Internal DLA and the Gaussian free field

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Abstract

Central limit theorem: In previous works, we showed that the internal DLA cluster on $\mathbb{Z}^d$ with t particles is a.s. spherical up to a maximal error of $O(\log t)$ if $d = 2$ and $O(\sqrt{\log t})$ if $d \geq 3$. This paper addresses "average error": in a certain sense, the average deviation of internal DLA from its mean shape is of *constant* order when $d = 2$ and of order $r^{1-d/2}$ (for a radius r cluster) in general. Appropriately normalized, the fluctuations (taken over time and space) scale to a variant of the Gaussian free field.

Smoother than lattice balls: In some ways, internal DLA clusters in high dimensions grow even more smoothly than the lattice balls $\mathbb{Z}^d \cap B_r(0)$. The fluctuations of the latter are related to famous problems in number theory (including Gauss's circle and ball problems). These number theoretic fluctuations (while very small) are much larger than those produced by the randomness associated to internal DLA.

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Contents

1 Introduction

1.1 Overview

We study the scaling limits of *internal diffusion limited aggregation* ("internal DLA"), a growth model introduced in [MD86, DF91]. In internal DLA, one inductively constructs an **occupied set** $A_t \subset \mathbb{Z}^d$ for each time $t \geq 0$ as follows: begin with $A_0 = \emptyset, A_1 = \{0\}$ and let A_{t+1} be the union of A_t and the first place a random walk from the origin hits $\mathbb{Z}^d \setminus A_t$.

The purpose of this paper is to study the growing family of sets A_t. Let $A_t^* \subset \mathbb{R}^d$ be the union of the unit cubes centered at points of A_t. Following the pioneering work of [LBG92], it is by now well known that, for large t, the set A_t^* approximates an origin-centered Euclidean ball $B_r(0)$ (where $r = r(t)$ is such that $B_r(0)$ has volume t). The authors recently showed that this is true in a fairly strong sense [JLS09, JLS10a, JLS10b]: the maximal distance of a point on ∂A_t^* from $\partial B_r(0)$ is a.s. $O(\log t)$ if $d = 2$ and $O(\sqrt{\log t})$ if $d \geq 3$. In fact, if C is large enough, the probability of an error of $C \log t$ (or $C\sqrt{\log t}$ when $d \geq 3$) decays faster than any fixed (negative) power of t. Some of these results are obtained by different methods in [AG10a, AG10b].

This paper will ask what happens if, instead of considering the maximal gap between ∂A_t^* from $\partial B_r(0)$ at time t, we consider the "average error" at time t (allowing inner and outer errors to cancel each other out). It turns out that in a distributional "average fluctuation" sense, the set A_t^* deviates from $B_r(0)$ by only a constant number of lattice spaces when $d = 2$ and by an even smaller amount when $d \geq 3$. Appropriately normalized, the fluctuations of A_t, taken over time and space, define a distribution on $\mathbb{R}^d$ that converges in law to a variant of the Gaussian free field (GFF): a random distribution on $\mathbb{R}^d$ that we will call the **augmented Gaussian free field**. (It comes from the GFF by replacing variances associated to spherical harmonics of degree ℓ by variances associated to spherical harmonics of degree $\ell + 1$; see Section 1.5.)

The "augmentation" is related (as discussed below) to the curvature of the sphere. (Though we do not prove this here, we expect that continuous time internal DLA on the half cylinder $[0, \ldots, m]^{d-1} \times \mathbb{Z}_+$, with particles started uniformly on the bottom level, produces clusters whose boundaries are approximately flat cross-sections of the cylinder, with fluctuations that scale to the ordinary GFF on the half cylinder as $m \to \infty$.) To our knowledge, this result has not been previously suggested in either the physics or the mathematics literature.

Nonetheless, the heuristic idea is easy to explain. Write a point $z \in \mathbb{R}^d$ as $r\theta$ where $|\theta| = 1$. Suppose that at each time t the boundary of A_t is approximately parameterized by $r_t(\theta)\theta$ for a function r_t on the unit disc. How do we expect the discrepancy $\bar{r}_t$ between r_t and its expectation to evolve in time? First, there is a smoothing effect coming from the fact that places where $\bar{r}_t$ is small are more likely to be hit by the random walks (hence more likely to grow in time). Second, there is another smoothing effect coming from the curvature of the sphere, which implies that even if particles hit all angles with equal probability, the magnitude of the fluctuations in $\bar{r}_t$ would shrink as t increased and these fluctuations were averaged over larger spheres. And finally, there is a space-time white noise term coming from the randomness of the particles.

The white noise should correspond to adding independent Brownian noise terms to the spherical Fourier modes of $\bar{r}_t$. The rate of smoothing in time should be approximately given by $\Lambda\bar{r}_t$ for some linear operator Λ. Since the random walks approximate Brownian motion (which is rotationally invariant), we would expect Λ to commute with orthogonal rotations, and hence have spherical harmonics as eigenfunctions. With the right normalization and parameterization, it is therefore natural to expect the spherical Fourier modes of $\bar{r}_t$ to evolve as independent Brownian motions subject to linear "restoration forces" depending on their degrees. It turns out that the restriction of the (augmented) GFF on $\mathbb{R}^d$ to a centered volume t sphere evolves in time in a similar way.

Of course, as stated above, the spherical Fourier modes of $\bar{r}_t$ have not really been defined (since the boundary of A_t generally *cannot* be parameterized by $r_t(\theta)\theta$). The key to our construction is to define related quantities that (in some sense) encode the Fourier modes of $\bar{r}_t$ and are easy to work with. These quantities will turn out to be martingales obtained by summing discrete harmonic polynomials over A_t.

1.2 FKG inequality statement

Before we set about formulating our central limit theorems precisely, we mention a previously overlooked fact. Suppose that we run internal DLA in continuous time by adding particles at Poisson random times instead of at integer times: this process we will denote by $A_{T(t)}$ (or often just A_T) where $T(t)$ is the counting function for a Poisson point process in the interval $[0,t]$ (so $T(t)$ is Poisson distributed with mean t). We then view the IDLA growth process as a (random) function on $[0,\infty) \times \mathbb{Z}^d$, which takes the value 1 or 0 on the pair (t,x) accordingly as $x \in A_{T(t)}$ or $x \notin A_{T(t)}$. Write $\mathcal{F}$ for the set of all functions $[0,\infty) \times \mathbb{Z}^d \to \{0,1\}$, endowed with the coordinate-wise partial ordering.

Theorem 1.1. (FKG inequality) *For any two increasing L^2 functions $F, G : \mathcal{F} \to \mathbb{R}$, the random variables $F(\{A_{T(t)}\}_{t\geq 0})$ and $G(\{A_{T(t)}\}_{t\geq 0})$ are nonnegatively correlated.*

One example of an increasing function is the total number of particles absorbed at a fixed time t. Another is -1 times the smallest t which all of the particles in some fixed set are occupied. Intuitively, Theorem 1.1 means that if one point is absorbed at an early time, then it is conditionally more likely for all other points to be absorbed early. The FKG inequality is an important feature of the discrete and continuous Gaussian free fields [She07], so it is interesting (and reassuring) that it appears in internal DLA at the discrete level.

Note that sampling a continuous time internal DLA cluster at time t is equivalent to first sampling a Poisson random variable T with expectation t and then sampling an ordinary internal DLA cluster with T particles. (By the central limit theorem, $|t-T|$ has order $\sqrt{t}$ with high probability.) Although using continuous time amounts to only a modest time reparameterization (chosen independently of everything else) it is sometimes aesthetically natural. Our use of "white noise" in the heuristic of the previous section implicitly assumed continuous time. (Otherwise the noise would have to be conditioned to have mean zero at each time.)

1.3 Main results in dimension two

For $x \in \mathbb{Z}^2$ write

$$F(x) := \inf\{t : x \in A_{T(t)}\}$$

and

$$L(x) := \sqrt{F(x)/\pi} - |x|.$$

In words, $L(x)$ is the difference between the radius of the area t disc — at the time t that x was absorbed into A_T — and $|x|$. It is a measure of how much later or earlier x was absorbed into A_T than it would have been if the sets $A_{T(t)}$ were *exactly* centered discs of area t. By the main result of [JLS10a],

$$|L(x)| = O(\log |x|),$$

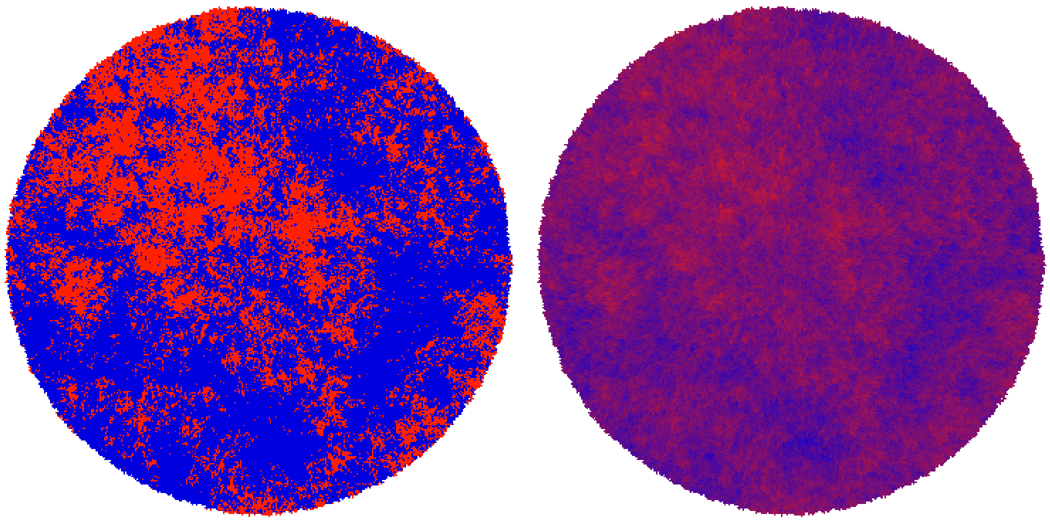

Figure 1: Left: Continuous-time IDLA cluster $A_{T(t)}$ for $t = 10^5$. Early points (where L is negative) are colored red, and late points (where L is positive) are colored blue. Right: The same cluster, with the function $L(x)$ represented by blue-red scaling.

almost surely.

The coloring in Figure 1(a) indicates the sign of the function $L(x)$, while Figure 1(b) provides a more nuanced illustration of $L(x)$. Note that the use of continuous time means that the average of $L(x)$ over x may differ substantially from 0. Indeed we see that — in contrast with the corresponding discrete-time figure of [JLS10a] — there are noticeably fewer early points than late points in Figure 1(a), which corresponds to the fact that in this particular simulation $T(t)$ was smaller than t for most values of t. Since for each fixed $x \in \mathbb{Z}^2$ the quantity $L(x)$ is a decreasing function of $A_t(x)$, the FKG inequality holds for L as well. The positive correlation between values of L at nearby points is readily apparent from the figure.

Identify $\mathbb{R}^2$ with $\mathbb{C}$ and let H_0 be the linear span of the set of functions on $\mathbb{C}$ of the form $\mathrm{Re}(az^k)f(|z|)$ for $a \in \mathbb{C}$, $k \in \mathbb{Z}_{\geq 0}$, and f smooth and compactly supported on $\mathbb{R}_{>0}$. The space H_0 is obviously dense in $L^2(\mathbb{C})$, and it turns out to be a convenient space of test functions. The augmented GFF (and its restriction to $\partial B_1(0)$) will be defined precisely in Section 1.5.

Theorem 1.2. (Weak convergence of the lateness function) *As $R \to \infty$, the rescaled functions on $\mathbb{R}^2$ defined by $G_R((x_1, x_2)) := L((\lfloor Rx_1 \rfloor, \lfloor Rx_2 \rfloor))$ converge to the augmented Gaussian free field h in the following sense: for each set of test functions $\phi_1, \ldots, \phi_k$ in H_0, the joint law of the inner products (ϕ_j, G_R) converges to the joint law of (ϕ_j, h).*

Theorem 1.3. (Fluctuations from circularity) *Let $A_t^* \subset \mathbb{R}^2$ be the union of the unit squares centered at points of A_t. Consider the random discrepancy function on $\mathbb{R}^2$ given by*

$$E_t := \sqrt{t}\big(1_{\sqrt{\pi/t}A_t^*} - 1_{B_1(0)}\big).$$

As $t \to \infty$, these functions converge (in the same sense as in Theorem 1.2) to the restriction of the augmented GFF to $\partial B_1(0)$. The latter restriction is absolutely continuous with respect to the restriction of the ordinary GFF on $\mathbb{R}^2$ to $\partial B_1(0)$ (where an additive constant for the latter is chosen so that the mean on $\partial B_1(0)$ is zero).

1.4 Main results in general dimensions

We do not expect the exact analog of Theorem 1.2 to be true for large d. The main reason for this is that the lattice balls themselves do not grow very smoothly in high dimensions. By classical number theory results, the size of $B_r(0) \cap \mathbb{Z}^d$ is approximately the volume of $B_r(0)$ — but with errors of order r^{d-2} in all dimensions $d \geq 5$. The errors in dimension $d = 4$ are of order r^{d-2} times logarithmic correction factors. It remains a famous open number theory problem to estimate the errors when $d \in \{2, 3\}$. (When $d = 2$ this is called Gauss's circle problem.) A recent and detailed survey of this subject appears in [IKKN04].

The results mentioned above imply that even if points were added to A_t precisely in order of their radius, we would find gaps between the radius of A_t and the radius of the ball $B_r(0)$ of volume t, gaps of order at least r^{-1} if $d \geq 4$. On the other hand, we will see that the kinds of fluctuations that emerge from internal DLA randomness are of the order that one would obtain by spreading an extra $\sqrt{t}$ particles over a constant fraction of the spherical boundary, which is also what one obtains by changing the radius (along some or all of the boundary) by $r^{1-d/2}$. This is of course much smaller than r^{-1} whenever $d > 4$.

Fortunately, there is another way of formulating a central limit theorem for internal DLA that is both natural and amenable to proof in any dimension. This formulation requires that we define and interpret the (augmented) Gaussian free field in a particular way.

Given smooth real-valued functions f and g on $\mathbb{R}^d$, write

$$(f, g)_\nabla = \int_{\mathbb{R}^d} \nabla f(z) \cdot \nabla g(z) dz. \tag{1}$$

Given a bounded domain in $\mathbb{R}^d$, let $H(D)$ be the Hilbert space closure in $(\cdot, \cdot)_\nabla$ of the set of smooth compactly supported functions on D. We define $H = H(\mathbb{R}^d)$ analogously except that the functions are taken modulo additive constants. The Gaussian free field (GFF) is defined formally by

$$h := \sum_{i=1}^{\infty} \alpha_i f_i, \tag{2}$$

where the f_i are any fixed $(\cdot,\cdot)_\nabla$ orthonormal basis for H and the α_i are i.i.d. mean zero, unit variance normal random variables. (One also defines the GFF on D similarly, using $H(D)$ in place of H.) The augmented GFF will be defined similarly below, but with a slightly different inner product.

Since the sum a.s. does not converge within H, one has to think a bit about how h is defined. Note that for any *fixed* $f = \sum \beta_i f_i \in H$, the quantity $(h, f)_\nabla := \sum(\alpha_i f_i, f)_\nabla = \sum \alpha_i \beta_i$ is almost surely finite, and has the law of a centered Gaussian with variance $\|f\|_\nabla = \sum |\beta_i|^2$. However, there a.s. exist some functions $f \in H$ for which the sum does not converge, and $(h,\cdot)_\nabla$ cannot be considered as a continuous functional on h. Rather than try to define $(h, f)_\nabla$ for all $f \in H$, it is often more convenient and natural to focus on some subset of f values (with dense span) on which $f \to (h, f)_\nabla$ is a.s. a continuous function (in some topology). Here are some sample approaches to defining a GFF on D:

1. h **as a random distribution:** For each smooth, compactly supported ϕ, write $(h,\phi) := (h, -\Delta^{-1}\phi)_\nabla$, which (by integration by parts) is formally the same as $\int h(z)\phi(z)dz$. This is almost surely well defined for all such ϕ and makes h a random distribution [She07]. (If $D = \mathbb{R}^d$ and $d = 2$, one requires $\int \phi(z)dz = 0$, so that (h,ϕ) is defined independently of the additive constant. When $d > 2$ one may fix the additive constant by requiring that the mean of h on $B_r(0)$ tends to zero as $r \to \infty$ [She07].)

2. h **as a random continuous** $(d + 1)$**-real-parameter function:** For each $\varepsilon > 0$ and $z \in \mathbb{R}^d$, let $h_\varepsilon(z)$ denote the mean value of h on $\partial B_\varepsilon(z)$. For each fixed z, this $h_\varepsilon(z)$ is a Brownian motion in time parameterized by $-\log\varepsilon$ in dimension 2, or $-\varepsilon^{2-d}$ in higher dimensions [She07]. For each fixed ε, h_ε can be thought of as a regularization of h (a point of view used extensively in [DS10]).

3. h **as a family of "distributions" on origin-centered circles:** For each polynomial function ψ on $\mathbb{R}^d$ and each time t, define $\Phi_h(\psi, t)$ to be the integral of $h\psi$ over $\partial B_r(0)$ where $B_r(0)$ is the origin-centered ball of volume t. We actually lose no generality in requiring ψ to be a harmonic polynomial on $\mathbb{R}^d$, since the restriction of any polynomial to $\partial B_r(0)$ agrees with the restriction of a (unique) harmonic polynomial.

The latter approach turns out to be particularly natural for our purposes. Using this approach, we will now give our first definition of the augmented GFF: it is the centered Gaussian function Φ_h for which

$$\mathrm{Cov}\big(\Phi_h(\psi_1, t_1), \Phi_h(\psi_2, t_2)\big) = \int_{B_r(0)} \psi_1(z)\psi_2(z)dz, \tag{3}$$

where $B_r(0)$ is the ball of volume $\min\{t_1, t_2\}$. In particular, taking $\psi_1 = \psi_2 = \psi$, then we find that

$$\mathrm{Var}\big(\Phi_h(\psi, t)\big) = \int_{B_r(0)} \psi(z)^2 dz. \tag{4}$$

Though not immediately obvious from the above, we will see in Section 1.5 that this definition is very close to that of the ordinary GFF. Now, for each integer m and harmonic polynomial ψ, we will write $\psi_m(x)$ for the discrete polynomial on $\frac{1}{m}\mathbb{Z}^d$ (defined precisely in Section 2.2) that approximates ψ in the sense that for each fixed ψ, we have $|\psi_m(x) - \psi(x)| = O(|x|^d/m^2)$. In particular, if we fix ψ and limit our attention to x in a fixed bounded subset of $\mathbb{R}^d$, then we have $|\psi_m(x) - \psi(x)| = O(1/m^2)$. Let $\mathcal{G}$ denote the grid comprised of the edges connecting nearest neighbor vertices of $\mathbb{Z}^d$. (As a set, $\mathcal{G}$ consists of the points in $\mathbb{R}^d$ with at most one non-integer coordinate.) As in [JLS10a], we extend the definition of ψ_m to $\mathcal{G}$ by linear interpolation.

Now write

$$\Phi_A^m(\psi, t) := \Big(m^{-d/2} \sum_{x \in \mathbb{Z}^d} \psi_m(x) A_{m^d t}(mx)\Big) - t\psi_m(0) \tag{5}$$

$$= m^{-d/2} \sum_{x \in A_t} \psi_m(x/m) - t\psi_m(0). \tag{6}$$

This is a way of measuring the deviation of $A_{m^d t}$ from circularity.

Theorem 1.4. *Let h be the augmented GFF, and Φ_h as discussed above. Then the random functions Φ_A^m converge in law to Φ_h (w.r.t. the smallest topology that makes $\Phi \to \Phi(\psi, t)$ continuous for each ψ and t). In other words, for each finite collection of pairs (ψ, t), the joint law of Φ_A^m on this set converges in law to the joint law of Φ_h evaluated on the same set.*

1.5 Comparing the GFF and the augmented GFF

We may write a general vector in $\mathbb{R}^d$ as $r\theta$ where $r \in [0, \infty)$ and $\theta \in S^{d-1} := \partial B_1(0)$. We write the Laplacian in spherical coordinates as

$$\Delta = r^{1-d} \frac{\partial}{\partial r} r^{d-1} \frac{\partial}{\partial r} + r^{-2} \Delta_{S^{d-1}}. \tag{7}$$

Let A_ℓ denote the space of all homogenous harmonic polynomials of degree ℓ in d variables, and let H_ℓ denote the space of functions on S^{d-1} obtained by restriction from A_ℓ. If $f \in H_\ell$, then we can write $f(r\theta) = g(\theta) r^\ell$ for a function $g \in H_\ell$, and setting (7) to zero at $r = 1$ yields

$$\Delta_{S^{d-1}} g = -\ell(\ell + d - 2) g,$$

i.e., g is an eigenfunction of $\Delta_{S^{n-1}}$ with eigenvalue $-\ell(\ell + d - 2)$. Note that (7) continues to be zero if we replace ℓ with the negative number $\ell' := -(d-2) - \ell$, since the expression $-\ell(\ell+d-2)$ is unchanged by replacing ℓ with ℓ'. Thus, $g(\theta)r^{\ell'}$ is also harmonic on $\mathbb{R}^d \setminus \{0\}$.

Now, suppose that g is normalized so that

$$\int_{S^{d-1}} g^2(z)dz = 0.$$

By scaling, the integral of f over $\partial B_R(0)$ is thus given by $R^{d-1}R^{2\ell}$. The L^2 norm on all of $B_R(0)$ is then given by

$$\int_{B_R(0)} f(z)^2 dz = \int_0^R r^{d-1}r^{2\ell}dr = \frac{R^{d+2\ell}}{d+2\ell}. \tag{8}$$

A standard identity states that the Dirichlet energy of g, as a function on S^{d-1}, is given by the L^2 inner product $(-\Delta g, g) = \ell(\ell+d-2)$. The square of $\|\nabla f\|$ is given by the square of its component along S^{d-1} plus the square of its radial component. We thus find that the Dirichlet energy of f on $B_R(0)$ is given by

$$\begin{aligned}
\int_{B_R(0)} \|\nabla f(z)\|^2 dz &= \ell(\ell+d-2)\int_0^R r^{d-1}r^{2(\ell-1)}dr + \int_0^R r^{d-1}r^{2(\ell-1)}\ell^2 dr \\
&= \frac{\ell(\ell+d-2)}{2\ell+d-2}R^{2\ell+d-2} + \frac{\ell^2}{2\ell+d-2}R^{2\ell+d-2} \\
&= \frac{2\ell^2+(d-2)\ell}{2\ell+(d-2)}R^{2\ell+d-2} \\
&= \ell R^{2\ell+d-2}.
\end{aligned}$$

Now suppose that we fix the value of f on $\partial B_R(0)$ as above but harmonically extend it outside of $B_R(0)$ by writing $f(r\theta) = R^{\ell-\ell'}g(\theta)r^{\ell'}$ for $r > R$. Then the Dirichlet energy of f outside of $B_R(0)$ can be computed as

$$R^{2(\ell-\ell')}\ell(\ell+d-2)\int_R^\infty r^{d-1}r^{2(\ell'-1)}dr + R^{2(\ell-\ell')}\int_R^\infty r^{d-1}r^{2(\ell'-1)}(\ell')^2 dr,$$

which simplifies to

$$\begin{aligned}
-\frac{\ell^2+\ell(d-2)+(\ell')^2}{2\ell'+(d-2)}R^{2\ell+d-2} &= -\frac{\ell^2+\ell(d-2)+\big(\ell+(d-2)\big)^2}{2(-\ell-(d-2))+(d-2)}R^{2\ell+d-2} \\
&= -\frac{2\ell^2+3\ell(d-2)+(d-2)^2}{-2\ell-(d-2)}R^{2\ell+d-2} \\
&= (\ell+d-2)R^{2\ell+d-2}.
\end{aligned}$$

Combining the inside and outside computations in the case $R=1$, we find that the harmonic extension $\tilde{f}$ of the function given by g on S^{d-1} has Dirichlet energy

$2\ell + (d-2)$. If we decompose the GFF into an orthonormal basis that includes this $\tilde{f}$, we find that the component of $\tilde{f}$ is a centered Gaussian with variance $\frac{1}{2\ell+(d-2)}$. If we replace $\tilde{f}$ with the harmonic extension of $g(R^{-1}\theta)$ (defined on $\partial B_R(0)$), then by scaling the corresponding variance becomes $\frac{1}{2\ell+(d-2)} R^{2-d}$.

Now in the augmented GFF the variance is instead given by (8), which amounts to replacing $\frac{1}{2\ell+(d-2)}$ with $\frac{1}{2\ell+d}$. Considering the component of $g(R^{-1}\theta)$ in a basis expansion the space of functions on $\partial B_R(0)$ requires us to divide (8) by $R^{2\ell}$ (to account for the scaling of f) and by $(R^{d-1})^2$ (to account for the larger integration area), so that we again obtain a variance of $\frac{1}{2\ell+d} R^{2-d}$ for the augmented GFF, versus $\frac{1}{2\ell+(d-2)} R^{2-d}$ for the GFF.

In light of Theorem 1.3, the following implies that (up to absolute continuity) the scaling limit of fixed-time A_t fluctuations can be described by the GFF itself.

Proposition 1.5. *When $d = 2$, the law ν of the restriction of the GFF to the unit circle (minus a constant, so that the mean is zero) is absolutely continuous w.r.t. the law μ of the restriction of the augmented GFF restricted to the unit circle.*

Proof. The relative entropy of a Gaussian of density $e^{-x^2/2}$ with respect to a Gaussian of density $\sigma^{-1} e^{-x^2/(2\sigma^2)}$ is given by

$$F(\sigma) = \int e^{-x^2/2} \left((\sigma^{-2} - 1) x^2/2 + \log \sigma \right) dx = (\sigma^{-2} - 1)/2 + \log \sigma.$$

Note that $F'(\sigma) = -\sigma^{-3} + \sigma^{-1}$, and in particular $F'(1) = 0$. Thus the relative entropy of a centered Gaussian of variance 1 with respect to a centered Gaussian of variance $1 + a$ is $O(a^2)$. This implies that the relative entropy of μ with respect to ν — restricted to the jth component α_j — is $O(j^{-2})$. The same holds for the relative entropy of ν with respect to μ. Because the α_j are independent in both μ and ν, the relative entropy of one of μ and ν with respect to the other is the sum of the relative entropies of the individual components, and this sum is finite. ☐

2 General dimension

2.1 FKG inequality: Proof of Theorem 1.1

We recall that increasing functions of a Poisson point process are non-negatively correlated [GK97]. (This is easily derived from the more well known statement that increasing functions of independent Bernoulli random variables are non-negatively correlated.) Let μ be the simple random walk probability measure on the space Ω of walks W beginning at the origin. Then the randomness for internal DLA is given by a rate-one Poisson point process on $\mu \times \nu$ where ν is Lebesgue measure on $[0, \infty)$. A realization of this process is a random collection of points in $\Omega \times [0, \infty)$. It is easy to see that adding an additional point (w, s) increases the value of A_t for all times t. The A_t are hence increasing functions of the Poisson point process, and are non-negatively correlated. Since F and L are increasing functions of the A_t, we conclude that increasing functions of these objects are also increasing functions of the point process — and are thus also non-negatively correlated.

2.2 Discrete Harmonic Polynomials

Let $\psi(x_1, \ldots, x_d)$ be a polynomial that is harmonic on $\mathbb{R}^d$, that is

$$\sum_{i=1}^{d} \frac{\partial^2 \psi}{\partial x_i^2} = 0.$$

In this section we give a recipe for constructing a polynomial ψ_1 that closely approximates ψ and is discrete harmonic on $\mathbb{Z}^d$, that is,

$$\sum_{i=1}^{d} D_i^2 \psi_1 = 0$$

where

$$D_i^2 \psi_1 = \psi_1(x + \mathbf{e}_i) - 2\psi_1(x) + \psi_1(x - \mathbf{e}_i)$$

is the symmetric second difference in direction $\mathbf{e}_i$. The construction described below is nearly the same as the one given by Lovász in [Lov04], except that we have tweaked it in order to obtain a smaller error term: if ψ has degree m, then $\psi - \psi_1$ has degree $m-2$ instead of $m-1$. Discrete harmonic polynomials have been studied classicaly, primarily in two variables: see for example Duffin [Duf56], who gives a construction based on discrete contour integration.

Consider the linear map

$$\Xi : \mathbb{R}[x_1, \ldots, x_d] \to \mathbb{R}[x_1, \ldots, x_d]$$

defined on monomials by

$$\Xi(x_1^{m_1} \cdots x_d^{m_d}) = P_{m_1}(x_1) \cdots P_{m_d}(x_d)$$

where we define

$$P_m(x) = \prod_{j=-(m-1)/2}^{(m-1)/2} (x + j).$$

Lemma 2.1. *If $\psi \in \mathbb{R}[x_1, \ldots, x_d]$ is a polynomial of degree m that is harmonic on $\mathbb{R}^d$, then the polynomial $\psi_1 = \Xi(\psi)$ is discrete harmonic on $\mathbb{Z}^d$, and $\psi - \psi_1$ is a polynomial of degree $m-2$.*

Proof. An easy calculation shows that

$$D^2 P_m = m(m-1)P_{m-2}$$

from which we see that

$$D_i^2 \Xi[\psi] = \Xi[\frac{\partial^2}{\partial x_i^2}\psi].$$

If ψ is harmonic, then the right side vanishes when summed over $i = 1, \dots, d$, which shows that $\Xi[\psi]$ is discrete harmonic.

Note that $P_m(x)$ is even for m even and odd for m odd. In particular, $P_m(x) - x^m$ has degree $m-2$, which implies that $\psi - \psi_1$ has degree $m-2$. □

To obtain a discrete harmonic polynomial ψ_R on the lattice $\frac{1}{R}\mathbb{Z}^d$, we let

$$\psi_R(x) = R^{-m}\psi_1(Rx),$$

where m is the degree of ψ.

2.3 General-dimensional CLT: Proof of Theorem 1.4

Proof of Theorem 1.4. For each fixed ψ, the value Φ_A^m is actually a martingale in t. Each time a new particle is added, we can imagine that it performs Brownian motion on the grid (instead of a simple random walk), which turns Φ_A^m into a continuous martingale, as in [JLS10a]. This martingale is a Brownian motion if we parameterize time by the quadratic variation, which we denote by s. We write $s(t) = s_m(t)$ for the quadratic variation time corresponding the time that the tth particle is added to A_t. To show that $\Psi_A^m(\psi, t)$ converges in law as $m \to \infty$ to a Gaussian (whose variance is some value depending on ψ and t), it suffices to show that when t is fixed, the random variable $s_m(t)$ converges in law to that value.

Let $V_t(\psi) := \mathrm{Var}\big(\psi(z)\big)$ where z is chosen uniformly on the sphere of volume of t. For later purposes, we also write $V_t(\psi_1, \psi_2) := \mathrm{Cov}\big(\psi_1(z), \psi_2(z)\big)$. We claim that the following limit holds in probability:

$$\lim_{m\to\infty} s_m(t) = \int_0^t V_u(\psi)du. \tag{9}$$

Indeed (9) is essentially immediate from the following bounds:

1. The fact that ψ and ψ_m agree up to an error of $O(1/m^2)$
2. The bounds in [JLS10a] and [JLS10b], which show that $A(t)$ is asymptotically spherical, up to an error a.s. bounded (for all t) by a constant times $\log t$ in dimension 2 and a constant times $\sqrt{\log t}$ in dimension $d > 2$. (Actually, any bound $o(t^{1/d})$, including the bounds in [LBG92], would suffice here.)

3. Kakutani's theorem, which implies that a Brownian motion on $\mathcal{G}$ can be coupled with Brownian motion on $\mathbb{R}^d$ up to an error of $\log s$.

Similarly, suppose we are given $0 = t_0, t_1 < t_2 < \ldots < t_\ell$ and distinct functions $\psi_1, \psi_2, \ldots \psi_\ell$. The same argument as above implies that $\sum_{i=j}^{\ell} \Phi_A^m(t_j, \psi_j)$ converges in law to a Gaussian with variance

$$\sum_{j=1}^{\ell} \int_{t_{j-1}}^{t_j} V_u(\sum_{i=j}^{\ell} \psi_i).$$

The theorem now follows from a standard fact about Gaussian random variables on a finite dimensional vector spaces (proved using characteristic functions): namely, a sequence of random variables on a vector space converges in law to a multivariate Gaussian if and only if all of the one-dimensional projections converge. The law of h is determined by the fact that it is a centered Gaussian with covariance

$$\mathrm{Cov}(\Phi_h(\psi_1, t_1), \Phi_h(\psi_2, t_2)) = \int_0^t V_u(\psi_1, \psi_2) du, \tag{10}$$

where $t = \min\{t_1, t_2\}$, which agrees with (3). □

3 Dimension two

3.1 Two dimensional CLT: Proof of Theorem 1.2

Recall that A_t for $t \in \mathbb{Z}_+$ denotes the discrete-time IDLA cluster with exactly t sites, and $A_T = A_{T(t)}$ for $t \in \mathbb{R}_+$ denotes the continuous-time cluster whose cardinality is Poisson-distributed with mean t.

Define

$$F_0(t) := \inf\{t : z \in A_t\}$$

and

$$L_0(z) := \sqrt{F_0(z)/\pi} - |z|.$$

Fix $N < \infty$, and consider a test function of the form

$$\varphi(re^{i\theta}) = \sum_{|k| \leq N} a_k(r) e^{ik\theta}$$

where the a_k are smooth functions supported in an interval $0 < r_0 \leq r \leq r_1 < \infty$. We will assume, furthermore, that φ is real-valued. That is, the complex numbers a_k satisfy

$$a_{-k}(r) = \overline{a_k(r)}$$

Theorem 3.1. *As $R \to \infty$,*

$$\frac{1}{R^2} \sum_{z \in (\mathbb{Z}+i\mathbb{Z})/R} L_0(Rz) \frac{\phi(z)}{|z|^2} \longrightarrow N(0, V_0)$$

in law, where

$$V_0 = \sum_{0<|k|\leq N} 2\pi \int_0^\infty \left| \int_\rho^\infty a_k(r)(\rho/r)^{|k|+1} \frac{dr}{r} \right|^2 \frac{d\rho}{\rho}.$$

This can be interpreted as saying that $L_0(Rz)$ tends weakly to a Gaussian random variable associated to the Hilbert space H^1_{nr} with norm

$$\|\eta\|_0^2 = \sum_{0<|k|<\infty} 2\pi \int_0^\infty [|r\partial_r \eta_k|^2 + (|k|+1)^2 |\eta_k|^2] \frac{dr}{r}$$

where

$$\eta_k(r) = \frac{1}{2\pi} \int_0^{2\pi} \eta(re^{i\theta}) e^{-ik\theta} d\theta$$

and $\eta_0(r) \equiv 0$. (The subscript nr means nonradial: H^1_{nr} is the orthogonal complement of radial functions in the Sobolev space H^1.)

If we use A_T and corresponding functions $F(z)$ and $L(z)$, then the a_0 coefficient figures in the limit formula as follows.

Theorem 3.2. *As $R \to \infty$,*

$$\frac{1}{R^2} \sum_{z \in (\mathbb{Z}+i\mathbb{Z})/R} L(Rz) \frac{\phi(z)}{|z|^2} \longrightarrow N(0, V)$$

in law, where

$$V = \sum_{|k|\leq N} 2\pi \int_0^\infty \left| \int_\rho^\infty a_k(r)(\rho/r)^{|k|+1} \frac{dr}{r} \right|^2 \frac{d\rho}{\rho}.$$

Theorem 3.2 is a restatement of Theorem 1.2. It can be interpreted as saying that $L(Rz)$ tends to a Gaussian distribution for the Hilbert space H^1 with the norm

$$\|\eta\|^2 = \sum_{k=-\infty}^{\infty} 2\pi \int_0^\infty [|r\partial_r \eta_k|^2 + (|k|+1)^2 |\eta_k|^2] \frac{dr}{r}.$$

By way of comparison, the usual Gaussian free field is the one associated to the Dirichlet norm

$$\int_{\mathbb{R}^2} |\nabla \eta|^2 dx dy = \sum_{k=-\infty}^{\infty} 2\pi \int_0^\infty [|r\partial_r \eta_k|^2 + k^2 |\eta_k|^2] \frac{dr}{r}.$$

Comparing these two norms, we see that the second term in $\|\eta\|$ has an additional $+1$, hence our choice of the term “augmented Gaussian free field.”

To prove Theorem 3.1, write

$$\begin{aligned} L_0(z) &= \frac{1}{2\sqrt{\pi}} \int_0^\infty (1 - 1_{A_t}) t^{1/2} \frac{dt}{t} - \frac{1}{2\sqrt{\pi}} \int_0^\infty (1 - 1_{\pi|z|^2 \le t}) t^{1/2} \frac{dt}{t} \\ &= \frac{1}{2\sqrt{\pi}} \int_0^\infty (1_{\pi|z|^2 \le t} - 1_{A_t}) t^{1/2} \frac{dt}{t} \end{aligned}$$

Let $p_0(z) = 1$, and for $k \ge 1$ let $p_k(z) = q_k(z) - q_k(0)$, where

$$q_k(z) = \Xi[z^k]$$

is the discrete harmonic polynomial associated to $z^k = (x_1 + ix_2)^k$ as described in §2.2.. Note that $p_1(z) = z$. We also set $p_{-k}(z) = \overline{p_k(z)}$.

Define

$$\psi(z,t,R) = \sum_{k=-N}^{N} a_k(\sqrt{t/\pi R^2}) p_k(z) (\sqrt{t/\pi})^{-|k|}$$

and

$$\psi_0(z,t,R) = \psi(z,t,R) - a_0(\sqrt{t/\pi R^2})$$

Lemma 3.3. *If* $c_1 R^2 \le t \le c_2 R^2$ *and* $||z| - \sqrt{t/\pi}| \le C \log R$, *then*

$$|\psi(z,t,R) - \phi(z/R)| \le C (\log R)/R$$

This lemma follows easily from the fact that the coefficients a_k are smooth and the bound $|p_k(z) - z^k| \le C|z|^{|k|-1}$.

3.2 Van der Corput bounds

Lemma 3.4. *(Van der Corput)*

a) $|\#\{z \in \mathbb{Z} + i\mathbb{Z} : \pi|z|^2 \le t\} - t| \le C t^{1/3}$

b) For $k \ge 1$,

$$\left| \sum_{z \in \mathbb{Z}+i\mathbb{Z}} z^k \, 1_{\pi|z|^2 \le t} \right| \le C t^{1/3}$$

c) For $k \ge 1$,

$$\left| \sum_{z \in \mathbb{Z}+i\mathbb{Z}} p_k(z) \, 1_{\pi|z|^2 \le t} \right| \le C t^{1/3}$$

Part (a) of this lemma was proved by van der Corput. Part (b) follows from the same method, as proved below. Part (c) follows from part (b) and the estimate $|p_k(z) - z^k| \le C|z|^{|k|-1}$.

We prove part (b) in all dimensions. Let P_k be a harmonic polynomial on $\mathbb{R}^d$ of homogeneous of degree k. Normalize so that

$$\max_{x \in B} |P_k(x)| = 1$$

where B is the unit ball. In this discussion k will be fixed and the constants are allowed to depend on k.

We are going to show that for $k \geq 1$,

$$\left| \frac{1}{R^d} \sum_{|x|<R,\, x \in \mathbb{Z}^d} P_k(x)/R^k \right| \leq R^{-1-\alpha}$$

where

$$\alpha = 1 - 2/(d+1)$$

For $d = 2$, $\alpha = 1/3$, and $R^d R^{-1-\alpha} = R^{2/3} \approx t^{1/3}$. This is the claim of part (b).

The van der Corput theorem is the case $k = 0$. It says

$$(1/R^d) \left| \# \left\{ x \in \mathbb{Z}^d : |x| < R \right\} - \mathrm{vol}\left(|\mathrm{x}| < \mathrm{R}\right) \right| \leq R^{-1-\alpha}$$

Let $\epsilon = 1/R^\alpha$.

Consider ρ a smooth, radial function on $\mathbb{R}^d$ with integral 1 supported in the unit ball. Then define $\chi = 1_B$ characteristic function of the unit ball. Denote

$$\rho_\epsilon(x) = \epsilon^{-d} \rho(x/\epsilon), \quad \chi_R(x) = R^{-d} \chi(x/R)$$

Then

$$\left| \sum_{x \in \mathbb{Z}^d} (\chi_R * \rho_\epsilon(x) - \chi_R(x)) P_k(x)/R^k \right| \leq R^{-1-\alpha}$$

This is because $\chi_R * \rho_\epsilon(x) - \chi_R(x)$ is nonzero only in the annulus of width 2ϵ around $|x| = R$ in which (by the van der Corput bound) there are $O(R^{d-1}\epsilon)$ lattice points.

The Poisson summation formula implies

$$\sum_{x \in \mathbb{Z}^d} \chi_R * \rho_\epsilon(x) P_k(x)/R^k = \sum_{\xi \in 2\pi \mathbb{Z}^d} [\hat{\chi}_R(\xi) \hat{\rho}_\epsilon(\xi)] * \hat{P}_k(\xi)/R^k$$

in the sense of distributions. The Fourier transform of a polynomial is a derivative of the delta function, $\hat{P}_k(\xi) = P_k(i\partial_\xi)\delta(\xi)$. Because $k \geq 1$ and $P_k(x)$ is harmonic, its average with repect to any radial function is zero. This is expressed in the dual variable as the fact that when $\xi = 0$,

$$P_k(i\partial_\xi)(\hat{\chi}_R(\xi)\hat{\rho}_\epsilon(\xi)]) = 0$$

So we our sum equals

$$\sum_{\xi\neq 0,\,\xi\in 2\pi\mathbb{Z}^d} [\hat{\chi}_R(\xi)\hat{\rho}_\epsilon(\xi)] * \hat{P}_k(\xi)/R^k$$

Next look at

$$\hat{\chi}_R(\xi) = \hat{\chi}(R\xi)$$

$$P_k(i\partial_\xi)\hat{\chi}(R\xi) = R^k \int_{|x|<1} P_k(x)e^{-iRx\cdot\xi}dx$$

All the terms in which fewer derivatives fall on $\hat{\chi}_R$ and more fall on ρ_ϵ give much smaller expressions: the factor R corresponding to each such differentiation is replaced by an ϵ.

The asymptotics of this oscillatory integral above are well known. For any fixed polynomial P they are of the same order of magnitude as for $P \equiv 1$, namely

$$|P_k(i\partial_\xi)\hat{\chi}(R\xi)|/R^k \leq C_k|R\xi|^{-(d+1)/2}$$

This is proved by the method of stationary phase and can also be derived from well known asymptotics of Bessel functions.

It follows that our sum is majorized by (replacing the letter d by n so that it does not get mixed up with the differential dr)

$$\begin{aligned}\int_1^\infty (Rr)^{-(n+1)}\frac{r^{n-1}dr}{(1+\epsilon r)^N} &\approx \int_1^{1/\epsilon} (Rr)^{-(n+1)} r^n \frac{dr}{r} \\ &\approx R^{-(n+1)/2}\epsilon^{-(n-1)/2} \\ &= R^{-1-\alpha}.\end{aligned}$$

3.3 The other 90% of the proof

Denote

$$X_R = \frac{1}{R^2}\sum_{z\in(\mathbb{Z}+i\mathbb{Z})/R} L_0(Rz)\frac{\phi(z)}{|z|^2}$$

Applying the formula above for L_0,

$$\begin{aligned}X_R &= \sum_{z\in\mathbb{Z}+i\mathbb{Z}} L_0(z)\frac{\phi(z/R)}{|z|^2} \\ &= \frac{1}{2\sqrt{\pi}}\int_0^\infty \sum_{z\in\mathbb{Z}+i\mathbb{Z}} (1_{\pi|z|^2\leq 1} - 1_{A_t})\frac{\phi(z/R)}{|z|^2} t^{1/2}\frac{dt}{t} \\ &= \frac{1}{2\sqrt{\pi}}\int_0^\infty \sum_{z\in\mathbb{Z}+i\mathbb{Z}} (1_{\pi|z|^2\leq 1} - 1_{A_t})\frac{\psi(z,t,R)}{t/\pi} t^{1/2}\frac{dt}{t} + E_R\end{aligned}$$

To estimate the error term E_R, note first that the coefficients a_k are supported in a fixed annulus, the integrand above is suported in the range $c_1 R^2 \leq t \leq c_2 R^2$. Furthermore, by [JLS10a], $1_{\pi|z|^2\leq 1} - 1_{A_t}$ is almost surely supported where $||z| - \sqrt{t/\pi}| \leq C \log R$. Thus, almost surely,

$$\sum_{z\in\mathbb{Z}+i\mathbb{Z}} |1_{\pi|z|^2\leq 1} - 1_{A_t}| \leq CR\log R$$

Moreover, Lemma 3.3 applies and

$$|E_R| \leq C\int_{c_1R^2}^{c_2R^2} (R\log R)\frac{\log R}{R} t^{-1/2}\frac{dt}{t} = O((\log R)/R)$$

Next, Lemma 3.4 a) says (since $\#A_t = t$)

$$\left|\sum_{z\in\mathbb{Z}+i\mathbb{Z}} |1_{\pi|z|^2\leq 1} - 1_{A_t}\right| \leq Ct^{1/3}$$

Thus replacing ψ by ψ_0 gives an additional error of size at most

$$C\int_{c_1R^2}^{c_2R^2} t^{1/3}t^{-1/2}\frac{dt}{t} = O(R^{-1/3})$$

In all,

$$X_R = \frac{\sqrt{\pi}}{2}\int_0^\infty \sum_{z\in\mathbb{Z}+i\mathbb{Z}} (1_{\pi|z|^2\leq 1} - 1_{A_t})\psi_0(z,t,R)t^{-1/2}\frac{dt}{t} + O(R^{-1/3}) \tag{11}$$

For $s = 0, 1, \ldots,$ define

$$M(s) = \frac{\sqrt{\pi}}{2}\int_0^\infty \sum_{z\in\mathbb{Z}+i\mathbb{Z}} (1_{\pi|z|^2\leq 1} - 1_{A(s\wedge t)})\psi_0(z,t,R)t^{-1/2}\frac{dt}{t}$$

Note that $M(s) \longrightarrow X_R$ as $s \to \infty$. Note also that Lemma 3.4 c) implies

$$M(0) = O(R^{-1/3})$$

Because p_k are discrete harmonic and $p_k(0) = 0$ for all $k \neq 0$, $M(s) - M(0)$ is a martingale. It remains to show that $M(s) - M(0) \longrightarrow N(0, V_0)$ in law. As outlined below, this will follow from the martingale convergence theorem (see, e.g., [Dur, p. 414]).

Note that $M(s+1) - M(s)$ is zero almost surely outside the range $c_1R^2 \leq s \leq c_2R^2$. Moreover, almost surely,

$$|M(s+1) - M(s)|^2 = O(1/R^2)$$

The only other thing we need to check is that almost surely

$$\sum_{s=0}^{\infty} |M(s+1) - M(s)|^2 = V_0 + O((\log R)/R) \tag{12}$$

Because A_t fills the lattice $\mathbb{Z} + i\mathbb{Z}$ as $t \to \infty$,

$$\begin{aligned}
\sum_{s=0}^{\infty}&|M(s+1) - M(s)|^2 \\
&= \sum_{z\in\mathbb{Z}+i\mathbb{Z}} \left| \frac{\sqrt{\pi}}{2} \int_{F_0(z)}^{\infty} \sum_{0<|k|\leq N} a_k(\sqrt{t/\pi R^2}) p_k(z)(t/\pi)^{-|k|/2} t^{-1/2} \frac{dt}{t} \right|^2
\end{aligned}$$

We prove (12) in three steps: replace $p_k(z)$ by z^k (or $\bar{z}^{|k|}$ if $k < 0$); replace the lower limit $F_0(z)$ by $\pi|z|^2$; replace the sum of z over lattice sites with the integral with respect to Lebesgue measure in the complex z-plane.

Begin by estimating the size of the integrand. Recall, as usual, that $c_1R^2 \leq t \leq c_2R^2$ is the only range in which the integrand is nonzero. By [JLS10a], almost surely,

$$\pi|z|^2 - C(\log R)R \leq F_0(z) \leq t \implies |z| \leq \sqrt{t/\pi} + C\log R$$

It follows that

$$|p_k(z)|(t/\pi)^{-|k|/2} \leq C_k$$

Moreover, the support properties of a_k imply that the integral is zero if $F_0(z) \geq cR^2$, so the terms of the sum are zero unless $|z| \leq CR$. There are only $O(R^2)$ such lattice points z. The size of each of these terms is majorized by

$$\left(\int_{c_1R^2}^{c_2R^2} t^{-1/2} \frac{dt}{t} \right)^2 = O(1/R^2)$$

The error term introduced by replacing p_k with z^k is

$$|p_k(z) - z^k|(t/\pi)^{-|k|/2} \leq C_k t^{-1} = O(1/R^2)$$

In the integral this is majorized by

$$\int_{c_1R^2}^{c_2R^2} t^{-1/2} \frac{dt}{t} \int_{c_1R^2}^{c_2R^2} \frac{1}{R^2} t^{-1/2} \frac{dt}{t} = O(1/R^4)$$

Since there are $O(R^2)$ such terms, this change contributed order $R^2/R^4 = 1/R^2$ to the sum.

Next, we change the lower limit from $F_0(z)$ to $\pi|z|^2$. Since $|F_0(z) - \pi|z|^2| \leq CR\log R$, the integral inside $|\cdots|^2$ is changed by

$$\int_{F_0(z)}^{\pi|z|^2} 1_{c_1R^2 \leq c_2R^2} t^{-1/2} \frac{dt}{t} = O((\log R)/R^2)$$

Thus the change in the whole expression is majorized by the order of the cross term

$$(1/R)(\log R)/R^2 = (\log R)/R^3$$

Again there are R^2 terms in the sum over z, so the sum of the errors is $O((\log R)/R)$.

Lastly, we replace the value at each site z_0 by the integral

$$\int_{Q_{z_0}} \left| \frac{\sqrt{\pi}}{2} \int_{\pi r^2}^{\infty} \sum_{0<|k|\le N} a_k(\sqrt{t/\pi R^2}) r^k e^{ik\theta} (t/\pi)^{-|k|/2} t^{-1/2} \frac{dt}{t} \right|^2 r dr d\theta$$

where Q_{z_0} is the unit square centered at z_0 and $z = re^{i\theta}$. Because the square has area 1, the term in the lattice sum is the same as this integral with $z = re^{i\theta}$ replaced by z_0 at each occurrence. Since $|z - z_0| \le \sqrt{2}$,

$$|z^k - z_0^k| \le 4k(|z| + |z_0|)^{k-1} = O(R^{k-1})$$

After we divide by $(\sqrt{t/\pi})^k$, the order of error is $1/R$. Adding all the errors contributes at most order $1/R$ to the sum. We must also take into account the change in the lower limit of the integral, $\pi|z_0|^2$ is replaced by $\pi|z|^2 = \pi r^2$. Since $|z - z_0| \le \sqrt{2}$,

$$||z|^2 - |z_0|^2| \le \sqrt{2}(|z| + |z_0|) \le CR$$

Recall that in the previous step we previously changed the lower limit by $O(R \log R)$. Thus by the same argument, this smaller change gives rise to an error of order $1/R$ in the sum over z_0.

The proof of (12) is now reduced to evaluating

$$\int_0^{2\pi} \int_0^{\infty} \left| \frac{\sqrt{\pi}}{2} \int_{\pi r^2}^{\infty} \sum_{0<|k|\le N} a_k(\sqrt{t/\pi R^2}) r^{|k|} e^{ik\theta} (t/\pi)^{-|k|/2} t^{-1/2} \frac{dt}{t} \right|^2 r dr d\theta$$

Integrating in θ and changing variables from r to $\rho = r/R$,

$$= \frac{\pi^2}{2} \sum_{0<|k|\le N} \int_0^{\infty} \left| \int_{\pi\rho^2 R^2}^{\infty} a_k(\sqrt{t/\pi R^2}) (R\rho)^{|k|+1} (t/\pi)^{-|k|/2} t^{-1/2} \frac{dt}{t} \right|^2 \frac{d\rho}{\rho}$$

Then change variables from t to to $r = \sqrt{t/\pi R^2}$ to obtain

$$= 2\pi \sum_{0<|k|\le N} \int_0^{\infty} \left| \int_{\rho}^{\infty} a_k(r) (\rho/r)^{|k|+1} \frac{dr}{r} \right|^2 \frac{d\rho}{\rho} = V_0$$

This ends the proof of Theorem 3.1.

The proof of Theorem 3.2 follows the same idea. We replace A_t by the Poisson time region A_T, and we need to find the limit as $R \to \infty$ of

$$\frac{\sqrt{\pi}}{2}\int_0^\infty (t - \#A_T) a_0(\sqrt{t/\pi R^2}) t^{-1/2} \frac{dt}{t} + \frac{\sqrt{\pi}}{2}\int_0^\infty \sum_{z \in \mathbb{Z}+i\mathbb{Z}} (1_{\pi|z|^2 \le 1} - 1_{A_T}) \psi_0(z,t,R) t^{-1/2} \frac{dt}{t}$$

The error terms in the estimation showing this quantity is within $O(R^{-1/3})$ of

$$\frac{1}{R^2} \sum_{z \in (\mathbb{Z}+i\mathbb{Z})/R} L(Rz) \frac{\phi(z)}{|z|^2}$$

are nearly the same as in the previous proof. We describe briefly the differences. The difference between Poisson time and ordinary counting is

$$|\#A_T - \#A_t| = |\#A_T - t| \le C t^{1/2} \log t = O(R \log R) \quad \text{almost surely}$$

if $t \approx R^2$. It follows that for $|z| \approx R$,

$$|F(z) - \pi |z|^2| \le R \log R \quad \text{almost surely}$$

as in the previous proof for $F_1(z)$. Further errors are also controlled since we then have the estimate analogous to the one above for A_t, namely

$$\sum_{z \in \mathbb{Z}+i\mathbb{Z}} |1_{\pi|z|^2 \le 1} - 1_{A_T}| \le C R \log R$$

We consider the continuous time martingale

$$M(s) = \frac{\sqrt{\pi}}{2}\int_0^\infty (s \wedge t - \#A_{s\wedge t}) a_0(\sqrt{t/\pi R^2}) t^{-1/2} \frac{dt}{t} + \frac{\sqrt{\pi}}{2}\int_0^\infty \sum_{z \in \mathbb{Z}+i\mathbb{Z}} (1_{\pi|z|^2 \le 1} - 1_{A_{s\wedge t}}) \psi_0(z,t,R) t^{-1/2} \frac{dt}{t}$$

Instead of using the martingale central limit theorem, we use the martingale representation theorem. This says that the martingale when reparameterized by its quadratic variation has the same law as Brownian motion. We must show that almost surely the quadratic variation of M on $0 \le s < \infty$ is $V + O(R^{-1/3})$.

$$\lim_{\epsilon \to 0} \mathbb{E}\left((M(s+\epsilon) - M(s))^2 | A(s) \right)/\epsilon = \frac{1}{2\pi}\int_0^{2\pi} \left| \frac{\sqrt{\pi}}{2}\int_s^\infty \sum_{|k| \le N} a_k(\sqrt{t/\pi R^2}) e^{ik\theta} (s/t)^{|k|/2} t^{-1/2} \frac{dt}{t} \right|^2 d\theta + O(R^{-1/3})$$

Integrating with respect to s gives the quadratic variation $V + O(R^{-1/3})$ after a suitable change of variable as in the previous proof.

3.4 Fixed time fluctuations: Proof of Theorem 1.3

This follows almost immediately from the $d = 2$ case of Theorem 1.4 and the estimates above.

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