Multiplicative Structures on Brown–Peterson Spectra at Odd Primes

by

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Abstract

We show that the odd-primary Brown-Peterson spectrum does not admit the structure of an $\mathbb{E}_{2(p^2+2)}$ ring spectrum and that there can be no map $\mathrm{MU} \to \mathrm{BP}$ of $\mathbb{E}_{2p+3}$ ring spectra for odd primes p. This extends results of Lawson at the prime 2.

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Contents

Chapter 1

Introduction

Two important themes in modern homotopy theory are the study of structured ring spectra, in particular $\mathbb{E}_\infty$ ring spectra, and chromatic homotopy theory, which had its genesis in computations with the Adams-Novikov spectral sequence based on the p-primary Brown-Peterson spectrum BP [18]. In [16], May asked about the interaction between these two programs:

Question 1.0.1. *Does the Brown-Peterson spectrum admit the structure of an* $\mathbb{E}_\infty$ *ring spectrum?*

This question has been seminal in the development of the theory of structured ring spectra. In an unpublished preprint [10], Kriz developed the theory of topological André-Quillen cohomology in an attempt to prove that BP does admit the structure of an $\mathbb{E}_\infty$ ring spectrum. While his attempt to apply his theory to BP did not ultimately succeed, the careful study of what exactly went wrong became the seed of a new attempt by Lawson to answer May's question in the negative; recently, this project reached maturity in Lawson's proof [12] that BP does not admit an $\mathbb{E}_\infty$ multiplication at the prime $p = 2$.

In this paper, we prove in Theorem 1.1.1 that BP does not admit an $\mathbb{E}_\infty$ multiplication at odd primes. Our technique is akin to Lawson's and relies on the computation of a certain secondary power operation in the dual Steenrod algebra. The key input to this computation is the calculation of a certain MU-power operation in MU_*.

For further motivation and background, we refer the reader to the introduction of [12].

1.1 Statement of the results

We prove two main results: one limiting the coherence of multiplicative structures on the Brown-Peterson spectrum and related spectra at odd primes, and another giving a stronger limitation on the coherence of complex orientations of such spectra.

Since the first theorem reduces to [12, Theorem 1.1.2] at the prime $p = 2$, we are able to state it for all primes.

Theorem 1.1.1. *Neither the Brown-Peterson spectrum* BP*, nor the truncated Brown-Peterson spectra* $\mathrm{BP}\langle n\rangle$ *for* $n \geq 4$*, nor any of their p-adic completions admit the structure of an* $\mathbb{E}_{2(p^2+2)}$ *ring spectrum.*

We will prove Theorem 1.1.1 at the end of Section 4.

Theorem 1.1.2. *Neither the Brown-Peterson spectrum* BP*, nor the truncated Brown-Peterson spectra* $\mathrm{BP}\langle n\rangle$ *for* $n \geq 3$*, nor any of their p-adic completions admit an* $\mathbb{E}_{2p+3}$*-map from the complex cobordism spectrum* MU*.*

We will prove Theorem 1.1.2 at the end of Section 2. Again, the $p = 2$ case of this theorem is due to Lawson [12, Remark 4.4.7].

1.2 Outline of the paper

In Section 2, we carry out the computations of MU-power operations that we will need. The main result Chapter 2 is Theorem 2.1.2. In Section 3, we generalize results of [12] to convert the MU power operations of Theorem 2 into Dyer-Lashof operations in $\pi_*(\mathrm{H}\mathbb{F}_p \wedge_{\mathrm{MU}} \mathrm{H}\mathbb{F}_p)$, thus obtaining Theorem 3.0.3. At the end of this section, we apply these results to obtain Theorem 1.1.2.

In Section 4.1, we state some relations satisfied by the action of the Dyer-Lashof operations on $\mathrm{H}_*(\mathrm{MU};\mathbb{F}_p)$ and $\mathrm{H}_*(\mathrm{H}\mathbb{F}_p;\mathbb{F}_p)$. In Section 4.2, we write down the relation defining the secondary operation of interest and show that it is defined on $-\xi_1 \in \mathrm{H}_*(\mathrm{H}\mathbb{F}_p; \mathbb{F}_p)$. Finally, in Section 4.3, we compute this secondary operation on $-\xi_1$ to be a nonzero multiple of τ_4 modulo the ξ_i by applying juggling formulae and a Peterson-Stein relation to reduce to Theorem 3.0.3. We then deduce Theorem 1.1.1.

1.3 Questions

Our work raises several interesting questions. While Theorems 1.1.1 and 1.1.2 provide upper bounds on the coherence of multiplicative structures on BP that are functions of p, the best known lower bounds [3] and [5], which state that BP is an $\mathbb{E}_4$-algebra and admits an $\mathbb{E}_2$ orientation $\mathrm{MU} \to \mathrm{BP}$, do not depend on the prime p. So one is led to ask whether these coherence bounds are independent of p.

Question 1.3.1. *Let* $\mathrm{coh}_{\mathrm{BP}}(p)$ *denote the largest integer* n *such that the* p*-primary* BP *admits the structure of an* $\mathbb{E}_n$ *ring spectrum. Is* $\mathrm{coh}_{\mathrm{BP}}(p)$ *constant in* p*? If not, how does it vary with* p*?*

In another direction, we may ask about $\mathbb{E}_\infty$ structures on the truncated Brown-Peterson spectra $\mathrm{BP}\langle n\rangle$. While Theorem 1.1.1 rules out the possibility of such structures for $n \geq 4$, the only known positive results state that $\mathrm{BP}\langle 1\rangle$ always admits an $\mathbb{E}_\infty$ structure (since it is the Adams summand) and that $\mathrm{BP}\langle 2\rangle$ admits an $\mathbb{E}_\infty$ structure at the primes 2 and 3 [7] [13]. What about the remaining cases?

Question 1.3.2. *At which of the primes* $p \geq 5$ *does the height* 2 *truncated Brown-Peterson spectrum* $\mathrm{BP}\langle 2\rangle$ *admit an* $\mathbb{E}_\infty$ *multiplication?*

Question 1.3.3. *At which primes does the height* 3 *truncated Brown-Peterson spectrum* $\mathrm{BP}\langle 3\rangle$ *admit an* $\mathbb{E}_\infty$ *multiplication?*

Remark 1.3.4. The above questions are not quite well-defined: there are many generalized truncated Brown-Peterson spectra $\mathrm{BP}\langle n\rangle$ which are not a priori equivalent. However, Angeltveit and Lind [1] have shown that all choices of $\mathrm{BP}\langle n\rangle$ are equivalent after p-completion, so that Question 1.3.2 and Question 1.3.3 are well-defined after p-completion.

1.3.1 Conventions

We work throughout at a fixed odd prime p. We will let H denote the mod p Eilenberg-MacLane specrum $\mathrm{H}\mathbb{F}_p$ and let $\mathrm{H}_*(X)$ denote mod p homology.

We let F denote the universal formal group law, defined over MU_*.

We will work freely with the language of ∞-cateogories and the notion of $\mathbb{E}_n$-ring native to this setting, as developed by Lurie [15] [14]. To translate between Lawson's framework [12, Section 1.6] and ours, we pass to the underlying ∞-category of the model categories considered by Lawson. The compatibility of this procedure with multiplicative structures is justified by [19, Theorem 7.10].

1.4 Generators of the homology and homotopy of MU

For the convenience of the reader, we review the relations between various sets of elements of $\pi_*(\mathrm{MU})$, $\mathrm{H}_*(\mathrm{MU};\mathbb{Z})$ and $\pi_*(\mathrm{MU})\otimes\mathbb{Q}$ that we will need to make use of.

The integral homology $\mathrm{H}_*(\mathrm{MU};\mathbb{Z})$ is generated by elements b_i which are the images of the duals of c_1^i under $\mathrm{H}_*(\mathbb{CP}^\infty;\mathbb{Z})\to\mathrm{H}_*(BU;\mathbb{Z})\cong\mathrm{H}_*(\mathrm{MU};\mathbb{Z})$. If we define the Newton polynomials in b_i inductively by $N_1(b)=b_1$ and

$$N_n(b)=b_1N_{n-1}(b)-t_2N_{n-2}(b)+\cdots+(-1)^{n-2}b_{n-1}N_1(b)+(-1)^{n-1}nb_n,$$

then $N_n(b)$ generates the group of primitive elements in $\mathrm{H}_{2n}(\mathrm{MU};\mathbb{Z})$. Furthermore, $N_n(b)\equiv(-1)^{n-1}nb_n$ modulo decomposables. As we will see in Section 4.1, there are convenient formulae for the action of the Dyer-Lashof operations on $N_n(b)$.

The homotopy $\pi_*(\mathrm{MU})$ of MU is generated by elements x_i whose images under the Hurewicz map are $h(x_i)\equiv qb_i$ modulo decomposables when $i=q^n-1$ for some prime q and $h(x_i)\equiv b_i$ modulo decomposables otherwise.

We may view the corbordism class of $\mathbb{CP}^n$ as an element $[\mathbb{CP}^n]$ of $\pi_{2n}(\mathrm{MU})$. Then, the $[\mathbb{CP}^n]$ do not generate $\pi_*(\mathrm{MU})$, though they are generators of $\pi_*(\mathrm{MU})\otimes\mathbb{Q}$.

Under the isomorphism $\pi_*(\mathrm{MU}) \otimes \mathbb{Q} \cong \mathrm{H}_*(\mathrm{MU}; \mathbb{Q})$ induced by the Hurewicz map, $[\mathbb{CP}^n] \equiv -(n+1)b_n$ modulo decomposables.

The logarithm of of the formal group F on $\pi_*(\mathrm{MU})$ may be expressed in terms of the $[\mathbb{CP}^n]$:

$$\ell_F(x) = \sum \frac{\left[\mathbb{CP}^{n-1}\right] x^n}{n}.$$

1.5 When are the Dyer-Lashof operations defined?

To obtain the precise bounds on $\mathbb{E}_n$ structures of Theorem 1.1.1 and Theorem 1.1.2, we need to know when a Dyer-Lashof operation Q^k is defined on an element $x \in \pi_n R$ for R an $\mathbb{E}_n$-H-algebra.

Theorem 1.5.1 ([4, Theorems III.3.1 and III.3.3])**.** *Let R be an $\mathbb{E}_n$-H-algebra. Then the operation Q^s is defined on an element $x \in \pi_n R$ when $2s - \deg(x) \leq n-1$; however, these operations only satisfy the expected properties (e.g. linearity, Cartan formula) when $2s - \deg(x) \leq n-2$.*

Chapter 2

Power operations in the homotopy of MU

2.1 Statement of results

Our goal in this section is to compute certain power operations in the homotopy of MU which will form the starting point of our proof that BP does not admit the structure of a $\mathbb{E}_{2(p^2+2)}$-ring.

We begin by recalling that the $\mathbb{H}_\infty^2$-structure on MU equips the even MU-cohomology of a space X with a power operation

$$P_{C_p} : \mathrm{MU}^{2*}(X) \to \mathrm{MU}^{2p*}(X \times BC_p).$$

Using the isomorphism

$$\mathrm{MU}^*(BC_p) \cong \mathrm{MU}^*[[\alpha]]/[p]_F(\alpha),$$

we may view this power operation applied to $X = *$ a point as a map

$$P_{C_p} : \mathrm{MU}^{2*} \to \mathrm{MU}^{2p*}[[\alpha]]/[p]_F(\alpha).$$

Let

$$r_* : \mathrm{MU}^*[[\alpha]]/[p]_F(\alpha) \to \mathrm{BP}^*[[\alpha]]/[p]_F(\alpha)$$

denote the map induced by the Quillen idempotent. Our goal in this section will be to compute the composition of $r_* \circ P$ applied to certain elements of MU^{2*}. We begin with the following piece of notataion.

Notation 2.1.1. Let

$$\chi = \prod_{i=1}^{p-1} [i]_F(\alpha) \in \mathrm{MU}^*(BC_p) \cong \mathrm{MU}^*[[\alpha]]/[p]_F(\alpha)$$

denote the MU-Euler class of the real reduced regular representation of C_p.

Theorem 2.1.2. *The follow equalities hold modulo* BP^**-decomposables:*

$$r_*\left(\chi^{2(p-1)} P_{C_p}\left(\left[\mathbb{CP}^{2(p-1)}\right]\right)\right) \equiv v_3 \alpha^{p^3-1-2(p-1)} + O(\alpha^{p^3}) \tag{2.1}$$

and

$$r_*\left(\chi^{p(p-1)} P_{C_p}\left(\left[\mathbb{CP}^{p(p-1)}\right]\right)\right) \equiv v_3 \alpha^{p^3-1-p(p-1)} + O(\alpha^{p^3}) \tag{2.2}$$

Remark 2.1.3. We will use (2.1) in the proof of Theorem 1.1.1 and (2.2) in the proof of Theorem 1.1.2. Equation (2.1) could also be used to prove a version of Theorem 1.1.2, but with a worse bound on the coherence.

We may deduce the following corollary.

Corollary 2.1.4. *Suppose* $f : \mathrm{MU}_{(p)} \to E$ *is a map of* $\mathbb{H}_\infty$*-ring spectra satisfying:*

1. f *factors through the Quillen idempotent* $\mathrm{MU}_{(p)} \to \mathrm{BP}$.

2. f *induces a Landweber exact* MU_**-module structure on* E_*.

Then the induced formal group on E_* *has height at most* 2*, i.e.* v_2 *is invertible in* $E_*/(p, v_1)$.

This corollary is similar to [8, Theorem 1.3], which differs from it in the following respect: [8, Theorem 1.3] shows the stronger result that E_* is a $\mathbb{Q}$-algebra, but only for primes $p \leq 13$.

Proof of Corollary 2.1.4. By [8, Theorem 1.3], we may as well assume that $p > 2$.

The map $\mathrm{MU} \to E$ automatically acquires an $\mathbb{H}^2_\infty$-structure by [8, Theorem 3.13]. Since $\chi^{2(p-1)}\left[\mathbb{CP}^{2(p-1)}\right]$ maps to zero in BP_* and thus E_*, it follows that

$$P_{C_p}\left(\chi^{2(p-1)}\left[\mathbb{CP}^{2(p-1)}\right]\right) = v_3\alpha^{p^3-1-2(p-1)} + O(\alpha^{p^3})$$

maps to zero in $E_*[[\alpha]]/[p](\alpha)$. Thus

$$v_3\alpha^{p^3-1-2(p-1)} + O(\alpha^{p^3}) = g(\alpha)\cdot[p](\alpha)$$

for some $g(\alpha) \in E_*[[\alpha]]$. Examining the coefficient of $\alpha^{p^3-1-2(p-1)}$, we see that v_3 is divisible by p in E_*, so that $v_3 = 0 \in E_*/(p, v_1, v_2)$. On the other hand, v_3 is regular in $E_*/(p, v_1, v_2)$ by Landweber exactness, so that we must have $E_*/(p, v_1, v_2) = 0$, as desired. □

We begin the proof of Theorem 2.1.2 with a reduction. Since we are working modulo BP^*-decomposables, the coefficients of $\alpha^{p^3-1-2(p-1)}$ (resp. $\alpha^{p^3-1-p(p-1)}$) in (2.1) (resp. (2.2)) can be taken to be some constant multiple of v_3 for degree reasons. Moreover, these are the first terms in (2.1) and (**??**) that can be nonzero modulo BP^*-decomposables. It therefore suffices to show that the (2.1) and (2.2) hold after composing with the map $q : \mathrm{BP}^* \to \mathbb{Z}_p[v_3]/(v_3^2)$ that sends v_3 to v_3 and v_i to 0 for $i \neq 3$. Here, we let v_i denote the (i)th Hazewinkel generator. In conclusion, to prove Theorem 2.1.2 it suffices to prove the following proposition.

Proposition 2.1.5. *There are equalities*

$$q \circ r_*\left(\chi^{2(p-1)}P_{C_p}(\mathbb{CP}^{2(p-1)})\right) = v_3\alpha^{p^3-1-2(p-1)}$$

and

$$q \circ r_* \left(\chi^{p(p-1)} P_{C_p}(\mathbb{CP}^{p(p-1)})\right) = v_3 \alpha^{p^3-1-p(p-1)}.$$

In the appendix of [12], Lawson shows how this computation may be made internally to $\mathbb{Z}_p[v_3]/(v_3^2)$ and the induced formal group law. Since this formal group law is much simpler than the formal group law of BP, the computation that we need to make simplifies dramatically and so becomes tractable.

2.2 Proof of Proposition 2.1.5

We begin by reviewing some basic facts about MU-power operations. This section is based on [12, Appendix A].

Notation 2.2.1. We let

$$\langle p \rangle_F(x) = \frac{[p]_F(x)}{x}.$$

Fact 2.2.2. *The power operation* $P_{C_p} : \mathrm{MU}^{2*}(X) \to \mathrm{MU}^{2p*}(X)[[\alpha]]/[p]_F(\alpha)$ *satisfies the following properties:*

1. $P_{C_p}(uv) = P_{C_p}(u) P_{C_p}(v)$
2. $P_{C_p}(u) = u^p$ *modulo* α
3. $P_{C_p}(u+v) = P_{C_p}(u) + P_{C_p}(v)$ *modulo* $\langle p \rangle_F(\alpha)$
4. *On the orientation class* $x \in \widetilde{\mathrm{MU}}^2(\mathbb{CP}^\infty)$,

$$P_{C_p}(x) = x \prod_{i=1}^{p-1} (x +_F [i]_F(\alpha)).$$

Notation 2.2.3. We let

$$g(x, \alpha) = x \prod_{i=1}^{p-1} (x +_F [i]_F(\alpha)),$$

viewed as an element of $\mathrm{MU}^*[[x, \alpha]]/[p]_F(\alpha)$, so that $P_{C_p}(x) = g(x, \alpha)$. Note that $\frac{\partial}{\partial x} g(0, \alpha) = \chi$.

Applying Fact 2.2.2 to the spaces $X = (\mathbb{CP}^\infty)^{\times n}$, we obtain the following proposition:

Proposition 2.2.4. *The composite*

$$\Psi : \mathrm{MU}^* \to \mathrm{MU}^*[[\alpha]]/\langle p\rangle_F(\alpha)$$

of P_{C_p} *with the quotient map* $\mathrm{MU}^*[[\alpha]]/[p]_F(\alpha) \to \mathrm{MU}^*[[\alpha]]/\langle p\rangle_F(\alpha)$ *is a ring homomorphism. Morover, the power series* $g(x, \alpha)$ *defines an isogeny* $F \to \Psi^* F$*.*

Let $\omega \in \mathbb{Z}_p$ denote a $(p-1)$st root of unity. We will find it convenient to express $g(x, \alpha)$ and χ in terms of $[\omega^i]_F(\alpha)$ instead of $[i]_F(\alpha)$, where $i = 1, \ldots, p-1$ on both sides. This is because we will eventually replace F with a p-typical formal group law G, and for any p-typical G we have the simple formula $[\omega^i]_G(x) = \omega^i x$.

To make sense of this, we must base change to the p-completion

$$\mathrm{MU}^*_p = \mathrm{MU}^* \otimes_{\mathbb{Z}} \mathbb{Z}_p.[1]$$

When base changed to MU^*_p, the formal group law F admits the structure of a $\mathbb{Z}_p$-module. In particular, if we let $\omega \in \mathbb{Z}_p$ denote a primitive $(p-1)$st root of unity, there are endomorphisms $[\omega^i]_F(x)$ of F. Since $\omega^1, \ldots, \omega^{p-1}$ form a set of representatives for $1, \ldots, p-1$ modulo p, we obtain the following lemma:

Lemma 2.2.5. *There are equalities*

$$\chi \equiv \prod_{i=1}^{p-1} [\omega^i]_F(\alpha) \mod [p]_F(\alpha)$$

and

$$g(x, \alpha) \equiv x \prod_{i=1}^{p-1} (x +_F [\omega^i]_F(\alpha)) \mod [p]_F(\alpha).$$

[1]Note that the p-completion may be described as the tensor product with $\mathbb{Z}_p$ because MU^n is finite dimensional over $\mathbb{Z}$ for each n.

Since MU^* and $\mathrm{MU}^*[[\alpha]]/\langle p\rangle_F(\alpha)$ are torsion-free, F and Ψ^*F admit logarithms

$$\ell_F(x) = \sum \frac{\left[\mathbb{CP}^{n-1}\right] x^n}{n}$$

and

$$\ell_{\Psi^*F}(x) = \sum \frac{\Psi\left(\left[\mathbb{CP}^{n-1}\right]\right) x^n}{n}.$$

This implies that we may compute $\Psi([\mathbb{CP}^n])$ as the coefficient of x^n in the derivative $\ell'_{\Psi^*F}(x)$ of $\ell_{\Psi^*F}(x)$ with respect to x. We will now describe a method for computing these coefficients. We begin with a lemma.

Lemma 2.2.6. *Let R^* denote a nonzero graded torsion-free ring and let $r : \mathrm{MU}_p^* \to R^*$ denote a map classifying a formal group law G over R^*. Then*

$$r(\chi) = \prod_{i=1}^{p-1} [\omega^i]_G(\alpha)$$

factors as $u\alpha^{p-1}$, where u is a unit. Moreover, α is not a zero divisor in $R^*[[\alpha]]/\langle p\rangle_G(\alpha)$, so neither is χ.

Proof. We have

$$\begin{aligned} r(\chi) &= \prod_{i=1}^{p-1} [\omega^i]_G(\alpha) \\ &= \prod_{i=1}^{p-1} (\omega^i\alpha + O(\alpha^2)) \\ &= \alpha^{p-1}(-1 + O(\alpha)), \end{aligned}$$

which implies that $r(\chi) = u \cdot \alpha^{p-1}$ for a unit u.

It remains to show that α is not a zero-divisor in $R^*[[\alpha]]/\langle p\rangle_G(\alpha)$. Suppose that $\alpha \cdot f(\alpha) = g(\alpha) \cdot \langle p\rangle_G(\alpha)$. We wish to show that α must divide $g(\alpha)$, or in other works that $g(\alpha)$ has trivial constant term. But this follows from the fact that $\langle p\rangle_G(\alpha)$ has constant term p, which is not a zero divisor in R^*. □

Definition 2.2.7. We fix an arbitrary lift $\Psi([\mathbb{CP}^n]) \in \mathrm{MU}^*[[\alpha]]$ of $\Psi([\mathbb{CP}^n]) \in$

$\mathrm{MU}^*[[\alpha]]/\langle p\rangle_F(\alpha)$. This determines a lift of $\ell_{\Psi^*F}(x)$ to $\mathrm{MU}^*[[x,\alpha]]$.

We also fix a lift of $g(x,\alpha)$ to $\mathrm{MU}_p^*[[x,\alpha]]$. Then $\frac{\partial}{\partial x}g(0,\alpha)$ is a lift of χ to $\mathrm{MU}_p^*[[\alpha]]$.

Notation 2.2.8. Define $k(y,\alpha)$ by $g(\chi y,\alpha) = \chi^2 k(y,\alpha)$. Then $k(y,\alpha)$ has leading term y, so we may let $k^{-1}(y,\alpha)$ denote a composition inverse.

Moreover, let $\ell_F'(x) = \frac{\partial}{\partial x}\ell_F(x)$, $\ell_{\Psi^*F}'(x) = \frac{\partial}{\partial x}\ell_{\Psi^*F}(x)$, $k'(y,\alpha) = \frac{\partial}{\partial y}k(y,\alpha)$ and $(k^{-1})'(y,\alpha) = \frac{\partial}{\partial y}k^{-1}(y,\alpha)$.

Proposition 2.2.9. *Let $f_n(\alpha)$ denote the coefficient of y^n in*

$$\ell_F'(\chi k^{-1}(y,\alpha)) \cdot (k^{-1})'(y,\alpha).$$

Then

$$\Psi([\mathbb{CP}^n])\chi^{2n} \equiv f_n(\alpha) \mod \langle p\rangle_F(\alpha).$$

Proof. Applying $\frac{\partial}{\partial y}$ to the equation

$$g(x,\alpha) +_{\Psi^*F} g(y,\alpha) \equiv g(x +_F y,\alpha) \mod \langle p\rangle(\alpha)$$

and evaluating at $y=0$, we obtain the equation

$$\frac{g'(0,\alpha)}{(\ell_{\Psi^*F})'(g(x,\alpha))} \equiv \frac{g'(x,\alpha)}{(\ell_F)'(x)} \mod \langle p\rangle_F(\alpha).$$

This implies that

$$g'(x,\alpha)\cdot(\ell_{\Psi^*F})'(g(x,\alpha)) = \chi\cdot(\ell_F)'(x) + h(x,\alpha)\cdot\langle p\rangle_F(\alpha)$$

for some $h(x,\alpha) \in \mathrm{MU}_p^*[[x,\alpha]]$. In the above equation, we have used the fact that $\chi = g'(0,\alpha)$. Plugging in $x=0$, we find that $h(0,\alpha)=0$, so that $h(x,\alpha) = x\widetilde{h}(x,\alpha)$ for some $\widetilde{h}(x,\alpha) \in \mathrm{MU}_p^*[[x,\alpha]]$.

Next, we make the substitution $x = \chi y$ and write $g(\chi y,\alpha) = \chi^2 k(y,\alpha)$ as in

Notation 2.2.8. Plugging in our substitution, we obtain

$$\begin{aligned}\chi \cdot k'(y,\alpha) \cdot (\ell_{\Psi^*F})'(\chi^2 k(y,\alpha)) &= \chi \cdot (\ell_F)'(\chi y) + h(\chi y, \alpha) \cdot \langle p \rangle_F(\alpha) \\ &= \chi \cdot (\ell_F)'(\chi y) + \chi y \cdot \widetilde{h}(\chi y, \alpha) \cdot \langle p \rangle_F(\alpha).\end{aligned}$$

Substituting $k^{-1}(y,\alpha)$ for y, applying the chain rule and dividing by χ (which is valid by Lemma 2.2.6), we obtain

$$(\ell_{\Psi^*F})'(\chi^2 y) = (\ell_F)'(\chi k^{-1}(y,\alpha)) \cdot (k^{-1})'(y,\alpha) + k^{-1}(y,\alpha) \cdot \widetilde{h}(y,\alpha) \cdot \langle p \rangle_F(\alpha).$$

Taking coefficients of y^n on both sides, we find that

$$\Psi(\mathbb{CP}^n)\chi^{2n} = f_n(\alpha) + \widetilde{h}_n(\alpha) \cdot \langle p \rangle_F(\alpha)$$

for some $\widetilde{h}_n(\alpha) \in \mathrm{MU}_p^*[[\alpha]]$, as desired. □

Finally, to compute $P_{C_p}([\mathbb{CP}^n])$, we have the following proposition.

Proposition 2.2.10. *There exists a unique polynomial $h_n(\alpha) \in \mathrm{MU}^*[\alpha]$ of degree $2n(p-1)$ with the property that*

$$f_n(\alpha) - h_n(\alpha) \cdot \langle p \rangle_F(\alpha) \equiv \chi^{2n}\, [\mathbb{CP}^n]^p \mod \alpha^{2n(p-1)+1}.$$

Furthermore,

$$P_{C_p}([\mathbb{CP}^n]) \equiv \chi^{-2n}(f_n(\alpha) - h_n(\alpha) \cdot \langle p \rangle_F(\alpha)) \mod [p]_F(\alpha).$$

Proof. By Proposition 2.2.9,

$$f_n(\alpha) \equiv \chi^{2n}\Psi([\mathbb{CP}^n]) \equiv \chi^{2n} P_{C_p}([\mathbb{CP}^n]) \mod \langle p \rangle_F(\alpha).$$

By Fact 2.2.2(2), this implies that

$$f_n(\alpha) \equiv [\mathbb{CP}^n]^p \mod (\langle p \rangle_F(\alpha), \chi^{2n}\alpha).$$

Combining the above with Lemma 2.2.6, we find that $h_n(\alpha)$ exists. Uniqueness follows from the fact that the constant term p of $\langle p \rangle_F(\alpha)$ is not a zero divisor in MU_p^*.

In particular, we find that $f_n(\alpha) - h_n(\alpha) \cdot \langle p \rangle_F(\alpha)$ is divisble by χ^{2n} and that

$$\chi^{-2n}(f_n(\alpha) - h_n(\alpha) \cdot \langle p \rangle_F(\alpha)) \equiv [\mathbb{CP}^n]^p \equiv P_{C_p}([\mathbb{CP}^n]) \mod \alpha$$

and

$$\chi^{-2n}(f_n(\alpha) - h_n(\alpha) \cdot \langle p \rangle_F(\alpha)) \equiv \Psi([\mathbb{CP}^n]) \equiv P_{C_p}([\mathbb{CP}^n]) \mod \langle p \rangle_F(\alpha).$$

Again using the fact that p is not a zero divisor in MU_p^*, this implies that

$$\chi^{-2n}(f_n(\alpha) - h_n(\alpha) \cdot \langle p \rangle_F(\alpha)) \equiv P_{C_p}([\mathbb{CP}^n]) \mod \langle p \rangle_F(\alpha),$$

as desired. □

Suppose now that we are given a graded torsion-free ring R^* and a homomorphism $r : \mathrm{MU}_p^* \to R^*$ classifying a formal group law G over R^*. Then we may define χ_G, $g_G(x, \alpha)$, $k_G(x, \alpha)$, $k_G^{-1}(x, \alpha)$ and $f_n^G(\alpha)$ as above, using the formal group law G on R^* in place of the formal group law F over MU^*.

Proposition 2.2.11. *Let R^* denote a graded torsion-free ring, and let $r : \mathrm{MU}_p^* \to R^*$ classify a formal group law G over R^*. Then there exists a unique polynomial $h_n^G(\alpha) \in R^*[\alpha]$ of degree $2n(p-1)$ with the property that*

$$f_n^G(\alpha) - h_n^G(\alpha) \cdot \langle p \rangle_G(\alpha) \equiv \chi^{2n} [\mathbb{CP}^n]^p \mod \alpha^{2n(p-1)+1}.$$

Moreover,

$$r(P_{C_p}([\mathbb{CP}^n])) \equiv \chi^{-2n}(f_n^G(\alpha) - h_n^G(\alpha) \cdot \langle p \rangle_G(\alpha)) \mod [p]_G(\alpha).$$

Proof. The first part follows exactly as in the proof of Proposition 2.2.10.

For the second part, we note that $f_n^G(\alpha) = r(f_n(\alpha))$ by the definitions, and that $h_n^G(\alpha) = r(h_n(\alpha))$ by uniqueness. The second part then follows from Proposition 2.2.10.

□

Proposition 2.2.12. *Consider the map* $q \circ r_* : \mathrm{MU}_p^* \to \mathbb{Z}_p[v_3]/(v_3^2)$ *and its induced formal group law* $G = (q \circ r_*)^* F$*. Then the following hold:*

1. $\ell_G(x) = x + \dfrac{v_3}{p} x^{p^3}$.

2. $x +_G y = x + y + \dfrac{v_3}{p}(x^{p^3} + y^{p^3} - (x+y)^{p^3})$.

3. $[p]_G(\alpha) = p\alpha - (p^{p^3-1} - 1)v_3\alpha^{p^3}$, *so that* $\langle p \rangle_G(\alpha) = p - (p^{p^3-1} - 1)v_3\alpha^{p^3-1}$.

4. $\chi_G = \displaystyle\prod_{i=1}^{p-1} \omega^i \alpha = -\alpha^{p-1}$.

5. $g_G(x, \alpha) \equiv \chi x + x^p + O(x^{p^2}) \mod [p]_G(\alpha)$.

6. $k_G(y, \alpha) \equiv y + \chi^{p-2} y^p + O(y^{p^2}) \mod [p]_G(\alpha)$.

7. $k_G^{-1}(y, \alpha) = y + \displaystyle\sum_{n=1}^{p} (-1)^n \frac{\binom{np}{n}}{n(p-1)+1} \chi^{n(p-2)} y^{n(p-1)+1} + O(y^{p^2})$.

8. $f_{i(p-1)}^G(\alpha) = (-1)^i \dbinom{ip}{i} \chi^{i(p-2)}$ *for* $1 \le i \le p$.

9. $h_{i(p-1)}^G(\alpha) = (-1)^i \dfrac{\binom{ip}{i}}{p} \chi^{i(p-2)}$ *for* $1 \le i \le p$.

Proof. Part (1) follows from the formula for the logarithm of the universal p-typical formal group law [20, Appendix A2]. Recall that we are using the Hazewinkel v_is. Parts (2) and (3) follow in a straightforward way from part (1). To establish part (4), we note that, since the formal group law G is p-typical, $[\omega^i]_G(x) = \omega^i x$. Therefore

$$\chi_G = \prod_{i=1}^{p-1} \omega^i \alpha = -\alpha^{p-1},$$

since p is odd. Moreover, we have

$$g_G(x, \alpha) = x \prod_{i=1}^{p-1} (x +_G (\omega^i \alpha)).$$

We then compute

$$\begin{aligned} g_G(x, \alpha) &= x \prod_{i=1}^{p-1} (x +_G (\omega^i \alpha)) \\ &= x \prod_{i=1}^{p-1} (x + \omega^i \alpha) \left[1 + \frac{v_3}{p} \sum_{j=1}^{p-1} \frac{x^{p^3} + (\omega^j \alpha)^{p^3} - (x + \omega^j \alpha)^{p^3}}{x + \omega^j \alpha} \right] \\ &\equiv x \prod_{i=1}^{p-1} (x + \omega^i \alpha) + O(x^{p^2}) \mod [p]_G(\alpha) \\ &= x(x^{p-1} - \alpha^{p-1}) + O(x^{p^2}) \\ &= \chi x + x^p + O(x^{p^2}), \end{aligned}$$

where we have used the fact that $pv_3\alpha = 0$ modulo $[p]_G(\alpha)$. This establishes part (5). Part (6) follows immediately from the defining equation $\chi^2 k_G(y, \alpha) = g_G(\chi y, \alpha)$.

To deduce part (7), we apply Lagrange inversion to part (6). Since

$$(\ell_G)'(x) = 1 + O(x^{p^3 - 1}),$$

we deduce that

$$(\ell_G)'(\chi k^{-1}(y, \alpha)(k^{-1})'(y, \alpha) = (k^{-1})'(y, \alpha) + O(y^{p^3-1}),$$

so we may read off (8) from (7).

Finally, (9) follows from (8) and the fact that $[\mathbb{CP}^n]^p = 0$ in $\mathbb{Z}[v_3]/v_3^2$. □

Corollary 2.2.13. *There is an equality*

$$q \circ r_* \left(\chi^{i(p-1)} P(\mathbb{CP}^{i(p-1)}) \right) \equiv -\frac{\binom{ip}{i}}{p} v_3 \alpha^{p^3-1-i(p-1)} \mod [p]_G(\alpha).$$

Proof. Using Proposition 2.2.11 and Proposition 2.2.12, we compute:

$$\begin{aligned} q \circ r_* \left(\chi^{i(p-1)} P(\mathbb{CP}^{i(p-1)}) \right) &\equiv \chi_G^{-i(p-1)} \cdot (f^G_{i(p-1)}(\alpha) - h^G_{i(p-1)}(\alpha) \cdot \langle p \rangle_G(\alpha)) \\ &\equiv \chi_G^{-i(p-1)} \cdot (-h^G_{i(p-1)}(\alpha)) \cdot (-(p^{p^3-1} - 1) v_3 \alpha^{p^3-1}) \\ &\equiv -h^G_{i(p-1)}(\alpha) v_3 \alpha^{p^3-1-i(p-1)^2} \\ &\equiv (-1)^{i+1} \frac{\binom{ip}{i}}{p} v_3 \alpha^{p^3-1-i(p-1)} \mod [p]_G(\alpha), \end{aligned}$$

where we have used the fact that $pv_3\alpha = 0$ modulo $[p]_G(\alpha)$. □

Proof of Proposition 2.1.5. Applying the congruences $\frac{\binom{2p}{2}}{p} \equiv -1$ and $\frac{\binom{p^2}{p}}{p} \equiv 1 \mod p$ to Corollary 2.2.13, we deduce that

$$q \circ r_* \left(\chi^{2(p-1)} P(\mathbb{CP}^{2(p-1)}) \right) \equiv v_3 \alpha^{p^3-1-2(p-1)} \mod [p]_G(\alpha)$$

and

$$q \circ r_* \left(\chi^{p(p-1)} P(\mathbb{CP}^{p(p-1)}) \right) \equiv v_3 \alpha^{p^3-1-p(p-1)} \mod [p]_G(\alpha),$$

as desired. □

Remark 2.2.14. Zeshen Gu has independently worked on computations similar to the above.

Chapter 3

A Dyer-Lashof operation in the MU-dual Steenrod algebra

In this section, we apply Theorem 2.1.2 to compute certain Dyer-Lashof operations in the MU-dual Steenrod algebra $\pi_*(\mathrm{H} \wedge_{\mathrm{MU}} \mathrm{H})$. We begin by determining the structure of $\pi_*(\mathrm{H} \wedge_{\mathrm{MU}} \mathrm{H})$ as an algebra.

Proposition 3.0.1. *The algebra* $\pi_*(\mathrm{H} \wedge_{\mathrm{MU}} \mathrm{H})$ *is isomorphic to an exterior algebra* $\Lambda_{\mathbb{F}_p}(\tau_i) \otimes \Lambda_{\mathbb{F}_p}(\sigma m_i \,|\, i \neq p^k - 1)$ *on classes* τ_i *for* $i \geq 0$ *and* σm_i *for* $i \geq 1$*. The degrees of these classes are* $|\tau_i| = 2p^i - 1$ *and* $|\sigma m_i| = 2i + 1$*.*

The natural map $\mathrm{H} \wedge \mathrm{H} \to \mathrm{H} \wedge_{\mathrm{MU}} \mathrm{H}$*, upon taking homotopy, induces a map*

$$\Lambda_{\mathbb{F}_p}(\tau_i) \otimes \mathbb{F}_p[\xi_i] \to \Lambda_{\mathbb{F}_p}(\tau_i) \otimes \Lambda_{\mathbb{F}_p}(\sigma m_i \,|\, i \neq p^k - 1)$$

sending τ_i *to* τ_i *and* ξ_i *to decomposable elements.*

Proof. By comparison of the Künneth spectral sequence

$$\mathrm{Tor}_{*,*}^{\mathrm{H}_*\mathrm{MU}}(\mathrm{H}_*, \mathrm{H}_*\mathrm{H}) \Rightarrow \pi_*\left((\mathrm{H} \wedge \mathrm{H}) \wedge_{\mathrm{H} \wedge \mathrm{MU}} (\mathrm{H} \wedge \mathrm{H})\right) = \pi_*(\mathrm{H} \wedge_{\mathrm{MU}} \mathrm{H})$$

with the other Künneth spectral sequence

$$\mathrm{Tor}_{*,*}^{\pi_*\mathrm{MU}}(\mathrm{H}_*, \mathrm{H}_*) \Rightarrow \pi_*(\mathrm{H} \wedge_{\mathrm{MU}} \mathrm{H}),$$

we find that the first Künneth spectral sequence collapses at the E^2-page. Since $\mathrm{Tor}_{*,*}^{\mathrm{H}_*\mathrm{MU}}(\mathrm{H}_*, \mathrm{H}_*\mathrm{H})$ is isomorphic to $\Lambda_{\mathbb{F}_p}(\tau_i) \otimes \Lambda_{\mathbb{F}_p}(\sigma m_i \mid i \neq p^k - 1)$, the description of $\pi_*(\mathrm{H} \wedge_{\mathrm{MU}} \mathrm{H})$ follows. The assertion about the map $\mathrm{H} \wedge \mathrm{H} \to \mathrm{H} \wedge_{\mathrm{MU}} \mathrm{H}$ follows from the naturality of the Künneth spectral sequence. □

Remark 3.0.2. Note that the second Künneth spectral sequence above gives an alternative description of $\pi_*(\mathrm{H} \wedge_{\mathrm{MU}} \mathrm{H})$ as $\Lambda_{\mathbb{F}_p}(\tau_0, \sigma x_i)$. Furthermore, Lawson [12] shows that for $x \in \pi_n R$ for $n \geq 1$, there is a distinguished choice of $\sigma x \in \pi_*(\mathrm{H} \wedge_R \mathrm{H})$: there is a map $\widetilde{\mathrm{H}}_*(SL_1(R)) \to \pi_{*+1}(\mathrm{H} \wedge_R \mathrm{H})$ which sends the Hurewicz image of $x \in \pi_n R \cong \pi_n SL_1(R)$ to a distinguished choice of σx.

Furthermore, this map $\sigma : \pi_n R \to \pi_{n+1}(\mathrm{H} \wedge_R \mathrm{H})$ annihilates decomposables. Whenever we write σx for $x \in \pi_n R$, we will be referring to this distinguished choice of σx.

We are now able to state the main theorem of this section.

Theorem 3.0.3. *In* $\pi_*(\mathrm{H} \wedge_{\mathrm{MU}} \mathrm{H})$*, we have*

$$Q^{p^2+p-1}\left(\sigma\left[\mathbb{CP}^{2(p-1)}\right]\right) = -\sigma x_{p^3-1}$$

and

$$Q^{p^2+1}\left(\sigma\left[\mathbb{CP}^{p(p-1)}\right]\right) = \sigma x_{p^3-1}.$$

This follows immediately from Theorem 2.1.2 and the following theorem:

Theorem 3.0.4. *Let* $y \in \pi_{2n}\mathrm{MU}$ *and suppose that*

$$\chi^n P(y) = \sum_{i=0}^{\infty} c_i \alpha^i$$

for some elements $c_i \in \pi_{2(n+i)}\mathrm{MU}$*. Then the action of the Dyer-Lashof operations on* $\pi_*(\mathrm{H} \wedge_{\mathrm{MU}} \mathrm{H})$ *are determined by the equation*

$$Q^k(\sigma y) = (-1)^k \sigma c_{k(p-1)}.$$

Our proof of this theorem will follow [12, Sections 3 and 4] rather closely. The idea will be to relate the power operation P_{C_p} to the action of the multiplicative Dyer-Lashof operations on the homology of the Ω-spectrum of MU, and to relate this in turn to the Dyer-Lashof operations on $\pi_*(\mathrm{H} \wedge_{\mathrm{MU}} \mathrm{H})$.

First, we need to introduce some notation from [12, Section 4].

Notation 3.0.5. We let

$$\mathrm{MU}_n = \Omega^\infty \Sigma^n \mathrm{MU}$$

denote the spaces in the Ω-spectrum for MU.

Since MU is a ring spectrum, the homology of the spaces MU_n is equipped with two products, making $\mathrm{H}_*(\mathrm{MU}_\bullet)$ into a Hopf ring. We denote the additive one, coming from the infinite loop space structure on MU_n, by

$$-\#- : \mathrm{H}_*(\mathrm{MU}_n) \otimes \mathrm{H}_*(\mathrm{MU}_n) \to \mathrm{H}_*(\mathrm{MU}_n),$$

and the multiplactive one, coming from the multiplication on MU, by

$$- \circ - : \mathrm{H}_*(\mathrm{MU}_n) \otimes \mathrm{H}_*(\mathrm{MU}_m) \to \mathrm{H}_*(\mathrm{MU}_{n+m}).$$

Since MU is an $\mathbb{E}_\infty$-ring spectrum, MU_0 is equipped with the structure of an $\mathbb{E}_\infty$-ring space. Its homology is therefore equipped with two actions of he Dyer-Lashof operations, an additive action coming the infinite loop space structcure on MU_0, and a multiplictive one coming from the $\mathbb{E}_\infty$-multiplication ono MU.

We denote the additive operations by

$$Q^k : \mathrm{H}_n(\mathrm{MU}_0) \to \mathrm{H}_{n+2(p-1)k}(\mathrm{MU}_0)$$

and the multiplicative operations by

$$\widehat{Q}^k : \mathrm{H}_n(\mathrm{MU}_0) \to \mathrm{H}_{n+2(p-1)k}(\mathrm{MU}_0).$$

Definition 3.0.6. The $\mathbb{H}^2_\infty$-algebra structure on MU implies the existence of based maps

$$\mathrm{MU}_{2n} \wedge (B\Sigma_p)_+ \to \mathrm{MU}_{2pn}$$

representing the power operation

$$P : \mathrm{MU}^{2n}(X) \to \mathrm{MU}^{2pn}(X \times B\Sigma_p).$$

We let

$$\mathcal{Q} : \mathrm{H}_*(\mathrm{MU}_{2n}) \to \mathrm{H}_*(\mathrm{MU}_{2pn})\widehat{\otimes}\mathrm{H}^*(B\Sigma_p)$$

denote the adjoint to the map

$$H_*(\mathrm{MU}_{2n}) \otimes \mathrm{H}_*(B\Sigma_p) \to \mathrm{H}_*(\mathrm{MU}_{2pn})$$

induced by the above map of spaces.

Multiplicativity of P implies the following:

Proposition 3.0.7 ([12, Proposition 4.2.2])**.** *The operation $\mathcal{Q}$ preserves the $\circ$-product: $\mathcal{Q}(x) \circ \mathcal{Q}(y) = \mathcal{Q}(x \circ y)$.*

Notation 3.0.8. Let $b_i \in \mathrm{H}_{2i}(\mathrm{MU}_2)$ denote the image under the orientation map $\mathbb{CP}^\infty \to \mathrm{MU}_2$ of the class in $\mathrm{H}_{2i}(\mathbb{CP}^\infty)$ dual to c_1^i. We let $b(s) = \sum_{i=1}^\infty b_i s^i$, viewed as a formal power series in s.

Remark 3.0.9. Since b_1 is the fundamental class of the unit map $S^2 \to \mathrm{MU}_2$, $- \circ b_1 : \mathrm{H}_*(\mathrm{MU}_{2n}) \to \mathrm{H}_*(\mathrm{MU}_{2n+2})$ corresponds to suspension.

Notation 3.0.10. Given a homotopy element $x \in \pi_{2n}(\mathrm{MU})$, we let $[x] \in \mathrm{H}_0(\mathrm{MU}_{2n})$ denote the image of the corresponding class in $\pi_0(\mathrm{MU}_{2n})$ under the Hurewicz map.

It follows from Remark 3.0.9 that $[x] \circ b_1^{\circ n} \in \mathrm{H}_{2n}(\mathrm{MU}_0)$ is the image of x, viewed as an element of $\pi_{2n}(\mathrm{MU}_0)$, under the Hurewicz map.

Definition 3.0.11. Given a based space X, there is a natural map

$$\Lambda : \mathrm{MU}^{2n}(X) = [X, \mathrm{MU}_n] \to \mathrm{Hom}(\mathrm{H}_*(X), \mathrm{H}_*(\mathrm{MU}_{2n})) = \mathrm{H}_*(\mathrm{MU}_{2n})\widehat{\otimes}\mathrm{H}^*(X)$$

which sends a homotopy class of map to its induced map on homology.

The groups $\mathrm{H}_*(\mathrm{MU}_{2n})\widehat{\otimes}\mathrm{H}^*(X)$ are equipped with products $\#$ and $\circ$, each induced by the corresponding product in $\mathrm{H}_*(\mathrm{MU}_{2n})$ and the cup product in $\mathrm{H}^*(X)$.

Proposition 3.0.12 ([12, Propositions 3.2.3 and 4.2.3])**.** *The map Λ satisfies the following properties:*

- $\Lambda(x+y) = \Lambda(x)\#\Lambda(y)$
- $\Lambda(xy) = \Lambda(x)\circ\Lambda(y)$
- $\Lambda([c]) = [c]\otimes 1$
- $(\mathcal{Q}\otimes 1)(\Lambda(x)) = \Lambda(P(x))$.

Notation 3.0.13. Recall that

$$\mathrm{H}^*(BC_p) \cong \mathbb{F}_p[w]\otimes\Lambda_{\mathbb{F}_p}(v),$$

where $|v| = 1$, $|w| = 2$, and w is the image of the generator c_1 of $H^2(\mathbb{CP}^\infty)$ under the map on cohomology induced by the canonical map

$$BC_p \to \mathbb{CP}^\infty.$$

Furthermore,

$$\mathrm{H}^*(B\Sigma_p) \cong \mathbb{F}_p[u]\otimes\Lambda_{\mathbb{F}_p}(z),$$

where, when pulled back to BC_p, $u = w^{p-1}$ and $z = vw^{p-2}$.

Remark 3.0.14 ([12, Remark 4.2.4])**.** Recall that $\mathrm{MU}^*(BC_p) \cong \mathrm{MU}^*[[\alpha]]/[p]_F(\alpha)$ for some element $\alpha \in \mathrm{MU}^2(BC_p)$. This element satifies the equation

$$\Lambda(\alpha) = b(w).$$

Lemma 3.0.15. *For a space X with pth extended power $D_p(X)$, the composite diagonal map*

$$\mathrm{H}_*(X) \otimes \mathrm{H}_*(B\Sigma_p) \to \mathrm{H}_*(X \times B\Sigma_p) \to \mathrm{H}_*(D_p(X))$$

on mod p homology is given by

$$x \otimes \beta_n \mapsto (-1)^n \sum_{j \geq 0} Q^{j+n}(P_j x)$$

and

$$x \otimes \gamma_n \mapsto (-1)^{n+|x|} \left(\sum_{j \geq 0} \beta Q^{j+n}(P_j x) - \sum_{j \geq 0} Q^{j+n}(P_j \beta x) \right)$$

where β_n is dual to u^n in $\mathrm{H}^*(B\Sigma_p) \cong \mathbb{F}_p[u] \otimes \Lambda_{\mathbb{F}_p}[z]$, γ_n is dual to $u^{n-1}z$, and P_j is the homology operation dual to P^j.

Proof. This follows from the definition of the Dyer-Lashof operations (cf. [17, Definition 2.2]) and [17, Proposition 9.1]. Note that an extra sign is introduced in the latter equation due to the fact that we have written the $B\Sigma_p$-action on the right and not the left. See also [11, Proposition 6]. □

The following corollary then follows from the definitions:

Corollary 3.0.16. *Suppose that $x \in \mathrm{H}_*(\mathrm{MU}_0)$. Then:*

$$\mathcal{Q}(x) = \sum_{n,j} (-1)^n \widehat{Q}^{j+n}(P_j x) u^n + (-1)^{n+|x|} \left(\beta \widehat{Q}^{j+n}(P_j x) - \widehat{Q}^{j+n}(\mathcal{P}_i \beta x) \right) u^{n-1} w.$$

In particular, if x is in the image of the Hurewicz map, then

$$\mathcal{Q}(x) = \sum_{n=0}^{\infty} (-1)^n \widehat{Q}^n(x) u^n + (-1)^{n+|x|} \beta \widehat{Q}^n(x) u^{n-1} w.$$

Proposition 3.0.17. *We have*

$$\mathcal{Q}(b_1) = b_1 \circ \Lambda(\chi).$$

Proof. This is just the second to last equation in the proof of [12, Proposition 4.3.1]. □

Proposition 3.0.18. *Let* $y \in \pi_{2n}\mathrm{MU}$ *and suppose that*

$$\chi^n P(y) = \sum_{i=0}^{\infty} c_i \alpha^i$$

for some elements $c_i \in \pi_{2(n+i)}\mathrm{MU}$.

Then, modulo #-decomposables and the $\circ$*-ideal generated by* $b_2, b_3, \ldots,$ *we have*

$$\widehat{Q}^k([y] \circ b_1^{\circ n}) \equiv (-1)^k [c_{(p-1)k}] \circ b_1^{\circ(p-1)k}.$$

Proof. We have:

$$\begin{aligned}
\mathcal{Q}([y] \circ b_1^{\circ n}) &= \mathcal{Q}([y]) \circ \mathcal{Q}(b_1)^{\circ n} \\
&= \Lambda(P(y)) \circ \Lambda(\chi)^{\circ n} \\
&= \Lambda(P(y)\chi^n) \\
&= \Lambda(\sum_{i=0}^{\infty} c_i \alpha^i) \\
&= \mathop{\#}_{i=0}^{\infty} [c_i] \circ b(w)^{\circ i} \\
&\equiv \sum_{i=0}^{\infty} [c_i] \circ (b_1)^i \circ w^i,
\end{aligned}$$

where we view $\mathcal{Q}([y] \circ b_1^{\circ n})$ as living inside of $\mathrm{H}_*(\mathrm{MU}_0)\widehat{\otimes}\mathrm{H}^*(BC_p)$ via the natural inclusion $\mathrm{H}^*(B\Sigma_p) \hookrightarrow \mathrm{H}^*(BC_p)$.

The result now follows from Corollary 3.0.16 and the fact that the operations P_j and $P_j\beta$ vanish on the spherical class $[y] \circ b_1^{\circ n}$. ☐

Proposition 3.0.19. *Let* p *be an odd prime. Then the multiplicative Dyer-Lashof operations in the Hopf ring of an* $\mathbb{E}_\infty$*-ring satisfy the following identity whenever* y *is in the homology of the path component of zero:*

$$\widehat{Q^s}([1]\#y) \equiv [1]\#\widehat{Q^s}(y)$$

modulo # and $\circ$ *decomposables.*

We first prove a lemma.

Lemma 3.0.20. *In the situation of Proposition 3.0.19, for any x there exist elements z_i for $0 < i < |x|$ such that the additive Dyer-Lashof operations satisfy*

$$Q^s(x) = Q^s[1] \circ x + \sum Q^{s_i}[1] \circ z_i.$$

Therefore $Q^s(x)$ is $\circ$-decomposable for any x and any $s > 0$.

Proof. This follows from the formula

$$Q^s[1] \circ x = \sum Q^{s+i}([1] \circ P_i x)$$

of [6, II.1.6] by inducting on the degree of x. □

Proof of Proposition 3.0.19. We apply the mixed Cartan formula, which states that

$$\widehat{Q^s}(x\#y) = \sum_{s_0+\cdots+s_p=s} \sum \widehat{Q_0^{s_0}}(x_0 \otimes y_0)\# \ldots \#\widehat{Q_p^{s_p}}(x_p \otimes y_p)$$

where

$$\Delta_{p+1}(x \otimes y) = \sum (x_0 \otimes y_0) \otimes \cdots \otimes (x_p \otimes y_p)$$

and where

$$\widehat{Q_0^s}(x \otimes y) = \widehat{Q^s}(\varepsilon(x)y),$$

$$\widehat{Q_p^s}(x \otimes y) = \widehat{Q^s}(x\varepsilon(y)),$$

and for $0 < i < p$ we put $m_i = \frac{1}{p}\binom{p}{i}$ so that

$$\widehat{Q_i^s}(x \otimes y) = [m_i] \circ \left(\sum Q^j(x_1 \circ \cdots \circ x_i \circ y_1 \circ \cdots \circ y_{p-i})\right)$$

where $\Delta_i x = \sum x_1 \otimes \ldots x_i$ and $\Delta_{p-i} y = \sum y_1 \otimes \cdots \otimes y_{p-i}$.

Applying this to the case that $x = [1]$ and y is in the homology of the path component of zero, we first note that this is #-decomposable and hence zero unless all of but one of the terms lies in degree 0, i.e. unless all of the $y_i = [0]$ and $s_i = 0$ for all but one i.

Using Lemma 3.0.20, we further deduce that all of the terms with $s_i \neq 0$ for some $0 < i < p$ are zero. Finally, we note that $\widehat{Q_p^s}([1] \otimes y) = \widehat{Q^s}([1]) = 0$ for $s > 0$, so that in fact the only term left is

$$\widehat{Q_0^s}([1] \otimes y) \# \widehat{Q_1^0}([1] \otimes [0]) \# \ldots \# \widehat{Q_p^0}([1] \otimes [0]) = \widehat{Q^s} y \# [1].$$

All that remains is to show that the multplicity of this term is one, i.e. that

$$([1] \otimes y) \otimes ([1] \otimes [0]) \otimes \cdots \otimes ([1] \otimes [0])$$

appears with coefficient one in $\Delta_{p+1}([1] \otimes y)$

That this term appears with coefficient $p + 1 \equiv 1$ in $\Delta_{p+1}([1] \otimes x)$ follows from the fact that $\Delta_{p+1}([1]) = [1] \otimes \cdots \otimes [1]$ and that $x \otimes [0] \otimes \cdots \otimes [0]$ appears in $\Delta_{p+1}(x)$ with coefficient one. □

We are now ready to prove Theorem 3.0.4.

Proof of Theorem 3.0.4. In [12, Section 3.3], a suspension map $\sigma : \widetilde{\mathrm{H}}_*(SL_1(\mathrm{MU})) \to \pi_{*+1}(\mathrm{H} \wedge_{\mathrm{MU}} \mathrm{H})$ is constructed. By the mod p analogs of [12, Corollary 3.3.6 & Proposition 3.3.7] (which are proved in exactly the same way as for $p = 2$), this map commutes with the Dyer-Lashof operations and kills #-decomposables, $\circ$-decomposables, and b_i for $i > 1$. Applying σ to Proposition 3.0.18 and Proposition 3.0.19, we obtain the desired result. □

Our next goal is to deduce Theorem 1.1.2 from Theorem 3.0.3 by noting that the Dyer-Lashof operations exhibited therein are incompatible with the existence of a highly structured map $\mathrm{H} \wedge_{\mathrm{MU}} \mathrm{H} \to \mathrm{H} \wedge_{\mathrm{BP}} \mathrm{H}$. We begin by showing that a highly structured map $\mathrm{MU} \to \mathrm{BP}$ would induce a (slightly less) highly structured map $\mathrm{H} \wedge_{\mathrm{MU}} \mathrm{H} \to \mathrm{H} \wedge_{\mathrm{BP}} \mathrm{H}$.

Proposition 3.0.21. *Let R be an $\mathbb{E}_\infty$-ring and let $A \to B$ denote a map of $\mathbb{E}_n$-rings augmented over R. Then there exists a natural map $R \wedge_A R \to R \wedge_B R$ of $\mathbb{E}_{n-1}$-$(R \wedge R)$-algebras.*

Proof. Let $\mathcal{C}$ denote the ∞-category $\mathrm{Alg}_R^{\mathbb{E}_{n-1}}$ of $\mathbb{E}_{n-1}$-R-algebras, equipped with the symmetric monoidal structure induced by that of Mod_R. Then the bar construction defines a functor $\mathrm{Bar} : \mathrm{Alg}(\mathcal{C})_{/R} \to \mathcal{C}$ by [14, Example 5.2.2.3]. By [14, Theorem 5.1.2.2], $\mathrm{Alg}(\mathcal{C})$ is equivalent to $\mathrm{Alg}_R^{\mathbb{E}_n}$, so that Bar defines a functor from augmented $\mathbb{E}_n$-R-algebras to $\mathbb{E}_{n-1}$-R-algebras.

Since the forgetful functor $\mathcal{C} \to \mathrm{Mod}_R$ preserves sifted colimits by [14, Proposition 3.2.3.1], Bar is computed in R-modules and so $\mathrm{Bar}(-) \cong R \wedge_- R$ as functors into R-modules.

This implies the existence of a natural map $R \wedge_{A \wedge R} R \to R \wedge_{B \wedge R} R$ of $\mathbb{E}_{n-1}$-R-modules. Applying the functor $- \wedge_R (R \wedge R)$ yields the desired map $R \wedge_A R \to R \wedge_B R$ of $\mathbb{E}_{n-1}$-$(R \wedge R)$-algebras. □

We are now ready to prove Theorem 1.1.2. In this proof, we allow p to be 2: in this case, Theorem 3.0.3 may be replaced by [12, Corollary 4.4.3]. At $p = 2$, Lawson indicated in [12, Remark 4.4.4] that the following argument should work in the case of BP.

Proof of Theorem 1.1.2. For the sake of simplicity of notation, we prove Theorem 1.1.2 for BP. The proof for $\mathrm{BP}\langle n\rangle$ with $n \geq 3$ is analogous. Taking the p-completion changes nothing because we are only using the mod p homology.

First note that the Künneth spectral sequences

$$\mathrm{Tor}_{*,*}^{\mathrm{H}_*\mathrm{BP}}(\mathrm{H}_*, \mathrm{H}_*\mathrm{H}) \Rightarrow \pi_*(\mathrm{H} \wedge_{\mathrm{BP}} \mathrm{H})$$

and

$$\mathrm{Tor}_{*,*}^{\pi_*\mathrm{BP}}(\mathrm{H}_*, \mathrm{H}_*) \Rightarrow \pi_*(\mathrm{H} \wedge_{\mathrm{BP}} \mathrm{H})$$

collapse at the E^2-term. So there are isomorphisms $\pi_*(\mathrm{H} \wedge_{\mathrm{BP}} \mathrm{H}) \cong \Lambda_{\mathbb{F}_p}(\tau_i)$ and $\pi_*(\mathrm{H} \wedge_{\mathrm{BP}} \mathrm{H}) \cong \Lambda_{\mathbb{F}_p}(\sigma v_i)$.

Suppose that there were a map of $\mathbb{E}_{2p+3}$-rings $\mathrm{MU} \to \mathrm{BP}$. By the naturality of Postnikov towers of $\mathbb{E}_{2p+3}$-rings, this is a map of $\mathbb{E}_{2p+3}$-algebras augmented over H. Then Proposition 3.0.21 implies that this induces a map $\mathrm{H} \wedge_{\mathrm{MU}} \mathrm{H} \to \mathrm{H} \wedge_{\mathrm{BP}} \mathrm{H}$ of $\mathbb{E}_{2p+2}$-$(\mathrm{H} \wedge \mathrm{H})$-algebras. Forgetting the action of the left H, we obtain a map of $\mathbb{E}_{2p+2}$-H-algebras.

We claim that the induced map $\Lambda_{\mathbb{F}_p}(\tau_0, \sigma x_i) \cong \mathrm{H} \wedge_{\mathrm{MU}} \mathrm{H} \to \mathrm{H} \wedge_{\mathrm{BP}} \mathrm{H} \cong \Lambda_{\mathbb{F}_p}(\sigma v_k)$ sends σx_{p^k-1} to a nonzero multiple of σv_k. Assuming this, we obtain a contradiction with the operation $Q^{p^2+1}\sigma x_{p(p-1)} = C_2 \sigma x_{p^3-1}$ of Theorem 3.0.3 because $\sigma x_{p(p-1)}$ goes to zero in $\Lambda_{\mathbb{F}_p}(\sigma v_k)$ for degree reasons. This operation is preserved by maps of $\mathbb{E}_{2p+2}$-H-algebras by Theorem 1.5.1.

To prove the claim, we use the fact that $\mathrm{Tor}_{*,*}^{\mathrm{H}_*\mathrm{BP}}(\mathrm{H}_*, \mathrm{H}_*\mathrm{H})$ is concentrated in homological degree zero and is therefore just $\mathrm{H}_* \otimes_{\mathrm{H}_*\mathrm{BP}} \mathrm{H}_*\mathrm{H}$. The induced map

$$\mathrm{H}_* \otimes_{\mathrm{H}_*\mathrm{MU}} \mathrm{H}_*\mathrm{H} \to \mathrm{H}_* \otimes_{\mathrm{H}_*\mathrm{BP}} \mathrm{H}_*\mathrm{H}$$

is automatically surjective; therefore the induced map of Künneth spectral sequences is surjective on the E^2 and therefore on the E^∞ term because it collapses at the E^2-term. We conclude that the map on indecomposables is surjective, which is equivalent to the claim. □

Chapter 4

A secondary power operation in the dual Steenrod algebra

In this section, we define and compute a secondary power operation in the dual Steenrod algebra and then show that Theorem 1.1.1 follows from this computation. We make free use of the formalism of Toda brackets in categories enriched over pointed topological spaces developed in [12, Section 2], including the juggling, additivity and Peterson-Stein formulae of [12, Propositions 2.3.5 and 2.4.3].

Notation 4.0.1. Given a set S of formal variables with gradings, we let $\mathbb{P}^n_{\mathrm{H}}(S)$ denote the free $\mathbb{E}_n$-H-algebra on the wedge of spheres $\bigvee_{x \in S} S^{|x|}$ and let $x \in \pi_{|x|}\left(\mathbb{P}^n_{\mathrm{H}}(S)\right)$ denote the homotopy element corresponding to the fundamental class $\iota_{|x|} \in \pi_{|x|}\left(S^{|x|}\right)$.

Let x be a formal variable with degree $2(p-1)$ and let $\mathbb{P}^{2(p^2+2)}_{\mathrm{H}}(x)$ denote the free $\mathbb{E}_{2(p^2+2)}$-H-algebra on x. Then we will let $\mathcal{D}$ denote the category $\left(\mathrm{Alg}_{\mathrm{H}}^{\mathbb{E}_{2(p^2+2)}}\right)_{\mathbb{P}^{2(p^2+2)}_{\mathrm{H}}(x)/}$ of $\mathbb{E}_{2(p^2+2)}$-H-algebras under $\mathbb{P}^{2(p^2+2)}_{\mathrm{H}}(x)$.

This is a topological category, so the category $\mathcal{C} = \mathcal{D}^{\pm}$ of possibly pointed or augmented objects [12, Definition 2.2.2] in this category is enriched over pointed topological spaces. The category $\mathcal{C}$ consists of augmented objects of $\mathcal{D}$, pointed objects of $\mathcal{D}$, and objects of $\mathcal{D}$ without a pointing or augmentation. Through casework, one is able to define pointed spaces of maps between these objects, making use of the pointings and augmentations in the expected way when present. We refer the reader to [12, Definition 2.2.2] for the somewhat lengthy details.

Whenever we take brackets in the below, it will be in the category $\mathcal{C}$. Given a set of graded elements S, we always view $\mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x, S)$ as an element of $\mathcal{C}$ via the augmentation $\mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x, S) \to \mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x)$ sending x to x and all of the elements of S to 0.

Notation 4.0.2. In the following, we will make our computations in the exterior quotient $\Lambda_{\mathbb{F}_p}(\tau_0, \tau_1, \dots)$ of the dual Steenrod algebra $\mathrm{H}_*\mathrm{H}$; we call this quotient $\mathcal{E}_*$.

4.1 Dyer-Lashof operations in $\mathrm{H}_*(\mathrm{MU})$ and $\mathrm{H}_*\mathrm{H}$

We will need to be able to compute Dyer-Lashof operations in $\mathrm{H}_*(\mathrm{MU})$ and $\mathrm{H}_*\mathrm{H}$. We will find the description of this action in terms of Newton polynomials convenient for our purposes, so we review how this works. Our choice to describe the action in this way was heavily influenced by [2, Section 5].

We define the mod p Newton polynomials $N_n(t) = N_n(t_1, \dots, t_n) \in \mathbb{F}_p[t_1, \dots, t_n]$ by setting $N_1(t) = t_1$ and inductively letting

$$N_n(t) = t_1 N_{n-1}(t) - t_2 N_{n-2}(t) + \cdots + (-1)^{n-2} t_{n-1} N_1(t) + (-1)^{n-1} n t_n.$$

Then the following useful relation holds:

$$N_{pn}(t) = (N_n(t))^p \mod p.$$

We let $N_n(b) \in \mathrm{H}_*\mathrm{MU}$ be defined by setting $t_n = b_n$, and let $N_n(\xi) \in \mathrm{H}_*\mathrm{MU}$ be defined by setting $t_{p^k-1} = \xi_k$ and the other t_n to zero. Writing out the recurrence for $N_{p^k-1}(\xi)$ shows that $N_{p^k-1}(\xi) = -\overline{\xi}_k$ where $x \mapsto \overline{x}$ is the conjugation in the Hopf algebra $\mathrm{H}_*\mathrm{H}$.

Kochman [9] showed that the action of the Dyer-Lashof operations on $N_n(b)$ is described by the formula:

$$Q^r N_n(b) = (-1)^{r+n} \binom{r-1}{n-1} N_{n+r(p-1)}(b).$$

Since the orientation $\mathrm{MU} \to \mathrm{H}$ maps b_{p^k-1} to ξ_k and the other b_n to zero, it maps $N_n(b)$ to $N_n(\xi)$ and so we also have:

$$Q^r N_n(\xi) = (-1)^{r+n} \binom{r-1}{n-1} N_{n+r(p-1)}(\xi).$$

Using $N_{p^k-1}(\xi) = -\overline{\xi}_k$, we get:

$$Q^r \overline{\xi}_k = (-1)^{r+1} \binom{r-1}{p^k-2} N_{p^k-1+r(p-1)}(\xi).$$

Using the above formulae, we may deduce the following two propositions by direct calculation.

Proposition 4.1.1. *In the dual Steenrod algebra* $\mathrm{H}_*\mathrm{H}$, *the following identities hold:*

$$Q^{p^2}\overline{\xi}_1 = (\overline{\xi}_1^{p-1})^p Q^p \overline{\xi}_1$$
$$Q^{p^2+i}\overline{\xi}_1 = 0 \text{ for } i = 1, \ldots, p-2$$
$$Q^{p^2+p-1}\overline{\xi}_1 = -(Q^p(\overline{\xi}_1))^p$$
$$Q^{p^2-p+1}(\overline{\xi}_1^{p-1}) = -(\overline{\xi}_1)^{p^2}$$
$$Q^{p^2+pi}Q^p\overline{\xi}_1 = 0 \text{ for } i = 1, \ldots, p-1$$
$$Q^{2p}\overline{\xi}_1 = -\overline{\xi}_1^p Q^p(\overline{\xi}_1)$$

Proposition 4.1.2. *The following identities hold in* $\mathrm{H}_*(\mathrm{MU})$*:*

$$Q^{p^2}N_{p-1}(b) = \frac{1}{2}Q^{p^2-1}N_{2(p-1)}(b)$$
$$Q^{p^2+i}N_{p-1}(b) = 0 \text{ for } i = 1, \ldots, p-2$$
$$Q^{p^2+p-1}N_{p-1}(b) = -(Q^p(N_{(p-1)}(b)))^p$$
$$Q^{p^2-p+1}(N_{p-1}(b)^{p-1}) = -(N_{p-1}(b))^{(p-2)p}(N_{2(p-1)}(b))^p$$
$$Q^{p^2+pi}Q^pN_{p-1}(b) = 0 \text{ for } i = 1, \ldots, p-1$$
$$Q^{2p}N_{p-1}(b) = -\frac{1}{2}Q^{2p-1}(N_{2(p-1)}(b))$$

4.2 A relation among power operations

We will define the secondary operation of interest to us in terms of the following relation between primary power operations.

Proposition 4.2.1. *Let* R *be an* $\mathbb{E}_{2(p^2+2)}$-H*-algebra and* $x \in \pi_{2(p-1)}(R)$*. Define classes* a_i*,* $i = 0, \ldots, p-1$*;* b*;* c_i*,* $i = 1, \ldots, p$ *in* $\pi_*(R)$ *by the following formulae:*

$$a_0 = Q^{p^2}x - (x^{p-1})^pQ^px$$
$$a_i = Q^{p^2+i}x \text{ for } i = 1, \ldots, p-2$$
$$a_{p-1} = Q^{p^2+p-1}x + (Q^px)^p$$
$$b = Q^{p^2-p+1}(x^{p-1}) + x^{p^2}$$
$$c_i = Q^{p^2+pi}Q^px \text{ for } i = 1, \ldots, p-1$$
$$c_p = Q^{2p}x + (Q^px)x^p$$

Then the following identity holds:

$$\begin{aligned}0 =& Q^{p^3+p}a_0 + \sum_{i=1}^{p-2}(-1)^iQ^{p^3+p-i}a_i + Q^{p^3+1}a_{p-1} + \\ & b^pQ^{p^2}Q^px + \sum_{i=1}^{p-1}(Q^{p^2-p-i+1}(x^{p-1}))^pc_i + (x^{p-1})^{p^2}Q^{2p^2-p}c_p\end{aligned}$$

Proof. This is defined for $\mathbb{E}_{2(p^2+2)}$-H-algebras by Theorem 1.5.1 because the operation which takes the greatest n to be defined on $\mathbb{E}_n$-H-algebras is the Q^{p^3+p} in in $Q^{p^3+p}a_0$. Since $|a_0| = 2(p-1)(p^2+1)$, we conclude that this is defined and satisfies the usual properties whenever

$$n \geq 2(p^3+p) - 2(p-1)(p^2+1) + 2 = 2(p^2+2).$$

The desired identity reduces to the following identities, which may be deduced from the Adem relations, the instability relations, and the Cartan formula:

$$\begin{aligned}
&Q^{p^3+p}Q^{p^2}x = \sum_{i=1}^{p-1}(-1)^{i+1}Q^{p^3+p-i}Q^{p^2+i}x \\
&Q^{p^3+1}((Q^p x)^p) = 0 \\
&Q^{p^3+p}((x^{p-1})^p Q^p x) = \sum_{i=0}^{p}(Q^{p^2-p-i+1}(x^{p-1}))^p Q^{p^2+pi}Q^p x \\
&Q^{2p^2}Q^p x = Q^{2p^2-p}Q^{2p}x \\
&Q^{2p^2-p}(x^p Q^p x) = x^{p^2}Q^{p^2}Q^p x.
\end{aligned}$$

□

Let the symbols a_i, $i = 0, \dots, p-1$; b; c_j, $j = 1, \dots, p$ have the gradings of the the elements in Proposition 4.2.1, and let d have the grading of the relation there described. Then the relation above determines maps

$$Q : \mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x, a_0, \dots, a_{p-1}, b, c_0, \dots, c_{p-1}) \to \mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x)$$

and

$$R : \mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x, d) \to \mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x, a_0, \dots, a_{p-1}, b, c_0, \dots, c_{p-1})$$

such that the composition $Q \circ R$ is nullhomotopic.

Proposition 4.2.2. *The bracket* $\langle \bar{\xi}_1, Q, R \rangle$ *is defined in* $\mathrm{H}_*\mathrm{H}$ *and has zero indeterminacy in the quotient* $\mathcal{E}_* = \Lambda_{\mathbb{F}_p}(\tau_0, \tau_1, \dots)$ *of* $\mathrm{H}_*\mathrm{H}$.

Proof. To show that the bracket is defined, we need to show that $Q(\overline{\xi}_1) = 0$. This is equivalent to Proposition 4.1.1.

The indeterminacy comes from degree $2p^3 + 2p^2 + 2p + 1$ homotopy operations applied to $\overline{\xi}_1$ and from the image of the suspended operation σR. All homotopy operations are generated by multiplication, addition, the operations Q^n and βQ^n and the Browder bracket. Since H is $\mathbb{E}_\infty$, the Browder bracket always vanishes. The rest of these operations preserve the subalgebra of $\mathrm{H}_*\mathrm{H}$ generated by the $\overline{\xi}_i$ and therefore the first sort of indeterminacy is trivial in $\mathcal{E}_*$.

Up to indecomposables, σR is equal to $Q^{p^3+p}\sigma a_0 + \sum_{i=1}^{p-2}(-1)^i Q^{p^3+p-i}\sigma a_i + Q^{p^3+1}\sigma a_{p-1}$, where the σa_i are variables in degree one higher than a_i. So $|\sigma a_i| = (p^2+i+1)(p-1)+1$, $i = 0, \ldots, p-1$. Since $\mathcal{E}_*$ is decomposable in these degrees, we conclude as above that the second sort of indeterminacy must be decomposable in $\mathcal{E}_*$. Since there are no nonzero deomposables in $\mathcal{E}_*$ in degree $2p^4 - 1$, we conclude that the indeterminacy must actually be trivial in $\mathcal{E}_*$.

□

4.3 Computation of the secondary operation

To compute this operation, we will first juggle it into a functional operation for the map $\mathrm{H} \wedge \mathrm{MU} \to \mathrm{H} \wedge \mathrm{H}$. To this end, we define maps:

$$\mu : \mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x, a_0, \ldots, a_{p-1}, b, c_0, \ldots, c_{p-1}) \to \mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x, y_{2(p-1)})$$
$$\overline{Q} : \mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x, z_{p^3-1}) \to \mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x, y_{2(p-1)})$$
$$\nu : \mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x, d) \to \mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x, z_{p^3-1})$$
$$\alpha : \mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x, d) \to \mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x, w_1, \ldots, w_{p-2}, c_1, \ldots c_{p-1}, z_{p^2(p-1)}, z_{(2p+1)(p-1)})$$
$$\beta : \mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x, w_1, \ldots, w_{p-2}, c_1, \ldots, c_{p-1}, z_{p^2(p-1)}, z_{(2p+1)(p-1)}) \to \mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x, y_{2(p-1)})$$

with the y_i and the z_i in grading $2i$ and the w_i in grading $2(p-1)(p^2+i+2)$, by:

$$
\begin{aligned}
\mu(a_0) &= Q^{p^2-1}y_{2(p-1)} - (x^{p-1})^p Q^p x \\
\mu(a_i) &= 0 \text{ for } i \neq 0 \\
\mu(b) &= -2x^{(p-2)p}y^p_{2(p-1)} + x^{p^2} \\
\mu(c_i) &= 0 \text{ for } i \neq p \\
\mu(c_p) &= -Q^{2p-1}y_{2(p-1)} + (Q^p x)x^p \\
\overline{Q}(z_{p^3-1}) &= Q^{p^2+p-1}y_{2(p-1)} \\
\nu(d) &= -Q^{p^3}z_{p^3-1} \\
\alpha(d) &= \sum_{i=1}^{p-2}\sigma_i Q^{p^3+p-(i+1)}w_i - z^p_{p^2(p-1)}Q^{p^2}Q^p x \\
&\quad - \sum_{i=1}^{p-1}(Q^{p^2-p-i+1}(x^{p-1}))^p c_i - (x^{p-1})^{p^2}Q^{2p^2-p}z_{(2p+1)(p-1)} \\
\beta(w_i) &= Q^{p^2+i}y_{2(p-1)} \\
\beta(c_i) &= Q^{p^2+pi}Q^p x \\
\beta(z_{p^2(p-1)}) &= \frac{1}{2}x^{p(p-2)}y^p_{2(p-1)} + Q^{p^2-p+1}(x^{p-1}) \\
\beta(z_{(2p+1)(p-1)}) &= Q^{2p-1}y_{2(p-1)} + Q^{2p}x
\end{aligned}
$$

Here we choose the σ_i such that

$$
Q^{p^3+p}Q^{p^2-1}y_{2(p-1)} = \sum_{i=1}^{p-2}\sigma_i Q^{p^3+p-(i+1)}Q^{p^2+i}y_{2(p-1)} - Q^{p^3}Q^{p^2+p-1}y_{2(p-1)}.
$$

The existence of such a relation follows from the Adem and instability relations.

Proposition 4.3.1. *There is an identity* $\mu R = \overline{Q}\nu + \beta\alpha$ *and a homotopy commutative diagram*

$$\begin{array}{ccccc} \mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x,a_0,\ldots,a_{p-1},b,c_0,\ldots,c_{p-1}) & \xrightarrow{Q} & \mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x) & & \\ \Big\downarrow \mu & & \Big\downarrow -N_{p-1}(b) & \searrow \bar{\xi}_1 & \\ \mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x,y_{2(p-1)}) & \xrightarrow{f} & \mathrm{H}\wedge\mathrm{MU} & \xrightarrow{p} & \mathrm{H}\wedge\mathrm{H} \end{array}$$

where f is the map defined by sending x to $-N_{p-1}(b)$ and $y_{2(p-1)}$ to $-\frac{N_{2(p-1)}(b)}{2}$.

Proof. The proof of the identity $\mu R = \overline{Q}\nu + \beta\alpha$ follows directly from the relations

$$Q^{p^3+p}Q^{p^2-1}y_{2(p-1)} = \sum_{i=1}^{p-2}\sigma_i Q^{p^3+p-(i+1)}Q^{p^2+i}y_{2(p-1)} - Q^{p^3}Q^{p^2+p-1}y_{2(p-1)}$$

and

$$Q^{p^3+p}((x^{p-1})^pQ^px) = \sum_{i=0}^{p}(Q^{p^2-p-i+1}(x^{p-1}))^pQ^{p^2+pi}Q^px.$$

The right triangle of the diagram commutes because $\bar{\xi}_1 = -N_{p-1}(\xi)$ and hence $p(-N_{p-1}(b)) = \bar{\xi}_1$. The left square commutes by Proposition 4.1.2.

□

Proposition 4.3.2. *There is an equality $\langle\bar{\xi}_1, Q, R\rangle \equiv -Q^{p^3}(\langle p, f, \overline{Q}\rangle)$ in $\mathcal{E}_*$.*

Proof. Exactly as in [12], the juggling relations for brackets imply the following sequence of identities because each term is defined

$$\begin{aligned}
\langle\bar{\xi}_1, Q, R\rangle &= \langle pN_{p-1}(b), Q, R\rangle \\
&\subset \langle p, N_{p-1}(b)Q, R\rangle \\
&= \langle p, f\mu, R\rangle \\
&\supset \langle p, f, \mu R\rangle \\
&= \langle p, f, \overline{Q}\nu + \beta\alpha\rangle \\
&\subset \langle p, f, \overline{Q}\nu\rangle + \langle p, f, \beta\alpha\rangle \\
&\supset \langle p, f, \overline{Q}\rangle\nu + \langle p, f, \beta\rangle\alpha.
\end{aligned}$$

To show that we have equality up to decomposables in $\mathcal{E}_*$, it suffices to show that the indeterminacy of "local maxima" $\langle p, N_{p-1}(b)Q, R\rangle$ and $\langle p, f, \overline{Q}\nu\rangle + \langle p, f, \beta\alpha\rangle$ are decomposable in $\mathcal{E}_*$. The total indeterminacy of these two brackets is made up of elements of three kinds. The first are in the image of $\mathrm{H}_*\mathrm{MU} \to \mathrm{H}_*\mathrm{H}$, which maps to zero in $\mathcal{E}_*$. The second are in the image of σR, which we already dealt with in the proof of Proposition 4.2.2. Finally, there are elements in the images of $\sigma(\overline{Q}\nu)$ and $\sigma(\beta\alpha)$. These are either decomposable or multiples of Dyer-Lashof operations applied to a class in degree $2(p-1)+1$; there are no nonzero indecomposables in $\mathcal{E}_*$ in this degree.

Finally, we note that α applied to any set of classes in $\mathrm{H}_*\mathrm{H}$ is decomposable in $\mathcal{E}_*$ because $\mathcal{E}_*$ has no nonzero indecomposables in the degrees of w_i, $i = 1, \ldots, p-1$. Therefore the second term is zero modulo decomposables.

Since there are no nonzero decomposables in degree $2p^4 - 1$ of $\mathcal{E}_*$, we conclude that this holds on the nose in $\mathcal{E}_*$. □

Finally, we compute the bracket $\langle p, f, \overline{Q}\rangle$ by means of Theorem 3.0.3.

Proposition 4.3.3. *There is an equality $\langle p, f, \overline{Q}\rangle \equiv C\tau_3$ in $\mathcal{E}_*$ for some nonzero $C \in \mathbb{F}_p$.*

Proof. By noting that each pair of maps in the diagram

$$\mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x, z_{p^3-1}) \xrightarrow{\overline{Q}} \mathbb{P}_{\mathrm{H}}^{2(p^2+2)}(x, y_{2(p-1)}) \xrightarrow{f} \mathrm{H} \wedge \mathrm{MU} \xrightarrow{p} \mathrm{H} \wedge \mathrm{H} \xrightarrow{i} \mathrm{H} \wedge_{\mathrm{MU}} \mathrm{H}$$

compose to a nullhomotopic map in $\mathcal{C}$, we find that we are allowed to apply the Peterson-Stein formula to obtain the equality

$$i\langle p, f, \overline{Q}\rangle = -\langle i, p, f\rangle\overline{Q}.$$

By [12, Proposition 2.7.5], $\sigma(-\frac{N_{2(p-1)}(b)}{2}) \in \langle i, p, f\rangle$. Since

$$-\frac{N_{2(p-1)}(b)}{2} \equiv b_{2(p-1)} \equiv -\frac{\mathbb{CP}^{2(p-1)}}{2p-1} = \mathbb{CP}^{2(p-1)}$$

modulo decomposables, where we view $\mathbb{CP}^n$ as an element of homology via the Hurewicz map, we have $\sigma(-\frac{N_{2(p-1)}(b)}{2}) = \sigma\mathbb{CP}^{2(p-1)}$. By Theorem 3.0.3, $\overline{Q}$ applied to this is $-Q^{p^2+p-1}\sigma\mathbb{CP}^{2(p-1)} = -\sigma v_3$. Since i is an isomorphism modulo decomposables in this degree, we conclude that $C\tau_3 \equiv \langle p, f, \overline{Q}\rangle$ modulo decomposables for some nonzero $C \in \mathbb{F}_p$, as desired.

As before, we upgrade this from a result modulo decomposables in $\mathcal{E}_*$ to a precise result in $\mathcal{E}_*$ by noting that there are no nonzero decomposables in degree $2p^3 - 1$ of $\mathcal{E}_*$. □

Corollary 4.3.4. *There exists a nonzero $C \in \mathbb{F}_p$ and an equality $\langle\overline{\xi}_1, Q, R\rangle \equiv C\tau_4$ in $\mathcal{E}_*$.*

Proof. Combine Propositions 4.3.2 and 4.3.3. □

Since maps of $\mathbb{E}_{2(p^2+2)}$-ring spectra must preserve secondary power operations by [12, Proposition 2.1.10], we immediately obtain the following corollary.

Corollary 4.3.5. *Let R be an $\mathbb{E}_{2(p^2+2)}$-ring spectrum and let $R \to \mathrm{H}$ be a map of $\mathbb{E}_{2(p^2+2)}$-ring spectra. Then if the induced map on homology $\mathrm{H}_*R \to \mathrm{H}_*\mathrm{H}$ is injective in degrees less than or equal to $(2p^2+1)(p-1)$ and contains $\overline{\xi}_1$ in its image, then τ_4 must also be in the image of the composite $\mathrm{H}_*R \to \mathrm{H}_*\mathrm{H} \to \mathcal{E}_*$.*

We conclude by deducing Theorem 1.1.1 from Corollary 4.3.5.

Proof of Theorem 1.1.1. Assume that BP were an $\mathbb{E}_{2(p^2+2)}$-ring spectrum. Since the Postnikov tower of an $\mathbb{E}_n$-ring spectrum naturally lifts to a tower of $\mathbb{E}_n$-ring spectra, there is a map of $\mathbb{E}_{2(p^2+2)}$-ring spectra

$$\mathrm{BP} \to \tau_{\leq 0}\mathrm{BP} \cong \mathrm{H}\mathbb{Z}_{(p)} \to \mathrm{H}$$

which induces the inclusion

$$\mathbb{F}_p[\xi_1, \xi_2, \dots] \hookrightarrow \Lambda_{\mathbb{F}_p}(\tau_0, \tau_1, \dots) \otimes \mathbb{F}_p[\xi_1, \xi_2, \dots]$$

upon taking homology. In particular, the map is injective and contains $\overline{\xi}_1$ in its image. However, τ_4 cannot be in the image of $\mathrm{H}_*\mathrm{BP} \to \mathrm{H}_*\mathrm{H} \to \mathcal{E}_*$ because this composite is zero.

The case of $\mathrm{BP}\langle n\rangle$ for $n \geq 4$ is analogous, using the fact that

$$\mathrm{H}_*(\mathrm{BP}\langle n\rangle) \cong \Lambda_{\mathbb{F}_p}(\tau_{n+1}, \tau_{n+2}, \dots) \otimes \mathbb{F}_p[\xi_1, \xi_2, \dots].$$

Finally, taking p-completions makes no difference because we are only working with mod p homology in the first place. □

Bibliography

[1] Vigleik Angeltveit and John A. Lind. Uniqueness of $BP\langle n\rangle$. *J. Homotopy Relat. Struct.*, 12(1):17–30, 2017.

[2] Andrew Baker. Power operations and coactions in highly commutative homology theories. *Publ. Res. Inst. Math. Sci.*, 51(2):237–272, 2015.

[3] Maria Basterra and Michael A. Mandell. The multiplication on BP. *J. Topol.*, 6(2):285–310, 2013.

[4] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger. *H_∞ ring spectra and their applications*, volume 1176 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[5] Steven Greg Chadwick and Michael A. Mandell. E_n genera. *Geom. Topol.*, 19(6):3193–3232, 2015.

[6] Frederick R. Cohen, Thomas J. Lada, and J. Peter May. *The homology of iterated loop spaces*. Lecture Notes in Mathematics, Vol. 533. Springer-Verlag, Berlin-New York, 1976.

[7] Michael Hill and Tyler Lawson. Automorphic forms and cohomology theories on Shimura curves of small discriminant. *Adv. Math.*, 225(2):1013–1045, 2010.

[8] Niles Johnson and Justin Noel. For complex orientations preserving power operations, p-typicality is atypical. *Topology Appl.*, 157(14):2271–2288, 2010.

[9] Stanley O. Kochman. Homology of the classical groups over the Dyer-Lashof algebra. *Trans. Amer. Math. Soc.*, 185:83–136, 1973.

[10] I. Kriz. Towers of E_∞ ring spectra with an application to BP. *Unpublished Preprint*, 1995.

[11] Anssi Lahtinen. Homology operations revisited. 2020. arXiv:2001.02781.

[12] Tyler Lawson. Secondary power operations and the Brown-Peterson spectrum at the prime 2. *Ann. of Math. (2)*, 188(2):513–576, 2018.

[13] Tyler Lawson and Niko Naumann. Commutativity conditions for truncated Brown-Peterson spectra of height 2. *J. Topol.*, 5(1):137–168, 2012.

[14] Jacob Lurie. *Higher algebra*.
Available at: http://www.math.harvard.edu/ lurie/papers/HA.pdf. Theorem numbers referenced follow the Sep. 18, 2017 edition.

[15] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[16] J. P. May. Problems in infinite loop space theory. In *Conference on homotopy theory (Evanston, Ill., 1974)*, volume 1 of *Notas Mat. Simpos.*, pages 111–125. Soc. Mat. Mexicana, México, 1975.

[17] J. Peter May. A general algebraic approach to Steenrod operations. In *The Steenrod Algebra and its Applications (Proc. Conf. to Celebrate N. E. Steenrod's Sixtieth Birthday, Battelle Memorial Inst., Columbus, Ohio, 1970)*, Lecture Notes in Mathematics, Vol. 168, pages 153–231. Springer, Berlin, 1970.

[18] Haynes R. Miller, Douglas C. Ravenel, and W. Stephen Wilson. Periodic phenomena in the Adams-Novikov spectral sequence. *Ann. of Math. (2)*, 106(3):469–516, 1977.

[19] Dmitri Pavlov and Jakob Scholbach. Admissibility and rectification of colored symmetric operads. *J. Topol.*, 11(3):559–601, 2018.

[20] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1986.