from the definition (19.11). Indeed if $0 \in \partial\Omega_1$ and (15.1) holds, Theorem 15.2 applies to $B(0,r)$ for r small, and says that

$$\liminf_{r\to 0} \fint_{B(0,r)} |\nabla u_{1,+}|^2 \geq c_3 \tag{20.37}$$

for some constant $c_3 > 0$. But

$$\begin{aligned}
\Phi^0_{1,1}(r) &= \frac{1}{r^2}\int_{B(0,r)} \frac{|\nabla u_{1,+}|^2}{|x|^{n-2}}\,dx \geq \frac{1}{r^2}\int_{B(0,r)\setminus B(0,\eta r)} \frac{|\nabla u_{1,+}|^2}{|x|^{n-2}} \\
&\geq \frac{\eta^{n-2}}{r^n}\int_{B(0,r)\setminus B(0,\eta r)} |\nabla u_{1,+}|^2 = \frac{\eta^{n-2}}{r^n}\Big\{\int_{B(0,r)} |\nabla u_{1,+}|^2 - \int_{B(0,\eta r)} |\nabla u_{1,+}|^2\Big\} \\
&\geq \frac{\eta^{n-2}}{r^n}\int_{B(0,r)} |\nabla u_{1,+}|^2 - C\eta^{n-2} > c
\end{aligned} \tag{20.38}$$

for r small if we choose $\eta > 0$ small enough, depending on c_3 and the Lipschitz constant in (18.5). This proves (20.36).

Now we select a first sequence. Notice that our function Ψ from (20.7) is bounded, so

$$L = \limsup_{\rho\to 0} \Psi(\rho) \tag{20.39}$$

is finite. We choose $\{r_k\}$, tending to 0, such that

$$\lim_{k\to+\infty} \Psi(r_k) = L. \tag{20.40}$$

Then we find a subsequence for which we can apply Corollary 18.3; this gives a domain Ω_∞ and a pair $(\mathbf{u}_\infty, \mathbf{W}_\infty)$ that satisfies (20.12). By Lemma 20.1,

$$\Psi_\infty(1) = \lim_{k\to+\infty} \Psi(r_k r) = L \quad\text{and}\quad \Psi_\infty(r) = \lim_{k\to+\infty} \Psi(r_k r) \leq L \text{ for } r > 0. \tag{20.41}$$

Now we want to use the proof of Theorem 1.2 in [We] to show that

$$\Psi_\infty \text{ is a nondecreasing function on } (0,+\infty). \tag{20.42}$$

There are a few differences with his statement and the present situation that we need to discuss. First we need to modify a little our definition of minimizer for J_∞. Set $w = \mathbf{u}_{1,\infty}$. We know from (20.13) that $w \geq 0$ and all the other components are null. Let $Q \geq 0$ be defined by

$$Q^2 = \big[\lambda_1 - \min(0, \lambda_2, \ldots, \lambda_N)\big]_+. \tag{20.43}$$

Thus $Q > 0$ when (20.3) holds and $Q = 0$ otherwise. It turns out that the assumptions (15.1) and (18.11) imply (20.3), so $Q > 0$, but we do not need to know this for the moment.

We claim that w is a <u>one-phase ACF minimizer</u> in Ω_∞ with coefficient Q^2, and we mean by this that if $w^* \in W^{1,2}_{loc}(\mathbb{R}^n)$ is such that $w^* = 0$ almost everywhere on $\mathbb{R}^n \setminus \Omega_\infty$, and there is a ball B such that $w^* = w$ almost everywhere on $\mathbb{R}^n \setminus B$, then

$$\int_B |\nabla w|^2 + Q^2\big|\{x \in B\,;\, w(x) > 0\}\big| \leq \int_B |\nabla w^*|^2 + Q^2\big|\{x \in B\,;\, w^*(x) > 0\}\big|. \tag{20.44}$$

The verification is easy. Given a competitor w^* for w, we construct a competitor for $(\mathbf{u}_\infty, \mathbf{W}_\infty)$, use the minimality of this pair, and conclude. That is, we observe that w^*_+ is at least as good as w^*, take $u_1^* = w^*_+$ and keep $u_i^* = 0$ for $i \geq 2$. We keep $\mathbf{W}^* = \mathbf{W}$ on $\mathbb{R}^n \setminus B$, and on B we distinguish cases. If $Q > 0$ and $\min(0, \lambda_2, \ldots, \lambda_N) = \lambda_j$ for some $j \geq 2$, we take $B \cap W_1^* = B \cap \{w^* > 0\}$, $W_j^* = B \setminus W_1^*$, and all the other $B \cap W_i^*$ empty. If $Q > 0$ and $\min(0, \lambda_2, \ldots, \lambda_N) = 0$ we take $B \cap W_1^* = B \cap \{w^* > 0\}$ and all the other ones empty, and if $Q = 0$ we take $B \cap W_1^* = B$ and all the other ones empty. It is easy to see that we could hardly do better, and that the minimality of $(\mathbf{u}_\infty, \mathbf{W}_\infty)$ yields (20.44); we spare the details.

This definition is almost the same as the one used in [We], and we are happy because Q is constant and our functional will be nondecreasing. It is true that Theorem 1.2 in [We] is stated when $\Omega_\infty = \mathbb{R}^n$, but its proof also works when Ω_∞ is an open cone (and changing Ω on a set of measure 0 does not change the fact that w is an AFC minimizer). The main ingredient of the proof consists in taking a ball B centered at the origin, and testing the competitor w^* which is equal to w on $\mathbb{R}^n \setminus B$, is continuous across ∂B, and is homogeneous of degree 1 on B. The proof uses the fact that w is Lipschitz (this simplifies the computations), that $w(0) = 0$, and now we need to notice that $w^*(x) = 0$ on $\mathbb{R}^n \setminus \Omega_\infty$ because Ω is a cone. The claim (20.42) follows.

From (20.41) and (20.42) we easily deduce that

$$\Psi_\infty(r) = L \text{ for } r \geq 1; \tag{20.45}$$

we want to use this to show that

$$\mathbf{u}_\infty(\lambda x) = \lambda \mathbf{u}_\infty(x) \ \text{ for } x \in \mathbb{R}^n \text{ and } r > 0. \tag{20.46}$$

In fact, it is enough to prove this for $w = (u_{1,\infty})_+$ and x in the cone Ω_∞ (because $\mathbf{u}_\infty = 0$ almost everywhere on $\mathbb{R}^n \setminus \Omega_\infty$).

Again we just follow the proof given in [We]. The proof of monotonicity shows that for $0 < s < r$,

$$\Psi(r) - \Psi(s) \geq A(s,r), \tag{20.47}$$

where the quantity

$$A(s,r) = \int_{t=s}^{r} t^{-3} \int_{\xi\in\Omega_\infty\cap\partial B(0,1)} \Big[t\int_{a=0}^{t}\Big|\frac{\partial w}{\partial\rho}(a\xi)\Big|^2 da - \Big\{\int_{a=0}^{t}\frac{\partial w}{\partial\rho}(a\xi)da\Big\}^2\Big] d\sigma(\xi)dt \tag{20.48}$$

is nonnegative by an application of Cauchy-Schwarz in the a-integral. Here we know that $\Psi(s) = \Psi(r)$ for $1 < s < r$, and we get that

$$t\int_{a=0}^{t}\Big|\frac{\partial w}{\partial\rho}(a\xi)\Big|^2 da = \Big\{\int_{a=0}^{t}\frac{\partial w}{\partial\rho}(a\xi)da\Big\}^2 \tag{20.49}$$

for almost every $t \in (s,r)$ and almost every $\xi \in \Omega_\infty \cap \partial B(0,1)$. Thus for such t and ξ, $\frac{\partial w}{\partial\rho}(a\xi)$ is (almost everywhere) constant on $[0,t]$; (20.46) easily follows from this (recall that w is Lipschitz). Theorem 20.2 follows. ∎

When $n \leq 3$ and $\Omega_\infty = \mathbb{R}^n$, the main theorem of [CJK2] says that the homogeneous minimizer $\mathbf{u}_\infty$ that we produced for Theorem 20.2 has the following simple form: there is a unit vector $e \in \mathbb{R}^n$ such that

$$u_{1,\infty} = Q\max(0,\langle x,e\rangle) \ \text{ for } x \in \mathbb{R}^n, \tag{20.50}$$

where Q is given by (20.43). Notice that this forces $Q > 0$ (which we could also have obtained by comparing (15.1) with (18.3)), because $\mathbf{u}_\infty$ is not null. Thus we obtained the following result.

Corollary 20.3 *Suppose that $n = 2$ or 3, that 0 is an interior point of Ω, that $0 \in \partial\Omega_1 = \partial\{u_1(x) > 0\}$, and that (20.2), (20.4), and (20.6) hold. Then there is a sequence $\{r_k\}$ that tend to 0 such that the blow-up limit $\mathbf{u}_\infty$ defined by $\{r_k\}$ exists and is such that $\mathbf{u}_{i,\infty} = 0$ for $i > 0$, and (20.50) holds for some unit vector e, with*

$$Q^2 = \lambda_1 - \min(0, \lambda_2, \dots, \lambda_N) > 0. \tag{20.51}$$

Recall that the λ_i come from (18.11). We removed the assumptions (20.5) and (20.35) because they are trivial when 0 is an interior point of Ω.

The description of $\mathbf{u}_\infty$ by (20.50) implies that the free boundary $\partial\Omega_1$ has some flat blow-up limits at the origin, as in the following.

Lemma 20.4 *If $\{r_k\}$ is as in Corollary 20.3, then for each $R < 0$ there exist numbers $\beta_k > 0$ such that $\lim_{k\to+\infty}\beta_k = 0$ and*

$$|\langle x,e\rangle| \leq \beta_k r_k \ \ \text{for } x \in B(0,r_kR)\cap\partial\Omega_1. \tag{20.52}$$

Proof. Let $R > 0$ be given, and set $\varepsilon_k = ||u_{1,\infty} - u_{1,k}||_{L^\infty(B(0,2R))}$ for $k \geq 0$. Thus ε_k tends to 0. Let $k \geq 0$ and $x \in B(0, r_k R) \cap \partial\Omega_1$ be given. Set $y = r_k^{-1} y \in B(0, R)$, and observe that $u_{1,k}(y) = r_k^{-1} u_1(x) = 0$. Thus $u_{1,\infty}(y) \leq \varepsilon_k$, and by (20.50) $\langle y, e\rangle \leq Q^{-1}\varepsilon_k$. Thus $\langle x, e\rangle \leq Q^{-1}\varepsilon_k r_k$.

Now let $\eta > 0$ be small, and apply Theorem 15.1 to the ball $B(x, \eta r_k)$. This is possible, because we assumed that (15.1) holds and as soon as r_k is small enough. We get that $\fint_{B(x,\eta r)} |u_{1,+}|^2 \geq c_1(\eta r_k)^2$, by (15.6). By Chebyshev, we can choose $z \in B(x, \eta r_k)$ such that $u_1(z) \geq c_1^{1/2}\eta r_k$. Set $w = r_k^{-1} y$; then $z \in r_k^{-1} B(x, \eta r_k) \subset B(0, 2R)$ and $u_{1,k}(w) = r_k^{-1} u_1(z) \geq c_1^{1/2}\eta$. If k is so large (depending on η) that $\varepsilon_k < c_1^{1/2}\eta$, we get that $u_{1,\infty}(w) > 0$, hence $\langle w, e\rangle > 0$ by (20.50) and $\langle z, e\rangle > 0$ too. Then $\langle x, e\rangle \geq -\eta r_k$ for k large. This holds for every $\eta > 0$; the lemma follows. ∎

Of course it would be better to know that all the blow-up limits of $\mathbf{u}$ are as $\mathbf{u}_\infty$ above, or that all the blow-up limits of $\partial\Omega_1$ are hyperplanes, but here the presence of the other components Ω_φ seems to make it harder to prove better estimates, even though their contribution is small by (20.6). The situation will be better in Section 21, because none of the other Ω_φ are allowed to touch 0.

Notice that the flatness of $\mathbf{u}$ and $\partial\Omega_1$ at some small scales is often the entry point for further regularity results, but we do not know whether we still can prove such regularity results without further assumptions on the other components. See the comments in Section 21.

There is another case when we can get the same control on the blow-up limit $\mathbf{u}_\infty$ as in Corollaries 19.4 and 20.3, and this is when we know that the free boundary is flat at the origin.

Proposition 20.5 *Suppose that $0 \in \partial\Omega_1$, and that (20.2), (20.4), and (20.5) hold. Let $\{r_k\}$ be a sequence that tends to 0 and such that*

$$v(x) = \lim_{k\to+\infty} [u_{1,k}(x)]_+ \quad \text{exists for } x \in \mathbb{R}^n. \tag{20.53}$$

Also suppose that there is a unit vector $e \in \mathbb{R}^n$ and, for each $R > 0$, numbers $\beta_k > 0$ such that $\lim_{k\to+\infty} \beta_k = 0$ and (20.52) holds. Then there is a constant $a \neq 0$ such that

$$v(x) = \max(0, \langle x, ae\rangle) \quad \text{for } x \in \mathbb{R}^n. \tag{20.54}$$

If in addition 0 is an interior point of Ω and the $\mathbf{u}_k(x)$ converge to some limit $\mathbf{u}_\infty(x)$, then conclusion of either Corollary 19.4 or Corollary 20.3 holds.

Proof. Let $\{r_k\}$ be as in the statement, and as usual we use Lemma 19.1 to replace it with a subsequence for which we can apply Corollary 18.3 (see the discussion at the beginning of Section 19). This gives a limit $\mathbf{u}_\infty$ and a N-uple $\mathbf{W}_\infty$ such that $(\mathbf{u}_\infty, \mathbf{W}_\infty)$ is a local minimizer for J, as in (20.12) and (18.40). Naturally, $v = [u_{1,\infty}]_+$, so we look for a good description of $\mathbf{u}_\infty$.

We want to use (20.52) and the nondegeneracy results of Section 15 to show that $v > 0$ on one side of the hyperplane $H = \{x \in \mathbb{R}^n\,;\, \langle x, e\rangle = 0\}$, and $v = 0$ on the other side.

Let $x \in \mathbb{R}^n \setminus H$ be given, and choose $R > 0$ such that $x \in B(0, R/2)$ and $\delta > 0$ such that $B(x, 2\delta) \subset B(0, R) \setminus H$. Then set $x_k = r_k y$. Thus $x_k \in B(0, r_k r)$ and $\operatorname{dist}(x_k, H) \geq 2\delta r_k$. If k is so large that $\beta_k < \delta$, (20.52) says that $B(x_k, \delta r_k)$ does not meet $\partial\Omega_1$. One possibility is that $B(x_k, \delta r_k) \subset \mathbb{R}^n \setminus \Omega_1$; then $u_1 \leq 0$ on $B(x_k, \delta r_k)$ and $u_{1,k} \leq 0$ on $B(x, \delta)$. If this happens for an infinite number of values of k, we get that $u_{1,\infty} \leq 0$ and $v = 0$ on $B(x, \delta)$.

Otherwise, for an infinite number of values of k, we get that $B(x_k, \delta r_k) \subset \Omega_1$. We want to apply Theorem 15.3 to the point x_k. The assumption (15.1) comes from (20.2), $\delta(x_k) = \operatorname{dist}(x_k, \mathbb{R}^n \setminus \Omega_1) \leq r_k R$ (see (15.39) and recall that $0 \notin \Omega_1$), so $\mathbf{u}$ is Lipschitz on $B(x, \delta(x_k))$, with a constant that does not depend on k, by (19.5). Since $\delta(x_k) \leq r_k R \leq \min(\delta^{1/n}, 1)$ for k large (recall that ε is a constant that comes from (15.1)), (15.40) says that $u_1(x_k) \geq c_5 \delta(x_k) \geq c_5 r_k \delta$. Then $u_{1,k}(x) = r_k^{-1} u_1(x_k) \geq c_5 \delta$. Since this happens for infinitely many k, we get that $u_{1,\infty}(x) \geq c_5\delta$ as well. In this second case, there is a small ball centered at x on which $u_{1,\infty} > 0$. So, on each component of $\mathbb{R}^n \setminus H$, $v = [u_{1,\infty}]_+$ is either always positive, always 0.

We claim that the case when $u_{1,\infty} \leq 0$ on both sides of H is impossible. Indeed $0 \in \partial\Omega_1$, so for k large we can apply Theorem 15.1 to the ball $B(x, r_k)$, and use (15.6) and Chebyshev to find $y_k \in B(x, r_k)$ such that $u_1(y_k) \geq c_1^{1/2} r_k$; we can then extract a new sequence so that $r_k^{-1} y_k$ has a limit $y_\infty \in \overline{B}(0,1)$, and then $u_{1,k}(y_\infty) \geq u_1(y_k) - C|y_\infty - y_k| \geq c_1^{1/2} r_k/2$ for k large, hence $u_{1,k}(y_\infty) > 0$.

The case when $u_{1,\infty} > 0$ on both sides of H is impossible as well. Indeed, if this happens, $W_{1,\infty} = \mathbb{R}^n$ modulo a set of measure 0, then 0 is an interior point of Ω (because otherwise (19.1) would give big chunks of $\mathbb{R}^n \setminus \Omega$ in each small ball $B(0, r_k)$, and this would stay true for the limit Ω_∞, by (18.8). Then replacing $\mathbf{u}_{1,\infty}$ by the harmonic extension of its values on $\partial B(0,1)$ would be licit, and make a strictly better competitor than $\mathbf{u}_{1,\infty}$ (notice that the two are really different, because $\mathbf{u}_\infty(0) = 0$), a contradiction with the minimality of $(\mathbf{u}_\infty, \mathbf{W}_\infty)$.

Denote by U_+ the component of $\mathbb{R}^n \setminus H$ where $\mathbf{u}_{1,\infty}(x) > 0$. We know that $v = [\mathbf{u}_{1,\infty}]_+$ is harmonic on U_+, because $(\mathbf{u}_\infty, \mathbf{W}_\infty)$ is a local minimizer for J, and also that it is Lipschitz and vanishes at 0. It is then easy to show that v is affine on U_+ (use a reflection to get a harmonic function $\widetilde{v}$ in $\mathbb{R}^n$, and then write the Poisson formula for $\nabla\widetilde{v}$ on huge balls to find out that $\nabla^2\widetilde{v} = 0$). So v is given by (20.54) for some $a \neq 0$.

We continue a little further. If (19.52) holds, we can apply Corollary 19.4, $\mathbf{u}_\infty$ satisfies its conclusions, and we are happy (because the function $\mathbf{u}_\infty$ that we study now coincides with the $\mathbf{u}_\infty$ that appears in the statement). Otherwise, for $\varphi_1 = (1,1)$ and each choice of $\varphi_2 \neq \varphi_1$ and $R > 0$,

$$\begin{aligned} 0 &= L(\varphi_1, \varphi_2) = \lim_{\rho \to 0} \Phi^0_{\varphi_1}(\rho) \Phi^0_{\varphi_2}(\rho) \\ &= \lim_{k \to +\infty} \Phi^0_{\varphi_1}(r_k R)\, \Phi^0_{\varphi_2}(r_k R) = \Phi_{\varphi_1,\infty}(R)\, \Phi_{\varphi_2,\infty}(R) \end{aligned} \tag{20.55}$$

by (19.19) and (19.14). Now $\Phi_{\varphi_1,\infty}(R) = C > 0$, by direct computation with (20.54), so we are left with $\Phi_{\varphi_2,\infty}(R) = 0$, and hence $u_{\varphi,\infty} = 0$ (see the definition (19.13)). That is, $\mathbf{u}_\infty$

is given by the same sort of formula as in Corollary 20.3, we just need to check that $|a|$ in (20.54) is the same as Q in (20.51), and this is where we need our extra assumption that 0 is an interior point of Ω.

Indeed, otherwise it could be that $\partial\Omega$ is smooth near 0, $\Omega_1 \cap B(0,r) = \Omega \cap B(0,r)$ for r small, and the normal derivative of u along $\partial\Omega$ is very large, due to a large pressure coming from outside of $B(x,r)$. In other words, we are not free to add space to Ω_1 on the other side of Ω, so the first variation computation that leads to (20.51) is not available when $0 \in \partial\Omega$.

So we assume that 0 is an interior point of Ω, and then fact that $(\mathbf{u}_\infty, \mathbf{W}_\infty)$ is a local minimizer for the functional J (as in (20.12) and (18.40)), and the discussion near (20.43), show that v is a one-phase ACF minimizer, associated to the possibly different $Q_+ = \max(0,Q)$ (see (20.43)), and to the domain $\Omega_\infty = \mathbb{R}^n$. In addition, $Q_+ > 0$ because (20.54) holds for some $a \neq 0$, and if $Q_+ = 0$ we can replace v by its harmonic extension near 0, win on the energy, and not lose on the volume. So $Q > 0$ too. Finally, the fact that $|a| = Q$ comes from the fact that v is a one-phase ACF minimizer and a classical first variation computation. See Section 22, near (22.10). ∎

Remark 20.6 *An obvious defect of Proposition 20.5 (compared with Corollary 20.3 for instance) is that we cannot be sure in advance that the assumption will be satisfied for a given origin x_0 (so far called 0). But Proposition 16.3 gives a lot of points x_0 that it could be applied to. Indeed, under the current assumptions (including (10.2) and (15.1) for $i = 1$), Proposition 16.3 implies that $\partial\Omega_1$ has a tangent plane at $\mathcal{H}^{n-1}$-almost every point x_0. This is not hard; it uses the fact that locally uniformly rectifiable sets have tangent planes at almost every point. We could also use (16.26), and the fact that if a locally Ahlfors-regular set has an approximate tangent plane at some point, it has a true tangent plane at that point. See for instance Exercise 41.21 in [D]. Anyway, Proposition 20.5 applies to almost every point $x_0 \in \partial\Omega_1 \setminus \partial\Omega_0$, and with any sequence $\{r_k\}$ for which the $\mathbf{u}_k$ converges.*

21 Local regularity when all the indices are good

In this section we suppose that all the concerned indices are good, and reduce the study of our minimizer $(\mathbf{u}, \mathbf{W})$ near a point (say, the origin) to the study of minor variants of the Alt, Caffarelli, and Friedman's free boundary problems, with just one or two phases. Most of the section here will follow at once from results of Sections 19 and 20.

More precisely, we shall still assume that

(21.1) $\mathbf{u}$ satisfies (18.3) and (18.5), the f_i and g_i satisfy (18.4), and F satisfies (18.11),

as in the previous sections, and now that, for each index i such that

(21.2) $$0 \text{ lies in the boundary of } \big\{x \in \mathbb{R}^n \,;\, u_i(x) \neq 0\big\},$$

there exist $\lambda > 0$ and $\varepsilon > 0$ such that

(21.3) the analogue of (15.1) for the index i holds.

This is more brutal than in the previous sections, but not a shocking thing to ask; for instance, this holds if F is given by (1.7) with $q_i \geq c > 0$, or by (1.6) with $a > 0$ and $b \geq 0$, where in both cases we can trade against the empty set. Of course this assumption will simplify our life, because we won't have to worry about phases of $\mathbf{u}$ that may be small near 0, but not really vanish.

When 0 is not an interior point of Ω, we shall also assume that

$$\Omega \text{ satisfies the assumptions of Lemma 19.1.} \tag{21.4}$$

With these assumptions, we can already reduce the number of phases that live near the origin. For $\varphi = (i, \varepsilon) \in I = [1, N] \times \{-1, +1\}$, set

$$\Omega_\varphi = \big\{x \in \mathbb{R}^n \,;\, \varepsilon u_i(x) > 0\big\}, \tag{21.5}$$

and the denote by $I(0)$ the set of $\varphi \in I$ such that $0 \in \partial\Omega_\varphi$.

Lemma 21.1 *Under the assumptions above, $I(0)$ has at most two elements if 0 lies in the interior of Ω, and at most one if $0 \in \partial\Omega$.*

As we were preparing this document, we were informed of a recent result of D. Bucur and B. Velichkov [BV] , which contains a result very similar to Lemma 21.1, although in a slightly different context. Their paper is based on a 3-phase monotonicity formula, which they manage to prove and use without knowing that their analogue of $\mathbf{u}$ is Hölder-continuous (left alone, Lipschitz).

Proof. Because of (21.3), we know from (20.36) that

$$\liminf_{r \to 0} \Phi^0_\varphi(r) > 0 \ \text{ for } \varphi \in I(0). \tag{21.6}$$

We may assume that $I(0)$ contains (at least) two elements φ_1 and φ_2. Notice that $L(\varphi_1, \varphi_2) > 0$ (see the definition (19.19)); then we can apply Corollary 19.4, and point (ii) says that $\lim_{r\to 0} \Phi^0_\varphi(r) = 0$ for $\varphi \in I \setminus \{\varphi_1, \varphi_2\}$. By (21.6) again, such φ lie out of $I(0)$; hence $I(0)$ has at most 2 elements. In addition, part (iii) of Corollary 19.4 says that 0 lies in the interior of Ω; this completes our proof of Lemma 21.1. ∎

Notice that since the sets $\overline{\Omega}_\varphi$ are closed, and touch 0 only when $\varphi \in I(0)$, there is a small radius $r_0 > 0$ such that

$$u_\varphi(x) = 0 \ \text{ for } x \in B(0, r_0) \text{ and } \varphi \in I \setminus I(0), \tag{21.7}$$

where $u_\varphi = (\varepsilon u_i)_+$ when $\varphi = (i, \varepsilon)$. Thus we get a small ball $B_0 = B(0, r_0)$ where $\mathbf{u}$ has at most two nonzero phases. We shall soon see that in this ball, $(\mathbf{u}, \mathbf{W})$ solves a simpler free boundary problem.

Remark 21.2 *Our proof of Lemma 21.1 does not seem to give lower bounds for r_0 above. But the following scheme, that occurred to us* after *discussing the results of [BV] with B. Velichkov, seems to give such lower bounds. Suppose that Ω_{φ_j} meets $B(0,\tau)$ for three different phases φ_j; by (15.7), $|\Omega_{\varphi_3} \cap B(0,r)| \geq c_2 r^n$ for $2\tau \leq r \leq C^{-1}\rho_0$, where ρ_0 is as in (18.5). We can use this to show that for many radii r, the two open sets $\Omega_{\varphi_1} \cap \partial B(0,r)$ and $\Omega_{\varphi_2} \cap \partial B(0,r)$ have a significantly smaller joint measure than $\partial B(0,r)$, which leads to a strict increase (with estimates) for the corresponding functional $\Phi^0_{\varphi_1,\varphi_2} = \Phi^0_{\varphi_1}\Phi^0_{\varphi_2}$ of Section 9. If τ is small enough, we get that $\Phi^0_{\varphi_1,\varphi_2}(2\tau)$ is so small that this contradicts (15.10). Notice that for this, the fact that $\mathbf{u}$ is Lipschitz and the quantitative nondegeneracy estimates of Section 9 would be needed.*

Remark 21.3 *In the last sections, we have taken the fact that $\mathbf{u}$ is Lipschitz as a assumption, but Theorem 10.1 (when 0 is an interior point) and Theorem 11.1 (when $0 \in \partial\Omega$) give sufficient conditions for this to happen.*

Let us now formalize our claim that the restriction of $(\mathbf{u}, \mathbf{W})$ to B_0 comes from a simpler free boundary problem. We start with the simpler case when there are two true phases.

Case 1. Let us assume, in addition to the hypotheses above, that $I(0) = \{\varphi_1, \varphi_2\}$, with $\varphi_j = (i_j, \varepsilon_j)$. Recall that in this case 0 is an interior point of Ω, and choose B_0 so small that $B_0 \subset \Omega$.

Lemma 21.4 *Set*

$$v = u_{\varphi_1} - u_{\varphi_2} = [\varepsilon_1 u_{i_1}]_+ - [\varepsilon_2 u_{i_2}]_+ \tag{21.8}$$

and denote by $\mathcal{F}_v$ the class of functions $w \in W^{1,2}_{loc}(\mathbb{R}^n)$ such that $w = v$ almost everywhere on $\mathbb{R}^n \setminus B_0$. Then v is a minimizer in $\mathcal{F}_v$ of the functional

$$J(w) = G(w) + \int_{B(0,r)} |\nabla w|^2 + (w_+)^2 f_{i_1} + (w_-)^2 f_{i_2} - w_+ g_{i_1} - w_- g_{i_2}, \tag{21.9}$$

where the f_i and the g_i are as in the definition of M in (1.4), $w_\pm = \max(0, \pm w)$ as usual, and the volume term $G(w)$ can be computed in terms of the sets

$$A_{w,\pm} = \big\{x \in B_0\,;\, \pm w(x) > 0\big\}. \tag{21.10}$$

See (21.13) below for the formula.

Notice that the real-valued function contains all the information about $\mathbf{u}$ in B_0, by definition of $I(0)$. The condition that $w = v$ on $\mathbb{R}^n \setminus B_0$ is our way of stating a Dirichlet constraint on ∂B_0. We left the computation of G for the proof, because the formula is a little ugly. But let us say now that when F is given by (1.7) with nonnegative functions q_i, we can take

$$G(w) = \int_{A_{w,+}} q_{i_1} + \int_{A_{w,-}} q_{i_2}. \tag{21.11}$$

Proof. Let us prove the lemma, and at the same time define G. The idea is simple: we associate to each $w \in \mathcal{F}_v$ a competitor $(\mathbf{u}_w, \mathbf{W}_v)$ for $(\mathbf{u}, \mathbf{W})$, test the minimality of $(\mathbf{u}, \mathbf{W})$ on this competitor, and hopefully we shall get the desired inequality.

Due to our definition of phases, we shall need to distinguish between two main cases. First assume that $i_1 \neq i_2$, and to simplify the notation, that $\varphi_1 = (1, +1)$ and $\varphi_2 = (2, +1)$. The N-uple of functions associated to w is just $\mathbf{u}_w = (w_+, w_-, u_3, \ldots, u_N)$ (notice that w is also defined on $\mathbb{R}^n \setminus B_0$), and a simple N-uple of sets $\mathbf{W}_w$ that we can take is given by

$$\begin{aligned} W_{w,1} &= A_{w,+} \cup (W_1 \setminus B_0), \\ W_{w,2} &= A_{w,-} \cup (W_2 \setminus B_0), \\ W_{w,i} &= W_i \setminus B_0 \quad \text{ for } i > 2. \end{aligned} \tag{21.12}$$

It is easy to see that $(\mathbf{u}_w, \mathbf{W}_v)$ is an acceptable pair (i.e., that $(\mathbf{u}_w, \mathbf{W}_v) \in \mathcal{F}(\Omega)$); in particular there is no gluing problem because $u_{w,i} = u_i$ for $i > 2$, and the $W_{w,i}$ are contained in Ω because $B_0 \subset \Omega$.

If F is a nondecreasing function of the W_i, this is the best that we could do. But in some cases, we may prefer to use another element of the class $\mathcal{H}(w)$, where $\mathcal{H}(w)$ is the set of N-uples $\mathbf{W}^* = (W_1^*, \ldots, W_N^*)$, where the W_i^* are disjoint, and each W_i^* contains $W_{w,i}$ and coincides with W_i and $W_{w,i}$ on $\mathbb{R}^n \setminus B_0$. Notice that $(\mathbf{u}_w, \mathbf{W}_v^*)$ also lies in $\mathcal{F}(\Omega)$ when $\mathbf{W}^* \in \mathcal{H}(w)$. We set

$$G(w) = \inf\big\{F(\mathbf{W}^*)\,;\, \mathbf{W}^* \in \mathcal{H}(w)\big\}. \tag{21.13}$$

Notice that $G(v) = F(\mathbf{W})$, because $(\mathbf{u}, \mathbf{W})$ is a minimizer, hence $F(\mathbf{W})$ also minimizes F in the class $\mathcal{H}(v)$.

We are ready to check the minimality of v. Let $w \in \mathcal{F}_v$ be given, and let $(\mathbf{u}_w, \mathbf{W}_v^*)$ be as above. Call M_0 the part of $M(\mathbf{u})$ that comes from integrating outside of B_0; then

$$\begin{aligned} J(v) &= G(v) + \int_{B(0,r)} |\nabla u_1|^2 + u_1^2 f_1 - u_1 g_1 + u_2^2 f_2 - u_2^2 g_2 \\ &= F(\mathbf{W}) + \int_{B(0,r)} |\nabla u_1|^2 + u_1^2 f_1 - u_1 g_1 + u_2^2 f_2 - u_2^2 g_2 \\ &= J(\mathbf{u}, \mathbf{W}) - M_0 \leq J(\mathbf{u}_w, \mathbf{W}_w) - M_0 = J(w) \end{aligned} \tag{21.14}$$

because $(\mathbf{u}, \mathbf{W})$ minimizes J, all the other components of $\mathbf{u}$ and $\mathbf{u}_w$ vanish on B_0, and $\mathbf{u}_w = \mathbf{u}$ on $\mathbb{R}^n \setminus B(0, R)$. This proves the minimality of v when $\varphi_1 = (1, +1)$ and $\varphi_2 = (2, +1)$.

When $\varphi_1 = (1, +1)$ and $\varphi_2 = (1, -1)$, we just need to define $\mathbf{u}_w$ and $\mathbf{W}_w$ slightly differently. We set $\mathbf{u}_w = (w, u_2, \ldots, u_N)$,

$$W_{w,1} = A_{w,+} \cup A_{w,-} \cup (W_1 \setminus B_0) \ \text{ and } \ W_{w,i} = W_i \setminus B_0 \text{ for } i > 1. \tag{21.15}$$

We then define $\mathcal{H}(w)$ and G and complete the argument as above. All the other cases can be treated like one of these two, and the lemma follows. ∎

The formula (21.13) is not very beautiful, but in many cases we may use the simpler formula

$$G(w) = F(\mathbf{W}_w), \text{ with } \mathbf{W}_w \text{ as in (21.12) or (21.15).} \tag{21.16}$$

This is the case when F is given by (1.7) with nonnegative functions q_i, and this is why we get (21.11) (we may add a constant to G without changing the result). But for instance if q_N is negative and smaller than the all the other q_i, the good choice of G is really $G(w) = \int_{A_{w,+}} (q_{i_1} - q_N) + \int_{A_{w,-}} (q_{i_2} - q_N)$.

The proof above shows that Lemma 21.4 is still true (but may be less precise) with G given by (21.16) as soon as $F(\mathbf{W}) = F(\mathbf{W}_v)$, and we claim that this is the case as soon as

$$|W_i \cap B_0| = 0 \text{ for } i \neq i_1, i_2. \tag{21.17}$$

To check this, let us even show that for each good index i,

$$W_i = \big\{x \in \mathbb{R}^n \,;\, u_i(x) \neq 0\big\} \tag{21.18}$$

modulo negligible sets. Indeed, call V the set on the right; we know that $V \subset W_i$ almost everywhere, because $u_i = 0$ almost everywhere on $\mathbb{R}^n \setminus W_i$. If $|W_i \setminus V| > 0$, we can take a subset of positive measure in $W_i \setminus V$, use (15.1) to sell part of it to some other region, throw the rest to the trash, and make a profit. This is impossible because $(\mathbf{u}, \mathbf{W})$ is a minimizer. So (21.18) holds.

We use this, with $i = i_1$ or i_2, and (21.17), to see that $\mathbf{W}_v = \mathbf{W}$ (almost everywhere) when we take $w = v$ in (21.12) or (21.15). The claim follows.

We also deduce from (21.18) that (21.17) is automatic when all the indices i (such that W_i meets B_0) satisfy (15.1).

In this case 1, the results of Section 19 are available. That is, all the blow-up limits of $\mathbf{u}$ at the origin are given by pairs of affine functions, as in (19.23) and Corollary 19.4, and the free boundary $\partial\Omega_{\varphi_1} \cup \partial\Omega_{\varphi_2}$ is flat near the origin, as in Proposition 19.5.

It can be expected that, modulo additional regularity and nondegeneracy assumptions on F, the method of [C1], [C2] yields a much stronger regularity result, namely that both free boundaries $\partial\Omega_{\varphi_1}$ and $\partial\Omega_{\varphi_2}$ coincide with $C^{1+\alpha}$ submanifolds in a neighborhood of the origin. In the special case when $\mathbf{u}$ (or equivalently, v above) is harmonic on the Ω_{φ_1}, F is given by (1.7) with nonnegative functions q_i such that q_{i_1} and q_{i_2} are both Lipschitz near 0, and, say $q_{i_1}(0) > q_{i_2}(0) > 0$, we can apply directly the results of [C1] and [C2], and the description of the blow-up limits given by Corollary 19.4 is more than enough to say that v above is a weak solution. Here we have a slightly different situation in a few respects, because v satisfies a slightly different equation (coming from (9.6)), and F is a little more general than in (1.7). We do not expect major differences to come from our different equation (9.6), and it can be imagined that if we replace (18.11) with a stronger form of approximation by volume functionals of type (1.7), Caffarelli's regularity results may go through. We shall not do the verification here, but at least we know that the difficulties, if they exist, do not come from the fact that J is a functional with many phases, or the fact that we would not have enough control in the Lipschitz properties of $\mathbf{u}$ or the description of the blow-up limits.

Even if this works, we may still wonder about what happens when we do not assume the additional condition that $q_{i_1}(0) > q_{i_2}(0) > 0$ (or similar ones on the λ_i from (18.11)), but just that all the indices are good (as in (18.11)). Also, we do not know whether the presence of additional phases φ_i, with low energy near 0 as in (19.53), may disturb the results above, even when F is as in (1.7) with smooth nonnegative (but not positive) coefficients q_i.

Case 2. Let us now assume that $I(0)$ has a unique element φ, and let us assume for definiteness that $\varphi = (1, +1)$. For the moment, let us authorize the case when $0 \in \partial\Omega$, and let B_0 be as in (21.7). We start with an analogue of Lemma 21.4.

Set $v = (u_1)_+$ and

$$\mathcal{F}(B_0, \Omega, v) = \big\{w \in W^{1,2}(\mathbb{R}^n)\,;\, w = v \text{ a.e. on } \mathbb{R}^n \setminus B_0 \text{ and } w = 0 \text{ a.e. on } \mathbb{R}^n \setminus \Omega\big\}. \tag{21.19}$$

We also add in (21.19) the requirement that $w \geq 0$, if we required that $u_1 \geq 0$ in the definition of $\mathcal{F}$ (see Definition 1.1); otherwise, this is not needed.

Lemma 21.5 *Suppose as above that (21.1)-(21.5) hold, and let B_0 and $\mathcal{F}(B_0, \Omega, v)$ be as in (21.7) and (21.19). Then v is a minimizer in $\mathcal{F}_v$ of the functional*

$$J_+(w) = G_+(w) + \int_{B(0,r)} |\nabla w|^2 + w^2 f_1 - w g_1 \tag{21.20}$$

where the f_i and the g_i are as in (1.4) and the volume term $G_+(w)$, which can be computed in terms of

$$A_w = \big\{x \in B_0\,;\, w(x) > 0\big\}, \tag{21.21}$$

is given by (21.22) below.

Proof. The proof is essentially the same as for Lemma 21.4. To each $w \in \mathcal{F}(B_0, \Omega, v)$, we associate $\mathbf{u}_w = (w, u_2, \ldots, u_N)$ and $\mathbf{W}_w = (A_w \cup (W_1 \setminus B_0), W_2 \setminus B_0, \ldots, W_N \setminus B_0)$ (as we did near (21.12)), and observe that $(\mathbf{u}_w, \mathbf{W}_w) \in \mathcal{F}(\Omega)$, where this time we also need to say that $w \geq 0$ if we required that $u_1 \geq 0$ in the definition of $\mathcal{F}(\Omega)$, and $\mathbf{u}_w = 0$ almost everywhere on $\mathbb{R}^n \setminus \Omega$, by definition of $\mathcal{F}(B_0, \Omega, v)$. Then we denote by $\mathcal{H}(w)$ the class of N-uples $\mathbf{W}^* = (W_1^*, \ldots, W_N^*)$ such that the W_i^* are disjoint and contained in Ω, and each W_i^* coincides with W_i on $\mathbb{R}^n \setminus \Omega$ and contains $W_{w,i}$. Finally we set

$$G(w) = \inf\{F(\mathbf{W}^*)\,;\, \mathbf{W}^* \in \mathcal{H}(w)\big\} \tag{21.22}$$

as we did before. Then we follow the proof of Lemma 21.4 and get the result. Also notice that when $|W_i \cap B_0| = 0$ for $i > 1$, we may simplify the definition of G, and merely take $G(w) = F(\mathbf{W}_w)$, as in (21.16). ∎

Again we are in a relatively good situation to continue our investigation, with the functional J_+ that has only one phase.

If we also assume that F is Lipschitz (as in (10.2)) and (if $0 \in \partial\Omega$) that the blow-up limits of Ω are open cones Ω_∞, as in (20.35), then Theorem 20.2 says that some blow-up limits of $\mathbf{u}$ at 0 are nontrivial homogeneous minimizers of a simple Alt-Caffarelli-Friedman functional (J_∞ as in (18.40)), but in the perhaps complicated cone Ω_∞.

If in addition $n \leq 3$ and 0 is an interior point of Ω, this blow-up limit of $\mathbf{u}_\infty$ is given by an affine function, as in Corollary 20.3 and (20.50), and we also get that the free boundary $\partial\Omega_1$ is flat at the origin (but only along the corresponding sequence), as in (20.52).

Again, the situation is not perfect, because in addition to the potential difficulties of Case 1 (connected to our slightly different equations for $\mathbf{u}$, the more general F, and other non-good phases that may float around), we may have complications due to the shape of Ω, and the fact that one-phase minimizers of the Alt-Caffarelli functional in $\mathbb{R}^n$, $n \geq 4$ are not well understood yet. But we shall nonetheless quit here for the moment and pretend that this is a general problem about free boundaries, and not a specific problem about $N \geq 3$.

Let us also mention that if we do not insist on studying $\mathbf{u}$ and the free boundary near a specific point, Proposition 16.3 gives lots of points of the $\partial\Omega_i$ where the blow-up limits of $\mathbf{u}$ and $\partial\Omega_i$ are controlled as in Corollaries 19.4 and 20.3. See Remark 20.6 and Proposition 20.5. This is true in all dimensions, and the points where we can do this are also good candidates for further local regularity, depending for instance on how the techniques of [C1] and [C2] adapt.

Remark 21.6 *Lemmas 21.1, 21.4, and 21.5 seem to be give a fast shortcut to many of the results above, especially if the technique of [BV] allows to prove Lemma 21.1 here. But this would not give exactly the same estimates. First we would still need to check that the special form of F that we have does not disturb the usual Lipschitz and nondegeneracy estimates in the one- or two-phase problems (they are badly needed), but also our estimates would depend on the small radius r_0 which, with fast proofs, we won't be able to estimate. Finally, we can always hope to understand better the situation when some of the indices i do not satisfy* (15.1)*, and some of the long proofs above may prepare the way.*

22 First variation and the normal derivative

We start this section with some first variation computations and the verification of the formulas (19.49), (19.50), and (20.50). We first do a computation with one phase, consider

$$v(x) = a\langle x, e\rangle_+ \tag{22.1}$$

for some unit vector e and some positive constant a, and try to find necessary conditions for v to define a local minimizer of our functional J_∞ in $\mathbb{R}^n$, or more simply to be a one-phase ACF minimizer in $\mathbb{R}^n$, as defined near (20.44).

The typical small variations that we would try in general would be to add a small function to v, but here it will be simpler to compose with a one parameter family of diffeomorphism, because it will be easier to keep track of the domains. Let us choose coordinates in $\mathbb{R}^n$ so that $e = (0, \dots, 1)$, and write the generic point of $\mathbb{R}^n$ as $x = (x', y)$, with $x' \in e^\perp \simeq \mathbb{R}^{n-1}$ and $t \in \mathbb{R}e \simeq \mathbb{R}$.

We pick a nonnegative smooth bump function φ and for small $t \in \mathbb{R}$ define a diffeomorphism Φ_t of $\mathbb{R}^n$ by $\Phi_t(x) = x + t\varphi(x)e$. Then we set

$$v_t(x) = v(\Phi_t(x)) = a\langle \Phi_t(x), e\rangle_+ \tag{22.2}$$

and compute the derivative of v_t. Set $\mathcal{O} = \big\{x \in \mathbb{R}^n\,;\, \langle x, e\rangle > 0\big\}$ and $\mathcal{O}_t = \Phi_t^{-1}(\mathcal{O}) = \big\{x \in \mathbb{R}^n\,;\, \langle \Phi_t(x), e\rangle > 0\big\}$; we just need to compute ∇v_t on $\mathcal{O}_t$, because it vanishes on $\mathbb{R}^n \setminus \mathcal{O}_t$. And for $x \in \mathcal{O}_t$,

$$\frac{\partial v_t}{dx_j} = a\langle e, \frac{\partial \Phi_t}{dx_j}\rangle = a\delta_{j,n} + at\frac{\partial \varphi}{dx_j}, \tag{22.3}$$

where we used the Kronecker symbol $\delta_{j,n}$. Let B be a ball that contains the support of φ; observe that by (22.3), $|\nabla v_t(x)|^2 = a^2[1 + 2t\frac{\partial\varphi}{\partial x_n}(x) + O(t^2)]$, where $O(t^2)$ is function which is less than Ct^2 and is supported in B. Then

$$\int_{2B} |\nabla v_t|^2 = a^2 \int_{2B\cap\mathcal{O}_t} |\nabla v_t|^2 = a^2 \int_{2B\cap\mathcal{O}_t} 1 + 2t\frac{\partial\varphi}{\partial x_n}(x) + O(t^2). \tag{22.4}$$

Notice that for t small, $2B = \varphi_t(2B)$; then we set $y = \Phi_t(x)$, with $y \in 2B \cap \mathcal{O}$, notice that $dy = J_{\Phi_t}(x)dx = 1 + \frac{\partial\varphi}{dx_n}(x)$ (because the matrix of $D\Phi_t$ is an identity matrix, plus a last column composed of the $\frac{\partial\varphi}{dx_j}$), so

$$\begin{aligned}
\int_{2B} |\nabla v_t|^2 &= a^2 \int_{2B\cap\mathcal{O}} [1 + 2t\frac{\partial\varphi}{\partial x_n}(\Phi_t^{-1}(y)) + O(t^2)][1 + t\frac{\partial\varphi}{\partial x_n}(\Phi_t^{-1}(y)]^{-1} dy \\
&= a^2 \int_{2B\cap\mathcal{O}} [1 + t\frac{\partial\varphi}{\partial x_n}(\Phi_t^{-1}(y)) + O(t^2)]dy = a^2 \int_{2B\cap\mathcal{O}} [1 + t\frac{\partial\varphi}{\partial x_n}(y) + O(t^2)]dy \\
&= a^2|2B\cap\mathcal{O}| + a^2 t \int_{2B\cap\mathcal{O}} \frac{\partial\varphi}{\partial x_n} + O(t^2).
\end{aligned} \tag{22.5}$$

Set $A = \int_{2B\cap\mathcal{O}} \frac{\partial\varphi}{\partial x_n}$; notice that $A = -\int_{2B\cap\partial\mathcal{O}} \varphi(x')dx'$, and so we can choose φ so that $A < 0$. The same computation as above also shows that

$$\begin{aligned}
|2B\cap\mathcal{O}_t| &= \int_{2B\cap\mathcal{O}_t} dx = \int_{2B\cap\mathcal{O}} [1 + t\frac{\partial\varphi}{\partial x_n}(\Phi_t^{-1}(y)]^{-1} dy \\
&= |2B\cap\mathcal{O}_t| - t\int_{2B\cap\mathcal{O}} \frac{\partial\varphi}{\partial x_n} + O(t^2) = |2B\cap\mathcal{O}_t| - tA + O(t^2).
\end{aligned} \tag{22.6}$$

If v is a one-phase ACF minimizer in $\mathbb{R}^n$, associated to the constant Q^2, (20.44) says that for t small

$$\int_{2B} |\nabla v|^2 + Q^2|2B\cap\mathcal{O}| \leq \int_{2B} |\nabla v_t|^2 + Q^2|2B\cap\mathcal{O}_t|, \tag{22.7}$$

and by (22.5) and (22.6), this means that

$$0 \leq a^2 tA - Q^2 tA + O(t^2). \tag{22.8}$$

Since this holds for t small (of both signs), we get that $a^2 = Q^2$.

Thus, in (20.50), we had no choice about the value of Q, it had to be given by (20.43), where the λ_j are the same as in the definition of J_∞ (see (18.40)), and the story is the same for the end of the proof of Proposition 20.5.

We can do a similar computation for 2-phase functions. That is, let $\mathbf{u}_\infty$ be a local minimizer in $\mathbb{R}^n$ of the functional J_∞ that shows up in (18.40), and suppose that we have two phases $\varphi_1 = (i_1, \varepsilon_1)$ and $\varphi_2 = (i_2, \varepsilon_2)$ such that

$$v_1(x) = a_1 \langle x, e \rangle_+ \ \text{ and } \ v_2(x) = a_2 \langle x, e \rangle_- \ \text{ for } x \in \mathbb{R}^n, \tag{22.9}$$

where we set $v_j = [\varepsilon_j u_{i_j,\infty}]_+$, and with coefficients $a_1, a_2 > 0$. Then there is no place left for the other phases of $\mathbf{u}_\infty$ (so they are null), and also our only choice is to take $W_{i_j} = \big\{x \in \mathbb{R}^n \,;\, (-1)^j \langle x, e \rangle > 0\big\}$ if $i_1 \neq i_2$, and $W_{i_1} = W_{i_2} = \mathbb{R}^n$ if $i_1 = i_2$.

We can compare $\mathbf{u}_\infty$ with $\mathbf{u}_\infty \circ \Phi_t$, where Φ_t is as above, and then use the definition (18.40) of a local minimizer. Again we have no other choice than taking the sets $\Phi_t^{-1}(W_{i_j})$. We can compute both pieces $\int_{2B} |\nabla v_j|^2 + \lambda_{i_j} |2B \cap \Phi_t^{-1}(W_{i_j})|$ as we did before, and then (18.40) yields

$$0 \leq a_1^2 At - \lambda_{i_1} At + a_2^2 At - \lambda_{i_2} At + O(t^2), \tag{22.10}$$

just as we obtained (22.8) above. That is, we obtain the necessary condition

$$a_1^2 - a_2^2 = \lambda_{i_1} - \lambda_{i_2}, \tag{22.11}$$

which is the same as (19.49).

But we can also let Φ_t operate on v_1 alone, and let v_2 as it is, provided that the domain $\mathcal{O}_t = \Phi_{-1}(\mathcal{O})$ associated to the modification of v_1 stays inside of $\mathcal{O}$, so that $(v_1)_t$ does not interfere with v_2. We can get that if $\varphi \geq 0$, by restricting to $t < 0$. This even leaves some free space (the sets $\mathcal{O} \setminus \mathcal{O}_t$), which we can attribute to the empty set, or any index i that we may find suitable. That is, set $\lambda_0 = \min(0, \lambda_1, \dots, \lambda_N)$; when we add $\mathcal{O} \setminus \mathcal{O}_t$ to some other W_i, or just drop it, we win $(\lambda_{i_1} - \lambda_0)|\mathcal{O} \setminus \mathcal{O}_t|$ in the volume term.

The same computation as for (22.8) now yields

$$0 \leq a_1^2 tA - (\lambda_{i_1} - \lambda_0) tA + O(t^2). \tag{22.12}$$

Now recall that $A < 0$, and also that we are only allowed to take $t < 0$; then we only get the inequality

$$a_1^2 \geq \lambda_{i_1} - \lambda_0 = \lambda_{i_1} - \min(0, \lambda_1, \dots, \lambda_N). \tag{22.13}$$

The same argument also yields $a_2^2 \geq \lambda_{i_2} - \lambda_0$. Those are the constraints that were noted in (19.50).

It is possible to do the same sort of first variation computations near a minimizer $(\mathbf{u}, \mathbf{W})$, assuming that everything is sufficiently smooth for us to make the computations. This leads to Euler-Lagrange equations at the boundary, that can be stated as follows. On a (smooth enough) piece of boundary that would separate two regions Ω_{φ_i}, the normal derivatives $\frac{\partial v_j}{\partial n}$ satisfy

$$(22.14) \qquad \Big(\frac{\partial v_1}{\partial n}\Big)^2 - \Big(\frac{\partial v_2}{\partial n}\Big)^2 = \lambda_{i_1} - \lambda_{i_2},$$

as in (22.11), where as usual we write $\varphi_j = (i_j, \varepsilon_j)$ and $v_j = [\varepsilon_j u_{i_j}]_+$. Similarly, one-sided variations lead to the constraint

$$(22.15) \qquad \Big(\frac{\partial v_j}{\partial n}\Big)^2 \geq \lambda_{i_j} - \min(0, \lambda_1, \dots, \lambda_N),$$

as in (22.13). And along a nice piece of $\partial\Omega_{\varphi_1}$, that would lie inside of Ω and separate Ω_{φ_1} from a region where $\mathbf{u} = 0$, we get that

$$(22.16) \qquad \Big(\frac{\partial v_1}{\partial n}\Big)^2 = \lambda_{i_1} - \min(0, \lambda_1, \dots, \lambda_N),$$

as in our estimate below (22.13).

Finally, along a (smooth enough) piece of $\partial\Omega \cap \partial\Omega_{\varphi_1}$ (where we would have Ω_{φ_1} on one side and $\mathbb{R}^n \setminus \Omega$ on the other side), we would get that $\big(\frac{\partial v_1}{\partial n}\big)^2 \geq \lambda_{i_1} - \min(0, \lambda_1, \dots, \lambda_N)$, by the same one-sided variations as for (22.15).

We do not do the computation here, because they are similar, but just more complicated than the computations above. But the following discussion will justify the equations (22.14)-(22.16) on an almost everywhere pointwise level.

Proposition 22.1 *Assume that 0 is an interior point of Ω, that $0 \in \partial\Omega_1$, and that (20.2) and (20.4) hold; in particular, $i = 1$ is a good index. Assume that we can find a blow-up limit $\mathbf{u}_\infty$ of $\mathbf{u}$ at the origin such that*

$$(22.17) \qquad u_{1,\infty}(x) = a\langle x, e\rangle \quad \text{for } x \in \mathbb{R}^n$$

for some choice of $a > 0$ and $e \in \partial B(0,1)$, and that the limit

$$(22.18) \qquad h(0) = \frac{1}{\omega_{n-1}} \lim_{r\to 0} r^{1-n} \langle \Delta u_{1,+}, \mathbb{1}_{B(0,r)}\rangle$$

exists. Then $h(0) = a$.

Recall that $h(0)$ is the same as in (16.30), that the limit $h(x)$ exists for $\mathcal{H}^{n-1}$-every point $\partial\Omega_1 \cap \mathrm{int}(\Omega)$, and that h is the Radon-Nikodym density of μ, the restriction of $\Delta u_{1,+}$ to $\partial\Omega_1$, with respect with the restriction of $\mathcal{H}^{n-1}$ to $\partial\Omega_1$. See Proposition 16.2.

The first assumption says that there is a sequence $\{r_k\}$ that tends to 0 such that the $\mathbf{u}_k$ defined by (18.2) tend to $\mathbf{u}_\infty$. In general, $\mathbf{u}_\infty$ and e may depend on the sequence $\{r_k\}$, but the proposition says that a does not. The case when $a = 0$ would not be allowed by our assumption (20.2), because (20.36) says that $\liminf_{r\to 0} \Phi^0_{1,1}(r) > 0$, and then $\Phi_{1,1,\infty}(1) = \lim_{k\to+\infty} \Phi^0_{1,1}(r_k) > 0$ by Lemma 19.2.

Finally observe that if (20.2) holds, the assumptions of Proposition 22.1 (with 0 replaced by x_0) are satisfied at $\mathcal{H}^{n-1}$-every point $x_0 \in \partial\Omega_1 \cap \mathrm{int}(\Omega)$ for which (20.4) holds, because $\partial\Omega_1$ has a tangent plane at x_0; see Remark 20.6 and Proposition 20.5. In this case, since $\partial\Omega_1$ has a tangent plane at x_0, we even get that e does not depend on the blow-up sequence, and that $u_{1,+}$ has a normal derivative at x_0, equal to a. [The verification would need a little bit of playing with (22.17), to get a coherent coherent choice of e, but this is easy and we leave the details.]

Proof. Since we intend to take limits, let us replace $\mathbb{1}_{B(0,r)}$ by smoother functions. For $\tau > 0$ small, choose a smooth radial function φ_τ such that

$$\mathbb{1}_{B(0,1-\tau)} \leq \varphi_\tau \leq \mathbb{1}_{B(0,1)}, \tag{22.19}$$

and then define $\varphi_{\tau,k}$ by $\varphi_{\tau,k}(x) = \varphi_\tau(r_k^{-1}x)$, where r_k comes from our blow-up sequence. We will use $\varphi_{\tau,k}$ to approximate $\mathbb{1}_{B(0,r_k)}$. Set $v = u_{1,+}$ and $v_k(x) = r_k^{-1}v(r_k x)$ to save notation, and compute

$$\begin{aligned}
A_k : &= r_k^{1-n}\langle \Delta u_{1,+}, \varphi_{\tau,k}\rangle = -r_k^{1-n}\langle \nabla u_{1,+}, \nabla\varphi_{\tau,k}\rangle = -r_k^{1-n}\langle \nabla v, \nabla\varphi_{\tau,k}\rangle \\
&= -r_k^{-n}\int \langle \nabla v(x), (\nabla\varphi_\tau)(r_k^{-1}x)\rangle dx = -\int \langle \nabla v(r_k y), \nabla\varphi_\tau(y)\rangle dy \\
&= -\int \langle \nabla v_k(y), \nabla\varphi_\tau(y)\rangle dy.
\end{aligned} \tag{22.20}$$

Then we apply Corollary 18.3 (no need to extract a subsequence or apply Lemma 20.1 here, because 0 is an interior point of Ω), and get that (18.17) holds for all R. Hence

$$\lim_{k\to+\infty} A_k = -\int \langle \nabla v_\infty, \nabla\varphi_\tau\rangle = -\int \langle \nabla u_{1,\infty,+}, \nabla\varphi_\tau\rangle. \tag{22.21}$$

Set $H = \big\{x \in \mathbb{R}^n\,;\, \langle x, e\rangle\big\} = 0$ and $H_+ = \big\{x \in \mathbb{R}^n\,;\, \langle x, e\rangle\big\} > 0$. Then (22.17) yields

$$\lim_{k\to+\infty} A_k = -a\int_{H_+} \frac{\partial\varphi_\tau}{\partial e} = a\int_H \varphi_\tau, \tag{22.22}$$

and by (22.19)

$$(1-\tau)^{n-1}a \leq \frac{1}{\omega_{n-1}} \lim_{k\to+\infty} A_k \leq a. \tag{22.23}$$

Now we estimate

$$\delta_k = r_k^{1-n}\langle \Delta u_{1,+}, \mathbb{1}_{B(0,r_k)}\rangle - A_k = r_k^{1-n}\langle \Delta u_{1,+}, (\mathbb{1}_{B(0,r_k)} - \varphi_{\tau,k})\rangle. \tag{22.24}$$

Recall from Proposition 16.2 that $\Delta u_{1,+} = \mu + [f_1 v - \frac{1}{2} g_1] \mathbb{1}_{\Omega_1}$, where μ is a positive measure on $\partial\Omega_1$ and (we just need to know that) $w = [f_1 v - \frac{1}{2} g_1] \mathbb{1}_{\Omega_1}$ is bounded. Thus

$$
\begin{aligned}
|\delta_k| &\leq r_k^{1-n} \int (\mathbb{1}_{B(0,r_k)} - \varphi_{\tau,k}) d\mu + r_k^{1-n} \int_{B(0,r_k)} |w| \\
&\leq r_k^{1-n} \mu(B(0,r_k) \setminus B(0,(1-\tau)r_k)) + C r_k
\end{aligned} \tag{22.25}
$$

by (22.19). Also notice that for r small,

$$
\left| r^{1-n} \langle \Delta u_{1,+}, \mathbb{1}_{B(0,r)} \rangle - r^{1-n} \mu(B(0,r)) \right| = r^{1-n} \left| \int_{B(0,r)} w \right| \leq Cr \tag{22.26}
$$

for the same reason, which means that

$$
h(0) = \frac{1}{\omega_{n-1}} \lim_{r \to 0} r^{1-n} \mu(B(0,r)) \tag{22.27}
$$

by (22.18). Then $r_k^{1-n} \mu(B(0,r_k) \setminus B(0,(1-\tau)r_k))$ tends to $[1-(1-\tau)^{n-1}]\omega_{n-1} h(0)$, and $|\delta_k| \leq C\tau$ for k large. We may now put things together. For k large,

$$
\omega_{n-1} |h(0) - a| \leq \left| \omega_{n-1} h(0) - r_k^{1-n} \langle \Delta u_{1,+}, \mathbb{1}_{B(0,r_k)} \rangle \right| + |\delta_k| + |A_k - \omega_{n-1} a| \leq C\tau, \tag{22.28}
$$

by (22.18) and (22.23). Since τ is as small as we want, we get that $h(0) = a$; Proposition 22.1 follows. ∎

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