Relations in the Homotopy of Simplicial Abelian Hopf Algebras

by

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Abstract

In this paper, we analyze the structure possessed by the homotopy groups of a simplicial abelian Hopf algebra over the field $\mathbb{F}_2$. Specifically, we review the higher-order structure that the homotopy groups of a simplicial commutative algebra and simplicial cocommutative coalgebra possess. We then demonstrate how these structures interact under the added assumptions present in a Hopf algebra.

Thesis Supervisor: Haynes R. Miller
Title: Professor of Mathematics

Acknowledgements

Since I get very few opportunities to express my gratitude toward those who have made a moment like this, i.e. obtaining a doctorate, both memorable and possible, I'll not refrain from, nor apologize for, being longwinded.

First, since it is only appropriate, let me begin with my colleagues. Foremost among them is Haynes Miller. I have learned many things from Haynes, both as a researcher and as a teacher, but in his role as an advisor I am most appreciative. Many times I have gotten way off track in my objectives (sometimes more than a graduate student should), but Haynes' advice always got me back on course. I have never once regretted asking him to act as my advisor.

As far as my work leading up, or directly related, to this document, there are few, but very important persons to acknowledge. First Bill Dwyer, whose previous work is integral to this paper, made a crucial observation about abelian Hopf algebras which became a lynchpin to establishing the main result. Also Paul Goerss, whose work also was of great value, enlightened me on many points concerning simplicial algebras. He also pointed out an error in an earlier version of this paper. Another person with whom I've enjoyed many conversations is Phil Hirschhorn. Phil has been crucial in helping me understand the intricacies of simplicial and cosimplicial spaces. He has also been an indispensable source of computer information and is responsible for the present appearance of this paper. Thanks Phil. Finally, since Mike Hopkins has become a permanent fixture here, I have had many confusing conversations with him. But, as usually happens, the truth of his deep insights become wonderful revelations days, sometimes weeks, later. For sharing these and his delightful humor, I thank him. I also thank him for his many tutorials on group cohomology.

During my formative years, as a graduate student, I had made many comrades in my efforts to understand the subtleties of teaching and homotopy theory. My most memorable moment at the very beginning was learning Quillen's book with Matthew Ando. It was my greatest influence to algebraic topology. Also, around this time, my duties as a tutor placed me in an office with Rick Scott and Farshid Hajir. For two years, I had a wonderful time learning from them methods of teaching, number theory, toric varieties, philosophy, dealing with red tape, dealing with faculty, the joys of Indian food, etc. Laura Anderson was another person who helped make my life at MIT enjoyable. I missed her the year she was away. Finally, I want to thank John Palmieri and Hal Sadofsky for being kind mentors and good role models during my early years.

Two and a half years ago, I experienced, for about two semesters, a loss of desire to pursue mathematics. While two of my most productive efforts occurred during this time, they happened during brief intervals. One person who was quite pivotal in reawakening this desire was Brooke Shipley. Since then she has been a powerful ally in pursuing several mathematical projects, as well as a good friend. I look forward to collaborating with her in the future.

At this point, I want to mention my e-mail buddy, Michele Intermont. A year ago I had the good fortune of meeting her at a conference in Toronto. Since then I have had many delightful correspondence with her, which, I hope, will continue.

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One of the most important places for a graduate student is the library. While I have never been happy with the set up there, I have been very happy with the staff. In all areas, they have been of great help while being both courteous and pleasant to talk to.

Before I move on I should mention Judy Romvos. While I've only made a brief acquaintance with her, without her this document would not have existed.

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Since I have now actually neared the end, it is time that I pay homage to the most important people in my life, my family. During the past twelve years, I have experienced the most trying and the most exciting times of my life. Through it all they have each given me their solid support and their valuable assistance. In particular, my parents have been pivotal in helping me achieve all my goals I've pursued. To all of them my love and my thanks.

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Contents

Introduction

The goal of this paper is to determine all the natural relations that occur in the homotopy groups of a simplicial abelian Hopf algebra over $\mathbb{F}_2$, the field of two elements. Here Hopf algebra means a unitary algebra and a counitary coalgebra for which certain diagrams commute (see (2.3.1)). An abelian Hopf algebra then is one which is commutative as an algebra and cocommutative as a coalgebra.

It is well-known that over $\mathbb{F}_2$, the homotopy groups of a simplicial commutative algebra possesses, in addition to an algebra structure, a compatible action of a certain operator ring. These operations are viewed as higher-order versions of divided squares. Dually, the homotopy groups of a simplicial cocommutative coalgebra possesses an operational action which extends the coalgebra structure. In fact these are just the Steenrod operations viewed as the dual of higher-order squaring operations. In each case, the higher-order structure exists because of the (co)commutativity. Thus the homotopy groups of a simplicial abelian Hopf algebra possesses both of these structures and the additional properties will produce relations between them.

These relations contribute to the understanding of the cohomology of iterated loop spaces with $\mathbb{F}_2$-coefficients. In particular, the cohomology of a cosimplicial iterated loop space is a simplicial abelian Hopf algebra. The E_2-term of the generalized Eilenberg-Moore spectral sequence (see, for example, [3]) associated to this cosimplicial space, is the homotopy groups of this particular simplicial algebra. Thus the relations assist in making computations. Further, theses operations play a role in understanding the action of the Steenrod and Dyer-Lashoff operations on the abutment of the spectral sequence (see [10], [18], [19], and [20]).

This paper is organized as follows. Chapter 1 is a review of relevant simplicial homotopy and symmetric group actions. Chapter 2 sets up the background for and makes the statement of the Main Theorem. In particular, section 2.1 reviews simplicial commutative algebras and the properties of their homotopy groups, as presented in [9]. Section 2.2 does a similar summary for simplicial cocommutative coalgebras following [12]. Finally, section 2.3 reviews Hopf algebras, establishes an abelian version of the Hopf condition, and states the Main Theorem which portrays the natural relations that occur in the homotopy of a simplicial abelian Hopf algebra.

Chapter 3 is devoted to proving the Main Theorem. We begin, section 3.1, by stating the Reduction. This is a theorem which computes the homotopy groups of a functor on simplicial commutative algebras. We immediately reduce the proof of this Reduction to computing the effect of a natural map, between two functors on simplicial vector spaces, in homotopy. This natural map arises from the abelian Hopf condition, established in section 2.3. In section 3.2, we use the Reduction to prove the Main Theorem. In section 3.3, we begin the proof of the computation for the natural map of section 3.1. We first fit this map into two commuting diagrams. This reduces our efforts further by allowing us to divide the computations between two new natural maps, each possessing properties amenable to calculations. In particular, in section 3.4, we recall a method developed in [9] which allows us to convert our simplicial calculations to ones in the cohomology of groups. Finally, in section 3.5, we make these group cohomological calculations, completing the proof of the Reduction.

Conventions

All groups throughout are finite.

Let R be a ring, G a group, and V a left R-module. Then V is a G-module if V is a left $R[G]$-module. On the category of G-modules there are two functors. The first functor

$$(-)^G\colon (G\text{-modules}) \to (R\text{-modules})$$

called the G-invariant functor, is defined by

$$V^G = \{x \in V : gx = x \quad \text{for all } g \in G\}.$$

The second functor

$$(-)_G\colon (G\text{-modules}) \to (R\text{-modules})$$

called the G-coinvariant functor, is defined by

$$V_G = V/\{(1-g)x : x \in V,\ g \in G\}.$$

Further, given a subgroup $H \leq G$, the inclusion induces a natural transformation, called restriction,

$$r(G,H)\colon V^G \to V^H.$$

Also, if $g_1, \dots, g_m$ are coset representatives of G/H, where $m = (H:G)$, then the action of the element $g_1 + \cdots + g_m \in R[G]$ on V^H induces a natural transformation, called transfer,

$$t(h,G)\colon V^H \to V^G.$$

The two transformations are related by

$$t(H,G)r(G,H)x = (H:G)x$$

for any $x \in V^G$.

For a fixed group G, we denote by i the inclusion

$$V^G \to V$$

and by ρ the projection

$$V \to V_G$$

Next we call V a graded R-module if $V = \{V_n\}_{n\geq 0}$ where each V_n is an R-module. If W is another graded R-module we define the graded tensor product $V \otimes W$ by

$$(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j$$

for all $n \geq 0$.

On the category of graded R-modules we have a functor

$$\Phi\colon \begin{pmatrix}\text{graded} \\ R\text{-modules}\end{pmatrix} \to \begin{pmatrix}\text{graded} \\ R\text{-modules}\end{pmatrix}$$

called the doubling, defined by

$$(\Phi V)_n = \begin{cases} 0 & n \text{ odd} \\ V_{\frac{n}{2}} & n \text{ even} \end{cases}.$$

For an element $x \in V$ we denote its associated element in ΦV by $\overline{x}$.

Finally, for $n \in \mathbb{Z}$ and $k \in \mathbb{N}$ define $\binom{n}{k}$ as the coefficient of x^k in the Taylor expansion of $(1+x)^n$. These numbers satisfy the general Pascal relation

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

Further, we have

$$\binom{n}{k} = \binom{-n+k-1}{k}.$$

Also we define for $i, j \geq 0$

$$(i,j) = \binom{i+j}{j}.$$

For the rest of this work $R = \mathbb{F}_2$, the field of two elements.

CHAPTER I

Preliminaries

1. Simplicial $\mathbb{F}_2$-modules

Define a simplicial $\mathbb{F}_2$-module V to be a graded $\mathbb{F}_2$-module together with maps of modules

$$d_j : V_n \to V_{n-1}$$

called face maps, and

$$s_j : V_n \to V_{n+1}$$

called degeneracies, for $0 \leq j \leq n$, satisfying standard identities (See [14]). A map $f: V \to W$ of simplicial $\mathbb{F}_2$-modules is a map of graded modules which commutes with the face and degeneracy maps. We denote the category of simplicial $\mathbb{F}_2$-modules by $s\mathbb{F}_2$.

Next given two simplicial $\mathbb{F}_2$-modules V and W we define the simplicial tensor product $V \otimes W$ by

$$(V \otimes W)_n = V_n \otimes W_n$$

such that for $x \otimes y \in (V \otimes W)_n$ then

$$d_j(x \otimes y) = d_j x \otimes d_j y \qquad s_j(x \otimes y) = s_j x \otimes s_j y$$

$0 \leq j \leq n$.

Now, define the normalization functor

$$N : s\mathbb{F}_2 \to \begin{pmatrix} \mathbb{F}_2\text{-chain} \\ \text{complexes} \end{pmatrix}$$

as follows: For V in $s\mathbb{F}_2$ define, for each $n \geq 0$, the submodule $D_n V \subseteq V_n$ by

$$D_n V = \operatorname{im} s_0 + \cdots + \operatorname{im} s_{n-1}.$$

From this define

$$N_n V = V_n / D_n V.$$

Further, define

$$\partial : N_n V \to N_{n-1} V$$

by

$$\partial = d_0 + \cdots + d_n.$$

As shown in [14], (NV, ∂) is a well-defined chain complex.

Moreover, N has a left adjoint

$$S\colon \begin{pmatrix} \mathbb{F}_2\text{-chain} \\ \text{complexes} \end{pmatrix} \to s\mathbb{F}_2.$$

As shown in [6], the adjoint pair (S, N) determine an equivalence of categories. We are now in a position to define, for V in $s\mathbb{F}_2$, the homotopy groups $\pi_* V$ by

$$\pi_n V = H_n(NV, \partial).$$

This defines a functor

$$\pi_*\colon s\mathbb{F}_2 \to \begin{pmatrix} \text{graded} \\ \mathbb{F}_2\text{-modules} \end{pmatrix}.$$

Moreover, this functor is corepresentable as follows:

For V and W in $s\mathbb{F}_2$ define its homotopy set of maps to be

$$[V, W] = \hom_{s\mathbb{F}_2}(V, W)/\sim$$

where $f \sim g$ if f is homotopic to g (see [15]) for $f, g \in \hom_{s\mathbb{F}_2}(V, W)$. Now, define, for $n \geq 0$, $K(n)$ in $s\mathbb{F}_2$ by $SC(n)$ where $C(n)$ is the chain complex such that

$$C(n)_q = \begin{cases} \mathbb{F}_2 & q = n \\ 0 & \text{otherwise.} \end{cases}$$

From the equivalence of categories we have

$$\pi_q K(n) = \begin{cases} \mathbb{F}_2 & q = n \\ 0 & \text{otherwise.} \end{cases} \tag{1.1.1}$$

Moreover, the correspondence

$$[K(n), V] \to \pi_n V \tag{1.1.2}$$

given by

$$[f] \to f_*(\iota)$$

where $\iota \in \pi_n K(n)$ is the generator, is a bijection (see [15]).

Remark. An equivalent definition of $K(n)$ is given as follows:

Let $\Delta[n]$ be the standard n-simplex and $\dot{\Delta}[n]$ the simplicial set generated by $d_j \iota_n$ where $\iota_n \in \Delta[n]_n$. Then $K(n) =$ free $\mathbb{F}_2$-module on $\Delta[n]/\dot{\Delta}[n]$.

2. Eilenberg-Zilber Theorem.

We begin by summarizing the Eilenberg-Zilber theorem as given in [14] and [9].

THEOREM 1.2.1. *Let V and W be two simplicial $\mathbb{F}_2$-modules. Then there exists a unique natural chain map*

$$D\colon N(V)\otimes N(W)\to N(V\otimes W)$$

which is the identity in dimension 0.

Moreover, there exists a natural chain map

$$E\colon N(V\otimes W)\to N(V)\otimes N(W)$$

such that

$$ED=1 \qquad DE\simeq 1.$$

In [9] it was noticed that since D is necessarily the shuffle map (see [14]) thus D possesses a symmetry. This symmetry was exploited by Dwyer to construct higher order versions of D which we now describe.

DEFINITION 1.2.2. For each $k\geq 0$, let

$$\phi_k\colon N(V)\otimes N(W)\to N(V\otimes W)$$

be the chain map such that for $x\in N(V)$ and $y\in N(W)$

$$\phi_k(x\otimes y)=\begin{cases} x\otimes y & |x|=k=|y|\\ 0 & \text{otherwise}\end{cases}$$

ϕ_k is called an admissible map.

Let T denote the switching map for either

$$N(V)\otimes N(W)\to N(W)\otimes N(V)$$

or

$$N(V\otimes W)\to N(W\otimes V).$$

THEOREM 1.2.3. *Let V and W be simplicial $\mathbb{F}_2$-modules. For each $k\geq 0$ there exists a natural chain map*

$$D^k\colon [N(V)\otimes N(W)]_m\to N(V\otimes W)_{m-k}$$

defined for $m\geq 2k$ and satisfying

1. $D^0+TD^0T+\phi_0=D$
2. $D^{k+1}+TD^{k+1}T+\phi_{k+1}=\partial D^k+D^k\partial$

REMARK. Dwyer showed in [9] that each D^k is unique in a certain sense.

3. Group Actions on Tensor Products.

Let V be an $\mathbb{F}_2$-module and define

$$V^{\otimes m} = \underbrace{V \otimes \cdots \otimes V}_{m\text{-times}}.$$

Then Σ_m, the symmetric group on m letters, acts on $V^{\otimes m}$ by permutation. Thus for any subgroup $G \leq \Sigma_m$, $V^{\otimes m}$ is a G-module. With this we define the G-symmetric invariant functor

$$S^G : m\mathbb{F}_2 \to m\mathbb{F}_2 \tag{1.3.1}$$

by

$$S^G(V) = (V^{\otimes m})^G$$

and the G-symmetric coinvariant functor

$$S_G : m\mathbb{F}_2 \to m\mathbb{F}_2 \tag{1.3.2}$$

by

$$S_G(V) = (V^{\otimes m})_G.$$

If $G = \Sigma_m$ then we denote (1.3.1) by S^m and (1.3.2) by S_m.

Now, let $\overline{N} \in \mathbb{F}_2[G]$ be defined by

$$\overline{N} = \sum_{g \in G} g \tag{1.3.3}$$

Then the action of $\overline{N}$ on $V^{\otimes m}$ defines a map which factors

$$\begin{array}{ccc} V^{\otimes m} & \xrightarrow{\overline{N}} & V^{\otimes m} \\ & {\scriptstyle \tau}\searrow \quad \nearrow{\scriptstyle i} & \\ & S^G V & \end{array} \tag{1.3.4}$$

but since, for any $x \in V^{\otimes m}$, $\tau(gx) = \tau(x)$ for any $g \in G$ then we have a further factorization.

$$\begin{array}{ccc} V^{\otimes m} & \xrightarrow{\tau} & S^G V \\ & {\scriptstyle \rho}\searrow \quad \nearrow{\scriptstyle N} & \\ & S_G V & \end{array} \tag{1.3.5}$$

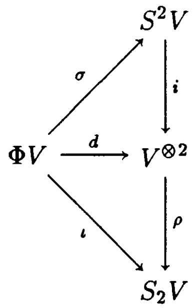
defining the norm map N.

Because of its importance later, we analyze the norm map N in the case $G = \Sigma_2$. First, we define the diagonal map

$$d : \Phi V \to V^{\otimes 2}$$

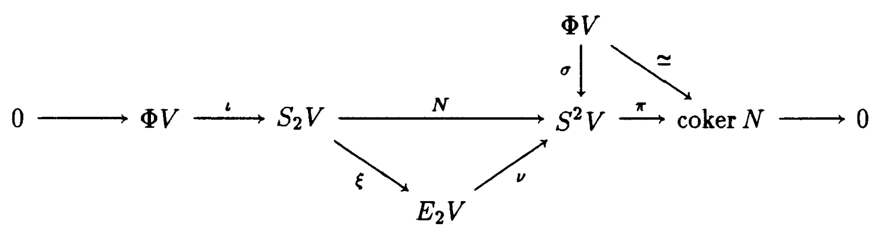
by $d\overline{x} = x \otimes x$. This is not a homomorphism, nonetheless we have a commuting diagram

(1.3.6)



σ is not a homomorphism, but ι is one. From this we define the exterior square functor E_2 by $E_2V = \operatorname{coker}\iota$. We now have the following commutative diagram

(1.3.7)



from which we have that $E_2V = \operatorname{im} N = \ker \pi$. Note that $\pi\sigma$ is a linear isomorphism.

As an application of (1.3.7) we have

PROPOSITION 1.3.8. *For any $\omega \in S^2V$ there exists $\alpha \in E_2V$ and $x \in V$, uniquely determined by ω, such that*

$$\omega = \nu(\alpha) + \sigma(\overline{x}).$$

PROOF. Let $\widetilde{\sigma}$: $\operatorname{coker} N \to S^2V$ be the composite $\sigma \cdot (\pi\sigma)^{-1}$ so that $\pi\widetilde{\sigma} = 1$. Then the self-map

$$1 + \widetilde{\sigma}\pi \colon S^2V \to S^2V$$

satisfies $\pi(1 + \widetilde{\sigma}\pi) = 0$. Thus since ν is injective, there exists $\alpha \in E_2V$ such that

$$\nu(\alpha) = \omega + \widetilde{\sigma}\pi(\omega).$$

Finally, let $x \in V$ be the element which satisfies

$$\overline{x} = (\pi\sigma)^{-1}(\pi\omega).$$

Conclusion follows. □

CHAPTER II

Simplicial Abelian Hopf Algebras

1. Simplicial Algebras and D-algebras.

Recall that a (graded) algebra is a triple (Λ, m, η) consisting of a (graded) vector space Λ and maps of (graded) vector spaces

(2.1.1) $$m\colon \Lambda \otimes \Lambda \to \Lambda,$$

called multiplication, and

(2.1.2) $$\eta\colon \mathbb{F}_2 \to \Lambda,$$

called the unit, such that the two diagrams

(2.1.3) $$\begin{array}{ccc} \Lambda\otimes\Lambda\otimes\Lambda & \xrightarrow{m\otimes 1} & \Lambda\otimes\Lambda \\ {\scriptstyle 1\otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ \Lambda\otimes\Lambda & \xrightarrow[m]{} & \Lambda \end{array}$$

and

(2.1.4) $$\begin{array}{ccc} \mathbb{F}_2\otimes\Lambda \simeq \Lambda \simeq \Lambda\otimes\mathbb{F}_2 & \xrightarrow{1\otimes\eta} & \Lambda\otimes\Lambda \\ {\scriptstyle \eta\otimes 1}\downarrow & \searrow^{1} & \downarrow{\scriptstyle m} \\ \Lambda\otimes\Lambda & \xrightarrow[m]{} & \Lambda \end{array}$$

commute.

We further call our algebra commutative if (2.1.1) factors as

(2.1.5) $$\begin{array}{ccc} \Lambda\otimes\Lambda & \xrightarrow{\quad m \quad} & \Lambda \\ & {\scriptstyle\rho}\searrow \quad \nearrow{\scriptstyle\mu} & \\ & S_2\Lambda & \end{array}$$

Notation. For brevity, we denote an algebra (Λ, m, η) by Λ. Also, for $x, y \in \Lambda$ we denote the image of $x \otimes y \in \Lambda \otimes \Lambda$ under m by $x \cdot y$.

Next, given algebras Λ and Λ', a linear map $f\colon \Lambda \to \Lambda'$ is a map of algebras if the diagrams

(2.1.6)
$$\begin{array}{ccc} \Lambda \otimes \Lambda & \xrightarrow{f \otimes f} & \Lambda' \otimes \Lambda' \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\ \Lambda & \xrightarrow[f]{} & \Lambda' \end{array}$$

and

(2.1.7)
$$\begin{array}{ccc} & & \Lambda \\ & \overset{\eta}{\nearrow} & \\ \mathbb{F}_2 & & \downarrow{\scriptstyle f} \\ & \underset{\eta'}{\searrow} & \\ & & \Lambda' \end{array}$$

commute.

Note that if Λ and Λ' are commutative then (2.1.6) can be replaced by

(2.1.8)
$$\begin{array}{ccc} S_2\Lambda & \xrightarrow{S_2(f)} & S_2\Lambda' \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu'} \\ \Lambda & \xrightarrow[f]{} & \Lambda' \end{array}$$

We denote the category of commutative algebras (respectively commutative graded algebras) by $\mathcal{A}$ (respectively $\mathcal{A}_*$).

Given a graded algebra Λ, let $I_s(\Lambda) \subseteq \Lambda$ $s \geq 0$ denote the ideal of elements x in Λ such that $|x| \geq s$.

DEFINITION 2.1.9. A Γ-algebra is a commutative graded algebra Λ together with a map

$$\gamma_2 \colon \Phi I_2 \to \Lambda$$

such that

1. I_1 is exterior under the product of Λ,
2. for $x, y \in I_2$
$$\gamma_2(\overline{x \cdot y}) = \gamma_2(\overline{x}) + \gamma_2(\overline{y}) + x \cdot y,$$
3. for $x, y \in \Lambda$ such that $x \cdot y \in I_2$
$$\gamma_2(\overline{x \cdot y}) = \begin{cases} 0 & x, y \in I_1 \\ (x \cdot x) \cdot \gamma_2(\overline{y}) & |x| = 0 \\ \gamma_2(\overline{x}) \cdot (y \cdot y) & |y| = 0. \end{cases}$$

We now make the following, as given in [12].

DEFINITION 2.1.10. A D-algebra Λ is a Γ-algebra together with maps

$$\delta_i \colon \Lambda_n \to \Lambda_{n+i}$$

for all $2 \le i \le n$ such that

1. δ_i is a homomorphism, for $i < n$, and $\delta_n = \gamma_2$,
2. for $x, y \in \Lambda$ such that $x \cdot y \in \Lambda_n$ then

$$\delta_i(x \cdot y) = \begin{cases} (x \cdot x) \cdot \delta_i(y) & |x| = 0 \\ \delta_i(x) \cdot (y \cdot y) & |y| = 0 \\ 0 & \text{otherwise,} \end{cases}$$

3. for $x \in \Lambda_n$ and $j < 2i$ then

$$\delta_j \delta_i x = \sum_{\frac{i+1}{2} \le s \le \frac{i+j}{3}} \binom{i-j+s-1}{i-s} \delta_{j+i-s} \delta_s x.$$

A map of D-algebras is a map in $\mathcal{A}_*$ that commutes with the δ_i. We denote the category of D-algebras by $\mathcal{AD}$.

We now define a simplicial algebra (Λ, m, η) so that (2.1.1)–(2.1.4) are satisfied with the caveat that (2.1.1) and (2.1.2) are now maps of simplicial modules ($\mathbb{F}_2$ is replaced with its constant simplicial alias). Further (2.1.5) is satisfied for simplicial commutative algebras with the factorization occurring in $s\mathbb{F}_2$. We denote the category of simplicial commutative algebras by $s\mathcal{A}$.

The following was proved in [9] and [13].

THEOREM 2.1.11. *Let Λ be a simplicial commutative algebra. Then $\pi_*\Lambda$ is naturally a D-algebra i.e. we have a functor*

$$\pi_* \colon s\mathcal{A} \to \mathcal{AD}.$$

REMARK. The operations δ_i in (2.1.10) were first discovered in [4]. Their properties were subsequently derived in [2] and [9]. In the latter, they were called higher divided squares.

We conclude this section by indicating why Theorem 2.1.11 completely determines the homotopy of a simplicial commutative algebra.

In light of (2.1.5), a computation of the homotopy of S_2V, for a simplicial module V, in terms of $\pi_* V$ would give a complete picture of the primary operator algebra for the homotopy of a simplicial commutative algebra. Such a description is known to exist by [8]. We now proceed to make this description explicit.

Fix a simplicial module V. For each $0 \le i \le n$ define

$$\Theta_i \colon N_n V \to N_{n+i} S_2 V \tag{2.1.12}$$

by

$$\Theta_i(a) = \rho D^{n-i}(a \otimes a) + \rho D^{n-i-1}(a \otimes \partial a) \tag{2.1.13}$$

where the D^s are from Theorem 1.2.3.

A computation gives us that

$$\partial \Theta_i = \Theta_i \partial.$$

Thus, for $2 \le i \le n$, Θ_i induces a natural map

$$\overline{\delta}_i : \pi_n V \to \pi_{n+i} S_2 V. \tag{2.1.14}$$

Also, the chain map

$$\rho D\colon N_s V \otimes N_t V \to N_{s+t} S_2 V$$

induces a homomorphism

$$\overline{m}\colon \pi_s V \otimes \pi_t V \to \pi_{s+t} S_2 V. \tag{2.1.15}$$

Combining the results of [4], [2], and [9] we are led to

PROPOSITION 2.1.16. *Let V be a fixed simplicial module. Define W to be the graded module with basis*

$$\begin{aligned} &\delta_i(x) \quad \text{for } x \in \pi_n V \text{ and } 2 \le i \le n, \\ &x \cdot y \quad \text{for } x \in \pi_s V \text{ and } y \in \pi_t V. \end{aligned}$$

Define a submodule B in W with basis

$$\begin{aligned} \delta_i(x+y) + \delta_i(x) + \delta_i(y) + &\begin{cases} 0 & 2 \le i < n \quad \text{for } x, y \in \pi_n V \\ x \cdot y & i = n, \end{cases} \\ x \cdot y + y \cdot x \quad & \text{for } x \in \pi_s V \text{ and } y \in \pi_t V, \\ x \cdot (y+z) + x \cdot y + x \cdot z \quad & \text{for } x \in \pi_s V \text{ and } y, z \in \pi_t V, \\ x \cdot x \quad & \text{for } x \in \pi_n V \text{ and } n > 0. \end{aligned}$$

Then the map $W \to \pi_* S_2 V$ given by

$$\begin{aligned} x \cdot y &\to \overline{m}(x \otimes y) \\ \delta_i x &\to \overline{\delta}_i x \end{aligned}$$

is natural and induces a linear isomorphism

$$W/B \simeq \pi_* S_2 V.$$

NOTE 2.1.17. Given (graded) algebras Λ and Λ' then $\Lambda \otimes \Lambda'$ is a (graded) algebra under the product

$$(\Lambda \otimes \Lambda') \otimes (\Lambda \otimes \Lambda') \xrightarrow{1 \otimes T \otimes 1} (\Lambda \otimes \Lambda) \otimes (\Lambda' \otimes \Lambda') \xrightarrow{m \otimes m'} \Lambda \otimes \Lambda'.$$

Further, if Λ and Λ' are Γ-algebras then we define

$$\gamma_2 \colon \Phi I_2 \to \Lambda \otimes \Lambda'$$

by demanding that the diagrams

$$\begin{array}{ccc} \Phi I_2(\Lambda) \simeq \Phi I_2(\Lambda \otimes \mathbb{F}_2) & \xrightarrow{\Phi I_2(1 \otimes \eta')} & \Phi I_2(\Lambda \otimes \Lambda') \\ \Big\downarrow \gamma_2 & & \Big\downarrow \gamma_2 \\ \Lambda \simeq \Lambda \otimes \mathbb{F}_2 & \xrightarrow[1 \otimes \eta']{} & \Lambda \otimes \Lambda' \end{array}$$

and

$$\begin{array}{ccc}
\Phi I_2(\Lambda') \simeq \Phi I_2(\mathbb{F}_2 \otimes \Lambda') & \xrightarrow{\Phi I_2(\eta\otimes 1)} & \Phi I_2(\Lambda \otimes \Lambda') \\
\Big\downarrow \gamma_2 & & \Big\downarrow \gamma_2 \\
\Lambda' \simeq \mathbb{F}_2 \otimes \Lambda' & \xrightarrow[\eta\otimes 1]{} & \Lambda \otimes \Lambda'
\end{array}$$

commute and then extending using 3. of Definition 2.1.9. Similarly, we define a D-algebra structure on $\Lambda \otimes \Lambda'$, when Λ and Λ' are D-algebras, by demanding that

$$\delta_i\colon (\Lambda \otimes \Lambda')_n \to (\Lambda \otimes \Lambda')_{n+i}$$

for $2 \le i \le n$, fits in the commuting diagrams

$$\begin{array}{ccc}
\Lambda_n \simeq (\Lambda \otimes \mathbb{F}_2)_n & \xrightarrow{1\otimes\eta'} & (\Lambda \otimes \Lambda')_n \\
\Big\downarrow \delta_i & & \Big\downarrow \delta_i \\
\Lambda_{n+i} \simeq (\Lambda \otimes \mathbb{F}_2)_{n+i} & \xrightarrow[1\otimes\eta']{} & (\Lambda \otimes \Lambda')_{n+i}
\end{array}$$

and

$$\begin{array}{ccc}
\Lambda'_n \simeq (\mathbb{F}_2 \otimes \Lambda')_n & \xrightarrow{\eta\otimes 1} & (\Lambda \otimes \Lambda')_n \\
\Big\downarrow \delta_i & & \Big\downarrow \delta_i \\
\Lambda'_{n+i} \simeq (\mathbb{F}_2 \otimes \Lambda')_{n+i} & \xrightarrow[\eta\otimes 1]{} & (\Lambda \otimes \Lambda')_{n+i}
\end{array}$$

and then extending using 2. of (2.1.10).

2. Simplicial Coalgebras and A-coalgebras.

Recall that a (graded) coalgebra is a triple (Π, Δ, ϵ) consisting of a (graded) module Π and maps of (graded) modules

(2.2.1) $$\Delta\colon \Pi \to \Pi \otimes \Pi,$$

called comultiplication, and

(2.2.2) $$\epsilon\colon \Pi \to \mathbb{F}_2,$$

called the counit, such that the diagrams

(2.2.3) $$\begin{array}{ccc}
\Pi & \xrightarrow{\Delta} & \Pi \otimes \Pi \\
\Delta \big\downarrow & & \big\downarrow \Delta\otimes 1 \\
\Pi \otimes \Pi & \xrightarrow{1\otimes\Delta} & \Pi \otimes \Pi \otimes \Pi
\end{array}$$

and

$$(2.2.4) \qquad \begin{array}{ccc} \Pi & \xrightarrow{\Delta} & \Pi \otimes \Pi \\ {\scriptstyle\Delta}\downarrow & \overset{1}{\searrow} & \downarrow{\scriptstyle 1\otimes\epsilon} \\ \Pi \otimes \Pi & \xrightarrow[\epsilon\otimes 1]{} & \mathbb{F}_2 \otimes \Pi \simeq \Pi \simeq \Pi \otimes \mathbb{F}_2 \end{array}$$

commute.

We further call our coalgebra cocommutative if (2.2.1) factors as

$$(2.2.5) \qquad \begin{array}{ccc} \Pi & \xrightarrow{\Delta} & \Pi \otimes \Pi \\ & {\scriptstyle\psi}\searrow \quad \nearrow{\scriptstyle i} & \\ & S^2\Pi & \end{array}$$

NOTATION. For brevity, we denote a coalgebra (Π, Δ, ϵ) by Π .

For two (graded) coalgebras Π and Π', a map $f\colon \Pi \to \Pi'$ of (graded) modules is a map of (graded) coalgebras if the two diagrams

$$(2.2.6) \qquad \begin{array}{ccc} \Pi & \xrightarrow{f} & \Pi' \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta'} \\ \Pi \otimes \Pi & \xrightarrow[f\otimes f]{} & \Pi' \otimes \Pi' \end{array}$$

and

$$(2.2.7) \qquad \begin{array}{ccc} \Pi & & \\ & \searrow{\scriptstyle\epsilon} & \\ {\scriptstyle f}\downarrow & & \mathbb{F}_2 \\ & \nearrow{\scriptstyle\epsilon'} & \\ \Pi' & & \end{array}$$

commute.

Note that for Π and Π' cocommutative, (2.2.6) can be replaced by

$$(2.2.8) \qquad \begin{array}{ccc} \Pi & \xrightarrow{f} & \Pi' \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi'} \\ S^2\Pi & \xrightarrow[S^2(f)]{} & S^2\Pi' \end{array}$$

We denote the category of cocommutative coalgebras (resp. cocommutative graded coalgebras) by $\mathcal{CA}$ (resp. $\mathcal{CA}_*$).

Next, given a cocommutative graded coalgebra Π we define the coalgebra map

$$(2.2.9) \qquad v\colon \Pi \to \Phi\Pi$$

called the verschiebung, as follows: Fix $x \in \Pi$. Then $\psi x \in S^2\Pi$. By Proposition 1.3.8 there exists unique $\alpha \in E_2\Pi$ and $\beta \in \Pi$ such that

$$\psi x = \nu(\alpha) + \sigma(\overline{\beta}).$$

From this we let $v(x) = \overline{\beta}$.

DEFINITION 2.2.10. An A-coalgebra is a cocommutative graded coalgebra Π together with homomorphisms

$$Sq^i : \Pi_n \to \Pi_{n-i}$$

for $i \geq 0$ such that for $x \in \Pi_n$ we have

1. $xSq^i = 0$ for $2i > n$ and $xSq^{\frac{n}{2}} = v(x)$,
2. if $\Delta x = \Sigma x' \otimes x''$ then
$$\Delta(xSq^i) = \sum_{s+t=i} \sum (x'Sq^s) \otimes (x''Sq^t),$$
3. for $j < 2i$ we have
$$xSq^j Sq^i = \sum_{2s \leq j} \binom{i-s-1}{j-2s} xSq^{i+j-s} Sq^s.$$

We define a map of A-coalgebras to be a map in $\mathcal{CA}_*$ which commutes with the Sq^i. Denote by $\mathcal{K}^*$ the category of A-coalgebras.

NOTE. A clearly denotes the Steenrod algebra.

We now define a simplicial coalgebra to be a triple (Π, Δ, ϵ) where Π is a simplicial module and satisfies (2.2.1)–(2.2.4) with the exception that all maps are maps of simplicial modules. Further, a simplicial cocommutative coalgebra also satisfies (2.2.5). (2.2.6)–(2.2.8) also define maps with the requirement that they be maps of simplicial modules.

We denote the category of simplicial cocommutative coalgebras by s$\mathcal{CA}$. A consequence of [7] (see also [12]) is the following

THEOREM 2.2.11. *Let Π be a simplicial cocommutative coalgebra. Then $\pi_*\Pi$ is naturally an A-coalgebra. That is, we have a functor*

$$\pi_* : s\mathcal{CA} \to \mathcal{K}^*.$$

We close this section by indicating why Theorem 2.2.11 completely determines the homotopy of a simplicial cocommutative coalgebra.

As in the algebra case, (2.2.5) indicates that it is sufficient to determine $\pi_* S^2 V$, for a simplicial module V, in terms of $\pi_* V$. This description exists by [8]. We now proceed to make this explicit.

Fix a simplicial module V. Consider the composite

$$N_n V \xrightarrow{\Theta_i} N_{n+i} S_2 V \xrightarrow{N_*} N_{n+i} S^2 V \tag{2.2.12}$$

of chain maps. Here Θ_i is from (2.1.12) and N is the norm map (1.3.7). This induces a natural map

$$\sigma_i : \pi_n V \to \pi_{n+i} S^2 V \tag{2.2.13}$$

for each $0 \le i \le n$. Also, the composite

$$(2.2.14) \qquad N_sV \otimes N_tV \xrightarrow{D} N_{s+t}(V \otimes V) \xrightarrow{\pi_*} N_{s+t}S_2V \xrightarrow{N_*} N_{s+t}S^2V$$

induces the homomorphism

$$(2.2.15) \qquad \tau\colon \pi_sV \otimes \pi_tV \to \pi_{s+t}S^2V$$

The following is given in [12].

Proposition 2.2.16. *Let V be a simplicial module. Let T be the graded vector space with basis*

$$\begin{array}{ll} \sigma_i(x) & \text{for } x \in \pi_nV \text{ and } 0 \le i \le n, \\ [x,y] & \text{for } x \in \pi_nV \text{ and } y \in \pi_mV, \quad n,m \ge 0. \end{array}$$

Let R be the submodule of T with basis

$$\begin{array}{rl} [x,y]+[y,x] & \text{for } x \in \pi_nV,\ y \in \pi_mV, \quad n,m \ge 0, \\ [x,y+z]+[x,y]+[x,z] & \text{for } x \in \pi_nV,\ y,z \in \pi_mV, \quad n,m \ge 0, \end{array}$$

$$\sigma_i(x+y)+\sigma_i(x)+\sigma_i(y)+\begin{cases} 0 & 0 \le i < n \\ [x,y] & i = n \quad \text{for } x,y \in \pi_nV, \end{cases}$$

$$[x,x] \quad \text{for } x \in \pi_nV,\ n \ge 0.$$

Then the map $T \to \pi_*S^2V$ defined by

$$\begin{array}{c} \sigma_i(x) \to \sigma_i(x) \\ [x,y] \to \tau(x \otimes y) \end{array}$$

induces a natural linear isomorphism

$$T/R \simeq \pi_*S^2V.$$

Moreover, if we let

$$e\colon \pi_*S^2V \to \pi_*V \otimes \pi_*V$$

be induced by the composition of chain maps

$$NS^2V \xrightarrow{i_*} N(V \otimes V) \xrightarrow{E} NV \otimes NV$$

(see Theorem 1.2.1) then for $x \in \pi_nV$ $y \in \pi_mV$ $n,m \ge 0$

$$e([x,y]) = x \otimes y + y \otimes x$$

and for $x \in \pi_nV$ $0 \le i \le n$

$$e(\sigma_i(x)) = \begin{cases} 0 & 0 \le i < n \\ x \otimes x & i = n. \end{cases}$$

We take a moment to note a corollary given in [12].

COROLLARY 2.2.17. *The effect of the homomorphism*

$$N_*\colon \pi_* S_2 V \longrightarrow \pi_* S^2 V$$

is given by

$$x \cdot y \longrightarrow [x, y]$$

for $x \in \pi_n V$, $y \in \pi_m V$, $n, m \geq 0$, *and*

$$\delta_i(x) \longrightarrow \sigma_i(x)$$

for $x \in \pi_n V$, $2 \leq i \leq n$. *Moreover, under the homomorphism* (1.3.6)

$$\iota_*\colon \Phi\pi_* V \longrightarrow \pi_* S_2 V$$

we have

$$\operatorname{im} \iota_* = \ker N_*.$$

Finally, given Π in $s\mathcal{CA}$, then for $x \in \pi_n \Pi$ Proposition 2.2.16 tells us that

$$\psi_* x = \Sigma[x', x''] + \Sigma\sigma_i(xSq^i) \tag{2.2.18}$$

which defines the action of the Steenrod operations. From this and Corollary 2.2.17 we conclude 1. of (2.2.10). Also, we define the coproduct

$$\Delta\colon \pi_*\Pi \longrightarrow \pi_*\Pi \otimes \pi_*\Pi \tag{2.2.19}$$

by $e\psi_*$ from Proposition 2.2.16.

3. Simplicial Hopf Algebras and Hopf D-algebras.

Recall that a (graded) Hopf algebra (in the sense of [16]) is a (graded) module H which is both a (graded) algebra and a (graded) coalgebra for which the two diagrams

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\Delta\otimes\Delta} & H \otimes H \otimes H \otimes H \\ & & \big\downarrow {\scriptstyle 1\otimes T\otimes 1} \\ {\scriptstyle m}\big\downarrow & & H \otimes H \otimes H \otimes H \\ & & \big\downarrow {\scriptstyle m\otimes m} \\ H & \xrightarrow[\Delta]{} & H \otimes H \end{array} \tag{2.3.1}$$

and

$$\begin{array}{ccc} \mathbb{F}_2 & \xrightarrow{\eta} & H \\ & {\scriptstyle 1}\searrow & \big\downarrow {\scriptstyle \epsilon} \\ & & \mathbb{F}_2 \end{array} \tag{2.3.2}$$

commute. A map of Hopf algebras is simply a map of algebras and a map of coalgebras. We further define a Hopf algebra to be abelian if it is commutative as an algebra and cocommutative as a coalgebra. Given an abelian Hopf algebra the diagram (2.3.1) possesses a modification. To describe it, we need some preliminaries.

LEMMA 2.3.3. *Let V be a (graded) module. Then there exist maps ϕ', ϕ'' of modules such that the following diagram commutes*

$$\begin{array}{ccccc} S^2V\otimes S^2V & \xrightarrow{i\otimes i} & V\otimes V\otimes V\otimes V & \xrightarrow{\rho} & S_2(V\otimes V) \\ \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle 1\otimes T\otimes 1} & & \downarrow{\scriptstyle \phi''} \\ S^2(V\otimes V) & \xrightarrow[i]{} & V\otimes V\otimes V\otimes V & \xrightarrow[\rho\otimes\rho]{} & S_2V\otimes S_2V \end{array}$$

PROOF. Define $\widehat{T}\colon V^{\otimes 4}\to V^{\otimes 4}$ by $\widehat{T}(a\otimes b\otimes c\otimes d)=c\otimes d\otimes a\otimes b$. Then on $V^{\otimes 4}$, we have the identity

$$(1\otimes T\otimes 1)(T\otimes T)=\widehat{T}(1\otimes T\otimes 1).$$

From this, the two composites

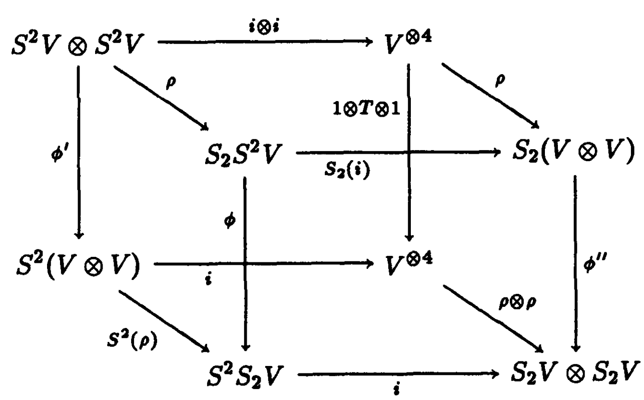
$$S^2V\otimes S^2V\xrightarrow{i\otimes i}V^{\otimes 4}\xrightarrow{1\otimes T\otimes 1}V^{\otimes 4}$$

and

$$V^{\otimes 4}\xrightarrow{1\otimes T\otimes 1}V^{\otimes 4}\xrightarrow{\rho\otimes\rho}S_2V\otimes S_2V$$

factors to give us the maps ϕ' and ϕ'' respectively. □

LEMMA 2.3.4. *For a module V there exists a map ϕ of modules such that the following cube commutes*



PROOF. The identity $(1\otimes T\otimes 1)(T\otimes T)=\widehat{T}(1\otimes T\otimes 1)$ from the proof of Lemma 2.3.3 tells us that the composite

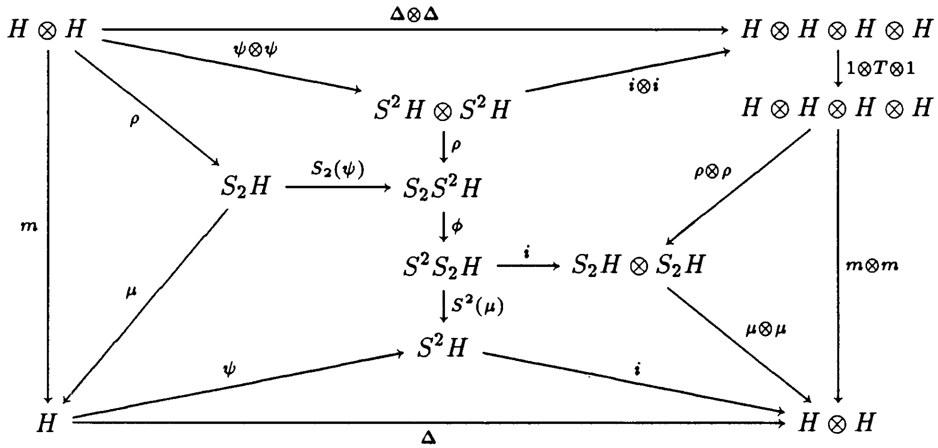
$$S^2V\otimes S^2V\xrightarrow{\phi'}S^2(V\otimes V)\xrightarrow{S^2(\rho)}S^2S_2V$$

factors to give us the desired map ϕ. The commutativity of the cube now follows from Lemma 2.3.3, the surjectivity of ρ, and the injectivity of i. □

PROPOSITION 2.3.5. *For an abelian Hopf algebra H, the following diagram commutes*

$$\begin{array}{ccccc} S_2H & \xrightarrow{S_2(\psi)} & S_2S^2H & \xrightarrow{\phi} & S^2S_2H \\ \downarrow{\scriptstyle\mu} & & & & \downarrow{\scriptstyle S^2(\mu)} \\ H & & \xrightarrow{\psi} & & S^2H \end{array}$$

PROOF. The diagram (2.3.1) can be expanded to give



which commutes by (2.1.5), (2.2.5), Lemma 2.3.4, the surjectivity of ρ, and the injectivity of i. □

We now pause to give a useful reinterpretation of Proposition 2.3.5.

Let Λ be a commutative algebra. Then $S^2\Lambda$ is a commutative algebra with product

$$S_2S^2\Lambda \xrightarrow{\phi} S^2S_2\Lambda \xrightarrow{S^2(\mu)} S^2\Lambda \tag{2.3.6}$$

and unit

$$\mathbb{F}_2 \simeq S^2(\mathbb{F}_2) \xrightarrow{S^2(\eta)} S^2\Lambda \tag{2.3.7}$$

COROLLARY 2.3.8. *For an abelian Hopf algebra H, the coproduct*

$$\psi\colon H \to S^2H$$

is a map of commutative algebras.

Also, if Λ is a Γ-algebra then by (2.1.17) $\Lambda\otimes\Lambda$ is a Γ-algebra. Moreover, from its definition we have

$$\gamma_2 T = T\gamma_2. \tag{2.3.9}$$

Thus $S^2\Lambda$ is also a Γ-algebra.

We denote by $\mathcal{H}$ (resp. $\mathcal{H}_*$) the category of abelian Hopf algebras (resp. abelian graded Hopf algebras).

DEFINITION 2.3.10. A Hopf Γ-algebra is a pair (H, γ_2) consisting of an abelian graded Hopf algebra H together with a map

$$\gamma_2 \colon \Phi I_2 \to H$$

satisfying 1. through 3. of Definition 2.1.9 along with the additional condition 4. for $x \in I_2$

$$\Delta \gamma_2 \overline{x} = \gamma_2(\overline{\Delta x}).$$

A map of Hopf Γ-algebras is just a map in $\mathcal{H}_*$ which is also a map of Γ-algebras.

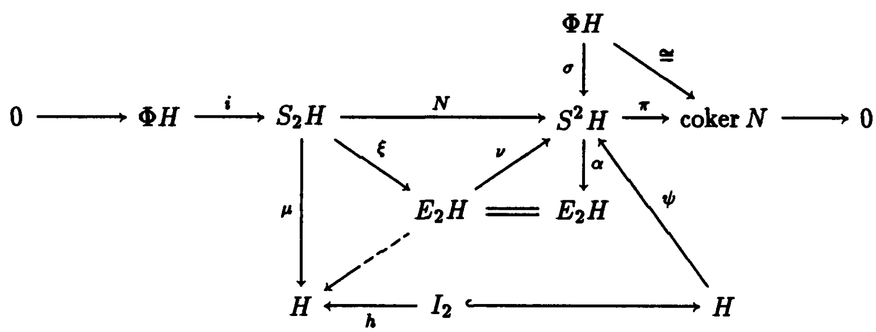
We pause here to record a basic relation on a Hopf Γ-algebra H. Our objective is to give a description of the composite

$$\Phi I_2 \xrightarrow{\gamma_2} H \xrightarrow{v} \Phi H.$$

To do so we define a map

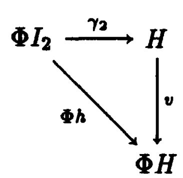
(2.3.11) $$h \colon I_2 \to H$$

which fits in the following expansion of (1.3.7)



Here α is the natural map determined by Proposition 1.3.8 and the dotted arrow exists in positive degrees by 1. of Definition 2.3.10.

PROPOSITION 2.3.12. *For a Hopf Γ-algebra H the diagram*



commutes.

To prove this, we note that since $i \colon S^2 H \to H \otimes H$ is a map of Γ-algebras then $\psi \colon H \to S^2 H$ is a map of Γ-algebras, by Corollary 2.3.8 and Definition 2.3.10. In light of this and Proposition 1.3.8 we are reduced to proving

LEMMA 2.3.13. *Let Λ be a Γ-algebra and $\omega \in S^2\Lambda$. Write $\omega = \nu(\alpha) + \sigma(\overline{\beta})$ as in Proposition* 1.3.8. *Then*

$$\pi(\gamma_2\omega) = \pi\sigma(\mu(\overline{\alpha}))$$

where $\overline{\alpha} \in S_2\Lambda$ satisfies $\xi(\overline{\alpha}) = \alpha$.

SKETCH OF PROOF. Since γ_2 is quadratic, we have

$$\gamma_2\omega = \gamma_2\nu(\alpha) + \gamma_2\sigma(\overline{\beta}) + \nu(\alpha)\cdot\sigma(\overline{\beta}).$$

Using the Γ-algebra map $i\colon S^2\Lambda \to \Lambda\otimes\Lambda$ we can compute $\pi(\nu(\alpha)\cdot\sigma(\overline{\beta})) = 0$. Also, since $\omega \in I_2$, $\gamma_2\sigma(\overline{\beta}) = 0$. We are thus left with computing $\gamma_2\nu(\alpha)$. Choose $z \in \Lambda\otimes\Lambda$ such that it maps to α under $\Lambda\otimes\Lambda \to E_2\Lambda$ and let $\overline{\alpha}$ be its image in $S_2\Lambda$. Then in $\Lambda\otimes\Lambda$

$$i\nu(\alpha) = (1+T)z$$

so that a computation using (2.1.5), (2.1.17), and (2.3.9) gives us

$$\begin{aligned} i\gamma_2\nu(\alpha) = \gamma_2 i\nu(\alpha) &= (1+T)\gamma_2 z + z\cdot Tz \\ &= i\nu(y) + i\sigma(\mu(\overline{\alpha})) \end{aligned}$$

for some $y \in E_2\Lambda$ (in fact y is the image of $\gamma_2 z$). □

DEFINITION 2.3.14. A Hopf D-algebra is a Hopf Γ-algebra H together with maps

$$\delta_i\colon H_n \to H_{n+i}$$

for all $2 \le i \le n$, satisfying conditions 1.–3. of Definition 2.1.10, and with maps

$$Sq^i\colon H_n \to H_{n-i}$$

for all $i \ge 0$, satisfying conditions 1.–3. of Definition 2.2.10, such that the following relations are satisfied for a fixed $x \in H_n$

1. for each $2 \le i \le n$

$$\Delta\delta_i x = \delta_i\Delta x$$

and for any $y \in H$, $j \ge 0$

$$(x\cdot y)Sq^j = \sum_{s+t=j}(xSq^s)\cdot(ySq^t)$$

2. for each $2 \le j < n$ and $i \ge 0$

$$(\delta_j x)Sq^i = \begin{cases} \sum_s (i-j, j-2i+2s-1)\delta_{j-i+s}(xSq^s) & i > j \\ v\gamma_2 x + \sum_{2s>j}\delta_s(xSq^s) & i = j \\ \sum_s (i-2s, j-2i+2s-1)\delta_{j-i+s}(xSq^s) & i < j \end{cases}$$

$$(\delta_n x)Sq^i = \begin{cases} 0 & i > n \\ v\gamma_2 x & i = n \\ \sum_{2s<i}(xSq^s)\cdot(xSq^{i-s}) & \\ \quad + \sum_s (i-2s, n-2i+2s-1)\delta_{n-i+s}(xSq^s) & i < n. \end{cases}$$

A map of Hopf D-algebras is simply a map of D-algebras and a map of A-coalgebras. We denote the category of Hopf D-algebras by $\mathcal{HD}$.

We now define a simplicial Hopf algebra to be both a simplicial algebra and a simplicial coalgebra which satisfies (2.3.1) and (2.3.2). Clearly, Proposition 2.3.5 applies to a simplicial abelian Hopf algebra. We denote by $s\mathcal{H}$ the category of simplicial abelian Hopf algebras.

We now come to the main theorem of this work, whose proof is postponed to Chapter 3.

THEOREM 2.3.15. *Let H be a simplicial abelian Hopf algebra. Then $\pi_* H$ is naturally a Hopf D-algebra. That is we have a functor*

$$\pi_* : s\mathcal{H} \to \mathcal{HD}.$$

On the proof: Consider the simplicial map (2.2.5)

$$\psi : H \to S^2 H.$$

In light of Corollary 2.3.8, if $x, y \in \pi_* H$ then we have the equations

$$\psi_*(x \cdot y) = (\psi_* x) \cdot (\psi_* y)$$

and

$$\psi_*(\delta_j x) = \delta_j(\psi_* x).$$

Thus by (2.2.18), we are reduced to understanding $\pi_* S^2 H$ as a D-algebra. This is the main focus of Chapter 3.

CHAPTER III

Proof of the Main Theorem

1. The Reduction

As we noted at the end of Chapter 2, the key to proving the main theorem (Theorem 2.3.15) is a complete understanding of the D-algebra $\pi_* S^2\Lambda$, where Λ is a simplicial commutative algebra. This is achieved in the following

THEOREM 3.1.1. *Let Λ be a simplicial commutative algebra. Then for the associated simplicial commutative algebra $S^2\Lambda$ the following relations hold in the D-algebra $\pi_* S^2\Lambda$*

a. *For $x \in \pi_n\Lambda$, $0 \leq i \leq n$, $2 \leq j \leq n+i$*

$$\delta_j \sigma_i(x) = \sum_{2s<j} \binom{s-j-1}{2s-j-1} \sigma_{i+j-s}\delta_s(x)$$

b. *For $x \in \pi_n\Lambda$, $y \in \pi_m\Lambda$, $2 \leq j \leq n+m$*

$$\delta_j[x,y] = \begin{cases} \sigma_j(x\cdot y) + [x\cdot x, \delta_j y] & \text{if } n=0 \\ \sigma_j(x\cdot y) + [\delta_j x, y\cdot y] & \text{if } m=0 \\ \sigma_j(x\cdot y) & \text{otherwise} \end{cases}$$

c. *For $x \in \pi_n\Lambda$, $y \in \pi_m\Lambda$, $0 \leq i \leq n$, $0 \leq j \leq m$*

$$\sigma_i(x)\cdot\sigma_j(y) = \sigma_{i+j}(x\cdot y)$$

d. *For $x \in \pi_n\Lambda$, $y,z \in \pi_*\Lambda$, $0 \leq i \leq n$*

$$\sigma_i(x)\cdot[y,z] = \begin{cases} [x\cdot y, x\cdot z] & \text{if } i=n \\ 0 & \text{otherwise} \end{cases}$$

e. *For $x,y,z,w \in \pi_*\Lambda$*

$$[x,y]\cdot[z,w] = [x\cdot z, y\cdot w] + [x\cdot w, y\cdot z].$$

To prove this theorem, we note that the algebra structure on $S^2\Lambda$ is completely determined from the one on Λ through the map of (2.3.4)

$$\phi\colon S_2S^2\Lambda \to S^2S_2\Lambda$$

by (2.3.6). Thus, we are reduced to computing this map in homotopy when Λ is an arbitrary simplicial module.

First, if we combine Proposition 2.1.16 and Proposition 2.2.16 then for V a simplicial module we have

PROPOSITION 3.1.2. *The following are generators for* $\pi_*S_2S^2V$:

a.	$\sigma_i\delta_j(x)$	for $x \in \pi_nV$, $0 \le i \le n$, $0 \le j \le n+i$
b.	$\delta_i[x,y]$	for $x \in \pi_nV$, $y \in \pi_mV$, $0 \le i \le n+m$
c.	$\sigma_i(x)\cdot\sigma_j(y)$	for $x \in \pi_nV$, $y \in \pi_mV$, $0 \le i \le n$, $0 \le j \le m$
d.	$\sigma_i(x)\cdot[y,z]$	for $x \in \pi_nV$, $y,z \in \pi_*V$, $0 \le i \le n$
e.	$[x,y]\cdot[z,w]$	for $x,y,z,w \in \pi_*V$

PROPOSITION 3.1.3. *The following are generators for* $\pi_*S^2S_2V$:

a.	$\sigma_i\delta_j(x)$	for $x \in \pi_nV$, $z \le j \le n$, $0 \le i \le n+j$
b.	$\sigma_i(x\cdot y)$	for $x \in \pi_nV$, $y \in \pi_mV$, $0 \le i \le n+m$
c.	$[\delta_i(x), y\cdot z]$	for $x \in \pi_nV$, $y,z \in \pi_*V$, $2 \le i \le n$
d.	$[\delta_i(x), \delta_j(y)]$	for $x \in \pi_nV$, $y \in \pi_mV$, $2 \le i \le n$, $2 \le j \le m$
e.	$[x\cdot y, z\cdot w]$	for $x,y,z,w \in \pi_*V$

We now arrive at the following which clearly implies Theorem 3.1.1.

PROPOSITION 3.1.4. *Let V be a simplicial module. Then the effect of the map*

$$\phi\colon S_2S^2V \to S^2S_2V$$

in homotopy is given by the following

a. *For* $x \in \pi_nV$, $0 \le i \le n$, $2 \le j \le n+i$

$$\phi_*\delta_j\sigma_i(x) = \sum_{2s<j} \binom{s-i-1}{2s-j-1} \sigma_{i+j-s}\delta_s(x)$$

b. *For* $x \in \pi_nV$, $y \in \pi_mV$, $2 \le i \le n+m$

$$\phi_*\delta_i[x,y] = \begin{cases} \sigma_i(x\cdot y) + [x\cdot x, \delta_i x] & \text{for } n=0 \\ \sigma_i(x\cdot y) + [\delta_i x, y\cdot y] & \text{for } m=0 \\ \sigma_i(x\cdot y) & \text{otherwise} \end{cases}$$

c. *For* $x \in \pi_nV$, $y \in \pi_mV$, $0 \le i \le n$, $0 \le j \le m$

$$\phi_*(\sigma_i(x)\cdot\sigma_j(y)) = \sigma_{i+j}(x\cdot y)$$

d. *For* $x \in \pi_n V$, $y, z \in \pi_* V$, $0 \leq i \leq n$

$$\phi_*(\sigma_i(x) \cdot [y, z]) = \begin{cases} [x \cdot y, x \cdot z] & i = n \\ 0 & \text{otherwise} \end{cases}$$

e. *For* $x, y, z, w \in \pi_* V$

$$\phi_*([x, y] \cdot [z, w]) = [x \cdot z, y \cdot w] + [x \cdot w, y \cdot z]$$

We end this section by taking a closer look at the map ϕ. Let V be a module. Then we have

Generators of $S_2 S^2 V$:

$$[x, y] \cdot [z, w]$$
$$\sigma(x) \cdot [y, z]$$
$$\sigma(x) \cdot \sigma(y)$$

for any $x, y, z, w \in V$.

Generators of $S^2 S_2 V$:

$$[x \cdot y, z \cdot w]$$
$$\sigma(x \cdot y)$$

for any $x, y, z, w \in V$.

Here σ is the map of (1.3.6).

The effect of

$$\phi \colon S_2 S^2 V \to S^2 S_2 V$$

is given by

$$\begin{aligned} [x, y] \cdot [z, w] &\to [x \cdot z, y \cdot w] + [x \cdot w, y \cdot z] \\ \sigma(x) \cdot [y, z] &\to [x \cdot y, x \cdot z] \\ \sigma(x) \cdot \sigma(y) &\to \sigma(x \cdot y). \end{aligned}$$

We can use this to compute the kernel and cokernel of ϕ. First, we have a map

$$\mathfrak{a} \colon V^{\otimes 4} \to S_2 S^2 V$$

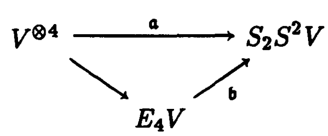
given by

$$a \otimes b \otimes c \otimes d \to [a, b] \cdot [c, d] + [a, c] \cdot [b, d] + [a, d] \cdot [b, c].$$

It is easy to see that

$$\phi \mathfrak{a} = 0.$$

Further, we have a factorization



Here E_4V is the 4^{th} exterior power of V i.e. the cokernel of the composite

$$(\Phi V) \otimes V^{\otimes 2} \xrightarrow{d\otimes 1} V^{\otimes 4} \longrightarrow S_4V$$

where d is from (1.3.6).

CLAIM. The induced map

$$\bar{\mathfrak{b}}\colon E_4V \longrightarrow \ker\phi$$

is a linear isomorphism.

PROOF. By naturality of $\mathfrak{b}$ and simplicity of the functor E_4, $\mathfrak{b}$ is injective, since it is nontrivial. To see surjectivity, we note that $\bar{\mathfrak{b}}$ is onto when $dimV \leq 4$. Thus, since E_4 is a polynomial functor of degree ≤ 4 the result follows. □

Now, an easy calculation shows

$$(S_2V)^* = S^2V^*$$

and

$$(S^2V)^* = S_2V^*.$$

From this and Lemma 2.3.4 we have

$$\phi^* = \phi.$$

Further $(E_4V)^* = E_4V^*$ so that the claim gives us an exact sequence

$$0 \to E_4V \to S_2S^2V \xrightarrow{\phi} S^2S_2V \to E_4V \to 0$$

which is natural as functors of modules. This defines a map

$$\mathbb{F}_2 \longrightarrow \operatorname{Ext}^2_{\mathcal{F}}(E_4, E_4)$$

where $\mathcal{F}$ is the category of endofunctors on the category of modules. L. Schwartz has shown (private communication) that this map is an injection.

2. Proof of Theorem 2.3.15.

First, by Theorem 2.1.11 and Theorem 2.2.11 π_*H is both a D-algebra and an A-coalgebra. Moreover, Δ is a map of simplicial commutative algebras by (2.1.17), (2.1.5), (2.1.8), and Lemma 2.3.3. By Theorem 1.2.1 and Theorem 2.1.11 we conclude π_*H is a Hopf Γ-algebra.

We now proceed to establish 1. and 2. of Definition 2.3.14. For the remainder of this section we fix $x \in \pi_nH$ and write

$$\psi_*x = \sum_k [x'_k, x''_k] + \sum_s \sigma_s(xSq^s)$$

as in (2.2.18).

1. The first part is an easy consequence of the fact that Δ is a map of simplicial commutative algebras. For the second part let $y \in \pi_m H$ and write

$$\psi_* y = \sum_\ell [y'_\ell, y''_\ell] + \sum_t \sigma_t(xSq^t).$$

By Theorem 3.1.1 we have

$$\begin{aligned}(\psi_* x)\cdot(\psi_* y) &\equiv \sum_{s,t} \sigma_s(xSq^s)\cdot\sigma_t(ySq^t)\\ &\equiv \sum_{i\geq 0}\sum_{s+t=i}\sigma_i(xSq^s\cdot ySq^t)\\ &\equiv \sum_{i\geq 0}\sigma_i(\sum_{s+t=i} xSq^s\cdot ySq^t)\end{aligned}$$

where, here and throughout, "$\equiv$" means "equal modulo [,]'s". By (2.2.18) we have

$$\psi_*(x\cdot y) \equiv \sum_{i\geq 0}\sigma_i((x\cdot y)Sq^i).$$

The conclusion follows from Corollary 2.3.7. □

2. Fix $2 \leq j < n$. By (2.2.18) we have

$$\psi_*\delta_j(x) \equiv \sum_i \sigma_i((\delta_j x)Sq^i).$$

Next, Theorem 3.1.1 gives us

$$\begin{aligned}\delta_j\psi_* x &\equiv \sum_k \sigma_j(x'_k\cdot x''_k) + \sum_s \delta_j\sigma_s(xSq^s)\\ &\equiv \sum_k \sigma_j(x'_k\cdot x''_k) + \sum_s\sum_{2\ell<j}\binom{\ell-s-1}{2\ell-j-1}\sigma_{j+s-\ell}\delta_\ell(xSq^s)\\ &\equiv \sum_k \sigma_j(x'_k\cdot x''_k) + \sum_s\sum_{2i-j<s}\binom{j-i-1}{j-2i+2s-1}\sigma_i\delta_{j-i+s}(xSq^s)\\ &\equiv \sum_k \sigma_j(x'_k\cdot x''_k) + \sum_i \sigma_i\Big(\sum_{2i-j<s}\binom{j-i-1}{j-2i+2s-1}\delta_{j-i+s}(xSq^s)\Big).\end{aligned}$$

When $i < j$ we immediately get the third equation. When $i > j$ the expression

$$\binom{m}{r} = \binom{-m+r-1}{r}$$

gives us the first equation. When $i = j$ we just need to verify

$$v\gamma_2 x = \sum x'_k \cdot x''_k$$

which is just a consequence of Proposition 2.3.12. Finally, combining Theorem 3.1.1 and Definition 2.1.9 we get

$$\begin{aligned}\delta_n\psi_*x &\equiv \sum_k \sigma_n(x'_k\cdot x''_k)+\sum_s \delta_n\sigma_s(xSq^s)+\sum_{s<t}\sigma_s(xSq^s)\cdot\sigma_t(xSq^t)\\ &\equiv \sum_k \sigma_n(x'_k\cdot x''_k)+\sum_s \delta_n\sigma_s(xSq^s)+\sum_{i\geq 0}\sum_{2s<i}\sigma_i(xSq^s\cdot xSq^{i-s})\\ &\equiv \sum_k \sigma_n(x'_k\cdot x''_k)+\sum_s \delta_n\sigma_s(xSq^s)+\sum_{i\geq 0}\sigma_i(\sum_{2s<i} xSq^s\cdot xSq^{i-s})\end{aligned}$$

and so proceeding as before gives us the remaining equations. The conclusion follows from Corollary 2.3.7. This completes the proof of Theorem 2.3.15. □

3. A Detection Scheme

In this section, we begin our assault upon the map

$$\phi_*:\pi_*S_2S^2V\to\pi_*S^2S_2V$$

with the objective of proving Proposition 3.1.4. Our method will be to divide and conquer. The key is that there exists a map

$$S^2S_2V\to(S_2V)^{\otimes 2}\oplus S^2S^2V$$

which is injective in homotopy.

We start by recalling from §3. of Ch. 1, that we have the norm map

$$N_V:S_2V\to S^2V$$

whose effect is

$$x\cdot y\to[x,y].$$

Consider now the maps

$$N_{S^2V}:S_2S^2V\to S^2S^2V$$

and

$$S^2N_V:S^2S_2V\to S^2S_2V.$$

It is well known that $S^2S^2V=S^{\Sigma_2\int\Sigma_2}V$ where $\Sigma_2\int\Sigma_2$ is the wreath product of Σ_2 and Σ_2 i.e. the subgroup of Σ_4 which fits into the split extension

(3.3.1) $$1\to\Sigma_2\times\Sigma_2\to\Sigma_2\int\Sigma_2\to\Sigma_2\to 1$$

where, in terms of transpositions, we have

$$\begin{aligned}\Sigma_2\times\Sigma_2&=\langle(1,2),(3,4)\rangle\\ \Sigma_2&=\langle(1,3)(2,4)\rangle.\end{aligned}$$

Moreover, it is well-known that $\Sigma_2\int\Sigma_2\simeq D_8$; the dihedral group of order 8. We thus have the identity

(3.3.2) $$S^2S^2V\simeq S^{D_8}V.$$

LEMMA 3.3.3. *There exists a natural idempotent map*

$$\alpha \colon S^{D_8}V \to S^{D_8}V$$

such that the diagram

$$\begin{array}{ccc} S_2S^2V & \xrightarrow{N_{S^2V}} & S^{D_8}V \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \alpha} \\ S^2S_2V & \xrightarrow[S^2N_V]{} & S^{D_8}V \end{array}$$

commutes. Explicitly

$$\alpha = 1 + r(\Sigma_4, D_8)t(D_8, \Sigma_4).$$

The proof will follow from the next lemma.

LEMMA 3.3.4. *There exists a natural map*

$$\alpha'' \colon S^2(V^{\otimes 2}) \to (S^2V)^{\otimes 2}$$

such that the diagram

$$\begin{array}{ccc} S_2(V^{\otimes 2}) & \xrightarrow{N_{V^{\otimes 2}}} & S^2(V^{\otimes 2}) \\ {\scriptstyle \phi''}\downarrow & & \downarrow{\scriptstyle \alpha''} \\ (S_2V)^{\otimes 2} & \xrightarrow[(N_V)^{\otimes 2}]{} & (S^2V)^{\otimes 2} \end{array}$$

commutes. Here ϕ'' is the map of Lemma 2.3.3. Indeed, we can take

$$\alpha'' = \epsilon t(\Sigma_2, \Sigma_2 \times \Sigma_2)$$

where the transfer is associated to the diagonal $\Sigma_2 \to \Sigma_2 \times \Sigma_2$ and ϵ is the isomorphism induced by $1 \otimes T \otimes 1 \colon V^{\otimes 4} \to V^{\otimes 4}$.

PROOF OF LEMMA 3.3.4. First, we have commuting diagrams

$$\begin{array}{ccc} V^{\otimes 4} & \xrightarrow{1+(1,3)(2,4)} & V^{\otimes 4} \\ {\scriptstyle \rho_{V^{\otimes 2}}}\downarrow & & \uparrow{\scriptstyle i_{V^{\otimes 2}}} \\ S_2(V^{\otimes 2}) & \xrightarrow[N_{V^{\otimes 2}}]{} & S^2(V^{\otimes 2}) \end{array}$$

and

$$\begin{array}{ccc} V^{\otimes 4} & \xrightarrow{(1+(1,2))\cdot(1+(3,4))} & V^{\otimes 4} \\ {\scriptstyle \rho_V^{\otimes 2}}\downarrow & & \uparrow{\scriptstyle i_V^{\otimes 2}} \\ (S_2V)^{\otimes 2} & \xrightarrow[N_V^{\otimes 2}]{} & (S^2V)^{\otimes 2}. \end{array}$$

An easy computation shows that the diagram

$$\begin{array}{ccc} V^{\otimes 4} & \xrightarrow{1+(1,3)(2.4)} & V^{\otimes 4} \\ \Big\downarrow{\scriptstyle (2,3)} & & \Big\downarrow{\scriptstyle (2,3)(1+(1,3))} \\ V^{\otimes 4} & \xrightarrow{(1+(1,2))(1+(3,4))} & V^{\otimes 4} \end{array}$$

commutes. Consider now the map

$$S^2(V^{\otimes 2}) \xrightarrow{i_{V^{\otimes 2}}} V^{\otimes 4}.$$

In the group ring $\mathbb{F}_2[\Sigma_4]$, we have the identity

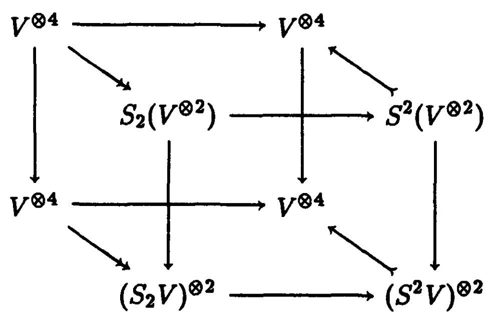
$$\begin{aligned}(2.3)(1+(1,3))(1,3)(2,4) &= (1+(1,2))(2,3)(1,3)(2,4)\\ &= (1+(1,2))(1,2)(3,4)(2,3)\\ &= ((1,2)(3,4)+(3,4))(2,3).\end{aligned}$$

This shows that the image of the above map is invariant under the action of $\langle(1,2),(3.4)\rangle$. We thus have a commuting diagram

$$\begin{array}{ccc} S^2(V^{\otimes 2}) & \xrightarrow{i_{V^{\otimes 2}}} & V^{\otimes 4} \\ {\scriptstyle \alpha''}\Big\downarrow & & \Big\downarrow{\scriptstyle (2,3)(1+(1,3))} \\ (S^2V)^{\otimes 2} & \xrightarrow[i_V^{\otimes 2}]{} & V^{\otimes 4} \end{array}$$

defining α''.

Combining these four diagrams and Lemma 2.3.3 gives us a cube



from which our desired commutative diagram results. The identification of α'' follows from our construction and the definition of transfer. □

Proof of Lemma 3.3.3. Consider the composite

$$S^2S^2V \xrightarrow{S^2(i_V)} S^2(V^{\otimes 2}) \xrightarrow{\alpha''} (S^2V)^{\otimes 2}.$$

From Lemma 3.3.4 and a computation we have

$$\alpha''(1,2)(3,4) = \epsilon(1,2)(3,4)t(\Sigma_2, \Sigma_2 \times \Sigma_2) = (1,3)(2,4)\alpha''.$$

Thus $(1,3)(2,4)\alpha'' S^2(i_V) = \alpha'' S^2(i_V)$. Hence we have a diagram

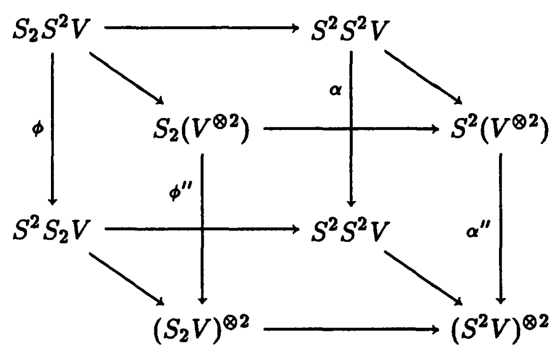
$$\begin{array}{ccc} S^2S^2(V) & \xrightarrow{S^2(i_V)} & S^2(V^{\otimes 2}) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha''} \\ S^2S^2(V) & \xrightarrow[i_{S^2V}]{} & (S^2V)^{\otimes 2}. \end{array}$$

By Lemma 2.3.4 and Lemma 3.3.4 our desired diagram commutes. From this and the identity $(2,3)(1,3) = (1,3)(1,2)$ we arrive at the commuting diagram

$$\begin{array}{ccc} S^2S^2(V) & \longrightarrow & V^{\otimes 4} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle(1,3)+(2,3)} \\ S^2S^2(V) & \longrightarrow & V^{\otimes 4} \end{array}$$

Clearly $1, (2,3), (1,3)$ are coset representatives for D_8 in Σ_4. Also $((2,3) + (1,3))^2 = (1,3)(1,2) + (2,3)(1,2)$ from the above and the identity $(1,3)(2,3) = (2,3)(1,2)$. Hence $\alpha^2 = \alpha$. □

COROLLARY 3.3.5. *The following cube commutes*



PROOF. This easily follows from Lemma 2.3.4, Lemma 3.3.3, Lemma 3.3.4, and naturality. □

NOTE. The effect of the map

$$\alpha : S^{D_8}V \to S^{D_8}V$$

on elements is

$$\begin{aligned} [[x,y],[z,w]] &\to [[x,z],[y,w]] + [[x,w],[y,z]] \\ [\sigma(x),[y,z]] &\to [[x,y],[x,z]] \\ [\sigma(x),\sigma(y)] &\to \sigma[x,y] \\ \sigma[x,y] &\to \sigma[x,y] \end{aligned}$$

from which we easily verify idempotence. We further note that the module of natural maps

$$(-)^{D_8} \to (-)^{D_8}$$

on the category of Σ_4-modules has as basis the set $\{1, \alpha\}$. In light of this, Lemma 3.3.3 should not be surprising.

Now, by Proposition 2.2.16 we have

PROPOSITION 3.3.6. *The following are generators of* $\pi_* S^2 S^2 V$:

a. $\sigma_j \sigma_i(x)$ for $x \in \pi_n V$, $0 \le i \le n$, $0 \le j \le n+i$
b. $\sigma_i[x, y]$ for $x \in \pi_n V$, $y \in \pi_m V$, $0 \le i \le n+m$
c. $[\sigma_i(x), \sigma_j(y)]$ for $x \in \pi_n V$, $y \in \pi_m V$, $0 \le i \le n$, $0 \le j \le m$
d. $[\sigma_i(x), [y, z]]$ for $x \in \pi_n V$, $y, z \in \pi_* V$, $0 \le i \le n$
e. $[[x, y], [z, w]]$ for $x, y, z, w \in \pi_* V$.

By Corollary 2.2.17, the effect of the map

$$(N_{S^2 V})_* \colon \pi_* S_2 S^2 V \to \pi_* S^2 S_2 V$$

is given by

$$\begin{aligned} \delta_j \sigma_i(x) &\to \sigma_j \sigma_i(x) \\ \delta_j[x, y] &\to \sigma_j[x, y] \\ \sigma_i(x) \cdot \sigma_j(y) &\to [\sigma_i(x), \sigma_j(y)] \\ \sigma_i(x) \cdot [y, z] &\to [\sigma_i(x), [y, z]] \\ [x, y] \cdot [z, w] &\to [[x, y], [z, w]]. \end{aligned}$$

Also, the effect of the map

$$(S^2 N_V)_* \colon \pi_* S^2 S_2 V \to \pi_* S^2 S^2 V$$

is given by

$$\begin{aligned} \sigma_i \delta_j(x) &\to \sigma_i \sigma_j(x) \\ \sigma_i(x \cdot y) &\to \sigma_i[x, y] \\ [\delta_i(x), \delta_j(y)] &\to [\sigma_i(x), \sigma_j(y)] \\ [\delta_i(x), y \cdot z] &\to [\sigma_i(x), [y, z]] \\ [x \cdot y, z \cdot w] &\to [[x, y], [z, w]]. \end{aligned}$$

Further, by Proposition 2.2.16, the effect of the map

$$(S_2 i_V)_* \colon \pi_* S_2 S^2 V \to \pi_* S_2(V^{\otimes 2})$$

is given by

$$\delta_j\sigma_i(x) \to \begin{cases} \delta_j(x\otimes x) & i=|x| \\ 0 & \text{otherwise} \end{cases}$$
$$\delta_j[x,y] \to \delta_j(x\otimes y + y\otimes x)$$
$$\sigma_i(x)\cdot\sigma_j(y) \to \begin{cases} (x\otimes x)\cdot(y\otimes y) & i=|x| \quad j=|y| \\ 0 & \text{otherwise} \end{cases}$$
$$\sigma_i(x)\cdot[y,z] \to \begin{cases} (x\otimes x)\cdot(y\otimes z + z\otimes y) & i=|x| \\ 0 & \text{otherwise} \end{cases}$$
$$[x,y]\cdot[z,w] \to (x\otimes y + y\otimes x)\cdot(z\otimes w + w\otimes z)$$

Also, the effect of the map

$$(i_{S_2V})_*\colon \pi_*S^2S_2V \to \pi_*(S_2V)^{\otimes 2}$$

is given by

$$\sigma_i\delta_j(x) \to \begin{cases} \delta_j(x)\otimes\delta_j(x) & i=|x|+j \\ 0 & \text{otherwise} \end{cases}$$
$$\sigma_i(x\cdot y) \to \begin{cases} (x\cdot y)\otimes(x\cdot y) & i=|x|+|y| \\ 0 & \text{otherwise} \end{cases}$$
$$[\delta_i(x),\delta_j(y)] \to \delta_i(x)\otimes\delta_j(y) + \delta_j(x)\otimes\delta_i(x)$$
$$[\delta_i(x), y\cdot z] \to \delta_i(x)\otimes(y\cdot z) + (y\cdot z)\otimes\delta_i(x)$$
$$[x\cdot y, z\cdot w] \to (x\cdot y)\otimes(z\cdot w) + (z\cdot w)\otimes(x\cdot y).$$

From this we conclude that the map

$$(S^2N_V)_* \oplus (i_{S_2V})_*\colon \pi_*S^2S_2V \to \pi_*S^2S^2V \oplus \pi_*(S_2V)^{\oplus 2}$$

is injective. We are thus reduced, by Corollary 3.3.5, to computing, in homotopy, the maps induced by α and ϕ''. For this we have

PROPOSITION 3.3.7. *Let V be a simplicial module. Then the effect of*

$$\alpha_*\colon \pi_*S^2S^2V \to \pi_*S^2S^2V$$

is given by

a. *For $x \in \pi_nV$, $0 \le i \le n$, $0 \le j \le n+1$*

$$\alpha_*\sigma_j\sigma_i(x) = \sum_{2sxj} \binom{s-i-1}{2s-j-1} \sigma_{i+j-s}\sigma_s(x)$$

b. *For $x \in \pi_nV$, $y \in \pi_mV$, $0 \le i \le n+m$*

$$\alpha_*\sigma_i[x,y] = \sigma_i[x,y]$$

c. *For $x \in \pi_nV$, $y \in \pi_mV$, $0 \le i \le n$, $0 \le j \le m$*

$$\alpha_*[\sigma_i(x),\sigma_j(y)] = \sigma_{i+j}[x,y]$$

d. *For* $x \in \pi_n V$, $y, z \in \pi_* V$, $0 \le i \le n$

$$\alpha_*[\sigma_i(x), [y,z]] = \begin{cases} [[x,y],[x,z]] & i = n \\ 0 & \text{otherwise} \end{cases}$$

e. *For* $x, y, z, w \in \pi_* V$

$$\alpha_*[[x,y],[z,w]] = [[x,z],[y,w]] + [[x,w],[y,z]].$$

PROPOSITION 3.3.8. *Let V be a simplicial module. Then the effect of*

$$\phi''_*: \pi_* S_2(V^{\otimes 2}) \to \pi_*(S_2 V)^{\otimes 2}$$

is given by

a. *For* $x \in \pi_n V$, $y \in \pi_n V$, $y \in \pi_m V$, $2 \le j \le n+m$

$$\phi''_* \delta_j(x \otimes y) = \begin{cases} \delta_j x \otimes y \cdot y & m = 0 \\ x \cdot x \otimes \delta_j y & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

b. *For* $x, y, z, w \in \pi_* V$

$$\phi''_*[(x \otimes y) \cdot (z \otimes w)] = (x \cdot z) \otimes (y \cdot w).$$

We will actually prove a much more general result then Proposition 3.3.8. To state it we first need the following set up.

Let V and W be modules. Then the map

$$1 \otimes T \otimes 1: V \otimes W \otimes V \otimes W \to V \otimes V \otimes W \otimes W$$

induces

$$\overline{\phi}'': S_2(V \otimes W) \to S_2 V \otimes S_2 W.$$

Following the proof of Lemma 3.3.4 verbatim gives us

LEMMA 3.3.9. *There exists a map*

$$\overline{\alpha}'': S^2(V \otimes W) \to S^2 V \otimes S^2 W$$

such that the diagram

$$\begin{array}{ccc} S_2(V \otimes W) & \xrightarrow{N_{V \otimes W}} & S^2(V \otimes W) \\ \overline{\phi}'' \downarrow & & \downarrow \overline{\alpha}'' \\ S_2 V \otimes S_2 W & \xrightarrow[N_V \otimes N_W]{} & S^2 V \otimes S^2 W \end{array}$$

commutes. Indeed we can take

$$\overline{\alpha}'' = \epsilon t(\Sigma_2, \Sigma_2 \times \Sigma_2)$$

as in Lemma 3.3.4.

PROPOSITION 3.3.10. *Let V and W be simplicial modules. Then the effect of*

$$\overline{\phi}'' : \pi_* S_2(V \otimes W) \to \pi_* S_2 V \otimes S_2 W$$

is given by

a. *For $x \in \pi_n V$, $y \in \pi_m V$, $2 \le j \le n+m$*

$$\overline{\phi}''_* \delta_j (x \otimes y) = \begin{cases} \delta_j x \otimes y \cdot y & m = 0 \\ x \cdot x \otimes \delta_j y & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

b. *For $x, z \in \pi_* V$, $y, w \in \pi_* W$*

$$\overline{\phi}''_* [(x \otimes y) \cdot (z \otimes w)] = (x \cdot z) \otimes (y \cdot w).$$

Clearly Proposition 3.3.10 implies Proposition 3.3.8. Finally, Proposition 3.1.4 follows from Lemma 3.3.3, Lemma 3.3.4, Proposition 3.3.7, and Proposition 3.3.8. The proof of Proposition 3.3.7 and Proposition 3.3.10 will be given in §5.

4. Dwyer's detection map and the cohomology of groups

In this section, we gather the tools necessary for proving Propositions 3.3.7 and 3.3.10. The key is the following theorem found in [9].

THEOREM 3.4.1. *Given a simplicial module V and a subgroup $G \le \Sigma_m$ there exists a natural homomorphism*

$$\Psi^G : \pi_i S^G V \to \bigoplus_{0 \le k} H^k(G; \pi_{i+k} V^{\otimes m})$$

such that for a subgroup $H \le G$

$$r(G,H)\Psi^G = \Psi^H r(G,H) \qquad t(H,G)\Psi^H = \Psi^G t(H,G).$$

We summarize the proof of this theorem. Let $G \le \Sigma_m$ and $\mathcal{C}_G$ the category of G-modules. Let

$$F : \mathcal{C}_G \to \mathcal{B}$$

be an additive functor to some abelian category. This induces a functor

$$F : ch\mathcal{C}_G \to ch\mathcal{B}$$

of bounded above chain complexes over these categories. For a fixed C in $ch\mathcal{C}_G$, there exists an injective resolution $C \to I$ i.e. an object I in $ch\mathcal{C}_G$ which is degree-wise injective, together with a quasi-isomorphism from C. Such an object is unique up to chain homotopy. Define, as in [17], the total right derived functor of F to be

$$\mathcal{R}F(C) = F(I)$$

which comes equipped with a natural map

$$F(C) \to \mathcal{R}F(C).$$

At this point, we should remark that given C in $ch\mathcal{C}_G$ we can construct an injective resolution $C \to I$ as follows: for each $k \in \mathbb{Z}$ we have an injective resolution $C_k \to I_{k,*}$ in $\mathcal{C}_G$ by homological algebra. The chain maps for C extend to give us a bichain complex I_{**}. Upon letting $I = \text{Tot } I_{**}$, the total chain complex, we immediately get a quasi-isomorphism

$$C \to I$$

which serves as an injective resolution. The advantage of this construction is that it gives us a spectral sequence

$$E^2_{i,j} = R^i F(H_{-j}C) \Longrightarrow \mathcal{R}^{i+j} F(C) \tag{3.4.2}$$

where $\mathcal{R}^k F(C) = H_{-k}\mathcal{R}F(C)$ and $R^k F(-)$ is the k^{th} derived functor of F on $\mathcal{C}_G$. This spectral sequence is constructed in ch. 17 of [5]. As an application, if C is a G-chain complex with trivial differential then (3.4.2) collapses to give us

$$\bigoplus_{k\geq 0} R^k F(C_{k-m}) \simeq \mathcal{R}^m F(C) \tag{3.4.3}$$

in $\mathcal{B}$. To define our desired map Ψ^G let F be the G-fixed point functor i.e. for M in $\mathcal{C}_G$

$$F(M) = M^G = H^0(G; M). \tag{3.4.4}$$

Now, let V be a simplicial module such that NV is bounded above. Then the Eilenberg-Zilber map provides us with a G-equivariant chain equivalence

$$(NV)^{\otimes m} \to N(V^{\otimes m}).$$

Moreover, since we are over a field, there is a chain equivalence

$$NV \to \pi_* V$$

where $\pi_* V$ has trivial differential, which induces a G-equivariant chain equivalence

$$(NV)^{\otimes m} \to (\pi_* V)^{\otimes m}.$$

By (3.4.2), we have quasi-isomorphisms

$$\mathcal{R}F(N(V^{\otimes m})) \leftarrow \mathcal{R}F(NV)^{\otimes m}) \to \mathcal{R}F((\pi_* V)^{\otimes m}).$$

By (3.4.3) and (3.4.4) we obtain

$$\mathcal{R}^n F(N(V^{\otimes m})) \simeq \bigoplus_{k\geq 0} H^k(G; \pi_{n+k} V^{\otimes m}).$$

Now, $H_* F(N(V^{\otimes m})) \simeq H_* NF(V^{\otimes m}) \simeq \pi_* S^G V$ by functoriality. Combining the above, we have a natural homomorphism

$$\pi_i S^G V \simeq H_i F(N(V^{\otimes m})) \to \mathcal{R}^{-i} F(N(V^{\otimes m})) \simeq \bigoplus_{k\geq 0} H^k(G; \pi_{i+k} V^{\otimes m})$$

which is what we call Ψ^G. The case of a general simplicial module V follows from a limit argument.

The relations involving restriction and transfer follow immediately from the naturality and equivariance of all maps involved.

The usefulness of the map of Theorem 3.4.1 is now made precise by the following

PROPOSITION 3.4.5. *For any simplicial module V, the natural homomorphism of Theorem 3.4.1 is injective for the group Σ_2.*

PROOF. We first prove the result for $V = K(n)$. From chapter 1 §1 we have

$$N_s(K(n) \otimes K(n)) = \begin{cases} 0 & s < n \\ \text{nonzero} & n \leq s \leq 2n \\ 0 & 2n < s. \end{cases}$$

Let C be the Σ_2-chain complex such that

$$C_s = \begin{cases} \mathbb{F}_2[\Sigma_2]\langle x_s \rangle & n < s \leq 2n \\ \mathbb{F}_2\langle y \rangle & n = s \\ 0 & \text{otherwise.} \end{cases}$$

If we write $\Sigma_2 = \{1, T\}$, then the differential ∂ on C is given by

$$\begin{aligned} \partial x_{s+1} &= (1+T)x_s \qquad n < s < 2n \\ \partial x_{n+1} &= y. \end{aligned}$$

Write $\pi_n\ K(n) = \mathbb{F}_2\langle a \rangle$ and define a map

$$f\colon C \to N(K(n) \otimes K(n))$$

by

$$\begin{aligned} x_s &\to D^{2n-s}(a \otimes a) \\ y &\to \phi_n(a \otimes a). \end{aligned}$$

By Theorem 1.2.3 this is a map of Σ_2-chain complexes. Moreover, it is a quasi-isomorphism. Let F be the functor $H^0(\Sigma_2; -)$. We wish to compute

$$H_n F(C) \to \mathcal{R}^{-n} F(C).$$

To do so define the complex $\widehat{C}$ by

$$\widehat{C}_s = \begin{cases} \mathbb{F}_2[\Sigma_2]\langle \widehat{x}_s \rangle & s \leq 2n \\ 0 & \text{otherwise} \end{cases}$$

with differential $\widehat{\partial}$ given by $\widehat{\partial}\widehat{x}_{s+1} = (1+T)\widehat{x}_s$. This is clearly a free Σ_2-chain complex and the map

$$C \to \widehat{C}$$

given by

$$\begin{aligned} x_s &\to \widehat{x}_s \\ y &\to (1+T)\widehat{x}_n \end{aligned}$$

is clearly a quasi-isomorphism. Thus $\mathcal{R}F(C) = F(\widehat{C})$ and an easy calculation gives that

$$H_s F(C) \to \mathcal{R}^{-s} F(C)$$

is an injection for all s.

To obtain the general case, we first take V so that NV is bounded above. Then we have a weak equivalence

$$\bigoplus_{\alpha} K(n_\alpha) \to V.$$

Thus it suffices to show that if Ψ^{Σ_2} is injective for W_1 and W_2 then it is injective for $W_1 \oplus W_2$. First, we have a decomposition of

$$N((W_1 \oplus W_2) \otimes (W_1 \oplus W_2))$$

as

$$N(W_1 \otimes W_1) \oplus N(W_2 \otimes W_2) \oplus N((W_1 \otimes W_2) \oplus (W_2 \otimes W_1)).$$

Since the last summand is Σ_2-free and since Ψ^{Σ_2} respects this decomposition, injectivity follows. A limit argument completes the proof. □

We now pause to record a useful property of total derived functors.

LEMMA 3.4.6. *Let G, H be finite groups and $\mathcal{B}$ an abelian category. Let $F_1 \colon \mathcal{C}_G \to \mathcal{C}_H$ and $F_2 \colon \mathcal{C}_H \to \mathcal{B}$ be additive functors such that F_1 preserves injectives. Then $\mathcal{R}(F_2 \circ F_1)$ is chain homotopic to $\mathcal{R}F_2 \circ \mathcal{R}F_1$. Moreover, the natural map*

$$F_2 \circ F_1 \to \mathcal{R}(F_2 \circ F_1)$$

is chain homotopic to the composite

$$F \circ F_1 \to (\mathcal{R}F_2) \circ F_1 \to \mathcal{R}F_2 \circ \mathcal{R}F_1.$$

As an application, we give a Corollary to Proposition 3.4.4.

COROLLARY 3.4.7. *For any simplicial module V, Ψ^G is injective for $G = \Sigma_2 \times \Sigma_2$ and $G = \Sigma_2 \int \Sigma_2$.*

PROOF. Let $F_1 = H^0(\Sigma_2; -)$ and $F_2 = H^0(\Sigma_2 \times \Sigma_2; -)$. A Kunneth theorem argument shows that

$$F_2 \to \mathcal{R}F_2$$

is equivalent to

$$F_1 \otimes F_1 \to \mathcal{R}F_1 \otimes \mathcal{R}F_1.$$

Thus injectivity follows from Proposition 3.4.5. Now F_1 and F_2 can be viewed as functors

$$F_2 \colon \mathcal{C}_{\Sigma_2 \int \Sigma_2} \to \mathcal{C}_{\Sigma_2}$$

and

$$F_1 \colon \mathcal{C}_{\Sigma_2} \to \text{(modules)}.$$

Since F_2 preserves injectives then using the fact that $F_1 \circ F_2 = H^0(\Sigma_2 \int \Sigma_2; -)$ our desired result follows from Proposition 3.4.5, Lemma 3.4.6, and the result for $\Sigma_2 \times \Sigma_2$. See [9] for details. □

We now proceed to recall some useful tools in group cohomology. See [11] or chapter 12 of [5] for details.

Lyndon-Serre-Hochschild Spectral Sequence. Consider the extension of finite groups

$$K \rightarrowtail G \twoheadrightarrow Q.$$

Let M be a G-module. Then we have a first quadrant spectral sequence

$$E_2^{*,*} = H^*(Q; H^*(K; M)) \Longrightarrow H^*(G; M). \tag{3.4.8}$$

Here $H^*(K; M)$ is a Q-module since we have the functor

$$H^0(K; -)\colon \mathcal{C}_G \to \mathcal{C}_Q.$$

To make this spectral sequence useful we have

LEMMA 3.4.9. *Given a diagram*

$$\begin{array}{ccccc} K & \rightarrowtail & G & \twoheadrightarrow & Q \\ \downarrow & & \downarrow & & \downarrow \\ K' & \rightarrowtail & G' & \twoheadrightarrow & Q' \end{array}$$

whose rows are extensions then the induced map

$$H^*(Q'; H^*(K'; M)) \to H^*(Q; H^*(K; M))$$

is a map of spectral sequences for a Q-module M. Moreover, the induced map on E_∞ is compatible with

$$H^*(G'; M) \to H^*(G; M).$$

Further, if the vertical maps are injective, then the map

$$H^*(Q; H^*(K; M)) \to H^*(Q'; H^*(K'; M))$$

induced from the associated transfers, becomes a map of spectral sequences. Again, the induced map on E^∞ is compatible with the associated transfer,

$$H^*(G; M) \to H^*(G'; M).$$

Double Coset Formula. Let H, K be subgroups of a finite group G. A double coset representation of G with respect to H and K is a subset $S \subset G$ such that

$$G = \bigcup_{\sigma \in S} H\sigma K$$

and is minimal among all such subsets. Next, if $x \in G$ and $J \leq G$ define the conjugation map

$$c_x : J \to xJx^{-1}$$

by $c_x(u) = xux^{-1}$.

PROPOSITION 3.4.10. *Let S be a double coset representation of G with respect to H and K and let M be a G-module. Then for $\alpha \in H^*(K;M)$*

$$r(G,H)t(K,G)(\alpha) = \sum_{x\in S} t(H\cap xKx^{-1},H)r(xKx^{-1},H\cap xKx^{-1})c_x(\alpha)$$
$$= \sum_{x\in S} t(H\cap xKx^{-1},H)c_x r(K,x^{-1}Hx\cap K)(\alpha)$$

holds in $H^*(H;M)$.

5. Proof of the Detection Scheme

In this section, we prove Proposition 3.3.7 and Proposition 3.3.10 using the methods of the previous section. First, we need some basic results to facilitate our computations.

Let $K(n)$ be the Eilenberg-MacLane module so that $\pi_* K(n) = \mathbb{F}_2\langle a\rangle$ where $|a| = n \geq 0$. Then by Proposition 2.2.16

$$\pi_* S^2 K(n) \simeq \begin{cases} \mathbb{F}_2\langle \sigma_i(a)\rangle & * = n+i \qquad 0\leq i\leq n \\ 0 & \text{otherwise.}\end{cases}$$

Also $H^*(\Sigma_2;\mathbb{F}_2) = \mathbb{F}_2[w]$ where w is dual to the generator $H_1(\Sigma_2;\mathbb{F}_2)\simeq \mathbb{F}_2$. We then have

PROPOSITION 3.5.1. *Under the homomorphism*

$$\Psi^{\Sigma_2}\colon \pi_* S^2K(n) \to H^*(\Sigma_2;\mathbb{F}_2)$$

of Theorem 3.4.1

$$\Psi^{\Sigma_2}\sigma_i(a) = w^{2n-i}$$

for all $0\leq i\leq n$.

PROOF. This follows easily from Proposition 3.4.5. □

Now, take $K(m)$ so that $\pi_* K(m)\simeq \mathbb{F}_2\langle b\rangle$ where $|b| = m\geq 0$.

PROPOSITION 3.5.2. *Let M be the Σ_2-submodule of $\pi_*(K(n)\times K(m))^{\otimes 2}$ generated by $a\otimes b$. Then*

$$H^i(\Sigma_2;M) = \begin{cases} 0 & i>0 \\ \mathbb{F}_2\langle \Psi^{\Sigma_2}[a,b]\rangle & i=0\end{cases}$$

PROOF. For $i>0$ this just follows from the fact that M is a free Σ_2-module. For $i=0$ we note that under the projections

$$S^2(K(n)\times K(m)) \to S^2K(n)$$
$$S^2(K(n)\times K(m)) \to S^2K(m)$$

$[a,b]$ projects to 0 in homotopy. Hence by naturality and Proposition 3.4.5 the result follows. □

Proposition 3.5.3. *Consider the extension*

$$\Sigma_2 \times \Sigma_2 \rightarrowtail D_8 \twoheadrightarrow \Sigma_2.$$

Then for a simplicial module V

$$H^*(D_8; \pi_* V^{\otimes 4}) \simeq H^*(\Sigma_2; H^*(\Sigma_2; \pi_* V^{\otimes 2})^{\otimes 2}).$$

Moreover, we have a factorization

$$\begin{array}{ccc}
\pi_* S^{D_8} V & \xrightarrow{\quad \Psi^{D_8} \quad} & H^*(D_8; \pi_* V^{\otimes 4}) \\
\| & & \| \\
\pi_* S^2 S^2 V & & \| \\
\Big\downarrow {\scriptstyle \Psi^{\Sigma_2}} & & \| \\
H^*(\Sigma_2; \pi_*(S^2 V)^{\otimes 2}) & \xrightarrow[H^*(\Sigma_2; (\Psi^{\Sigma_2})^{\otimes 2})]{} & H^*(\Sigma_2; H^*(\Sigma_2; \pi_* V^{\otimes 2})^{\otimes 2})
\end{array}$$

Proof. Define functors

$$F_1 \colon \mathcal{C}_{D_8} \to \mathcal{C}_{\Sigma_2}$$

and

$$F_2 \colon \mathcal{C}_{\Sigma_2} \to \text{(modules)}$$

by $F_1 = H^0(\Sigma_2 \times \Sigma_2; -)$ and $F_2 = H^0(\Sigma_2; -)$. Then F_1 preserves injectives and $F_2 \circ F_1 = H^0(D_8; -)$. So by Lemma 3.4.6,

$$\mathcal{R}(F_2 \circ F_1) \simeq \mathcal{R}F_2 \circ \mathcal{R}F_1.$$

Thus it suffices to compute $H_*(\mathcal{R}F_2 \circ \mathcal{R}F_1)$ for $NV^{\otimes 4}$. Since we have an equivariant equivalence

$$NV^{\otimes 4} \to (\pi_* V)^{\otimes 4}$$

and since $\mathcal{R}F_1$ is Σ_2-equivalent to $\mathcal{R}F_2 \otimes \mathcal{R}F_2$ we conclude that we have a Σ_2-equivalence

$$\mathcal{R}F_1(NV^{\otimes 4}) \to H^*(\Sigma_2; \pi_* V^{\otimes 2})^{\otimes 2}$$

so by (3.4.3)

$$H_*(\mathcal{R}F_2 \circ \mathcal{R}F_1(NV^{\otimes 4})) \simeq H^*(\Sigma_2; H^*(\Sigma_2; \pi_* V^{\otimes 2})^{\otimes 2}).$$

The identification of Ψ^{D_8} follows from the 2nd part of Lemma 3.4.6. □

Note. The identification in Proposition 3.5.3 can also be worded to say that the spectral sequence (3.4.8) collapses at the E^2-term. We also note that this identification gives us a choice of representatives for the generators for the cohomology of D_8, but we will see that in most cases the spectral sequence (3.4.8) has only one nontrivial column or row at E^2, forcing our hand.

Before proving Proposition 3.3.7, we note that by Lemma 3.3.3 and Proposition 3.4.1 we have

$$\alpha_* \Psi^{D_8} = \Psi^{D_8} \alpha_* \tag{3.5.4}$$

Also, combining Lemma 3.3.3 and Proposition 3.4.10, we have

PROPOSITION 3.5.5. *Let $\Delta \le D_8$ be the subgroup $(2,3)D_8(2,3) \cap D_8$. Then the map*

$$\alpha_* : H^*(D_8; \pi_* V^{\otimes 4}) \to H^*(D_8; \pi_* V^{\otimes 4})$$

satisfies the identity

$$\alpha_* = t(\Delta, D_8) c_{(2,3)} r(D_8, \Delta).$$

Now, we proceed to prove Proposition 3.3.7. To do so we exploit naturality using (1.1.2) and reduce to universal examples. To this end we fix the following throughout

$$\begin{aligned} \pi_* K(m) &= \mathbb{F}_2\langle a\rangle & |a| &= m \\ \pi_* K(n) &= \mathbb{F}_2\langle b\rangle & |b| &= n \\ \pi_* K(p) &= \mathbb{F}_2\langle d\rangle & |d| &= p \\ \pi_* K(q) &= \mathbb{F}_2\langle e\rangle & |e| &= q \end{aligned}$$

where $m, n, p, q \ge 0$.

Proof of Proposition 3.3.7 part a: First, since $\Delta = \Sigma_2 \times \Sigma_2$, $H^*(\Delta; \mathbb{F}_2) \simeq \mathbb{F}_2[v_1, v_2]$ where $v_1, v_2 \in H^1(\Delta; \mathbb{F}_2)$ is dual to the elements of $H_1(\Delta; \mathbb{F}_2)$ associated to the generators of Δ. We now summarize a result in [9].

PROPOSITION 3.5.6. *There exist elements $x, y \in H^1(D_8; \mathbb{F}_2)$ and $z \in H^2(D_8; \mathbb{F}_2)$ such that*

1. $$H^*(D_8; \mathbb{F}_2) \simeq \mathbb{F}_2[x, y, z]/(xy)$$

2. $$\begin{aligned} r(D_8, \Delta)x &= v_2 \\ r(D_8, \Delta)y &= 0 \\ r(D_8, \Delta)z &= v_1(v_1 + v_2) \end{aligned}$$

3. $$\begin{aligned} t(\Delta, D_8)v_1^m &= \sum_{0 \le 2\ell < m} \binom{m - \ell - 1}{\ell} x^{m-2\ell} z^{\ell} \\ t(\Delta, D_8)v_2^m &= 0. \end{aligned}$$

PROOF. 1. follows from Proposition 3.5.3 plus a determination of extensions which is performed in [1].

2. Is another calculation done in [1].

3. Is a computation performed in [9]. □

PROPOSITION 3.5.7. *Under the homomorphism*

$$\Psi^{D_8} : \pi_* S^{D_8} K(m) \to H^*(D_8; \mathbb{F}_2),$$
$$\Psi^{D_8} \sigma_j \sigma_i(a) = x^{m+i-j} z^{m-i}.$$

PROOF. As shown in [9], under the identification of Proposition 3.5.3, x^r is the element w^r in $H^r(\Sigma_2, H^0(\Sigma_2; \pi_{2m} K(m)^{\otimes 2})^{\otimes 2})$ and z^r is the element $w^r \otimes w^r$ in $H^0(\Sigma_2; H^r(\Sigma_2; \pi_{2m} K(m)^{\otimes 2})^{\otimes 2})$. The result now follows from Proposition 3.5.1 and 3.5.3. □

Before getting to our main computation, we need

LEMMA 3.5.8. *Let $N \in \mathbb{Z}$ and $a \geq r \geq 0$. Then*

$$\sum_{0 \leq \ell \leq r} \binom{r}{\ell} \binom{N}{s-\ell} = \binom{N+r}{s}.$$

PROOF. This follows from an easy induction on r using the general Pascal's identity. □

Now, combining (3.5.4), Propositions 3.5.5, 3.5.6, and 3.5.1 we have

$$\begin{aligned}
\Psi^{D_8}\alpha_*\sigma_j\sigma_i(a) &= \alpha_*(x^{m+i-j}z^{m-i}) \\
&= t(\Delta, D_8)c_{(2,3)}r(D_8,\Delta)(x^s z^t) \\
&= t(\Delta, D_8)c_{(2,3)}(v_2^s v_1^t(v_1+v_2)^t) \\
&= t(\Delta, D_8)(v_1^s v_2^t(v_1+v_2)^t) \\
&= t(\Delta, D_8)\left(\sum_{0\leq k\leq t}\binom{t}{k} v_1^{s+t-k}v_2^{t+k}\right).
\end{aligned} \tag{3.5.9}$$

Here we have the identity $S = m+i-j$ and $t = m-i$. We have also slipped in $c_{(2,3)}v_1 = v_2$.

Now, by p. 257 of [5] and Proposition 3.5.6 (3.5.9) becomes

(3.5.10)

$$\begin{aligned}
&\sum_{0\leq k\leq t}\binom{t}{k} t(\Delta, D_8)v_1^{s+t-k} \\
&= \sum_{0\leq k\leq t}\binom{t}{k} x^{t+k}\left(\sum_{0\leq 2\ell< s+t-k}\binom{s+t-k-\ell-1}{\ell} x^{s+t-k-2\ell}z^{\ell}\right) \\
&= \sum_{0\leq 2\ell< s+t}\left(\sum_{0\leq k< s+t-2\ell}\binom{t}{k}\binom{s+t-k-\ell-1}{\ell}\right) x^{s+2t-2\ell}z^{\ell} \\
&= \sum_{0\leq 2\ell}\left(\sum_{0\leq k\leq t}\binom{t}{k}\binom{s+t-k-\ell-1}{\ell}\right) x^{s+2t-2\ell}z^{\ell}
\end{aligned}$$

where the last equality follows since $k < s+t-2\ell$. Now, for each k

$$\binom{s+t-k-\ell-1}{\ell} = \binom{s+t-k-\ell-1}{s+t-k-2\ell-1} = \binom{-\ell-1}{s+t-2\ell-k-1}.$$

Applying Lemma 3.5.8, we obtain

$$\sum_{0\leq k\leq t}\binom{t}{k}\binom{s+t-k-\ell-1}{\ell} = \binom{t-\ell-1}{s+t-2\ell-1}.$$

Thus (3.5.10) becomes

$$\sum_{0\leq 2\ell<s+t}\binom{t-\ell-1}{s+t-2\ell-1}x^{s+t-2\ell}z^{\ell}=\sum_{j<2s}\binom{s-i-1}{2s-j-1}x^{n-i-j+2s}z^{n-s}$$

upon letting $\ell = n-s$.

Combining Proposition 3.4.1, (3.5.4), Propositions 3.5.5, and 3.5.7 we arrive at our desired result.

Proof of Proposition 3.3.7 parts b and c: As before, it is sufficient to prove it for the case $V = K(m) \times K(n)$.

Let N be the Σ_4-submodule of $\pi_*(K(m)\times K(n))^{\otimes 4}$ generated by $a\otimes a\otimes b\otimes b$. As such N is a direct summand of $\pi_*(K(m)\times K(n))^{\otimes 4}$ as a Σ_4-module. Further, as a D_8-module

$$N = N_1 \otimes N_2$$

where N_1 is generated by $a\otimes a\otimes b\otimes b$ and N_2 is generated by $a\otimes b\otimes a\otimes b$. Now, writing the extension of (3.3.1) as

$$B \rightarrowtail D_8 \twoheadrightarrow \Sigma_2$$

where $B = \langle(1,2),(3,4)\rangle \simeq \Sigma_2\times\Sigma_2$, then N is a direct sum of two trivial B-modules. Thus by the Kunneth theorem

$$H^*(B;N_1)\simeq H^*(B,\mathbb{F}_2)\oplus H^*(B;\mathbb{F}_2)\simeq \mathbb{F}_2[\zeta_1',\zeta_2']\oplus\mathbb{F}_2[\zeta_1'',\zeta_2''].$$

Here Σ_2 acts by exchanging summands, which is a free Σ_2-action. Hence (3.4.8) tells us that

$$H^*(D_8;N_1)\simeq H^0(\Sigma_2;H^*(B;N_1))\simeq\mathbb{F}_2[\zeta_1,\zeta_2]$$

where ζ_1 corresponds to $\zeta_1'\oplus\zeta_1''$ and ζ_2 corresponds to $\zeta_2'\oplus\zeta_2''$, $|\zeta_1| = 1 = |\zeta_2|$.

Next, N_2 is a free B-module so by (3.4.8)

$$H^*(D_8;N_2)\simeq H^*(\Sigma_2;H^0(B;N_2))\simeq\mathbb{F}_2[\xi]$$

with $|\xi| = 1$.

Now, we have an extension

$$\Sigma_2 \rightarrowtail \Delta \twoheadrightarrow \Sigma_2$$

so that

$$H^*(\Delta;N_1)\simeq H^0(\Sigma_2;H^*(\Sigma_2;N_1))\simeq\mathbb{F}_2[\eta]\qquad |\eta|=1$$

since N_1 is a direct sum of two trivial Σ_2-modules with respect to the inner Σ_2-action and so proceed as above. Now, N_2 factors into $N_2'\oplus N_2''$ as Δ-modules where N_2' is generated by $a\otimes b\otimes a\otimes b$ and N_2'' is generated by $a\otimes b\otimes b\otimes a$.

Thus

$$H^*(\Delta;N_2)\simeq\mathbb{F}_2[\lambda_1]\oplus\mathbb{F}_2[\lambda_2]\qquad |\lambda_i|=1\quad i=1,2$$

by a computation as above.

PROPOSITION 3.5.11. 1. *Under the map* $r(D_8, \Delta)\colon H^*(D_8; N) \to H^*(\Delta; N)$

$$\begin{aligned} \zeta_1 &\to \eta \\ \zeta_2 &\to \eta \\ \xi &\to \lambda_1 \oplus \lambda_2 \end{aligned}$$

2. *Under the map* $c_{(2,3)}\colon H^*(\Delta; N) \to H^*(\Delta; N)$

$$\begin{aligned} \eta &\to \lambda_1 \\ \lambda_2 &\to \lambda_2. \end{aligned}$$

3. *Under the map* $t(\Delta, D_8)\colon H^*(\Delta; N) \to H^*(D_8; N)$

$$\begin{aligned} \eta^r &\to 0 \\ \lambda_1^r &\to \zeta^r \\ \lambda_2^r &\to \xi^r \end{aligned}$$

for all $r > 0$.

PROOF. 1. Consider the diagram of extensions

$$\begin{array}{ccccc} \Sigma_2 & \rightarrowtail & \Delta & \twoheadrightarrow & \Sigma_2 \\ {\scriptstyle\delta}\downarrow & & \downarrow & & \| \\ B & \rightarrowtail & D_8 & \twoheadrightarrow & \Sigma_2 \end{array}$$

where δ is the diagonal map. This induces

$$H^0(\Sigma_2; H^*(B; N_1)) \to H^0(\Sigma_2; H^*(\Sigma_2; N_1))$$

and

$$H^*(\Sigma_2; H^0(B; N_2)) \to H^*(\Sigma_2; H^0(\Sigma_2; N_2)).$$

These are the restriction maps

$$H^*(D_8; N_i) \to H^*(\Delta; N_i)$$

for $i = 1, 2$, by our above computations and Lemma 3.4.9. The first restriction is an easy computation. For the second restriction we have $H^0(B; N_2) \simeq \mathbb{F}_2$ and $H^0(\Sigma_2; N_2) \simeq \mathbb{F}_2 \oplus \mathbb{F}_2$ so that the induced map $\mathbb{F}_2 \to \mathbb{F}_2 \oplus \mathbb{F}_2$ is the diagonal map.

2. This is an easy consequence of the fact that

$$\begin{aligned} c_{(2,3)}N_1 &= N_2' \\ c_{(2,3)}N_2'' &= N_2''. \end{aligned}$$

3. First, N_2 is a free B-module so that $r(D_8, B)$ is trivial on $H^*(D_8; N_2)$ in positive degrees. Next, N_1 is a direct sum of two trivial B-modules thus

$$H^*(B; N_2) \simeq \mathbb{F}_2[\gamma_1, \gamma_2] \oplus \mathbb{F}_2[\varphi_1, \varphi_2]$$

where $|\gamma_i| = 1 = |\varphi_i|$ $i = 1, 2$. From the diagram of extensions

$$\begin{array}{ccccc} B & \rightarrowtail & B & \twoheadrightarrow & 1 \\ \| & & \downarrow & & \downarrow \\ B & \rightarrowtail & D_8 & \twoheadrightarrow & \Sigma_2 \end{array}$$

and Lemma 3.4.9, the restriction map $r(D_8, B)$ on $H^*(D_8; N_1)$ is equal to the inclusion

$$H^0(\Sigma_2; H^*(B; N_1)) \to H^*(B; N_1).$$

Thus

$$r(D_8; B)\zeta_1^s \zeta_2^t = \gamma_1^s \gamma_2^t \oplus \varphi_1^s \varphi_2^t.$$

We now pause to bring in the transfer

CLAIM.

$$r(D_8, B)t(\Delta, D_8) = 0$$

PROOF. By Proposition 3.4.10

$$r(D_8, B)t(\Delta, D_8) = t(I, B)r(\Delta, I)$$

where

$$I = \Delta \cap B.$$

Since I is a factor of B, $r(B, I)$ is onto, but $t(I, B)r(B, I) = 0$ so that $t(I, B) = 0$. □

From this claim and our computations, we conclude that

$$t(\Delta, D_8)\lambda_i^r = c_i \xi^r$$

$c_i \in \mathbb{F}_2$, $i = 1, 2$. From Proposition 3.4.10, we have

$$r(D_8, \Delta)t(\Delta, D_8) = 1 + c_{(1,2)}.$$

Since

$$(1, 2)N_1 = N_1$$
$$(1, 2)N_1' = N_1''$$

we get that under $r(D_8, \Delta)t(\Delta, D_8)$

$$\lambda_i^r \to \lambda_1^r \oplus \lambda_2^r.$$

So $c_i = 1$ for $i = 1, 2$. Finally, $t(\Delta, D_8)\eta^r = 0$ since η^r is in the image of $r(D_8, \Delta)$. □

Now, the relevance of the module N comes from

PROPOSITION 3.5.12. 1. *For* $0 \le i \le m$, $0 \le j \le n$

$$\Psi^{D_8}[\sigma_i(a), \sigma_j(b)] = \zeta_1^{m-i}\zeta_2^{n-j} \in H^*(D_8; N_1)$$

2. *For* $0 \le i \le m + n$

$$\Psi^{D_8}\sigma_i[a, b] = \xi^{n+m-i} \in H^*(D_8; N_2).$$

PROOF. These follow from Propositions 3.5.1, 3.5.2, and 3.5.3. □

Combining Corollary 3.4.7, (3.5.4), Propositions 3.5.5, 3.5.11, and 3.5.12 gives us our desired result.

Proof of Proposition 3.3.7 part d: Again it is sufficient to prove the result for $V = K(m) \times K(n) \times K(p)$. Let N be the Σ_4-submodule of $\pi_* V^{\otimes 4}$ generated by $a \otimes a \otimes b \otimes d$. As such it is a summand of the Σ_4-module $\pi_* V^{\otimes 4}$.

PROPOSITION 3.5.13. *For all* $0 \leq i \leq m$

$$\Psi^{D_8}\left[\sigma_i(a), [b, d]\right] \in H^{m-i}(D_8; N).$$

PROOF. Again, this is a computation utilizing Propositions 3.5.1, 3.5.2, and 3.5.3. □

Now, since N is a free Δ-module, then by (3.5.4) and Proposition 3.5.5 the result follows from a computation utilizing Proposition 2.2.16 and Corollary 3.3.5.

Proof of Proposition 3.3.7 part e: Let $V = K(m) \times K(m) \times K(p) \times K(q)$ and N the Σ_4-submodule of $\pi_* V^{\otimes 4}$ generated by $a \otimes b \otimes d \otimes e$.

PROPOSITION 3.5.14.

$$\Psi^{D_8}\left[[a, b], [d, e]\right] \in H^*(D_8; N).$$

PROOF. Combine Proposition 3.5.2 and 3.5.3. □

N is Σ_4-free so another computation using Proposition 2.2.16 and Corollary 3.3.5 gives us our result.

This completes the proof of Proposition 3.3.7.

PROOF OF PROPOSITION 3.3.10. a. It is sufficient to prove the result for $V = K(n)$ and $W = K(n)$. Suppose $n, m > 0$. Then

$$(N_V \otimes N_W)_* \colon \pi_* S_2 V \otimes S_2 W \to \pi_* S^2 V \otimes S^2 W$$

is injective. Thus it suffices to compute $\overline{\alpha}''_*$. By Theorem 3.4.1 and Lemma 3.3.9 our conclusion follows from $t(\Sigma_2, \Sigma_2 \times \Sigma_2) = 0$ since $r(\Sigma_2 \times \Sigma_2, \Sigma_2)$ is onto $H^*(\Sigma_2; \pi_*(V \otimes W)^{\otimes 2})$. Suppose $n = 0$. Define

$$i_1 \colon (S_2 V) \otimes W \to S_2(V \otimes W)$$

as the unique simplicial map such that

$$(xy) \otimes b \to (x \otimes b)(y \otimes b).$$

Also define

$$i_2 \colon (S_2 V) \otimes W \to (S_2 V) \otimes (S_2 W)$$

as $1 \otimes \iota$ (see (1.3.6)). Then the diagram

$$\begin{array}{ccc} & & S_2(V \otimes W) \\ & \nearrow^{i_1} & \\ (S_2V) \otimes W & & \Big\downarrow \overline{\phi}'' \\ & \searrow_{i_2} & \\ & & S_2V \otimes S_2W \end{array}$$

commutes. A computation gives the result. The case of $m = 0$ is the same.

b. This is an easy computation using the diagram

$$\begin{array}{ccc} (V \otimes W)^{\otimes 2} & \longrightarrow & S_2(V \otimes W) \\ {\scriptstyle 1 \otimes T \otimes 1}\Big\downarrow & & \Big\downarrow \overline{\phi}'' \\ (V^{\otimes 2}) \otimes (W^{\otimes 2}) & \longrightarrow & S_2V \otimes S_2W \end{array} .$$

□

Bibliography

[1] J. Adem, *The relations on Steenrod powers of cohomology classes*, Algebraic Geometry and Topology, Princeton NJ (1957), 191–238.

[2] A.K. Bousfield, *Operations on derived functors for non-additive functors*, manuscript, Brandeis University (1967).

[3] ———, *On the homotopy spectral sequence of a cosimplicial space*, Amer. J. Math. **109** (1987), 361–394.

[4] H. Cartan, *Algebras d'Eilenberg-MacLane et homotopie*, in Seminaire Henri Cartan (1954–1955).

[5] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton NJ (1956).

[6] A. Dold, *Homology of symmetric products and other functors of complexes*, Ann. Math. **68** (1958), 54–80.

[7] ———, *Uber die Steenrodschen kohomologieoperationen*, Ann. Math. **73** (1961), 258–294.

[8] A. Dold and D. Puppe, *Homologie nitch-additiver functoren, Anwendungen*, Ann. Inst. Fourier **11** (1961), 201–312.

[9] W.G. Dwyer, *Homotopy operations for simplicial commutative algebras*, Trans. Amer. Math. Soc. **260** (1980), 421–435.

[10] ———, *Higher divided squares in second quadrant spectral sequences*, Trans. Amer. Math. Soc. **260** (1980), 437–447.

[11] L. Evens, *The Cohomology of Groups*, Oxford University Press, New York (1991).

[12] P.G. Goerss, *The André-Quillen cohomology of commutative* $\mathbf{F}_2$*-algebras*, Asterisque **186** (1990).

[13] P.G. Goerss and T. Lada, *Relations among homotopy operations for simplicial commutative algebras*, preprint, University of Washington (1993).

[14] S. Mac Lane, *Homology*, Springer-Verlag, New York (1967).

[15] J.P. May, *Simplicial Objects in Algebraic Topology*, Van Nostrand, New York (1967).

[16] J.W. Milnor and J.C. Moore, *On the structure of Hopf algebras*, Ann. Math. **81** (1965), 211–264.

[17] D.G. Quillen, *Homotopical Algebra*, Lect. Notes in Math. vol. 43, Springer-Verlag, Berlin, 1967.

[18] J.M. Turner, *Dyer-Lashoff operations in second quadrant spectral sequences I*, prepreprint, MIT (1992).

[19] ———, *Dyer-Lashoff operations in second quadrant spectral sequences II*, in preparation.

[20] ———, *Looping Bousfield-Kan towers*, in preparation.