EXISTENCE AND REGULARITY OF HIGHER CRITICAL POINTS IN ELLIPTIC FREE BOUNDARY PROBLEMS

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Abstract. Existence and regularity of minimizers in elliptic free boundary problems have been extensively studied in the literature. We initiate the corresponding study of higher critical points by considering a superlinear free boundary problem related to plasma confinement. The associated energy functional is nondifferentiable, and therefore standard variational methods cannot be used directly to prove the existence of critical points. Here we obtain a nontrivial generalized solution u of mountain pass type as the limit of mountain pass points of a suitable sequence of C^1-functionals approximating the energy. We show that u minimizes the energy on the associated Nehari manifold and use this fact to prove that it is nondegenerate. We use the nondegeneracy of u to show that it satisfies the free boundary condition in the viscosity sense. Moreover, near any free boundary point that has a measure-theoretic normal, the free boundary is a smooth surface, and hence the free boundary condition holds in the classical sense.

1. Introduction

The existence and regularity of minimizers in elliptic free boundary problems have been extensively studied for over four decades (see, e.g., [1, 2, 3, 5, 6, 7, 8, 9, 11, 12, 13, 15, 18, 19, 20, 25, 26] and the references therein). Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$. A typical two-phase free boundary problem seeks a minimizer of the variational integral

$$\int_\Omega \left[\frac{1}{2}\,|\nabla u|^2 + \chi_{\{u>0\}}(x)\right] dx$$

among all functions $u \in H^1(\Omega) \cap C^0(\Omega)$ with prescribed values on some portion of the boundary $\partial\Omega$, where $\chi_{\{u>0\}}$ is the characteristic function of the set $\{u > 0\}$. A

2010 *Mathematics Subject Classification.* 35R35, 35J20.

Key words and phrases. superlinear elliptic free boundary problems, higher critical points, existence, nondegeneracy, regularity, variational methods, Nehari manifold.

The first author was supported by NSF grant DMS 1069225 and the Stefan Bergman Trust.

This work was completed while the second-named author was visiting the Department of Mathematics at the Massachusetts Institute of Technology, and he is grateful for the kind hospitality of the department.

local minimizer u satisfies

$$\Delta u = 0$$

except on the free boundary $\partial \{u > 0\} \cap \Omega$, and

$$|\nabla u^+|^2 - |\nabla u^-|^2 = 2$$

on smooth portions of the free boundary, where $\nabla u^\pm$ are the limits of ∇u from $\{u > 0\}$ and $\{u \leq 0\}^\circ$, respectively. The existence and regularity of local minimizers for this problem have been studied, for example, in Alt and Caffarelli [1], Alt, Caffarelli and Friedman [2], Weiss [25, 26], Caffarelli, Jerison and Kenig [12].

In the present paper we initiate the corresponding study of higher critical points by considering a superlinear free boundary problem related to plasma confinement (see, e.g., [10, 14, 16, 22, 23, 24]). We consider the functional

$$J(u) = \int_\Omega \left[\frac{1}{2}\, |\nabla u|^2 + \chi_{\{u>1\}}(x) - \frac{1}{p}\, (u-1)_+^p\right] dx, \quad u \in H_0^1(\Omega),$$

for $2 < p < \infty$ if $N = 2$ and $2 < p < 2N/(N-2)$ if $N \geq 3$. Here $H_0^1(\Omega)$ is the usual Sobolev space with the norm given by

$$\|u\|^2 = \int_\Omega |\nabla u|^2\, dx.$$

The functional J is nondifferentiable and therefore standard variational methods cannot be used directly to obtain a higher critical point. We will obtain our solution as the limit of mountain pass points of a suitable sequence of C^1-functionals approximating J. The crucial ingredient in the passage to the limit is the uniform Lipschitz continuity result, proved in Caffarelli, Jerison, and Kenig [11] (see Proposition 2.8).

The mountain pass solution u that we construct is Lipschitz continuous in Ω and satisfies the following Euler-Lagrange equations in a generalized sense (see Definition 3.1).

$$\begin{cases} -\Delta u = (u-1)_+^{p-1} & \text{in } \Omega \setminus \partial\{u > 1\} \\ |\nabla u^+|^2 - |\nabla u^-|^2 = 2 & \text{on } \partial\{u > 1\} \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $w_\pm = \max\{\pm w, 0\}$ are the positive and negative parts of w, respectively, $\nabla u^\pm$ are the limits of ∇u from $\{u > 1\}$ and $\{u \leq 1\}^\circ$, respectively.

It will follow from integration by parts that u belongs to the Nehari-type manifold

$$\mathcal{M} = \left\{u \in H_0^1(\Omega) : \int_{\{u>1\}} |\nabla u|^2\, dx = \int_{\{u>1\}} (u-1)^p\, dx > 0\right\},$$

We will deduce that the solution u is non-degenerate in the following sense.

Definition 1.1. A generalized solution u of problem (1.1) is nondegenerate if there exist constants r_0, $c > 0$ such that if $x_0 \in \{u > 1\}$ and $r := \operatorname{dist}(x_0, \{u \leq 1\}) \leq r_0$, then $u(x_0) \geq 1 + c\,r$.

This nondegeneracy makes it possible to apply the regularity results in Lederman and Wolanski [20] showing that u satisfies the free boundary condition in the viscosity sense and that near points of the free boundary with a measure-theoretic normal, the free boundary is a smooth surface and hence the free boundary condition holds in the classical sense.

To formulate our results more precisely, recall the definition of the mountain pass solution.

Definition 1.2 (Hofer [17])**.** We say that $u \in H^1_0(\Omega)$ is a mountain pass point of J if the set $\{v \in U : J(v) < J(u)\}$ is neither empty nor path connected for every neighborhood U of u.

Let

$$\Gamma = \big\{\gamma \in C([0,1], H^1_0(\Omega)) : \gamma(0) = 0,\ J(\gamma(1)) < 0\big\}$$

be the class of continuous paths from 0 to the set $\{u \in H^1_0(\Omega) : J(u) < 0\}$, and denote

$$c^* = c^*(\Omega) := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} J(u).$$

Our main result is the following theorem.

Theorem 1.3. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$. Let $2 < p < \infty$ if $N = 2$ and $2 < p < 2N/(N-2)$ if $N \geq 3$. Then*

a) $c^* = c^*(\Omega) > 0$.

b) There is a mountain pass point u for the functional J satisfying $J(u) = c^*$, which is Lipschitz continuous and satisfies (1.1) *in the generalized sense of Definition 3.1.*

c) u minimizes $J|_{\mathcal{M}}$ and is nondegenerate.

Corollary 1.4 (Lederman and Wolanski [20])**.** *Let u be a nondegenerate generalized solution to problem* (1.1) *as in Theorem 1.3. Then*

a) u satisfies the free boundary condition in the viscosity sense and

b) In a neighborhood of every free boundary point where the measure-theoretic normal exists, the free boundary is a $C^{1,\alpha}$-surface, and hence u satisfies the free boundary condition in the classical sense.

We point out that the existence of a mountain pass solution is by no means routine due to the severe lack of smoothness of J. Indeed, J is not even continuous, much less of class C^1. Note that for the functional in which the discontinuous term $\chi_{\{u>1\}}$ is removed, there is no difficulty in applying the mountain pass theorem (see Flucher and Wei [14] and Shibata [22]). We believe that Theorem 1.3 is the first result in the literature that establishes the existence of a higher critical point and verifies its nondegeneracy in a free boundary problem of the type (1.1).

The paper is organized as follows. In Section 2 we construct a putative mountain pass solution u as a limit of solutions to a regularized problem. We prove that u is a generalized solution in Section 3. In Section 4 we show that u belongs to the Nehari-type manifold and that minimizers on the Nehari manifold have the nondegeneracy property. In Section 5 we prove our main theorem by showing that our solution u does minimize over the Nehari manifold. We deduce Corollary 1.4 and mention further questions about partial regularity of the free boundary.

2. Limits of mountain pass solutions

We approximate J by C^1-functionals as follows. Let $\beta : \mathbb{R} \to [0, 2]$ be a smooth function such that $\beta(t) = 0$ for $t \le 0$, $\beta(t) > 0$ for $0 < t < 1$, $\beta(t) = 0$ for $t \ge 1$, and $\int_0^1 \beta(s)\, ds = 1$. Then set

$$\mathcal{B}(t) = \int_0^t \beta(s)\, ds,$$

and note that $\mathcal{B} : \mathbb{R} \to [0, 1]$ is a smooth nondecreasing function such that $\mathcal{B}(t) = 0$ for $t \le 0$, $\mathcal{B}(t) > 0$ for $0 < t < 1$, and $\mathcal{B}(t) = 1$ for $t \ge 1$. For $\varepsilon > 0$, let

$$J_\varepsilon(u) = \int_\Omega \left[\frac{1}{2}\, |\nabla u|^2 + \mathcal{B}\left(\frac{u-1}{\varepsilon}\right) - \frac{1}{p}\, (u-1)_+^p\right] dx, \quad u \in H_0^1(\Omega)$$

and note that J_ε is of class C^1.

If u is a critical point of J_ε, then u is a weak solution of

$$\left\{\begin{aligned} \Delta u &= \frac{1}{\varepsilon}\, \beta\left(\frac{u-1}{\varepsilon}\right) - (u-1)_+^{p-1} && \text{in } \Omega\\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right. \tag{2.1}$$

and hence also a classical solution by elliptic regularity theory. By the maximum principle, either $u > 0$ everywhere or u vanishes identically. If $u \le 1$ everywhere, then u is harmonic in Ω and hence vanishes identically again. So, if u is a nontrivial critical point, then $u > 0$ in Ω and $u > 1$ in a nonempty open subset of Ω.

Lemma 2.1. *J_ε satisfies the Palais-Smale compactness condition* (PS)*, i.e., every sequence $(u_j) \subset H_0^1(\Omega)$ such that $J_\varepsilon(u_j)$ is bounded and $J_\varepsilon'(u_j) \to 0$ has a convergent subsequence.*

Proof. It suffices to show that (u_j) is bounded by a standard argument. We have

$$J_\varepsilon(u_j) = \int_\Omega \left[\frac{1}{2}\, |\nabla u_j|^2 + \mathcal{B}\left(\frac{u_j-1}{\varepsilon}\right) - \frac{1}{p}\, (u_j-1)_+^p\right] dx = \mathrm{O}(1) \tag{2.2}$$

and

$$J_\varepsilon'(u_j)\, v = \int_\Omega \left[\nabla u_j \cdot \nabla v + \frac{1}{\varepsilon}\, \beta\left(\frac{u_j - 1}{\varepsilon}\right) v - (u_j - 1)_+^{p-1}\, v\right] dx = \mathrm{o}(1)\, \|v\|\,, \quad v \in H_0^1(\Omega). \tag{2.3}$$

Since $\mathcal{B}$ and β are bounded, writing

$$u_j = (u_j - 1)_+ + [1 - (u_j - 1)_-] =: u_j^+ + u_j^-$$

in (2.2) gives

$$\int_\Omega \left[|\nabla u_j^+|^2 + |\nabla u_j^-|^2 - \frac{2}{p}\, (u_j^+)^p\right] dx = \mathrm{O}(1),$$

and taking $v = u_j^+$ in (2.3) and using the Sobolev imbedding theorem gives

$$\int_\Omega \left[|\nabla u_j^+|^2 - (u_j^+)^p\right] dx = \mathrm{O}(1)\, \|u_j^+\|.$$

Combining the last two equations now gives

$$\left(1 - \frac{2}{p}\right) \|u_j^+\|^2 + \|u_j^-\|^2 = \mathrm{O}(1) \left(\|u_j^+\| + 1\right),$$

which implies that $\|u_j^\pm\|$, and hence $\|u_j\|$, is bounded. □

Since $p < 2N/(N-2)$, the Sobolev imbedding theorem implies

$$J_\varepsilon(u) \ge \int_\Omega \left[\frac{1}{2}\, |\nabla u|^2 - \frac{1}{p}\, |u|^p\right] dx \ge \frac{1}{2}\, \|u\|^2 - C\, \|u\|^p \quad \forall u \in H_0^1(\Omega)$$

for some constant C depending on Ω. Since $p > 2$, then there exists a constant $\rho > 0$ such that

$$\|u\| \le \rho \implies J_\varepsilon(u) \ge \frac{1}{3}\, \|u\|^2.$$

Moreover

$$J_\varepsilon(u) \le \int_\Omega \left[\frac{1}{2}\, |\nabla u|^2 + 1 - \frac{1}{p}\, (u - 1)_+^p\right] dx$$

and hence, again because $p > 2$, there exists a function $u_0 \in H_0^1(\Omega)$ such that $J_\varepsilon(u_0) < 0 = J_\varepsilon(0)$. Therefore the class of paths

$$\Gamma_\varepsilon = \left\{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0,\ J_\varepsilon(\gamma(1)) < 0\right\}$$

is nonempty and

$$c_\varepsilon := \inf_{\gamma \in \Gamma_\varepsilon}\ \max_{u \in \gamma([0,1])}\ J_\varepsilon(u) \ge \frac{\rho^2}{3}. \tag{2.4}$$

Lemma 2.2. *J_ε has a (nontrivial) critical point u^ε at the level c_ε.*

Proof. If not, then there exists a constant $0 < \delta \le c_\varepsilon/2$ and a continuous map $\eta : \{J_\varepsilon \le c_\varepsilon + \delta\} \to \{J_\varepsilon \le c_\varepsilon - \delta\}$ such that η is the identity on $\{J_\varepsilon \le 0\}$ by the first deformation lemma (see, e.g., Perera and Schechter [21, Lemma 1.3.3]). By the definition of c_ε, there exists a path $\gamma \in \Gamma_\varepsilon$ such that $\max_{\gamma([0,1])} J_\varepsilon \le c_\varepsilon + \delta$. Then $\widetilde{\gamma} := \eta \circ \gamma \in \Gamma_\varepsilon$ and $\max_{\widetilde{\gamma}([0,1])} J_\varepsilon \le c_\varepsilon - \delta$, a contradiction. □

Lemma 2.3. *We have*

$$c_\varepsilon \le c^*$$

In particular, by (2.4), $c^* > 0$, *and Theorem 1.3 (a) holds.*

Proof. Since $\mathcal{B}((t-1)/\varepsilon) \le \chi_{\{t>1\}}$ for all t, $J_\varepsilon(u) \le J(u)$ for all $u \in H^1_0(\Omega)$. So $\Gamma \subset \Gamma_\varepsilon$ and

$$c_\varepsilon \le \max_{u \in \gamma([0,1])} J_\varepsilon(u) \le \max_{u \in \gamma([0,1])} J(u) \quad \forall \gamma \in \Gamma. \quad \square$$

For $0 < \varepsilon \le 1$, u^ε have the following uniform regularity properties.

Lemma 2.4. *There exists a constant* $C > 0$ *such that, for* $0 < \varepsilon \le 1$,

$$\|u^\varepsilon\| \le C.$$

Proof. By Lemma 2.3,

$$\int_\Omega \left[\frac{1}{2}\,|\nabla u^\varepsilon|^2 + \mathcal{B}\left(\frac{u^\varepsilon - 1}{\varepsilon}\right) - \frac{1}{p}\,(u^\varepsilon - 1)_+^p\right] dx \le c$$

and hence

$$\frac{1}{2}\int_\Omega |\nabla u^\varepsilon|^2\, dx \le c + \frac{1}{p}\int_{\{v_\varepsilon > 0\}} v_\varepsilon^p\, dx, \tag{2.5}$$

where $v_\varepsilon = u^\varepsilon - 1$. Testing (2.1) with $(u^\varepsilon - 1 - \varepsilon)_+$ gives

$$\int_{\{u^\varepsilon > 1+\varepsilon\}} |\nabla u^\varepsilon|^2\, dx = \int_{\{v_\varepsilon > \varepsilon\}} v_\varepsilon^{p-1}\,(v_\varepsilon - \varepsilon)\, dx. \tag{2.6}$$

Fix $\lambda > 2/(p-2)$. Multiplying (2.6) by $(\lambda+1)/p\lambda$ and subtracting from (2.5) gives

$$\begin{aligned} \frac{1}{2}\int_{\{u^\varepsilon \le 1+\varepsilon\}} |\nabla u^\varepsilon|^2\, dx &+ \frac{(p-2)\lambda - 2}{2p\lambda}\int_{\{u^\varepsilon > 1+\varepsilon\}} |\nabla u^\varepsilon|^2\, dx \\ &\le c + \frac{1}{p}\int_{\{0 < v_\varepsilon \le \varepsilon\}} v_\varepsilon^p\, dx + \frac{1}{p\lambda}\int_{\{v_\varepsilon > \varepsilon\}} v_\varepsilon^{p-1}\left[(\lambda+1)\,\varepsilon - v_\varepsilon\right] dx. \end{aligned} \tag{2.7}$$

The last integral is less than or equal to $\int_{\{\varepsilon < v_\varepsilon < (\lambda+1)\,\varepsilon\}} v_\varepsilon^{p-1}\left[(\lambda+1)\,\varepsilon - v_\varepsilon\right] dx$ and hence (2.7) gives

$$\min\left\{\frac{1}{2}, \frac{(p-2)\lambda - 2}{2p\lambda}\right\}\int_\Omega |\nabla u^\varepsilon|^2\, dx \le c + \frac{\varepsilon^p\,|\Omega|}{p}\left[1 + (\lambda+1)^{p-1}\right].$$

Since $\varepsilon \leq 1$, the conclusion follows. $\square$

Lemma 2.5. *There exists a constant $C > 0$ such that, for $0 < \varepsilon \leq 1$,*

$$\max_{x \in \Omega} u^\varepsilon(x) \leq C. \tag{2.8}$$

Proof. Since $p < 2N/(N-2)$, we have $N(p-2)/2 < 2N/(N-2)$. Fix $N(p-2)/2 < q < 2N/(N-2)$. Since

$$-\Delta u^\varepsilon = (u^\varepsilon - 1)_+^{p-1} - \frac{1}{\varepsilon}\,\beta\left(\frac{u^\varepsilon - 1}{\varepsilon}\right) \leq (u^\varepsilon)^{p-1},$$

there exists a constant $C > 0$ such that

$$\|u^\varepsilon\|_\infty \leq C\,\|u^\varepsilon\|_q^{2q/(2q-N(p-2))}$$

by Bonforte et al. [4, Theorem 3.1]. Since u^ε is bounded in $L^q(\Omega)$ by the Sobolev imbedding theorem and Lemma 2.4, the conclusion follows. $\square$

By Lemma 2.5, $(u^\varepsilon - 1)_+^{p-1} \leq A_0$ for some constant $A_0 > 0$. Let $\varphi_0 > 0$ be the solution of

$$\begin{cases} -\Delta \varphi_0 = A_0 & \text{in } \Omega \\[2ex] \varphi_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 2.6. *For $0 < \varepsilon \leq 1$,*

$$u^\varepsilon(x) \leq \varphi_0(x) \quad \forall x \in \Omega,$$

in particular, $\{u^\varepsilon \geq 1\} \subset \{\varphi_0 \geq 1\} \subset\subset \Omega$.

Proof. Since $\beta(t) \geq 0$ for all t,

$$-\Delta u^\varepsilon \leq (u^\varepsilon - 1)_+^{p-1} \leq A_0 = -\Delta \varphi_0,$$

so $u^\varepsilon \leq \varphi_0$ by the maximum principle. $\square$

Lemma 2.7. *There is a constant C such that for $r > 0$ and $0 < \varepsilon \leq 1$, if $B_r(x_0) \subset \Omega$, then*

$$\max_{x \in B_{r/2}(x_0)} |\nabla u^\varepsilon(x)| \leq C/r$$

Proof. Since $\beta(t) \leq 2$ for all t,

$$\Delta u^\varepsilon \leq \frac{1}{\varepsilon}\,\beta\left(\frac{u^\varepsilon - 1}{\varepsilon}\right) \leq \frac{2}{\varepsilon}\,\chi_{\{|u^\varepsilon - 1| < \varepsilon\}}(x),$$

and since $\beta(t) \geq 0$ for all t,

$$-\Delta u^\varepsilon \leq (u^\varepsilon - 1)_+^{p-1} \leq A_0,$$

so

$$\pm\Delta u^\varepsilon \leq \max\{2, A_0\}\left(\frac{1}{\varepsilon}\,\chi_{\{|u^\varepsilon-1|<\varepsilon\}}(x)+1\right).$$

Since u^ε is also uniformly bounded in $L^2(\Omega)$ by Lemma 2.5, the conclusion follows from the following result of Caffarelli, Jerison and Kenig.

Proposition 2.8 ([11, Theorem 5.1])**.** *Suppose that u is a Lipschitz continuous function on $B_1(0) \subset \mathbb{R}^N$ satisfying the distributional inequalities*

$$\pm\Delta u \leq A\left(\frac{1}{\varepsilon}\,\chi_{\{|u-1|<\varepsilon\}}(x)+1\right)$$

for some constants $A > 0$ and $0 < \varepsilon \leq 1$. Then there exists a constant $C > 0$, depending on N, A and $\int_{B_1(0)} u^2\,dx$, but not on ε, such that

$$\max_{x\in B_{1/2}(0)} |\nabla u(x)| \leq C.$$

□

3. The limit is a generalized solution

Definition 3.1. We say a locally Lipschitz function u defined on Ω is a generalized solution to (1.1) if $u \in C^0(\bar{\Omega}) \cap H_0^1(\Omega) \cap C^2(\Omega \setminus \partial\{u > 1\})$ and satisfies

$$-\Delta u = (u-1)_+^{p-1} \quad \text{on} \quad \Omega \setminus \partial\{u > 1\}$$

in the classical sense, and $u = 0$ on $\partial\Omega$. Moreover, for all $\varphi \in C_0^1(\Omega, \mathbb{R}^N)$ such that $u \neq 1$ a.e. on the support of φ,

$$\lim_{\delta^+\searrow 0}\int_{\{u=1+\delta^+\}}\left(|\nabla u|^2-2\right)\varphi\cdot n^+\,d\sigma - \lim_{\delta^-\searrow 0}\int_{\{u=1-\delta^-\}}|\nabla u|^2\,\varphi\cdot n^-\,d\sigma = 0,$$

where $n^\pm$ are the outward unit normals to $\partial\{u > 1 \pm \delta^\pm\}$. (The sets $\{u = 1 \pm \delta^\pm\}$ are smooth hypersurfaces for a.a. $\delta^\pm > 0$ by Sard's theorem and the above limits are taken through such $\delta^\pm$.)

In particular, a generalized solution u to (1.1) satisfies the free boundary condition in the classical sense on any smooth portion of the free boundary $\partial\{u > 1\}$.

Let $\varepsilon_j \searrow 0$, let u_j be the critical point of J_{ε_j} obtained in Lemma 2.2, and set $c_j = c_{\varepsilon_j} = J_{\varepsilon_j}(u_j)$.

Lemma 3.2. *There exists a locally Lipschitz continuous function $u \in C^0(\overline{\Omega}) \cap H_0^1(\Omega)$ such that, for a suitable sequence ε_j,*

(*i*) *$u_j \to u$ uniformly on $\overline{\Omega}$,*
(*ii*) *$u_j \to u$ strongly in $H_0^1(\Omega)$,*
(*iii*) *$J(u) \leq \varliminf c_j \leq \varlimsup c_j \leq J(u) + |\{u = 1\}|$.*

Moreover, u is a nontrivial generalized solution of problem (1.1)*.*

Proof. First we prove (i). Since (u_j) is bounded in $H^1_0(\Omega)$ by Lemma 2.4, we may choose ε_j so that u_j converges weakly in $H^1_0(\Omega)$ to some u. Because, by Lemmas 2.5 and 2.7, (u_j) is uniformly bounded and uniformly locally Lipschitz, we may also choose the sequence so that $u_j \to u$ uniformly on compact subsets of Ω, and u is locally Lipschitz continuous. Since $0 < u_j \le \varphi_0$ by Lemma 2.6, we have $0 \le u \le \varphi_0$, and hence $|u_j - u| \le \varphi_0$. Thus u extends continuously to $\overline{\Omega}$ with zero boundary values and $u_j \to u$ uniformly on $\overline{\Omega}$.

Next we show that u satisfies the equation $-\Delta u = (u-1)_+^{p-1}$ in $\{u \ne 1\}$. Let $\varphi \in C^\infty_0(\{u > 1\})$. Then $u \ge 1 + 2\,\varepsilon$ on the support of φ for some $\varepsilon > 0$. For all sufficiently large j, $\varepsilon_j < \varepsilon$ and $|u_j - u| < \varepsilon$ by (i). Then $u_j \ge 1 + \varepsilon_j$ on the support of φ, so testing

$$-\Delta u_j = (u_j - 1)_+^{p-1} - \frac{1}{\varepsilon_j}\,\beta\left(\frac{u_j - 1}{\varepsilon_j}\right) \tag{3.1}$$

with φ gives

$$\int_\Omega \nabla u_j \cdot \nabla\varphi\, dx = \int_\Omega (u_j - 1)^{p-1}\,\varphi\, dx.$$

Passing to the limit gives

$$\int_\Omega \nabla u \cdot \nabla\varphi\, dx = \int_\Omega (u - 1)^{p-1}\,\varphi\, dx \tag{3.2}$$

since u_j converges to u weakly in $H^1_0(\Omega)$ and uniformly in Ω. This then holds for all $\varphi \in H^1_0(\{u > 1\})$ by density, and hence u is a classical solution of $-\Delta u = (u-1)^{p-1}$ in $\{u > 1\}$. A similar argument shows that u is harmonic in $\{u < 1\}$.

Now we show that u is harmonic in $\{u \le 1\}^\circ$. Testing (3.1) with any nonnegative $\varphi \in C^\infty_0(\Omega)$ gives

$$\int_\Omega \nabla u_j \cdot \nabla\varphi\, dx \le A_0 \int_\Omega \varphi\, dx$$

since $\beta(t) \ge 0$ for all t and $(u_j - 1)_+^{p-1} \le A_0$, and passing to the limit gives

$$\int_\Omega \nabla u \cdot \nabla\varphi\, dx \le A_0 \int_\Omega \varphi\, dx.$$

So

$$-\Delta u \le A_0 \quad \text{in } \Omega \tag{3.3}$$

in the weak sense. On the other hand, since u is harmonic in $\{u < 1\}$, $\mu := \Delta(u-1)_-$ is a nonnegative Radon measure supported on $\Omega \cap \partial\{u < 1\}$ by Alt and Caffarelli [1, Remark 4.2], so

$$-\Delta u = \mu \ge 0 \quad \text{in } \{u \le 1\}. \tag{3.4}$$

It follows from (3.3) and (3.4) that $u \in W^{2,q}_{\mathrm{loc}}(\{u \le 1\}^\circ)$, $1 < q < \infty$ and hence μ is actually supported on $\Omega \cap \partial \{u < 1\} \cap \partial \{u > 1\}$, so u is harmonic in $\{u \le 1\}^\circ$.

Since u_j tends weakly to u in $H^1_0(\Omega)$, $\|u\| \le \liminf \|u_j\|$. Thus to prove (ii), it suffices to show that $\limsup \|u_j\| \le \|u\|$. The majorant φ_0 shows that $\{u_j \ge 1\}$ is a fixed distance from $\partial\Omega$, uniformly in j. It follows from standard regularity arguments that u_j is uniformly in C^2 in a sufficiently small, fixed neighborhood of $\partial\Omega$. Therefore, after replacing u_j with a subsequence, we may assume that $\partial u_j/\partial n \to \partial u/\partial n$ uniformly on $\partial\Omega$, where n is the outward unit normal. Multiplying (3.1) by $u_j - 1$, integrating by parts, and noting that $\beta((t-1)/\varepsilon_j)\,(t-1) \ge 0$ for all t gives

$$\int_\Omega |\nabla u_j|^2\, dx \le \int_\Omega (u_j - 1)_+^p\, dx - \int_{\partial\Omega} \frac{\partial u_j}{\partial n}\, d\sigma \to \int_\Omega (u-1)_+^p\, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n}\, d\sigma. \tag{3.5}$$

Fix $0 < \varepsilon < 1$. Taking $\varphi = (u - 1 - \varepsilon)_+$ in (3.2) gives

$$\int_{\{u > 1+\varepsilon\}} |\nabla u|^2\, dx = \int_\Omega (u-1)_+^{p-1}\,(u - 1 - \varepsilon)_+\, dx, \tag{3.6}$$

and integrating $(u - 1 + \varepsilon)_-\, \Delta u = 0$ over Ω gives

$$\int_{\{u < 1-\varepsilon\}} |\nabla u|^2\, dx = -(1-\varepsilon) \int_{\partial\Omega} \frac{\partial u}{\partial n}\, d\sigma. \tag{3.7}$$

Adding (3.6) and (3.7), and letting $\varepsilon \searrow 0$ gives

$$\int_\Omega |\nabla u|^2\, dx = \int_\Omega (u-1)_+^p\, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n}\, d\sigma.$$

This together with (3.5) gives

$$\limsup \int_\Omega |\nabla u_j|^2\, dx \le \int_\Omega |\nabla u|^2\, dx.$$

To prove (iii), write

$$J_{\varepsilon_j}(u_j) = \int_\Omega \left[\frac{1}{2}\,|\nabla u_j|^2 + \mathcal{B}\left(\frac{u_j - 1}{\varepsilon_j}\right) \chi_{\{u \ne 1\}}(x) - \frac{1}{p}\,(u_j - 1)_+^p\right] dx$$
$$+ \int_{\{u=1\}} \mathcal{B}\left(\frac{u_j - 1}{\varepsilon_j}\right) dx.$$

Since $\mathcal{B}((u_j - 1)/\varepsilon_j)\,\chi_{\{u \ne 1\}}$ converges pointwise to $\chi_{\{u > 1\}}$ and is bounded by 1, the first integral converges to $J(u)$ by (i) and (ii). Since $0 \le \mathcal{B}(t) \le 1$ for all t,

$$0 \le \int_{\{u=1\}} \mathcal{B}\left(\frac{u_j - 1}{\varepsilon_j}\right) dx \le |\{u = 1\}|\,.$$

(iii) follows.

By (iii) and (2.4),

$$J(u) + |\{u = 1\}| \geq \frac{\rho^2}{3} > 0$$

and hence u is nontrivial.

Finally we show that u satisfies the generalized free boundary condition, i.e., for all $\varphi \in C_0^1(\Omega, \mathbb{R}^N)$ such that $u \neq 1$ a.e. on the support of φ,

$$\lim_{\delta^+ \searrow 0} \int_{\{u=1+\delta^+\}} \left(2 - |\nabla u|^2\right) \varphi \cdot n \, d\sigma - \lim_{\delta^- \searrow 0} \int_{\{u=1-\delta^-\}} |\nabla u|^2 \, \varphi \cdot n \, d\sigma = 0, \tag{3.8}$$

where n is the outward unit normal to $\{1 - \delta^- < u < 1 + \delta^+\}$. Multiplying (3.1) by $\nabla u_j \cdot \varphi$ and integrating over $\{1 - \delta^- < u < 1 + \delta^+\}$ gives

$$\begin{aligned}
0 \;&=\; \int_{\{1-\delta^- < u < 1+\delta^+\}} \left[-\Delta u_j + \frac{1}{\varepsilon_j}\, \beta\left(\frac{u_j - 1}{\varepsilon_j}\right) - (u_j - 1)_+^{p-1}\right] \nabla u_j \cdot \varphi \, dx \\
&=\; \int_{\{1-\delta^- < u < 1+\delta^+\}} \left[\operatorname{div}\left(\frac{1}{2}\, |\nabla u_j|^2\, \varphi - (\nabla u_j \cdot \varphi)\, \nabla u_j\right) + \nabla u_j\, D\varphi \cdot \nabla u_j\right. \\
&\qquad \left. - \frac{1}{2}\, |\nabla u_j|^2 \, \operatorname{div} \varphi + \nabla \mathcal{B}\left(\frac{u_j - 1}{\varepsilon_j}\right) \cdot \varphi - \frac{1}{p}\, \nabla (u_j - 1)_+^p \cdot \varphi\right] dx \\
&=\; \frac{1}{2} \int_{\{u=1+\delta^+\} \bigcup \{u=1-\delta^-\}} \left[|\nabla u_j|^2\, \varphi - 2\, (\nabla u_j \cdot \varphi)\, \nabla u_j + 2\, \mathcal{B}\left(\frac{u_j - 1}{\varepsilon_j}\right) \varphi\right] \cdot n \, d\sigma \\
&\qquad - \frac{1}{p} \int_{\{u=1+\delta^+\} \bigcup \{u=1-\delta^-\}} (u_j - 1)_+^p \, \varphi \cdot n \, d\sigma \\
&\qquad - \int_{\{1-\delta^- < u < 1+\delta^+\}} \left[\mathcal{B}\left(\frac{u_j - 1}{\varepsilon_j}\right) - \frac{1}{p}\, (u_j - 1)_+^p\right] \operatorname{div} \varphi \, dx \\
&\qquad + \int_{\{1-\delta^- < u < 1+\delta^+\}} \left(\nabla u_j \, D\varphi \cdot \nabla u_j - \frac{1}{2}\, |\nabla u_j|^2 \, \operatorname{div} \varphi\right) dx \\
&=: \; \frac{I_1}{2} - \frac{I_2}{p} - I_3 + I_4.
\end{aligned}$$

Since $u_j \to u$ uniformly on $\overline{\Omega}$, strongly in $H_0^1(\Omega)$, and locally in $C^1(\{u \neq 1\})$,

$$\begin{aligned} I_1 \;&\to\; \int_{\{u=1+\delta^+\} \bigcup \{u=1-\delta^-\}} \big(|\nabla u|^2\, \varphi - 2\, (\nabla u \cdot \varphi)\, \nabla u\big) \cdot n\, d\sigma + \int_{\{u=1+\delta^+\}} 2\, \varphi \cdot n\, d\sigma \\ &=\; \int_{\{u=1+\delta^+\}} \big(2 - |\nabla u|^2\big)\, \varphi \cdot n\, d\sigma - \int_{\{u=1-\delta^-\}} |\nabla u|^2\, \varphi \cdot n\, d\sigma \end{aligned}$$

since $n = \pm \nabla u/|\nabla u|$ on $\{u = 1 \pm \delta^\pm\}$, and

$$I_2 \to \int_{\{u=1+\delta^+\}} (u-1)_+^p\, \varphi \cdot n\, d\sigma = (\delta^+)^p \int_{\{u=1+\delta^+\}} \varphi \cdot n\, d\sigma = (\delta^+)^p \int_{\{u<1+\delta^+\}} \operatorname{div} \varphi\, dx,$$

which goes to zero as $\delta^+ \searrow 0$. Since $|\mathcal{B}((u_j - 1)/\varepsilon_j)| \le 1$,

$$\begin{aligned} |I_3| \;&\le\; \int_{\{1-\delta^-<u<1+\delta^+\}} \left[1 + \frac{1}{p}\,(u_j - 1)_+^p\right] |\operatorname{div} \varphi|\, dx \\ &\to\; \int_{\{1-\delta^-<u<1+\delta^+\}} \left[1 + \frac{1}{p}\,(u - 1)_+^p\right] |\operatorname{div} \varphi|\, dx, \end{aligned}$$

and

$$I_4 \to \int_{\{1-\delta^-<u<1+\delta^+\}} \left(\nabla u\, D\varphi \cdot \nabla u - \frac{1}{2}\, |\nabla u|^2\, \operatorname{div} \varphi\right) dx.$$

The last two integrals go to zero as $\delta^\pm \searrow 0$ since $|\{u = 1\} \cap \operatorname{supp} \varphi| = 0$, so first letting $j \to \infty$ and then letting $\delta^\pm \searrow 0$ gives (3.8). □

4. The Nehari manifold and non-degeneracy

Lemma 4.1. *Every nonzero generalized solution u to* (1.1) *belongs to the Nehari manifold $\mathcal{M}$ and satisfies $J(u) > 0$.*

Proof. If u is a generalized solution of problem (1.1), then by the maximum principle, the set $\{u < 1\}$ is connected and either $u > 0$ everywhere or u vanishes identically. If $u \le 1$ everywhere, then u is harmonic in Ω and hence vanishes identically again. So if u is nontrivial, then $u > 0$ in Ω and $u > 1$ in a nonempty open subset of Ω, where it satisfies $-\Delta u = (u-1)^{p-1}$. As in the proof of Lemma 3.2 (ii), multiplying this equation by $u - 1$ and integrating over the set $\{u > 1\}$ shows that u lies on $\mathcal{M}$. Finally, if $u \in \mathcal{M}$, then

$$J(u) = \frac{1}{2} \int_{\{u<1\}} |\nabla u|^2\, dx + \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\{u>1\}} |\nabla u|^2\, dx + |\{u > 1\}| > 0,$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^N$. □

For $u \in H^1_0(\Omega)$, set

$$u^+ = (u-1)_+, \quad u^- = 1 - (u-1)_-; \qquad u = u^- + u^+.$$

Let

$$W = \left\{ u \in H^1_0(\Omega) : u^\pm \neq 0 \right\}$$

Then $\mathcal{M} \subset W$, and for $u \in W$, we define the curve

$$\zeta_u(s) = \begin{cases} (1+s)\, u^-, & s \in [-1,0] \\ u^- + s\, u^+, & s \in (0,\infty), \end{cases}$$

which passes through u at $s = 1$. For $s \in [-1,0]$,

$$J(\zeta_u(s)) = \frac{(1+s)^2}{2} \int_{\{u<1\}} |\nabla u|^2\, dx$$

is increasing in s. There is a discontinuity in J at $s = 0$:

$$\lim_{s \searrow 0} J(\zeta_u(s)) = J(\zeta_u(0)) + |\{u > 1\}| > J(\zeta_u(0)).$$

For $s \in (0,\infty)$,

$$J(\zeta_u(s)) = \frac{1}{2} \int_{\{u<1\}} |\nabla u|^2\, dx + \frac{s^2}{2} \int_{\{u>1\}} |\nabla u|^2\, dx - \frac{s^p}{p} \int_{\{u>1\}} (u-1)^p\, dx + |\{u > 1\}| \tag{4.1}$$

and

$$\frac{d}{ds}\, J(\zeta_u(s)) = s \left[\int_{\{u>1\}} |\nabla u|^2\, dx - s^{p-2} \int_{\{u>1\}} (u-1)^p\, dx \right].$$

Define

$$s_u = \left[\frac{\displaystyle\int_{\{u>1\}} |\nabla u|^2\, dx}{\displaystyle\int_{\{u>1\}} (u-1)^p\, dx} \right]^{1/(p-2)}.$$

Thus we see that $J(\zeta_u(s))$ increases for $s \in [-1, s_u)$, attains its maximum at $s = s_u$ and decreases for $s \in (s_u, \infty)$, and

$$\lim_{s \to \infty} J(\zeta_u(s)) = -\infty. \tag{4.2}$$

Proposition 4.2. *We have*

$$c^* \leq \inf_{u \in \mathcal{M}} J(u). \tag{4.3}$$

If $u \in \mathcal{M}$ and $J(u) = c^*$, then u is a mountain pass point of J.

Proof. For each $u \in \mathcal{M}$, (4.2) implies that we may choose $\bar{s} > 1$ sufficiently large that that $J(\zeta_u(\bar{s})) < 0$. Note that $s_u = 1$. Therefore,

$$\gamma_u(t) = \zeta_u((\bar{s} + 1)\, t - 1), \quad t \in [0, 1]$$

defines a path $\gamma_u \in \Gamma$ such that

$$\max_{v \in \gamma_u([0,1])} J(v) = J(\zeta_u(s_u)) = J(u),$$

so $c^* \le J(u)$. (4.3) follows.

Now suppose $J(u) = c^*$ and let U be a neighborhood of u. The path γ_u passes through u at $t = 2/(\bar{s} + 1) =: \bar{t}$ and $J(\gamma_u(t)) < c$ for $t \neq \bar{t}$. By the continuity of γ_u, there exist $0 < t^- < \bar{t} < t^+ < 1$ such that $\gamma_u(t^\pm) \in U$, in particular, the set $\{v \in U : J(v) < c\}$ is nonempty. If it is path connected, then this set contains a path η joining $\gamma_u(t^\pm)$, and reparametrizing $\gamma_u|_{[0,t^-]} \cup \eta \cup \gamma_u|_{[t^+,1]}$ gives a path in Γ on which $J < c^*$, contradicting the definition of c^*. So the set is not path connected, and u is a mountain pass point of J. $\square$

For $u \in W$, ζ_u intersects $\mathcal{M}$ exactly at one point, namely, where $s = s_u$, and $s_u = 1$ if $u \in \mathcal{M}$. So we can define a continuous projection $\pi : W \to \mathcal{M}$ by

$$\pi(u) = \zeta_u(s_u) = u^- + s_u\, u^+.$$

Lemma 4.3. *For $u \in W$,*

$$J(\pi(u)) = \frac{1}{2} \int_{\{u<1\}} |\nabla u|^2\, dx + \left(\frac{1}{2} - \frac{1}{p}\right) s_u^2 \int_{\{u>1\}} |\nabla u|^2\, dx + |\{u > 1\}|\,.$$

In particular, for $u \in \mathcal{M}$, since $\pi(u) = u$,

$$J(u) = \frac{1}{2} \int_{\{u<1\}} |\nabla u|^2\, dx + \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\{u>1\}} |\nabla u|^2\, dx + |\{u > 1\}|\,.$$

Proof. For $u \in W$, $J(\pi(u))$ is given by (4.1) with $s = s_u$, and

$$s_u^2 \int_{\{u>1\}} |\nabla u|^2\, dx = s_u^p \int_{\{u>1\}} (u - 1)^p\, dx. \quad \square$$

Proposition 4.4. *If u is a locally Lipschitz continuous minimizer of $J|_{\mathcal{M}}$, then u is nondegenerate (Definition 1.1).*

Proof. Suppose that $B_r(x_0) \subset \{x \in \Omega : u(x) > 1\}$ and there is $x_1 \in \partial B_r(x_0)$ such that $u(x_1) = 1$. Define

$$v(y) = \frac{1}{r}(u(x_0 + ry) - 1).$$

Our goal is to show that

$$\alpha := v(0) \ge c > 0$$

We begin by oberving that

$$0 < v(y) = \frac{1}{r}(u(x_0 + ry) - u(x_1)) \leq \frac{L}{r}|x_0 - x_1 + ry| \leq 2L \quad \forall y \in B_1(0), \tag{4.4}$$

where L is the Lipschitz constant of u in $\{u \geq 1\}$. Therefore,

$$-\Delta v = r^p \, v^{p-1} \quad \text{in } B_1(0),$$

Define h by

$$-\Delta h = r^p \, v^{p-1} \text{ in } B_1(0), \quad h = 0 \text{ on } \partial B_1(0).$$

Then $|h| \leq C L^{p-1} r^p$ and applying the Harnack inequality to $v - h + \max h$, there is a constant C depending on L and dimension such that

$$v(y) \leq C\,(\alpha + r^p) \quad \forall y \in B_{2/3}(0),$$

Take a smooth cutoff function $\psi : B_1(0) \to [0,1]$ such that $\psi = 0$ in $\overline{B_{1/3}(0)}$, $0 < \psi < 1$ in $B_{2/3}(0) \setminus \overline{B_{1/3}(0)}$ and $\psi = 1$ in $B_1(0) \setminus B_{2/3}(0)$, let

$$w(y) = \begin{cases} \min\left\{v(y), C\,(\alpha + r^p)\,\psi(y)\right\}, & y \in B_{2/3}(0) \\ v(y), & \text{otherwise}, \end{cases}$$

and set $z(x) = 1 + rw((x - x_0)/r)$. Since u is a minimizer of $J|_{\mathcal{M}}$,

$$J(u) \leq J(\pi(z)).$$

Since $z^- = u^-$, $z = 1$ in $\overline{B_{r/3}(x_0)}$, and $\{z > 1\} = \{u > 1\} \setminus \overline{B_{r/3}(x_0)}$, Lemma 4.3 implies this inequality can be rewritten as

$$\left(\frac{1}{2} - \frac{1}{p}\right) \int_{\{u>1\}} |\nabla u|^2\, dx + \left|B_{r/3}(x_0)\right| \leq \left(\frac{1}{2} - \frac{1}{p}\right) s_z^2 \int_{\{u>1\}} |\nabla z|^2\, dx\,.$$

Let $y = (x - x_0)/r$ and define

$$\mathcal{D} := \left\{x \in B_{2r/3}(x_0) : v(y) > C\,(\alpha + r^p)\,\psi(y)\right\}$$

Because $z = u$ outside $\mathcal{D}$, this last inequality implies

$$s_z^2 \int_{\mathcal{D}} |\nabla z|^2\, dx + \left(s_z^2 - 1\right) \int_{\{u>1\} \setminus \mathcal{D}} |\nabla u|^2\, dx \geq \frac{2p}{p-2}\left|B_{1/3}(0)\right| r^N. \tag{4.5}$$

Since $\{z > 1\} = \{u > 1\} \setminus \overline{B_{r/3}(x_0)}$ and $z = 1$ in $\overline{B_{r/3}(x_0)}$,

$$s_z^{p-2} = \frac{\displaystyle\int_{\{z>1\}} |\nabla z|^2\, dx}{\displaystyle\int_{\{z>1\}} (z-1)^p\, dx} = \frac{\displaystyle\int_{\{u>1\}} |\nabla z|^2\, dx}{\displaystyle\int_{\{u>1\}} (z-1)^p\, dx}.$$

Since $z = u$ in $\{u > 1\} \setminus \mathcal{D}$, we have

$$s_z^{p-2} \le \frac{\displaystyle\int_{\{u>1\}} |\nabla u|^2\, dx + \int_{\mathcal{D}} |\nabla z|^2\, dx}{\displaystyle\int_{\{u>1\}} (u-1)^p\, dx - \int_{\mathcal{D}} (u-1)^p\, dx} = \frac{A_1 + \int_{\mathcal{D}} |\nabla z|^2\, dx}{A_1 - \int_{\mathcal{D}} (u-1)^p\, dx},$$

where, since $u \in \mathcal{M}$,

$$A_1 = \int_{\{u>1\}} |\nabla u|^2\, dx = \int_{\{u>1\}} (u-1)^p\, dx$$

It follows as in (4.4), $0 < u - 1 < 2L\, r$ in $\mathcal{D}$, and $|\mathcal{D}| = \mathrm{O}(r^N)$ as $r \to 0$. Thus

$$\int_{\mathcal{D}} (u-1)^p\, dx = \mathrm{O}(r^{p+N}).$$

It follows that

$$s_z^{p-2} \le 1 + \frac{1}{A_1} \int_{\mathcal{D}} |\nabla z|^2\, dx + \mathrm{O}(r^{p+N}). \tag{4.6}$$

We have

$$\int_{\mathcal{D}} |\nabla z|^2\, dx = C^2\, (\alpha + r^p)^2\, r^N \int_{\{y : x \in \mathcal{D}\}} |\nabla \psi|^2\, dy. \tag{4.7}$$

The right-hand side is $\mathrm{O}(r^N)$ since $0 < \alpha < 2L$ by (4.4). Consequently, so (4.6) gives

$$s_z^2 \le 1 + \frac{2}{(p-2) A_1} \int_{\mathcal{D}} |\nabla z|^2\, dx + \mathrm{O}(r^{q+N}),$$

where $q = \min\{p, N\} \ge 2$. Using this estimate in (4.5) now gives

$$\frac{1}{r^N} \int_{\mathcal{D}} |\nabla z|^2\, dx + \mathrm{O}(r^q) \ge 2 \left| B_{1/3}(0) \right|.$$

In view of (4.7), we find that there are r_0, $c > 0$ such that $r \le r_0$ implies $\alpha \ge c$. This concludes the proof of nondegeneracy. $\square$

5. Proof of the main theorem and further boundary regularity

Proof of Theorem 1.3. We can now conclude the proof of our main theorem. Let u be the nontrivial generalized solution of problem (1.1) obtained in Lemma 3.2. Since $u \in \mathcal{M}$,

$$\inf_{\mathcal{M}} J \le J(u).$$

By Lemma 3.2 (*iii*), Lemma 2.3, and Proposition 4.2, we also have

$$J(u) \le \varliminf c_j \le \varlimsup c_j \le c^* \le \inf_{\mathcal{M}} J.$$

So

$$J(u) = c^* = \inf_{\mathcal{M}} J$$

and $c_j \to c$. Then u is a mountain pass point of J by Proposition 4.2, minimizes $J|_{\mathcal{M}}$, and is therefore nondegenerate by Proposition 4.4. □

Definition 5.1. We say that $u \in C(\Omega)$ satisfies the free boundary condition $|\nabla u^+|^2 - |\nabla u^-|^2 = 2$ in the viscosity sense if whenever there exist a point $x_0 \in \partial \{u > 1\}$, a ball $B \subset \{u > 1\}$ (resp. $\{u \le 1\}^\circ$) with $x_0 \in \partial B$, and α (resp. γ) ≥ 0 such that

$$u(x) \ge \alpha \langle x - x_0, \nu \rangle_+ + \mathrm{o}(|x - x_0|) \;\; (\text{resp. } u(x) \le -\gamma \langle x - x_0, \nu \rangle_- + \mathrm{o}(|x - x_0|))$$

in B, where ν is the interior (resp. exterior) unit normal to ∂B at x_0, we have

$$u(x) < -\gamma \langle x - x_0, \nu \rangle_- + \mathrm{o}(|x - x_0|) \;\; (\text{resp. } u(x) > \alpha \langle x - x_0, \nu \rangle_+ + \mathrm{o}(|x - x_0|))$$

in B^c for any γ (resp. α) ≥ 0 such that $\alpha^2 - \gamma^2 >$ (resp. $<$) 2.

Definition 5.2. We say that the point $x_0 \in \partial \{u > 1\}$ is regular if there exists a unit vector $\nu \in \mathbb{R}^N$, called the interior unit normal to the free boundary $\partial \{u > 1\}$ at x_0 in the measure theoretic sense, such that

$$\lim_{r \to 0} \frac{1}{r^N} \int_{B_r(x_0)} \left| \chi_{\{u>1\}}(x) - \chi_{\{\langle x - x_0, \nu \rangle > 0\}}(x) \right| dx = 0.$$

Proof of Corollary 1.4. Let $\varepsilon_j \searrow 0$ be a suitable sequence and let u_j be the solution of (2.1) obtained in Lemma 2.2. Then u_j is uniformly bounded on Ω by Lemma 2.5 and u_j converges uniformly on Ω to u by Lemma 3.2 (i). Set

$$f_j(x) = -(u_j(x) - 1)_+^{p-1}, \quad f(x) = -(u(x) - 1)_+^{p-1}, \quad x \in \Omega.$$

Since $p > 2$, $f_j \to f$ uniformly on Ω. Since, by Theorem 1.3, u is nondegenerate, the corollary follows from now follow from Corollaries 7.1, 7.2 and Theorem 9.2, respectively, of Lederman and Wolanski [20]. □

We expect that, at least in dimension 2, the free boundary has a measure-theoretic normal at all points and hence is smooth. But this is an open question, and it would even be nice to show that the measure-theoretic normal exists at "most" free boundary points.

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