Gröbner Bases in Rational Homotopy Theory

by

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Abstract

The Mayer-Vietoris sequence in cohomology has an obvious Eckmann-Hilton dual that characterizes the homotopy of a pullback, but the Eilenberg-Moore spectral sequence has no dual that characterizes the homotopy of a pushout. The main obstacle is the lack of an Eckmann-Hilton dual to the Künneth theorem with which to understand the homotopy of a coproduct.

This difficulty disappears when working rationally, and we dualize Rector's construction of the Eilenberg-Moore spectral sequence to produce a spectral sequence converging to the homotopy of a pushout. We use Gröbner-Shirshov bases, an analogue of Gröbner bases for free Lie algebras, to compute directly the E^2 term for pushouts of wedges of spheres. In particular, for a cofiber sequence $A \to X \to C$ where A and X are wedges of spheres, we use this calculations to generalize a result of Anick by giving necessary and sufficient conditions for the map $X \to C$ to be surjective in rational homotopy. More importantly, we are able to avoid the use of differential graded algebra and minimal models, and instead approach simple but open problems in rational homotopy theory using a simplicial perspective and the combinatorial properties of Gröbner-Shirshov bases.

Thesis Supervisor: Haynes R. Miller
Title: Professor of Mathematics

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Contents

Chapter 1

Introduction

While the Mayer-Vietoris sequence in cohomology has an Eckmann-Hilton dual in the homotopical Mayer-Vietoris sequence [EH64, DR80], the Eilenberg-Moore spectral sequence, which characterizes the cohomology of a pullback, appears to have no dual describing the homotopy of a pushout. Creating a usable "homotopical EMSS" is difficult because there is no Eckmann-Hilton dual to the Künneth theorem with which to compute the homotopy of a wedge—we have only the Hilton-Milnor theorem [Hil55] and the Stover spectral sequence [Sto90]. It is hard then, for example, to relate the homotopy of a space X, the homotopy class $[f]$ of a map $f \in \pi_n X$, and the homotopy of $X \cup_f e^{n+1}$ to each other.

Ignoring those difficulties temporarily, our starting point is a homotopy spectral sequence that given a cellular simplicial space or Reedy cofibrant simplicial space D, converges to $\pi_*|D|$ with $E^1 = \pi_* D$ and $E^2 = \pi_*\pi_* D$. In order to be able to compute the E^2 term directly, we need the homotopy of the levelwise spaces and face maps of D to be tractable. Furthermore, we want D to be non-constant and have interesting geometric realization.

Given a pushout square

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow \\ Y & \dashrightarrow & C \end{array} \tag{1.1}$$

we construct a simplicial space $\mathcal{D}$ with geometric realization homeomorphic to

$$|\mathcal{D}| \cong X \cup_{A\times\{0\}} (A \times I) \cup_{A\times\{1\}} Y$$

by taking the Eckmann-Hilton dual of Rector's geometric cobar construction [Rec70]. Applying the above spectral sequence, we get a homotopical EMSS that converges to the homotopy of a pushout.

Levelwise, $\mathcal{D}$ is a coproduct of copies of A, X, and Y, and the face maps are that are sums of f, g, and fold maps, so computing $E^2 = \pi_*\pi_*\mathcal{D}$ is hard. In the integral case, the Hilton-Milnor theorem helps us write down (though not explicitly) E^1 when A, X, and Y are all wedges of spheres, but computing E^2 directly is not possible as the Hilton-Milnor decomposition is not functorial. Furthermore, we do not understand in homotopy any non-trivial fold map. The only solution is to work rationally, and the homotopy of fold maps and coproducts becomes simple in this setting. The computation of $\pi_*\pi_*\mathcal{D}$ becomes a problem in Lie algebras, and one key property of this construction is that the resulting simplicial space is smaller than the one coming from the more general homotopy colimit construction.

There is already a very effective perspective on rational homotopy theory. Quillen showed in [Qui69] that rationalization discards enough information that there is an equivalence between, among other things, the rational homotopy category of simply-connected spaces, and the homotopy category of 0-connected differential graded Lie algebras over $\mathbb{Q}$. The study of rational homotopy theory has since been heavily based on the theory of differential-graded algebra and minimal models [BG76, BL77]. However, a simple problem in rational homotopy theory with a simple equivalent in the language of minimal models is often still difficult to answer. In fact, despite the power of Quillen's "differential-graded" algebraization, many simple problems in rational homotopy theory remain open [FHT01, §39].

So the EMSS is a way to avoid differential-graded algebra and minimal models, and is instead a simplicial perspective on simple but difficult open problems in rational homotopy theory. One question to consider is:

1. Let $f : A \to X$ be a map between simply-connected wedges of spheres, and C a two-cone be its cofiber. When is f an inert map, that is, the map $X \to C$ a surjection in homotopy?

For example, $[\iota_1, \iota_2] : S^3 \to S^2 \vee S^2$ has cofiber $S^2 \times S^2$ and is inert, while $[\iota, \iota] : S^3 \to S^2$ has cofiber $\mathbb{C}P^2$ and is not. And while there are many characterizations of inert cell attachments, the only explicit result—that is, given explicit attaching maps, is the attachment inert?—is a partial one due to Anick in [Ani82, Theorem 3.2]. We answer this question completely for two-cones in Corollary 4.42. We also extend Anick's result to pushouts of wedges of spheres in Theorem 4.21.

By Theorem 4.33, there is an explicit dgL model for C with n generators, where n is the number of spheres in $X \vee A$. And yet, perhaps surprisingly, the question is non-trivial. Indeed, understanding the homology of a dgL beyond doing brute force calculation is hard, even when the underlying graded Lie algebra is free. While the naïve algorithm for computing homology of dgLs is very simple—a dgL is just a chain complex—the behavior of the differential can be hard to analyze. We usually have the value of the differential on generators, and have to extend the differential inductively, and the derivation property implies that the differential on a monomial can have many terms. Lie algebras are also in general hard to work with, because the Jacobi identity implies that no basis is closed under the Lie bracket.

The intractability of dgLs is reflected in the large number of open questions in rational homotopy theory that are easy to state. For example, for every $N > 0$ and finite simply-connected CW complex X with $\dim X < N$, is there a finite rationally elliptic CW complex Y such that X is the N-skeleton of Y? Or, if X is a rationally hyperbolic finite simply connected CW complex, does L_X contain a free Lie algebra on two generators?

We can use Quillen's correspondence between dgLs and simplicial Lie algebras, because Lie algebra face maps might be easier to deal with, and calculate the homotopy of simplicial Lie algebras and not the homology of dgLs. However, directly computing the homotopy of these simplicial Lie algebras appears to be just as hard. One reason is that the Dold-Kan construction creates extremely large simplicial vector spaces, since given an element of degree d, there will be a copy of d in simplicial dimension n for every surjective map $[n] \to [d]$.

The dgL models of two-cones are bigraded, and we establish a graded version of Quillen's correspondence between differential *bigraded* Lie algebras and simplicial *graded* Lie algebras. This graded setup places generators in degree 0 and 1, with an additional internal grading. Since there are only n surjective maps $[n] \to [1]$, the Dold-Kan correspondence produces a smaller and more manageable simplicial vector space, making the use of Gröbner-Shirshov bases easier. In fact, that $\mathcal{D}$ is smaller than the simplicial construction of hocolim is extremely helpful.

All we do, then, is analyze the simplicial graded Lie algebra (E^1, d_1)—which is levelwise free—using Gröbner-Shirshov methods. For A and X wedges of spheres and Y a point, we show in Theorem 4.34 that the simplicial graded Lie algebra corresponding to the dgL model of C is exactly the E^1 term of the EMSS, and so the spectral sequence collapses at E^2. This result is unfortunately limited to two-cones, because the dgL model of a three-cone may not have a non-trivial bigrading.

In the case of two-cones, the failure of a cell attachment to be inert is measured by the failure of the kernels of the face maps $d_* : \mathcal{D}_n \to \mathcal{D}_{n-1}$ to be orthogonal, as described in Theorem 4.15. By Corollary 4.35, this is the case if and only if cell attachment is inert. Intuitively, we show that these kernels are orthogonal if and only if the attaching maps are "sufficiently unrelated," while Anick's result states that cell attachment is inert if attaching maps are "completely unrelated." In Example 4.26, we show that for x, y, z the fundamental classes of three spheres S^a, S^b, S^c of odd dimension, the attaching maps

$$\{[x,[x,y]] + [x,[x,z]], [[x,y],y] + [[x,z],z]\}$$

to the wedge $S^a \vee S^b \vee S^c$ is an inert cell attachment, despite the attaching maps not meeting Anick's criteria of being combinatorially free.

There is a rational dual to the Serre spectral sequence for a cofiber sequence $A \longrightarrow X \longrightarrow C$, with $E^2 = \pi_* A \coprod \pi_* C$ converging to $\pi_* X$. Since we are interested in the homotopy of a two-cone, we use the Barratt-Puppe construction to get the cofiber sequence $X \longrightarrow C \longrightarrow \Sigma A$ where A and X are wedges of spheres and $C = X \cup_f A$ a two-cone. Computing the E^∞ term of this spectral sequence mounts to computing the homology of the dgL model for $X \cup_f A$, which we are deliberately avoiding.

We have seen that the two-cone C has a simple dgL model. But the colimit of the pushout diagram in Equation 1.1 may not, even when A, X, and Y are wedges of spheres, because cofibrant replacement in the category of dgLs is unwieldy. This makes using models ineffective for following question:

2. When is the map $\pi_* X \coprod \pi_* Y \longrightarrow \pi_* C$ surjective?

We give a sufficient condition in Theorem 4.21.

As simple as rational homotopy can be, we have to be careful not to set our sights too high. The combinatorics of models is complex enough to have the following negative complexity result, though it is not known whether the corresponding decision problem is NP-hard:

Theorem ([Ani89, Theorem 5.6]). *Let $\mathcal{C}$ be the set of finite two-cones with cells in dimensions 2 and 4. Then the counting problem $r : \mathbb{N} \times \mathcal{C} \longrightarrow \mathbb{N}$, defined by $(n, X) \mapsto \dim \pi_n(X; \mathbb{Q})$, is $\sharp$P-hard.*

This limitation is reflected in the fact that our results are somehow bounded by the computational complexity of Gröbner-Shirshov bases, as opposed to overcoming them. We see that any attempt to describe ranks of homotopy groups in terms of the dgL model is as hard as figuring out, for example, how many sublists of a list of integers sum to zero. If Anick's negative result can be extended to hold for inert two-cones, then we can conclude that counting the number of elements in a Gröbner-Shirshov basis is again $\sharp$P-hard.

Chapter 2

The Eilenberg-Moore spectral sequence

2.1 The spectral sequence

We can attempt to understand the homotopy of the geometric realization of a simplicial space D using the spectral sequences of Theorem 2.3 and Theorem 2.5, though they are hard to use in the actual computation of homotopy groups. The difficulty lies in finding a simplicial space that has a geometric realization of interest, that satisfies certain cofibrance conditions, and has a tractable E^2 term. The last condition requires that the levelwise spaces of D and the face maps between them be tractable in homotopy. Finally, the higher differentials of these spectral sequences are not understood at all.

Let $\mathbf{T}_*$ be the category of pointed spaces, $s\mathcal{C}$ be the category of simplicial objects over a category $\mathcal{C}$, and $\mathrm{sk}_{n-1}X_n$ and $\mathrm{ck}_{n-1}X_n$ be the skeleton and coskeleton construction on $X \in s\mathcal{C}$.

Definition 2.1. A **cellular simplicial space** is a simplicial space with levelwise CW-complexes and each degeneracy map is an inclusion of a subcomplex.

Definition 2.2 ([DKS93, 3.3]). A map in $s\mathbf{T}_*$ will be called an E^2 cofibration if it is a retract of an S^1-free map, where a map $X \longrightarrow Y \in s\mathbf{T}_*$ is called S^1-free if, for every $n \geq 0$, there exist

1. A CW-complex $Z_n \in \mathbf{T}_*$ which has the homotopy type of a wedge of spheres S^i ($i \geq 1$), and

2. A map $Z_n \to Y_n \in \mathbf{T}_*$ such that the induced map

$$(X_n \coprod_{\mathrm{sk}_{n-1}X_n} \mathrm{sk}_{n-1}Y_n) \coprod Z_n \to Y_n \in \mathbf{T}_*$$

is a trivial cofibration.

Theorem 2.3 ([Sto90, Theorem 3.4]). *There is a first quadrant spectral sequence, functorial in cellular simplicial spaces $Y_\bullet$, converging strongly to the homotopy groups of the realization $\pi_*\Delta Y_\bullet$, and such that $E^2_{p,q} = \pi_p\pi_q Y_\bullet$.*

By taking the E^2 cofibrant replacement of a constant simplicial pointed connected CW-complex X, Stover constructs a cellular simplicial space $V_\bullet X$ whose geometric realization is homotopy equivalent to X. Levelwise, $V_\bullet X$ is a wedge of spheres and depends only on $\pi_* X$, so $V_\bullet X$ can be thought of as a simplicial resolution of X by spheres. Since the geometric realization of $V_\bullet X \vee V_\bullet Y$ is homotopy equivalent to $X \vee Y$, there is a spectral sequence [Sto90, 1.1] converging to $\pi_*(X \vee Y)$, but whose E^2 term depends only on the groups $\pi_* X$ and $\pi_* Y$. However, it is unclear how to extend this idea to construct the simplicial resolution by spheres of C using f and f.

Definition 2.4 ([Ree, Theorem A]). A map $X \to Y$ in $s\mathcal{C}$ is said to be a **Reedy cofibration** (with respect to a chosen model structure on $\mathcal{C}$) if for all n, the map

$$X_n \coprod_{\mathrm{sk}_{n-1}X_n} \mathrm{sk}_{n-1}Y_n \to Y_n$$

is a cofibration in $\mathcal{C}$. The Reedy cofibrant replacement of a constant space has levelwise spaces that are somewhat unwieldy. The factorization of a map $X \to Y$ into a Reedy cofibration $X \to Z$ followed by a trivial Reedy fibration $Z \to Y$ is defined inductively: Z_n is a space that factors

$$\begin{array}{ccc} X_n \vee_{\mathrm{sk}_{n-1}X_n} \mathrm{sk}_{n-1}Z_n & \rightarrowtail & Z_n \\ & \searrow & \downarrow\!\!\!\downarrow \cong \\ & & \mathrm{ck}_{n-1}Z_n \times_{\mathrm{ck}_{n-1}Y_n} Y_n. \end{array}$$

In $s\mathbf{T}_*$, E^2 cofibrant objects are Reedy cofibrant.

Theorem 2.5 ([DKS95, Proposition 8.3]). *For Reedy cofibrant $X \in s\mathbf{T}_*$, there is a first quadrant spectral sequence with $E^2_{i,j} = \pi_i\pi_j X$, converging strongly to $\pi_*|X^b|$.*

Rector showed that the EMSS is a spectral sequence of modules over the Steenrod algebra, but the dual approach gives a homotopy spectral sequence with d_2, a map of Π-algebras, being the boundary map in the spiral exact sequence. Of course, rationally, Π-algebras are merely Lie algebras.

2.2 The bar construction

The bar construction on a pushout (Equation 1.1) yields a simplicial space that is levelwise a wedge of A, X, and Y, making the E^1 and d^1 easy to write down rationally and the calculation of the E^2 term tractable. This analysis is the basis of Chapter 4.

Definition 2.6. Given a pushout square in Equation 1.1, we define the pointed simplicial space $\mathcal{D}$ to be

$$\mathcal{D}(X, A, Y)_n = X \vee A_1 \vee A_2 \vee \cdots \vee A_n \vee Y \tag{2.1}$$

where A_i are copies of A. The face maps $d_0 : \mathcal{D}_n \to \mathcal{D}_{n-1}$ are:

$$\begin{array}{ccc} X \vee A_1 & & A_2 \vee \cdots \vee A_n \vee Y \\ {\scriptstyle (1,f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ X & & A_1 \vee \cdots \vee A_{n-1} \vee Y \end{array} \tag{2.2}$$

The face maps $d_i : \mathcal{D}_n \to \mathcal{D}_{n-1}$ for $0 < i < n$ are:

$$\begin{array}{ccccc} X \vee A_1 \vee \cdots \vee A_{i-1} & & A_i \vee A_{i+1} & & A_{i+2} \vee \cdots \vee A_n \vee Y \\ {\scriptstyle \mathrm{id}}\big\downarrow & & {\scriptstyle \nabla}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow \\ X \vee A_1 \vee \cdots \vee A_{i-1} & & A_i & & A_{i+i} \vee \cdots \vee A_{n-1} \vee Y \end{array} \tag{2.3}$$

The face maps $d_n : \mathcal{D}_n \to \mathcal{D}_{n-1}$ are:

$$\begin{array}{ccc} X \vee A_1 \vee \cdots \vee A_{n-1} & & A_n \vee Y \\ {\scriptstyle \mathrm{id}}\big\downarrow & & {\scriptstyle (g,1)}\big\downarrow \\ X \vee A_1 \vee \cdots \vee A_{n-1} & & Y \end{array} \tag{2.4}$$

The degeneracy maps $s_i : \mathcal{D}_n \to \mathcal{D}_{n+1}$ for $0 \leq i \leq n$ are:

$$\begin{array}{ccccc} X \vee A_1 \vee \cdots \vee A_i & & & & A_{i+1} \vee \cdots \vee A_n \vee Y \\ {\scriptstyle \mathrm{id}}\big\downarrow & & & & {\scriptstyle \mathrm{id}}\big\downarrow \\ X \vee A_1 \vee \cdots \vee A_i & & A_{i+1} & & A_{i+2} \vee \cdots \vee A_{n+1} \vee Y \end{array}$$

Since $\mathcal{D}$ is a 1-skeleton, its geometric realization is homeomorphic to the geometric realization of its 1-truncation $\mathcal{D}^{(1)}$. Notice that none of these properties depends on f or g. To see that we get $X \cup_{A\times\{0\}} (A \times I) \cup_{A\times\{1\}} Y$. observe that s_0 collapses $(X \vee A) \times I$ to $X \vee (A \times I)$, d_0 identifies $A \times \{0\}$ to Y via g and d_1 identifies $A \times \{1\}$ to X via f. Suppose A, X, and Y are wedges of spheres. Then $\mathcal{D}$ is clearly cellular. Furthermore, $\mathcal{D}^{(1)}$ is E^2 and Reedy cofibrant, and therefore so is $\mathcal{D}$.

The bar construction $\mathcal{D}(X, *, Y)$ is the constant simplicial space with $X \vee Y$ levelwise, and using $\hat{E}^n$ to denote the terms of the associated EMSS, we get $\hat{E}^2_{0,*} = \pi_*(X \vee Y)$ and $\hat{E}^2_{n,*} = 0$ for $n > 0$. Then the map of pushout diagrams

$$\begin{array}{ccccc} Y & \longleftarrow & * & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ Y & \xleftarrow{g} & A & \xrightarrow{f} & X \end{array}$$

gives a map $\hat{E}^\infty \to E^\infty$ coinciding with $\pi_*(X \vee Y) \to \pi_* C$. So $E^\infty_{0,*}$ is the image of $\pi_*(X \vee Y)$ in $\pi_* C$.

There are several ways to view the bar construction of Definition 2.6, some of which are specialized to the case of a cofiber sequence, i.e. $Y = *$. Checking that the simplicial spaces constructed below are all isomorphic is not difficult.

2.2.1 Geometric cobar construction

Let $\mathbf{T}/B$ be the category of spaces over B, whose objects are morphisms from spaces to B and whose morphisms are commuting triangles. But since $X \times_B Y$ is a product in this category,

$$\begin{array}{ccc} X \times_B Y & \dashrightarrow & X \\ \vdots & & \downarrow f \\ Y & \xrightarrow[g]{} & B \end{array} \tag{2.5}$$

the geometric Eilenberg-Moore spectral sequence can be thought of as a Künneth theorem in $\mathbf{T}/B$ [Smi69]. By applying Eckmann-Hilton duality, we reverse the solid arrows in (2.5) and get a pushout square:

$$\begin{array}{ccc} X \cup_A Y & \dashleftarrow & X \\ \vdots & & \uparrow f \\ Y & \xleftarrow[g]{} & A \end{array} \tag{2.6}$$

so a "co-Künneth" theorem in $(A/\mathbf{T})^{op}$ could conceivably converge to the homotopy of the pushout. Because arrows have been reversed, we have an approach that can be thought of as the Eckmann-Hilton dual of Rector's geometric cobar construction: the treatment of Equation 2.5 using cosimplicial techniques in [Rec70] may be translated to a treatment of Equation 2.6 using simplicial techniques, injective resolutions become projective resolutions, and so on. (The other dual to Rector's geometric cobar construction is the geometric bar construction on H-spaces. The latter construction will have an Eckmann-Hilton dual on co-H-spaces.)

The geometric cobar construction on Equation 2.5 is a cosimplicial space $C(X, B, Y)$ whose totalization is homeomorphic to the pullback $X \times_B Y$. The Eckmann-Hilton dual of this construction on Equation 2.6 is the simplicial space $\mathcal{D}(X, A, Y)$ whose geometric realization is homeomorphic to $X \cup_A Y$. Recalling that the diagonal map $\triangle : A \longrightarrow A \times A$ is dual to the fold map $\nabla : A \vee A \longrightarrow A$, we get the face and degeneracy maps of $D(X, A, Y)$ by dualizing coface and codegeneracy maps of $C(X, B, Y)$.

2.2.2 Dold-Kan correspondence

The Dold-Kan correspondence is a way to construct a simplicial abelian group associated to a chain complex. But suppose we have a "chain complex" $\mathcal{X}$ of pointed topological spaces

$$\cdots \longrightarrow * \longrightarrow X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{d_1} X_0$$

where $d_j \circ d_{j-1}$ is not just nullhomotopic but trivial. Then we get a simplicial pointed topological space by following the Dold-Kan construction.

Let Δ be the simplicial category whose objects are totally ordered sets $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$, and whose morphisms are order-preserving maps. Then we construct a pointed simplicial space D associated to $\mathcal{X}$ where

$$D_i = \coprod_{0 \leq j \leq n} \coprod_{\eta} (X_j)_\eta$$

with η ranging over surjective maps $[i] \longrightarrow [j]$. To specify the map $D_i \longrightarrow D_{i'}$ corresponding to a map $\alpha : [i'] \longrightarrow [i]$, we consider its action on $(X_j)_\eta$, $\eta : [i] \longrightarrow [j]$. Let the unique epi-monic factorization of $\eta \circ \alpha$ be

$$\begin{array}{ccc} [i'] & \xrightarrow{\alpha} & [i] \\ \downarrow{\eta'} & & \downarrow{\eta} \\ [j'] & \xrightarrow{\alpha'} & [j]. \end{array}$$

If $j' = j$ then α takes $(X_j)_\eta$ to $(X_j)_{\eta'}$ identically. If $j' = j - 1$ and α' is the inclusion $\{0, \ldots, j-1\} \hookrightarrow \{0, \ldots, j\}$, then α takes $(X_j)_\eta$ to $(X_{j-1})_{\eta'}$ via d_j. Otherwise, α takes $(X_j)_\eta$ to the basepoint in $D_{i'}$.

To understand the geometric realization of the resulting simplicial space D, let $M(f)$ be the mapping cone of f, and let $\tilde{d}_{n-1}$ be the map $M(d_n) \to X_{n-2}$ and inductively define

$$\tilde{d}_i : M(\tilde{d}_{i+1}) \to X_{i-1}, \quad 0 < i < n-1.$$

Then $|D|$ is homeomorphic to the mapping cone $M(\tilde{d}_1)$ of the map $\tilde{d}_1 : M(\tilde{d}_2) \to X_0$. If we shift the chain complex up in dimension by one, the resulting simplicial space will have geometric realization homeomorphic to $\Sigma|D|$.

Notice that any map $f : A \to X$ vacuously lies in the complex

$$\cdots \longrightarrow * \longrightarrow A \xrightarrow{f} X$$

with A in dimension 1 and X in dimension 0, and the associated simplicial pointed topological space is isomorphic to $\mathcal{D}$ in Definition 2.6 with $Y = *$. Its geometric realization is then $M(f)$.

2.2.3 Coproducts in $s\mathbf{T}_*$

Let $CA = (\Delta[1] \otimes A)/(\Delta^0[1] \otimes A) \in s\mathbf{T}_*$ as in [DKS95, 1.5(ii), 3.6(i)]. Then $|CA|$ is homeomorphic to the cone on A, and there is an E^2 Reedy cofibration $A \to CA$ where A is regarded as a constant simplicial space. Since $\cdot \leftarrow \cdot \rightarrow \cdot$ is a Reedy category, and if f is a cofibration, the diagram

$$CA \longleftarrow A \longrightarrow X$$

is Reedy cofibrant and we can take the colimit levelwise, giving exactly $\mathcal{D}(X, A, *)$.

2.3 The E^2 term

When $g : A \to Y$ is of the form $i : A \to A \vee W$, the pushout becomes quite simple:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow \\ A \vee W & \longrightarrow & X \vee W \end{array}$$

and as Lie algebras under $\pi_* A$, the map

$$h : \pi_* X \coprod_{\pi_* A} \pi_* Y \to \pi_*(X \cup_A Y)$$

is an isomorphism. We want to use these objects $i : A \to A \vee W$ as models to resolve an arbitrary map $g : A \to Y$. This resolution will the analogue of a free resolution, and as with free resolutions in other categories, it will be a measure of how close h is to an isomorphism when $g : A \to Y$ is not a model. These models are free if we consider the adjoint pair

$$A/\mathbf{T} \underset{\mathrm{F}}{\overset{\mathrm{G}}{\rightleftarrows}} \mathbf{T} \qquad [\mathrm{F}(X), A \to Y]_{A/\mathbf{T}} \equiv [X, \mathrm{G}(A \to Y)]_{\mathbf{T}}$$

with the forgetful functor G taking $f : A \to X$ to X and the free functor F taking X to $i : A \to A \vee X$. The composite $\mathrm{F} \circ \mathrm{G}$ yields a cotriple and therefore a simplicial resolution of an object in $A/\mathbf{T}$. Applying this construction to $g : A \to Y$, we get a free resolution and a simplicial space under A whose geometric realization is Y. This is the one-sided bar construction. We can then take the coproduct levelwise with $f : A \to X$ to get $\mathcal{D}$. So we get

$$E^2_{p,q} = \pi_p \pi_q \mathcal{D} = L_p(- \coprod_{\pi_q A} \pi_q Y)(\pi_q X) \Longrightarrow \pi_{p+q}(X \cup_A Y),$$

with the map $\pi_*(X \vee Y) \to \pi_* C$ factoring through h.

Chapter 3

Bases of free Lie algebras

No linear basis of free Lie algebras is as simple as the monomial basis of polynomial rings; after all, the Jacobi identity implies that no basis is closed under the Lie bracket, and unlike multiplication in polynomial rings, the Lie bracket of two non-zero elements can be zero. However, the Lyndon-Shirshov linear basis [CFL58, Šir58] has a rich combinatorial structure that makes the combinatorics of the Lie bracket tractable enough to define the Gröbner-Shirshov basis of a Lie ideal [Šir62], the Lie analogue of the Gröbner basis of a polynomial ideal.

We first describe the construction of the more general Hall-Viennot basis in §3.1, and then detail the combinatorial structure particular to the Lyndon-Shirshov basis in §3.2, and use it to construct Gröbner-Shirshov bases in §3.3.

3.1 Hall-Viennot bases

Hall-Viennot bases generalize constructions due to M. Hall [Hal50] and A. Shirshov [Šir58], and are sometimes called Hall bases or Hall-Shirshov bases.

Let X a set of ungraded indeterminates and $\Gamma(X)$ be the free magma on X, writing (u, v) for the non-associative product of u and v. Magma elements correspond in an obvious way to rooted full planar binary **trees** whose leaves are labeled by indeterminates, and so we will refer to them as such. If a tree t is not a leaf, then $t = (u, v)$ for unique left and right subtrees u and v.

Definition 3.1. A **pre-Hall-Viennot set** is a subset $H \subset \Gamma(X)$ with a total order $\prec_H$ satisfying:

1. $X \subset H$,

2. A tree $h = (u, v) \in \Gamma(X)$ is in H if and only if the following hold:

(a) $u, v \in H$,

(b) $u \succ_H v$,

(c) If $u = (u', u'')$ then $u'' = v$ or $u'' \prec_H v$.

Let $\mathrm{L}(X)$ be the free Lie algebra on graded indeterminates X and $\varphi : \Gamma(X) \to \mathrm{L}(X)$ be the natural map taking (u,v) to $[u,v]$. Then the above conditions make φ injective on H and the image $\varphi(H)$ linearly independent: given $x, y, z \in H$, (2b) implies that not both $[x,y]$ and $[y,x]$ are in $\varphi(H)$, and (2c) implies not all $[x,[y,z]]$, $[y,[z,x]]$, and $[z,[x,y]]$ are in $\varphi(H)$. The only issue is whether $\varphi(H)$ spans $\mathrm{L}(X)$.

Definition 3.2. A **Hall-Viennot order** on a subset $H \subset \Gamma(X)$ is a total order $\prec_H$ such that

$$u,\, v,\, (u,v) \in H \implies (u,v) \prec_H u. \tag{3.1}$$

A pre-Hall-Viennot set is a **Hall-Viennot set** if $\prec_H$ is a Hall-Viennot order. Elements of a Hall-Viennot set are called **Hall-Viennot trees**.

Theorem 3.3 ([Vie78, Theorem 1.2]). *Given a pre-Hall-Viennot set $H \subset \Gamma(X)$, $\varphi(H)$ is a linear basis of $\mathrm{L}(X)$ if and only if H is a Hall-Viennot set. In this case, the restriction $\varphi : H \to \mathrm{L}(X)$ is an injection, so $\varphi(H)$ is ordered by $\prec_H$.*

Let $A(X)$ be the free monoid on X. An element of $A(X)$ is called a **word**, and the product of $u, v \in A(X)$ is denoted by concatenation as uv. There is a natural map $\widetilde{-} :$ $\Gamma(X) \to A(X)$ that is the identity on indeterminates and sends $(u,v) \mapsto \widetilde{u}\widetilde{v}$; we call $\widetilde{t}$ the **foliage** of the tree t.

Theorem 3.4 ([Vie78]). *For a Hall-Viennot set H, the map $\widetilde{-}$ is injective on H.*

Now suppose X is a set of positively-graded indeterminates. For a Hall-Viennot set $H \subset \Gamma(X)$, the elements of $\varphi(H)$ remain linearly independent in the free *graded* Lie algebra $\mathrm{L}(X)$, but do not necessarily form a basis as Lie squares can be non-zero but are excluded from pre-Hall-Viennot sets by (2b) in Definition 3.1. But clearly with **super-Hall-Viennot trees** $H' = H \cup \{(h,h) \mid h \text{ odd}\}$, we get that $\varphi(H')$ is a linear basis of $\mathrm{L}(X)$. We do not extend the order $\prec_H$ to H' here, but will do so in the next section for super-Lyndon-Shirshov trees.

3.2 Lyndon-Shirshov bases

3.2.1 Construction

Let X be a positively graded set with a fixed total order $<$. From this order we get the lexicographical order $<$ on $A(X)$ by declaring that $x < 1$ for all $x \in X$. Here, a word is smaller than its proper prefixes, so given any $H \subset \Gamma(X)$ on which $\widetilde{-}$ is injective, the order $\prec_H$ defined by

$$u \prec_H v \Longleftrightarrow \widetilde{u} < \widetilde{v}$$

is a Hall-Viennot order.

Definition 3.5 (cf. Definition 3.1). A tree $t \in \Gamma(X)$ is a **Lyndon-Shirshov tree** if $t \in X$ or, writing $t = (u, v)$:

1. Both u and v are Lyndon-Shirshov trees,
2. $\widetilde{u} > \widetilde{v}$, and
3. If $u = (u', u'')$ then $\widetilde{u''} = \widetilde{v}$ or $\widetilde{u''} \leq \widetilde{v}$.

Theorem 3.6 ([CFL58]). *The map $\widetilde{-}$ is injective on the set H of Lyndon-Shirshov trees, and forms a Hall-Viennot set with $\prec_H$.*

A **Lyndon-Shirshov word** is the foliage of a LS tree, and a **Lyndon-Shirshov monomial** is a Lie monomial of the form $\varphi(t)$, where t is a LS tree.

Corollary 3.7. Lyndon-Shirshov trees, monomials, and words are in bijection with each other under φ and $\widetilde{-}$.

Given a word in $A(X)$, we can check if it is Lyndon-Shirshov, and construct its associated LS tree.

Theorem 3.8 ([CFL58]). *A non-trivial word $w \in A(X)$ is a Lyndon-Shirshov word if and only if $w > ba$ for every non-trivial factorization $w = ab$. If $w = ab$ is a non-trivial factorization such that such that b is the longest proper suffix of w that is again an LS word, then the LS tree associated to w is the product of the LS trees associated to a and b.*

Example 3.9. Let $w = xzyzz$. Then $w_1 = xz$ and $w_2 = yzz$. We further split w_2 into $w_2 = yz \cdot z$, so $[[x, z], [[y, z], z]]$ is the LS monomial associated to w.

Corollary 3.10. Let m_1 and m_2 be super-LS monomials with $m_1 > m_2$. Then though $p = [m_1, m_2]$ may not be a super-LS monomial, we have $\widetilde{p} = \widetilde{m_1}\widetilde{m_2}$.

The super-Hall-Viennot trees in this case are called **super-Lyndon-Shirshov trees**. The foliage of a super-LS tree is called a **super-Lyndon-Shirshov word**, and its image under φ is called a **super-Lyndon-Shirshov monomial**.

Corollary 3.11. Super-LS trees, monomials, and words are in bijection under φ and $\widetilde{-}$. The lexicographical order on super-LS objects is therefore well-defined, and respects the lexicographical order $<$ on LS objects. Lexicographical orders are in bijection with total orders on X, and give get ordered linear bases of $\mathrm{L}(X)$.

Super-LS monomials are a basis for a free graded Lie algebra, so with any order on super-LS monomials, a non-zero element $p \in \mathrm{L}(X)$ can be written $p = \lambda_0 \overline{p} + p'$, where $p' = \sum \lambda_i m_i$ and $\overline{p}$ and m_i are super-LS monomials and $\overline{p} > m_i$. Notice that any order $<$ on super-LS monomials yields a partial order on $\mathrm{L}(X)$, which we denote again by $<$.

Definition 3.12. The super-LS monomial $\overline{p}$ is the **leading monomial** of p , its associated super-LS word is called the **leading word** of p and is written $\widetilde{p}$. The **second-leading monomial** of p, if p' is nonzero, is the leading monomial of p'.

We emphasize that while "monomials" and "words" are not necessarily super-Lyndon-Shirshov, *leading* monomials and words are. We extend this notation to commutative polynomials.

3.2.2 Bracketings

We describe here the properties of super-Lyndon-Shirshov monomials that make them suitable for the construction of Gröbner-Shirshov bases.

Definition 3.13. Let $\Gamma^n(X)$ be the set of trees in $\Gamma(X \cup \{\iota_1, \iota_2, \ldots, \iota_n\})$ such that for $1 \leq j \leq n$, there is exactly one leaf labeled ι_j. We interpret $t \in \Gamma^n(X)$ as a map $t : \Gamma(X)^n \to \Gamma(X)$ where $t(s_1, \ldots, s_n)$ is the tree obtained from t by replacing ι_j with s_j. We will use t again to denote the multilinear map $\mathrm{L}(X)^n \to \mathrm{L}(X)$ where $t(s_1, \ldots, s_n)$ is the image of $\widetilde{t}$ under the map $\mathrm{L}(X \cup \{\iota_1, \iota_2, \ldots, \iota_n\}) \to \mathrm{L}(X)$ that takes X to X identically and ι_i to s_i.

Lemma 3.14. *Let $u, v_1, \ldots, v_n$ be super-LS trees. If the words $\widetilde{v}_j$ occur as disjoint subwords in $\widetilde{u}$, then there exists a tree $t \in \Gamma^n(X)$ such that the leading monomial of $\varphi(t(v_1, \ldots, v_j)) \in \mathrm{L}(X)$ is the leading monomial of $\varphi(u)$.*

Proof. The proof when $n = 1$ [MZ95, Lemma 7.1] is easily generalized. □

Definition 3.15. Let w be a super-LS word, and $p \in \mathrm{L}(X)$ such that $\widetilde{p}$ is a subword of w. If $w = \widetilde{p}$, we define $[w]_p = p$. Otherwise, let u and v be the super-LS trees associated to w and $\widetilde{p}$. Let t be as in Definition 3.13, and $t' : \mathrm{L}(X) \longrightarrow \mathrm{L}(X)$ be the multiple of $\varphi(t)$ such that $t'(v)$ is monic. Then we define $[w]_p = t'(p)$.

Notice that $[w]_p$ depends on the choice of t, but we do not indicate it.

Lemma 3.16 ([MZ95, Lemma 5.22]). *Let $e_1, e_2, e_3 \in A(X)$ be non-trivial and not all equal. If e_1e_2 and e_2e_3 are super-LS words, then $e_1e_2e_3$ is also a super-LS word.*

3.3 Gröbner-Shirshov bases

Gröbner-Shirshov bases, like Gröbner bases, are computed with respect to a choice of a linear basis and an order on that basis. We use super-LS monomials with a monomial order:

Definition 3.17. A **monomial order** $<$ on super-LS monomials is a total order satisfying

1. The order $<$ is a well-order, to guarantee the termination of the reduction algorithm in Definition 3.18,
2. $\mathrm{LC}(p,q)_w < w$, from Definition 3.28, for the Composition Lemma (Theorem 3.34) to hold, and
3. If $[m_1, m_2]$ and $[m_1, m_3]$ are super-LS monomials, then $[m_1, m_2] < [m_1, m_3]$ if and only if $m_2 < m_3$, as in Corollary 3.11.

We note that lexicographical order $<$ as defined in Corollary 3.11 and the monomial orders constructed in Theorem 4.5 do not satisfy the first condition, which is needed to guarantee the termination of the reduction algorithm in Definition 3.18. For example, the sequence

$$\{[x,y], [[x,y],y], [[[x,y],y],y], \ldots\} \subset \mathrm{L}(x,y)$$

is decreasing. One solution is to modify $<$ so that it respects total degree, such as deglex or degrevlex in the commutative case. But we consider only Lie ideals generated by homogeneous elements, and there are only finitely many super-LS monomials of a given degree, so this issue becomes irrelevant.

Definition 3.18. Let A be a free Lie algebra or polynomial ring. A **reduction algorithm** for an ideal $I \in A$ is one that given $f \in A$, will either

1. Conclude that $f \notin I$, or

2. Produce an element $f' \in A$ such that $f' < f$ and $f \in I$ if and only if $f' \in I$.

For both free Lie algebras and polynomial rings, the reduction algorithm will rely on the computation of an element $h = f - f' \in I$ with the same leading term as f. (If no such h exists, we conclude $f \notin I$.) Gröbner and Gröbner-Shirshov bases, then, characterize the leading terms of elements in a finitely-generated ideal.

We get an algorithm to determine ideal membership by iterating the reduction algorithm. That this process terminates is guaranteed by the decreasing sequence condition above. We are otherwise not interested in the computational feasibility of these algorithms, but instead use them to understand intersections of Lie ideals.

Let $\langle S \rangle$ be the ideal generated by the elements of S. We write $\langle s \rangle$ for $\langle \{s\} \rangle$.

3.3.1 Commutative bases

We sketch here the ideas of Gröbner bases in order to draw parallels to the ideas of Gröbner-Shirshov bases. Let $k[X]$ be a polynomial ring with the monomial basis and a degree-respecting monomial order such as deglex or degrevlex. Considering the idea of a reduction algorithm, a natural definition is:

Definition 3.19 (cf. Definition 3.26). Given $f, g \in k[X]$ non-zero, we say that g **reduces** f if there exists $h \in \langle g \rangle$ such that the leading terms of f and h are equal. The difference $f - h$ is called a **reduction** of f by g.

We have that a reduction $f - h$ is in $\langle g \rangle$ if and only if f is in $\langle g \rangle$. and since we are using a monomial order, every reduction of f is smaller than f. Of course, determining whether a polynomial reduces another is easy.

Proposition 3.20 (cf. Definition 3.26). *Let $f, g \in k[X]$ be non-zero. Then g reduces f if and only if the leading monomial of g is a factor of the leading monomial of f. The leading term of h in Definition 3.19 is determined by the leading terms of f and g, and h is otherwise arbitrary.*

Clearly, g reduces every element of $\langle g \rangle$, but reducibility is more complicated with ideals generated by more than one element. For example, if a multiple of g_1 happens to have the same leading term as a multiple of g_2, their difference, which is in $\langle g_1, g_2 \rangle$, may not be reducible by g_1 or g_2. It is easy to show, though, that for m_1 and m_2 monomials, $m_1 g_1 + m_2 g_2$ is reducible by at least one of g_1, g_2, and $S(g_1, g_2)$.

Definition 3.21 (Buchberger S-polynomials, cf. Definition 3.28). Let $f, g \in k[X]$ be monic. The **S-polynomial** of f and g is defined to be

$$S(f,g) = \frac{m}{\overline{f}} \cdot f - \frac{m}{\overline{g}} \cdot g$$

where m is the least common multiple of the leading monomials of f and g.

To a unique notion of remainder, we choose h to be precisely that leading term described in Proposition 3.20.

Definition 3.22. When g reduces f, we declare $S(f,g)$ to be the (unique) **reduction** of f by g.

We can think of m as the smallest monomial that can be reduced by both f and g, and $S(f,g)$ is the difference in reductions. We will also see $S(f,g)$ as an element in $\langle f,g\rangle$ but not necessarily reducible by f and g.

Definition 3.23 (cf. Definition 3.31). Let $f \in k[X]$ and $B \subset k[X]$. We say that f **reduces to f' modulo** B if there exists $b_i \in B$ and $f_i \in k[X]$ such that $f_0 = f$, $f_n = f'$, and $f_{i+1} = S(f_i, b_i)$.

Definition 3.24 (cf. Definition 3.32). Given an ideal $I \subset k[X]$, a **Gröbner basis** $B \subset I$ of I is a set such that for every $f \in I$ there exists $b \in B$ that reduces f.

Any ideal I is trivially a Gröbner basis of itself, but what we really want is a small Gröbner basis to make the ideal membership algorithm computationally tractable. The generators $G = \{g_1, \ldots, g_n\}$ may not be a Gröbner basis of $I = \langle G\rangle$. A basic result of Gröbner theory is that for $f \in I$, we can construct—starting with the generators G and using only the Buchberger S construction—an element $g \in I$ that reduces f. Therefore anything that is closed under S is a Gröbner basis.

Theorem 3.25 (cf. Theorem 3.34). *The set $B \subset k[X]$ is a Gröbner basis of $\langle B\rangle$ if and only if for every $b_1, b_2 \in B$, the Buchberger S-polynomial $S(b_1, b_2)$ reduces to zero modulo B.*

3.3.2 Lie compositions

Given $p, q \in L$, it is unclear how to construct (if possible) an element $r \in \langle q\rangle$ such that $\overline{p} = \overline{r}$, so a direct translation of Definition 3.19 and Proposition 3.20 to free Lie algebras is difficult. (The commutative case is of course trivial by Proposition 3.20.) Some cases are trivial, such as when $\overline{q}$ is a submonomial of $\overline{p}$. But the leading monomials of $\langle q\rangle$ may not have $\overline{q}$ as a submonomial in two ways. (Again, the commutative case is much simpler: the leading monomial of a product is the product of the leading monomials.)

1. While commutative monomials are closed under multiplication, the product of two LS monomials is not necessarily another LS monomial. For example, the element $[[x,y],z] = [x,[y,z]] + [[x,z],y] \in \langle [x,y]\rangle$ has leading monomial $[x,[y,z]]$, of which $[x,y]$ is not a submonomial.

2. The product of two super-LS monomials may be zero, e.g. if $q \in \mathrm{L}(X)$ has a leading monomial that is a LS square $\overline{q} = [r, r]$, then the leading monomial of $[r, q]$ will be the leading monomial of $[r, q']$, where q' is the second-leading monomial (Definition 3.12) of q.

In fact, the leading monomials of $\langle q \rangle$ can depend on every term of q, not just its leading monomial. We do have a partial answer from Lemma 3.14:

Definition 3.26 (cf. Definition 3.19, Proposition 3.20)**.** Let $p, q \in \mathrm{L}(X)$ be monic. If $\widetilde{q}$ is a subword of $\widetilde{p}$, we shall say that q **reduces** p. Writing $w = \widetilde{p}$, we call $p - [w]_q$ a **reduction** of p by q.

Notice that the above condition is easy to check, and that by Definition 3.15, $[w]_q$ can be chosen to have the be of the form $t'(q)$. These two properties are crucial in Lemma 3.14. For convenience, we add

Definition 3.27. Let $U \subset \mathrm{L}(X)$ be any subset. We say that U **reduces** $p \in \mathrm{L}(X)$ if there exists $q \in U$ such that q reduces p, i.e. $\widetilde{q}$ is a subword of $\widetilde{p}$.

We will see in Corollary 3.36 that the converse is true when the leading monomial of q is not a square. Otherwise, let $q = [x, x] + y$. Then q does not reduce $[x, q] = [x, y] \in \langle q \rangle$ as xx is not a subword of xy.

The Buchberger S-polynomial $S(f, g)$ can be thought of as the difference in reductions of the smallest monomial reducible by both $f, g \in k[X]$. The Lie analogue, called the Lie composition, is also constructed as the difference of two reductions. The elements $p, q \in \mathrm{L}(X)$ will both reduce an element $m \in \mathrm{L}(X)$ if and only if $\widetilde{p}$ and $\widetilde{q}$ both appear as subwords in $w = \widetilde{m}$. We are interested in two particular cases:

1. With $\widetilde{p} = e_1 e_2$ and $\widetilde{q} = e_2 e_3$ as in Lemma 3.16, as subwords in the LS word $w = e_1 e_2 e_3$, or
2. The word $\widetilde{q}$ appears in the super-LS word $w = \widetilde{p}$.

These two cases correspond to

Definition 3.28 (cf. Definition 3.21, [BKLM99])**.** Let $p, q \in L = \mathrm{L}(X)$ be monic.

1. If there exists e_1, e_2, e_3 as above such that $w = \widetilde{p} e_3 = e_1 \widetilde{q}$, then we define a **composition of intersection** (with respect to w) to be

$$\mathrm{LC}(p, q)_w = [w]_p - [w]_q.$$

This composition depends on the choice of $[w]_p$ and $[w]_q$, though we do not explicitly indicate it. The elements p and q may therefore have more than one composition of intersection.

2. If there exists $a, b \in A(X)$ such that $w = \widetilde{p} = a\widetilde{q}b$, then we define a **composition of inclusion** to be

$$\mathrm{LC}(p,q)_w = [w]_p - [w]_q = p - [w]_q.$$

This composition depends on the choice of a (and hence b), and $[w]_q$, though we do not explicitly indicate it. The elements p and q may therefore have more than one composition of inclusion.

An element of $\mathrm{LC}(p,q)_w$ is called a **Lie composition** of p and q.

Example 3.29. Let $p = [[x,z],[y,[y,z]]] + 2[[[[x,z],z],y],y]$ and $q = [[x,z],y] + [[x,z],z]$. Then the composition of inclusion $\mathrm{LC}(p,q)_{xzyyz}$ exists. Lemma 3.14 implies that $t = (\star,(y,z))$, and we have

$$\begin{aligned}
[xzyyz]_q &= t'(q) = [q,[y,z]] \\
&= [[x,z],[y,[y,z]]] + [[[x,z],[y,z]],y] + [[[x,z],z],[y,z]] \\
\mathrm{LC}(p,q)_{xzyyz} &= p - [xzyyz]_q \\
&= -[[[x,z],[y,z]],y] + 2[[[[x,z],z],y],y] - [[[x,z],z],[y,z]].
\end{aligned}$$

Example 3.30. Let $p = [[x,z],y]$ and $q = [y,[y,z]]$. Both p and q reduce the LS monomial $[[x,z],[y,[y,z]]]$, so with $w = xzyyz$, the composition of intersection $\mathrm{LC}(p,q)_w$ exists. The unlabeled tree associated to $[w]_p$ is $(\star,(y,z))$, and the unlabeled tree associated to $[w]_q$ is $((x,z),\star)$. Then $\mathrm{LC}(p,q)_w = [p,[y,z]] - [[x,z],q] = [[[x,z],[y,z]],y]$.

3.3.3 Lie bases

Recall that the defining feature of commutative Gröbner bases is its closure under S, where S is a way of constructing elements with "new" leading terms. Similarly, the Lie composition is essentially the only way to construct elements of $\langle B\rangle$ that are not reduced by some $b \in B$. Recall that δ is the inclusion of a free Lie algebra into its universal enveloping algebra.

Definition 3.31 (cf. Definition 3.23). Let $p, q \in \mathrm{L}(X)$ and $B \subset \mathrm{L}(X)$. We say that $p \equiv q \bmod (B, w)$ if $\delta(p-q)$ can be written

$$\delta(p-q) = \sum \lambda_i a_i \delta(b_i) c_i \in \mathbb{Q}\langle X\rangle$$

with $\lambda_i \in \mathbb{Q}$, $a_i, c_i \in A(X)$, and $a_i\delta(b_i)c_i < w$ lexicographically.

Definition 3.32 (cf. Theorem 3.25)**.** A set $B \subset \mathrm{L}(X)$ is a Gröbner-Shirshov basis if for all $p, q \in B$, every Lie composition of intersection satisfies $\mathrm{LC}(p,q)_w \equiv 0 \bmod (B,w)$, and if $p \neq q$, the Lie composition of inclusion does not exist.

Proposition 3.33. *Let $\{p_i\} \subset \mathrm{L}(X)$ have distinct leading monomials that are all indeterminates. Then $\{p_i\}$ is a Gröbner-Shirshov basis.*

Proof. By Definition 3.28, there are no Lie compositions of intersection, and a Lie composition of inclusion $\mathrm{LC}(p_i, p_j)$ exists only if $p_i = p_j$. □

Checking whether $p \equiv 0 \bmod (B,w)$ for arbitrary p, B, and w is difficult. However, this difficulty is not an issue when constructing a GS basis: we simply start with the generators of an ideal and add all possible Lie compositions. At the same time, we may discard compositions that reduce to zero, since if q reduces p, then $p \equiv \mathrm{LC}(p,q)_{\widetilde{p}} \bmod (\{q\}, \widetilde{p})$.

Theorem 3.34 (Composition Lemma, [BKLM99, Lemma 2.5], cf. Definition 3.24)**.** *If $B \subset \mathrm{L}(X)$ is a Gröbner-Shirshov basis, then for every $p \in \langle B \rangle$ there exists $b \in B$ that reduces p.*

Corollary 3.35. Let $I \subset \mathrm{L}(X)$ be an ideal with Gröbner-Shirshov basis $S \subset \mathrm{L}(X)$. Then I reduces $p \in \mathrm{L}(X)$ if and only if S reduces p.

If the leading monomial of $q \in \mathrm{L}(X)$ is not a square, then by Theorem 3.8 no composition of intersection $\mathrm{LC}(q,q)_w$ exists. Therefore $\{q\}$ is a Gröbner-Shirshov basis, and we have

Corollary 3.36. Let $q \in \mathrm{L}(X)$ have a leading monomial that is a Lyndon-Shirshov monomial. Then q reduces any element $p \in \langle q \rangle$, i.e. $\widetilde{q}$ is a subword of $\widetilde{p}$.

Chapter 4

Inert and non-inert pushouts

We work rationally in this chapter. All Lie algebras and indeterminates are graded, and all wedges of spheres are simply-connected and finite; a generalization to the locally-finite case is easy. We consider throughout this chapter the spectral sequence of Theorem 2.5 for the bar construction.

Let $\mathrm{B}(X)$ be a wedge of spheres, indexed by the elements of X, of dimension one greater than the dimensions of the indeterminates in X. Writing π_* with an implied dimension shift for the functor from spaces to Lie algebras with the Samelson product, we have a natural isomorphism $\pi_*(\mathrm{B}(X)) \cong \mathrm{L}(X)$.

Definition 4.1. The pushout diagram

$$\begin{array}{ccc} \mathrm{B}(A) & \xrightarrow{\bar{f}} & \mathrm{B}(X) \\ {\scriptstyle \bar{g}}\downarrow & & \vdots \\ \mathrm{B}(Y) & \cdots\cdots\to & C \end{array} \tag{4.1}$$

is **inert** if the map $\mathrm{L}(X) \vee \mathrm{L}(Y) \to \pi_* C$ is surjective, or equivalently, considering the edge homomorphism, $E^\infty_{n,*} = 0$ for $n > 0$. The diagram is **sharp inert** if $E^2_{n,*} = 0$ for $n > 0$.

Notice that we reusing the names A, X, and Y from Equation 1.1.

The goal of this chapter is to understand when the above diagram is inert or sharp inert.

4.1 The bar construction

Recall from Definition 2.6 that

$$\mathcal{D}_n = \mathrm{B}(X) \vee \mathrm{B}(A_1) \vee \cdots \vee \mathrm{B}(A_n) \vee \mathrm{B}(Y)$$

so the E^1 term is a simplicial graded Lie algebra which is levelwise free:

$$E^1_{n,*} = \pi_*(\mathcal{D}_n) = \mathrm{L}(X \cup A_1 \cup \cdots \cup A_n \cup Y). \tag{4.2}$$

We write $f = \pi_*\bar{f} : \mathrm{L}(A) \to \mathrm{L}(X)$, $g = \pi_*\bar{g} : \mathrm{L}(A) \to \mathrm{L}(Y)$, $A = \{a_1, \ldots, a_d\}$, $f_j = f(a_j)$, and $g_j = g(a_j)$. Since A_i is a copy of A, we write $A_i = \{a_{i,1}, \ldots, a_{i,d}\}$ to be explicit. For face maps $d_k : E^1_{n,*} \to E^1_{n-1,*}$, we have

$$\ker d_i = \begin{cases} \langle f_j - a_{1,j} \rangle & i = 0 \\ \langle a_{i,j} - a_{i+1,j} \rangle & 0 < i < n \\ \langle a_{n,j} - g_j \rangle & i = n. \end{cases} \tag{4.3}$$

We can compute $E^2 = \pi_*\pi_*\mathcal{D}$ by considering E^1 as a simplicial graded vector space. By the Dold-Kan correspondence, $E^2 = H_*(\mathcal{C}, \partial)$, where $\mathcal{C}$ is a bigraded chain complex with $\mathcal{C}_n = \ker d_1 \cap \cdots \cap \ker d_n \subset E^1_{n,*}$ with differential $\partial = d_0$. We use the theory of Gröbner-Shirshov bases to characterize $\ker d_k$. But unlike commutative Gröbner bases for polynomial rings, given Gröbner-Shirshov bases of Lie ideals I and J, it is not known how to find a Gröbner-Shirshov basis of $I \cap J$.

4.2 Monomial orders

Let $W \subset X$ and $p \in \mathrm{L}(X)$ nonzero. We fix an order on X, and let M be the set of super-LS monomials in $\mathrm{L}(X)$ or the set of super-LS words in $A(X)$.

Definition 4.2. The W**-count** of $m \in M$ is the number of times an indeterminate of W occurs in m, and the W**-weight** of m is the cumulative degree of those occurrences. An element p has **leading** W**-count** n (resp. **leading** W**-weight** n) if $\overline{p}$ has W-count (resp. W-weight) n, and p has W**-count** n if p is a linear combination of super-LS-monomials of W-count n. We denote the preorders on M induced by W-count and W-weight by ct^W and wt^W.

Definition 4.3. A preorder P **respects** a preorder R if $m_1 \leq_R m_2$ implies $m_1 \leq_P m_2$.

Definition 4.4. Given preorders P and Q on M, we define the preorder PQ as follows: $m_1 \leq m_2$ if and only if $m_1 \leq_P m_2$ and one of the following holds:

1. $m_2 \not\leq_P m_1$, or
2. $m_2 \leq_P m_1$ and $m_1 \leq_Q m_2$.

Let T be the lexicographical order on M (Corollary 3.11). We construct monomial orders (Definition 3.17) needed in Theorem 4.22 and Theorem 4.37.

Theorem 4.5 ([MZ95]). *Let $-P$ be the preorder opposite to P. Let $W_i \subset X$ be disjoint, and let P_i be either ct^{W_i} or wt^{W_i}. Then the total order $(-P_1)\cdots(-P_n)T$ is a monomial order on $\mathrm{L}(X)$.*

4.3 Intersections of Lie ideals

Let $\mathcal{I} = \{I_0, \ldots, I_n\}$ be a sequence of ideals in $\mathrm{L}(X)$, not necessarily distinct. Let $S_i \subset \mathrm{L}(X)$ be the Gröbner-Shirshov basis of I_i.

Definition 4.6. Let $\bigsqcap \mathcal{I} = I_1 \sqcap \cdots \sqcap I_n \subset \mathrm{L}(X)$ be the ideal generated by elements $t(p_1, \ldots, p_n)$ for $t \in \Gamma^n(X)$, $p_i \in I_i$.

Note that the ideals $\langle a\rangle \sqcap \langle b\rangle \sqcap \langle c\rangle$, $(\langle a\rangle \sqcap \langle b\rangle) \sqcap \langle c\rangle$, and $\langle a\rangle \sqcap (\langle b\rangle \sqcap \langle c\rangle)$ are all different in $\mathrm{L}(a, b, c)$.

Corollary 4.7. Let $p, q_1, \ldots, q_n \in \mathrm{L}(X)$ and each $\widetilde{q}_i$ occur as disjoint subwords in $\widetilde{p}$. Then $\bigsqcap \langle q_i\rangle$ reduces p, that is, there exists an element $q \in \bigsqcap \langle q_i\rangle$ such that $\overline{p} = \overline{q}$.

Proof. Let u and v_i be the super-LS trees associated to $\widetilde{p}$ and $\widetilde{q}_i$ respectively. By Lemma 3.14, there is a tree $t \in \Gamma^n(X)$ such that the leading monomial of $\varphi(t(v_1, \ldots, v_n))$ is $\overline{p}$. Then by the third point in Definition 3.17, the leading monomial of $t(q_1, \ldots, q_n) \in \bigsqcap \langle q_i\rangle$ is again $\overline{p}$. □

Proposition 4.8. *If f is a surjective Lie algebra map, then $f(\bigsqcap \mathcal{I}) = \bigsqcap f(\mathcal{I})$.*

Definition 4.9. The sequence $\mathcal{I}$ is **orthogonal** if $\bigcap \mathcal{I} = \bigsqcap \mathcal{I}$. If the ideals $\mathcal{I}$ are distinct, we will simply say that $I_1, \ldots, I_n$ are orthogonal.

For example, the ideals $\langle a\rangle, \langle b\rangle \subset \mathrm{L}(a, b)$ are orthogonal, as $\langle a\rangle \cap \langle b\rangle$ and $\langle a\rangle \sqcap \langle b\rangle$ have a linear basis of super-LS monomials with positive a-count and b-count. On the other hand, the sequence $\{\langle a\rangle, \langle a\rangle\} \subset \mathrm{L}(a, b)$ is not orthogonal as $a \notin \langle a\rangle \sqcap \langle a\rangle$.

Definition 4.10. We say that $\mathcal{I}$ is **incomposable** (with respect to a monomial order) if the Lie composition $\mathrm{LC}(p, q)$ does not exist for all $p \in S_i$, $q \in S_j$, $i \neq j$.

While it is unclear whether a subsequence of an orthogonal sequence is again orthogonal, a subsequence of an incomposable sequence is clearly again incomposable.

Proposition 4.11. *If $\mathcal{I}$ is incomposable, then $\mathcal{I}$ is orthogonal.*

We do not know of any other method to determine when ideals are orthogonal.

Proof. Let $p \in \bigcap \mathcal{I} \setminus \bigsqcap \mathcal{I}$ have minimal leading monomial, so $\widetilde{p}$ has subwords $\widetilde{s_i}$ for $s_i \in S_i$. Since $\mathcal{I}$ is is incomposable, the subwords $\widetilde{s_i}$ appear in $\widetilde{p}$ without overlapping. Then $\bigsqcap \mathcal{I}$ reduces p by Corollary 4.7, contradicting its minimality. □

Definition 4.12. Let $W \subset X$. A subset $U \subset \mathrm{L}(X) \setminus \{0\}$ is W**-leading** if $\overline{u} \in \mathrm{L}(W)$ for all $u \in U$. An ideal $I \subset \mathrm{L}(X)$ is W**-closed** if its Gröbner-Shirshov basis is W-leading. The **leading indeterminates** W of U is the smallest subset $W \subset X$ such that U is W-leading.

Let $B_i \subset X$ be the leading indeterminates of $S_i \subset \mathrm{L}(X)$.

Corollary 4.13. If the sets B_j are pairwise disjoint, then the ideals $\mathcal{I}$ are incomposable, and hence orthogonal.

4.4 Inert pushouts

Fix $n > 0$, and let $I_i = \ker d_i \subset E^1_{n,*}$ and $\mathcal{I} = \{I_0, \dots, I_n\}$. When a monomial order is specified, let S and S_i be the Gröbner-Shirshov bases of $\langle f_j \rangle$ and I_i respectively.

Proposition 4.14. *If $\mathcal{J} \subsetneq \mathcal{I}$, then there is a monomial order making $\mathcal{J}$ incomposable, and therefore $\mathcal{J}$ is orthogonal.*

Proof. The subsequence of an incomposable sequence is again incomposable, so let $\mathcal{J} = \{I_0, \cdots \hat{I}_k, \cdots I_n\}$. We use any monomial order that satisfies $A_k > \cdots > A_1 > X$ and $A_{k+1} > \cdots > A_n > Y$. By Proposition 3.33, the generators of $\ker d_i$ are already a Gröbner-Shirshov basis. So I_i is A_{i+1}-closed for $i < k$, and A_i-closed for $i > k$, and for $i \neq k$, Then $\mathcal{J}$ is incomposable by Corollary 4.13. □

The failure of $\mathcal{I}$ to be orthogonal is exactly the E^2 term.

Theorem 4.15. *$E^2_{n,*} = \bigcap \mathcal{I} / \bigsqcap \mathcal{I}$ and $E^2_{0,*}$ is the quotient of $\mathrm{L}(X) \vee \mathrm{L}(Y)$ by $\langle f_j - g_j \rangle$.*

Proof. For clarity, we write d^n_k for $d_k : \mathcal{D}_n \to \mathcal{D}_{n-1}$. By the Dold-Kan correspondence,

$$E^2_{n,*} = \frac{\bigcap \mathcal{I}}{d^{n+1}_0(\ker d^{n+1}_1 \cap \cdots \cap \ker d^{n+1}_{n+1})}.$$

Simplicial identities imply $d^{n+1}_0(\ker d^{n+1}_{i+1}) = \ker d^n_i$, and we get

$$\begin{aligned} d^{n+1}_0(\ker d^{n+1}_1 \cap \cdots \cap \ker d^{n+1}_{n+1}) &= d^{n+1}_0(\ker d^{n+1}_1 \sqcap \cdots \sqcap \ker d^{n+1}_{n+1}) \\ &= d^{n+1}_0(\ker d^{n+1}_1) \sqcap \cdots \sqcap d^{n+1}_0(\ker d^{n+1}_{n+1}) \\ &= \ker d^n_0 \sqcap \cdots \sqcap \ker d^n_n. \end{aligned}$$

Finally, $E^2_{0,*} = \mathrm{L}(X) \vee \mathrm{L}(Y)/d^1_0(\ker d^1_1)$. Since d^1_0 is surjective, we have $d^1_0(\ker d^1_1) = d^1_0(\langle a_{1,j} - g_j\rangle) = \langle d^1_0(a_{1,j} - g_j)\rangle = \langle f_j - g_j\rangle$. □

Example 4.16 (cf. Example 4.24)**.** Attaching a cell to a bouquet of two spheres via the Whitehead product is an inert cell attachment, because the map $S^m \vee S^n \longrightarrow S^m \times S^n$ is surjective in rational homotopy. Corollary 4.35 implies that the cell attachment is sharp inert, and hence $E^2_{1,*} = 0$. Then $\langle [x_1, x_2] - a\rangle$ and $\langle a\rangle$ are orthogonal by Theorem 4.15.

Example 4.17 (cf. Example 4.25)**.** We have $[x, a] \in \langle [x, x] - a\rangle \cap \langle a\rangle \setminus \langle [x, x] - a\rangle \sqcap \langle a\rangle$, by Corollary 4.35 the cell attachment $[\iota, \iota] : S^3 \longrightarrow S^2$ is not inert.

Corollary 4.18. The E^2 term has a vanishing line of slope $\min_j |a_j|$.

Proof. For a homogeneous ideal I, let $\min I$ be the smallest integer such that there exists a homogeneous element of that degree in I. Let $\mathcal{I}' = \{I_1, \ldots, I_n\}$. Then

$$\min \bigcap \mathcal{I} \geq \min \bigcap \mathcal{I}' = \min \bigsqcap \mathcal{I}' \geq \sum \min \mathcal{I}' = n(\min_j |a_j|).$$ □

The generators of $\ker d_0$ and $\ker d_n$ motivates the following definition.

Definition 4.19. Let $U = \{u_1, \ldots, u_n\} \subset \mathrm{L}(X)$ be a sequence of homogeneous elements, and ι_j be an indeterminate of the same degree as u_j. We say that U is **indeterminately closed** if the ideal

$$\langle u_1 - \iota_1, \ldots, u_n - \iota_n\rangle \subset \mathrm{L}(X \cup \{\iota_1, \ldots, \iota_n\})$$

is X-closed. We may say that a subset $S \subset \mathrm{L}(X)$ is indeterminately closed.

So $\{f_j\}$ is indeterminately closed if and only if I_0 is X-closed.

Definition 4.20. A map $f : \mathrm{L}(A) \longrightarrow \mathrm{L}(X)$ is **indeterminately closed** if f is non-zero and injective on A and $f(A) \subset \mathrm{L}(X)$ is an indeterminately closed subset.

Theorem 4.21. *Let f be indeterminately closed, with g and Y arbitrary in Equation 4.1. Then that pushout is sharp inert, and $\pi_* C = E^2_{0,*}$ as Lie algebras.*

Proof. Since f is indeterminately closed, there is a lexicographical order on $X \cup A_1$ that makes I_0 X-closed. We extend it to another lexicographical order satisfying $X \cup A_1 > A_2 > \cdots > A_n > Y$ as in the proof of Proposition 4.14, so I_i is A_i-closed for $i > 0$, and therefore $\mathcal{I}$ is incomposable. □

Let $n = 1$.

Theorem 4.22. *If f and g are injective, then $E^2_{1,*} = 0$.*

Proof. Let T be a lexicographical order satisfying $(X \cup Y) > A_1$. Instead of using T, we use a monomial order that respects $-\mathrm{wt}^{A_1}$, such as $(-\mathrm{wt}^{A_1})T$ by Theorem 4.5. The maps f and d_0 agree on A_1, so d_0 is also injective on $\mathrm{L}(A_1)$. Let $p \in S_0$, so $p \notin \mathrm{L}(A_1)$. The preorder $-\mathrm{wt}^{A_1}$ implies $\overline{p} \notin \mathrm{L}(A_1)$. Therefore $\overline{p}$ contains an indeterminate from X, and by Theorem 3.8, $\widetilde{p}$ begins with an indeterminate from X. Similarly, the leading word of any $q \in S_1$ begins with an indeterminate from Y. Since X and Y are disjoint, neither $\mathrm{LC}(p,q)$ nor $\mathrm{LC}(q,p)$ exist, so I_0 and I_1 are incomposable. $\square$

Definition 4.23. A subset $U \subset \mathrm{L}(X)\backslash\{0\}$ is **incomposable** if for all $p, q \in U$, the Lie composition $\mathrm{LC}(p,q)$ does not exist.

Clearly, an incomposable subset is both indeterminately closed and a Gröbner-Shirshov basis. The converses are not true.

Example 4.24 (cf. Example 4.16)**.** The set $S = \{s_1\} = \{[x_1,x_2]\} \subset \mathrm{L}(x_1,x_2)$ is indeterminately closed, because the Gröbner-Shirshov basis of $\langle s_1 - \iota_1 \rangle$ is $\{[x_1,x_2] - \iota_1\}$, which is $\{x_1,x_2\}$-leading.

Example 4.25 (cf. Example 4.17)**.** The set $U = \{u_1\} = \{[x,x]\} \subset \mathrm{L}(x)$ is a Gröbner-Shirshov basis of $\langle [x,x] \rangle$, but is neither indeterminately closed nor incomposable, as the composition of intersection $LC(u_1 - \iota_1, u_1 - \iota_1) = [x,\iota_1]$ exists and is in the Gröbner-Shirshov basis $\{[x,x],[x,\iota_1]\}$ of $\langle u_1 - \iota_1 \rangle \subset \mathrm{L}(x,\iota_1)$.

Example 4.26. Let $X = \{x,y,z\}$ be indeterminates of even degree, $p_1 = [x,[x,y]] + [x,[x,z]]$ and $p_2 = [[x,y],y] + [[x,z],z]$ in $\mathrm{L}(X)$. Then $S = \{p_1,p_2\}$ is not incomposable for any lexicographical order. For example, using the order $x > y > z$, the Lie composition of intersection $\mathrm{LC}(p_1,p_2)_{xxyy}$ exists.

We show that S is indeterminately closed by computing the Gröbner-Shirshov basis of $S' = \{p_1 - \iota_1, p_2 - \iota_2\}$. We use the lexicographical order extending $x > y > z > \iota_1 > \iota_2$.

$$[p_1 - \iota_1, y] = [x,[[x,y],y]] + [x,[[x,z],y]] + [[x,y],[x,z]] + [y,\iota_1]$$
$$[x, p_2 - \iota_2] = [x,[[x,y],y]] + [x,[[x,z],z]] - [x,\iota_2]$$

so the Lie composition of intersection $p_3 = \mathrm{LC}(p_1 - \iota_1, p_2 - \iota_2)_{xxyy}$ is

$$\begin{aligned} p_3 &= [p_1 - \iota_1, y] - [x, p_2 - \iota_2] \\ &= [x,[[x,z],y]] - [x,[[x,z],z]] + [[x,y],[x,z]] + [x,\iota_2] + [y,\iota_1]. \end{aligned}$$

Since no other Lie compositions exist in $\{p_1 - \iota_1, p_2 - \iota_2, p_3\}$, it is a Gröbner-Shirshov basis of $\langle S' \rangle$. So S is indeterminately closed.

Definition 4.27. A map $f : \mathrm{L}(A) \to \mathrm{L}(X)$ is **incomposable** if f is non-zero and injective on A and $f(A) \subset \mathrm{L}(X)$ is an incomposable subset.

By taking Y to be a point in Theorem 4.21, we get

Corollary 4.28. An indeterminately closed map is a sharp inert cell attachment.

An incomposable map is shown to be inert in [Ani82, Theorem 3.2]. This result is more general in two ways. First, indeterminately closed maps are not necessarily incomposable, as in Example 4.26. Second, Anick's result assumes a specific ordering on the words in $A(X)$, whereas we can use any ordering that allows us to define the compositions of inclusion and intersection.

4.5 Differential-graded models

Though the homotopy groups $\pi_* \mathfrak{g} = H_* N\mathfrak{g}$ of a simplicial graded Lie algebra $\mathfrak{g}$ depend only on $\mathfrak{g}$ as a simplicial vector space, we can use the levelwise Lie brackets on $\mathfrak{g}$ to construct a global multiplication $[\![-,-]\!]$ on $N\mathfrak{g}$ that descends to $\pi_*\mathfrak{g}$. This product gives $N\mathfrak{g}$ the structure of a differential graded Lie algebra whose the underlying chain complex is $\operatorname{Tot} N\mathfrak{g}$, but most importantly, an element $x \in N\mathfrak{g}_{p,s}$ has degree $p+s$ despite having degree s in $N\mathfrak{g}_p$.

Using this multiplication, we establish a correspondence between certain bigraded dgLs and simplicial graded Lie algebras, which will be a graded version of Quillen's correspondence between dgLs and simplicial (ungraded) Lie algebras. Quillen's correspondence will be a special case by considering a dgL as a bigraded dgL with second dimension 0, with the underlying chain complex $\operatorname{Tot} N\mathfrak{g} = N\mathfrak{g}$ as expected.

Let V and W be simplicial graded vector spaces.

Definition 4.29. The **(graded) shuffle product** $\zeta : V_p \otimes W_q \to (V \otimes W)_{p+q}$ is the linear extension of the map

$$x \otimes y \mapsto (-1)^{qs} \sum_{(\mu,\nu)} \varepsilon(\mu,\nu) s_\nu x \otimes s_\mu y, \quad x \in V_{p,s},\ y \in W_{q,t}.$$

Proposition 4.30.

1. *The differential d on NV respects the shuffle product with respect to total degree, i.e. for $x \in N_p V^s$, we have $d(\zeta(x,y)) = \zeta(dx,y) + (-1)^{p+s}\zeta(x,dy)$.*

2. *The shuffle product is associative, i.e.* $\xi \circ (1 \otimes \xi) = \xi \circ (\xi \otimes 1)$.

3. *With usual sign conventions, the square*

$$\begin{array}{ccc} N_p V^s \otimes N_q W^t & \xrightarrow{T} & NW \otimes NV \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle\xi} \\ N(V \otimes W) & \xrightarrow{NT} & N(W \otimes V) \end{array}$$

commutes up to $(-1)^{(p+s)(q+t)}$.

4. *The following square commutes,*

$$\begin{array}{ccc} N_p V^s \otimes N_q W^t & \xrightarrow{\xi} & N_{p+q}(V \otimes W)^{s+t} \\ {\scriptstyle d\otimes 1+(-1)^{p+s}\otimes d}\downarrow & & \downarrow{\scriptstyle N(d_V \otimes d_W)} \\ (NV \otimes NW)^{s+t}_{p+q-1} & \xrightarrow{\xi} & N_{p+q-1}(V \otimes W)^{s+t} \end{array}$$

and the shuffle product ξ *descends to a map of bigraded chain complexes*

$$\xi : NV \otimes NW \to N(V \otimes W)$$

that is a chain homotopy equivalence..

Proof. Since our definition of the shuffle product differs from Quillen's only in sign, these claims follow easily from [Qui69, p. 220]. □

Definition 4.31. The product $[\![-,-]\!]$ is the composition

$$[\![-,-]\!] : \mathfrak{g}_p \otimes \mathfrak{g}_q \xrightarrow{\xi} \mathfrak{g}_{p+q} \otimes \mathfrak{g}_{p+q} \xrightarrow{\beta} \mathfrak{g}_{p+q} \tag{4.4}$$

of the shuffle product ξ with the Lie bracket map β.

Proposition 4.32. *The product* $[\![-,-]\!]$ *turns* $N\mathfrak{g}$ *into a graded Lie algebra with respect to total grading.*

Proof. Since $\mathfrak{g}$ is levelwise a graded Lie algebra, we have $\beta \circ (1+T) = 0$. Then

$$\begin{aligned}
[\![x,y]\!] &= (N\beta \circ \xi)(x \otimes y) \\
&= (N\beta \circ \xi \circ T)(y \otimes x) \\
&= (-1)^{|x|\cdot|y|}(N\beta \circ NT \circ \xi)(y \otimes x) \\
&= (-1)^{|x|\cdot|y|}(N(\beta \circ T) \circ \xi)(y \otimes x) \\
&= -(-1)^{|x|\cdot|y|}(N\beta \circ \xi)(y \otimes x) \\
&= -(-1)^{|x|\cdot|y|}[\![y,x]\!].
\end{aligned}$$

By naturality of ξ, we have $(Nf \otimes Ng) \circ \xi = \xi \circ (f \otimes g)$. Writing $w = x \otimes y \otimes z$,

$$\begin{aligned}
[\![x,[\![y,z]\!]]\!] &= (N(\beta \circ (1 \otimes \beta)) \circ \xi \circ (1 \otimes \xi))(w) \\
[\![z,[\![x,y]\!]]\!] &= (N\beta \circ \xi \circ (1 \otimes N\beta) \circ (1 \otimes \xi))(z \otimes x \otimes y) \\
&= (N\beta \circ N(1 \otimes \beta) \circ \xi \circ (1 \otimes \xi))(z \otimes x \otimes y) \\
&= (N(\beta \circ (1 \otimes \beta)) \circ \xi \circ (\xi \otimes 1) \circ (T \otimes 1))(x \otimes z \otimes y) \\
&= (-1)^{|x|\cdot|z|}(N\beta \circ (1 \otimes \beta) \circ \xi \circ (NT \otimes 1) \circ (\xi \otimes 1) \circ (1 \otimes T))(w) \\
&= (-1)^{|x|\cdot|z|}(N\beta \circ (1 \otimes \beta) \circ N(T \otimes 1) \circ \xi \circ (\xi \otimes 1) \circ (1 \otimes T))(w) \\
&= (-1)^{|x|\cdot|z|}(N(\beta \circ (1 \otimes \beta) \circ (T \otimes 1)) \circ \xi \circ (1 \otimes \xi) \circ (1 \otimes T))(w) \\
&= (-1)^{|x|\cdot|z|+|y|\cdot|z|}(N(\beta \circ (1 \otimes \beta) \circ (T \otimes 1)) \circ \xi \circ (1 \otimes NT) \circ (1 \otimes \xi))(w) \\
&= (-1)^{|x|\cdot|z|+|y|\cdot|z|}(N(\beta \circ (1 \otimes \beta) \circ (T \otimes 1)) \circ N(1 \otimes T) \circ \xi \circ (1 \otimes \xi))(w) \\
&= (-1)^{|x|\cdot|z|+|y|\cdot|z|}(N(\beta \circ (1 \otimes \beta) \circ (T \otimes 1) \circ (1 \otimes T)) \circ \xi \circ (1 \otimes \xi))(w) \\
[\![y,[\![z,x]\!]]\!] &= (-1)^{|z|\cdot|x|+|y|\cdot|x|}(N(\beta \circ (1 \otimes \beta) \circ (1 \otimes T) \circ (T \otimes 1)) \circ \xi \circ (1 \otimes \xi))(w).
\end{aligned}$$

Since the Jacobi identity in $\mathfrak{g}$ is

$$\beta \circ (1 \otimes \beta) \circ (1 + (1 \otimes T) \circ (T \otimes 1) + (T \otimes 1) \circ (1 \otimes T)) = 0,$$

we immediately get the Jacobi identity in $N\mathfrak{g}$. □

Let C be the two-cone associated to Equation 4.1 with $Y = \varnothing$. The following is well-known.

Theorem 4.33. *The dgL*

$$(L, d_f) = (\mathrm{L}(X \cup \Sigma A), d_f), \quad d_f(X) = 0,\ d_f(\sigma a_j) = f_j$$

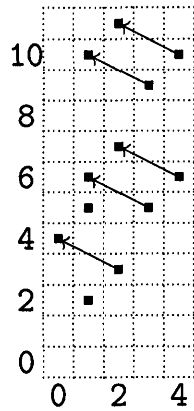

Figure 4-1: E^2 term for the cofiber sequence $S^2 \to \mathbf{C}P^2 \to S^4$.

is a dgL model of C*, and* $\pi_* C \cong H_*(L, d_f)$.

We can give this dgL a bigrading: an element $x \in X$ has bigrading $(|x|, 0)$ and σa_j has bigrading $(|a_j|, 1)$. The differential, which has bigrading $(0, -1)$, has grading -1 with respect to total grading. Using the graded translation of [Qui69, §I.4, §II.5], we get that $H_*(L, d_f)$ isomorphic to the homotopy of some levelwise free simplicial graded Lie algebra $\mathfrak{g}$. The following can be verified directly:

Theorem 4.34. *There is an isomorphism* $E^1 \cong \mathfrak{g}$.

Here, the E^1 term refers to the EMSS with $\mathcal{D}(X, A, *)$.

Corollary 4.35. The Eilenberg-Moore spectral sequence collapses at E^2 for a cofiber sequence of wedges of spheres, and such a cofiber sequence is inert if and only if it is sharp inert.

Proof. By the graded correspondence, the homology of the dgL model (L, d_f) is isomorphic to the $\pi_* \mathfrak{g}$. But

$$E^2 = \pi_* E^1 \cong \pi_* \mathfrak{g} \cong H_*(L, d_f) \cong \pi_* C \Longrightarrow \pi_* C,$$

so the spectral sequence collapses at E^2. □

The EMSS does not collapse at E^2 for the cofiber sequence $S^2 \to \mathbf{C}P^2 \to S^4$. Figure 4-1 shows $E^2_{p,q}$ for $p \leq 5$ and $q \leq 11$, with a bullet indicating a single generator. Since the cofiber is S^4, we conclude that only the generators at $E^2_{1,2}$ and $E^2_{1,5}$ survive, and by Corollary 4.18 all the remaining indicated elements must be killed by d_2.

4.6 Non-inert cell attachments

Let $n > 0$, $Y = \emptyset$, and we retain the notation of §4.4. We use any monomial order respecting $(-\mathrm{ct}^{A_1}) \cdots (-\mathrm{ct}^{A_n})$.

Definition 4.36. An element $p \in E^1_{n,*}$ is **minimal** if the following conditions hold:

1. p is in the Moore complex $\mathcal{C}$, that is, $p \in \bigcap \mathcal{I}$,

2. S does not reduce p (Definition 3.27), and

3. $\overline{p}$ has A_i-count 1 for $1 \leq i \leq n$.

Theorem 4.37. *If p is minimal, then p is not zero in $E^2_{n,*}$.*

Proof. We give the proof when $n = 2$; the general case is similar. Suppose p is nullhomologous, that is, $p \in \bigsqcap \mathcal{I} = \langle f_j - a_{1,j} \rangle \sqcap \langle a_{1,j} - a_{2,j} \rangle \sqcap \langle a_{2,j} \rangle$, and $\overline{p}$ has A_i-count 1. We show that S reduces p.

Recall that $\langle S \rangle = \langle f_j \rangle$ and $A_i = \{a_{i,j}\}$, and so p can be written as the sum of $2^n = 4$ elements

$$p_1 \in \langle S \rangle \sqcap \langle A_1 \rangle \sqcap \langle A_2 \rangle \qquad p_3 \in \langle A_1 \rangle \sqcap \langle A_1 \rangle \sqcap \langle A_2 \rangle$$
$$p_2 \in \langle S \rangle \sqcap \langle A_2 \rangle \sqcap \langle A_2 \rangle \qquad p_4 \in \langle A_1 \rangle \sqcap \langle A_2 \rangle \sqcap \langle A_2 \rangle.$$

No super-LS monomial in p_2, p_3, p_4 has A_i-count 1 for all i. So $\overline{p}$ appears in p_1, and by the monomial order, $\overline{p} = \overline{p_1}$. But $p_1 \in \langle S \rangle$, so S reduces p_1 and therefore p. □

Corollary 4.38. Two minimal elements have distinct leading monomials if and only if they are non-homologous.

We now consider the case $n = 1$, which is of particular interest because $E^2_{1,*}$ is trivial if and only if the cell attachment is inert. Recall that S_0 is the Gröbner-Shirshov basis of $I_0 = \ker d_0$.

Proposition 4.39. *Let the Lie algebra map $h : \mathrm{L}(X, A) \to \mathrm{L}(X)$ be the identity on X and zero on A. Then $S = h(S_0) \backslash \{0\}$.*

Proof. This is trivial if we consider an algorithm to compute Gröbner-Shirshov bases [MZ95, Algorithm 22.1]. □

Theorem 4.40. *If $p \in S_0$ has leading A_1-count 1, then p is minimal.*

Proof. Since the monomial order respects $-\mathrm{ct}^{A_1}$, we have $p \in \langle A_1 \rangle = I_1$, so $p \in I_0 \cap I_1$. Now suppose $q \in S$ reduces p. By Proposition 4.39, there exists $q_0 \in S_0$ such that $h(q_0) = q$. Since q_0 has leading A_1-count 0, we conclude that $p \neq q_0$ and q_0 reduces p, contradicting Definition 3.24. □

Theorem 4.41. *If f is not inert, then there exists an element of S_0 with leading A_1-count 1.*

Proof. Let the $(L, d_f) = (\mathrm{L}(X \cup \Sigma A), d_f)$ be as in Theorem 4.33. By [Ani82, Theorem 2.9], there exists a non-nullhomologous homogeneous element $q \in L$ having A-count 1, corresponding via Theorem 4.34 to a nonzero element $E^1_{1,*}$ that is non-nullhomologous and has homogeneous A_1-count 1.

By the above, there exists p be the smallest element in $\bigcap \mathcal{I}$ that is non-nullhomologous and has leading A_1-count 1. Since $p \in I_0$, there exists some $s_0 \in S_0$ that reduces it. If the leading monomial of s_0 has A_1-count 0, then $s = h(s_0) \in S$ reduces p. So $\widetilde{s}$ and an element of A_1 appear disjointly in $\widetilde{p}$, which by Corollary 4.7 contradicts the minimality of p. Therefore s_0 has A_1-count positive, and since p has leading A_1-count 1, we conclude that $s_0 \in S_0$ has leading A_1-count 1. □

Combining the previous two theorems, we get

Corollary 4.42. The map f is inert if and only if no element of S_0 has leading A_1-count 1.

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