Fusion action systems

by

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Abstract

The study of fusion first arose in the local theory of finite groups. Puig abstracted the fusion data of a finite group to the notion of fusion system, an object that reflects local data in more abstract algebraic settings, such as the block theory of finite groups. Martino and Priddy conjectured that the algebraic data of a fusion system of a finite group should have a topological interpretation, which result was proved by Oliver using the notion of $\mathfrak{p}$-local finite group introduced by the team of Broto, Levi, and Oliver. The study of fusion systems and $\mathfrak{p}$-local finite groups thus provides a bridge between algebraic fields related to local group theory and algebraic topology.

In this thesis we generalize the notion of abstract fusion system to model the local structure of a group action on a finite set. The resulting *fusion action systems* can be seen as a generalization of the notion of abstract fusion system, though we describe other possible interpretations as well. We also develop the notion of a $\mathfrak{p}$-*local finite group action*, which allows for connections between fusion action system theory and algebraic topology.

Thesis Supervisor: Haynes Miller
Title: Professor of Mathematics

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Contents

List of Figures

Chapter 1

Introduction

1.1 Foundations

Let $\mathfrak{p}$ be a prime. The goal of this thesis is to explore the $\mathfrak{p}$-local structure of finite groups from both algebraic and topological perspectives.

The algebraic $\mathfrak{p}$-local structure of finite groups has long been a fruitful area of research. For a finite group theorist, the term "$\mathfrak{p}$-local" roughly refers to the study of how the $\mathfrak{p}$-subgroups of a finite group G are embedded in G, with particular emphasis given to understanding how the ambient group acts on its $\mathfrak{p}$-subgroups by conjugation. Consequently, the $\mathfrak{p}$-local structure of a finite group is closely related to an understanding of the normalizers and centralizers of its $\mathfrak{p}$-subgroups. The *fusion system* of a finite group is an algebraic structure that encodes such $\mathfrak{p}$-local data for all $\mathfrak{p}$-subgroups, together with their relations to one another.

As the $\mathfrak{p}$-local study of finite groups progressed, interest in related algebraic structures rose. A prime area of research was the structure of blocks of a finite group, the indecomposable chunks of the representation theory associated to that group. A block naturally gives rise to the sort of fusion structure previously associated only to actual finite groups.[1] As in any case where similar structures occur in different situations, a natural question arises: Is it possible to study fusion data as an algebraic structure in its own right, divorced not only from finite groups but also from blocks or other mo-

[1] Cf. [AB] for more details.

tivating examples? Could such an abstraction provide insight to the original sources of fusion systems?

Puig provided the insight necessary to achieve this level of abstraction. In [Pui1][2] he describes the notion of an abstract *Frobenius category*, an algebraic structure that generalizes both the fusion systems of finite groups and blocks of finite groups to a setting where no ambient algebraic object must be mentioned. Later authors, for us most relevantly the team of Broto-Levi-Oliver, would use the term *abstract fusion system* to describe essentially the same data, and make use of this level of abstraction. As I learned this material first by reading [BLO2], I shall use the terminology of that document, and we shall henceforth use the term "fusion system" to describe the new algebraic object.

While all this work was going on mostly in the world of pure algebra and finite group theory, it was clear that the study of fusion data in whatever form would have deep implications for other mathematical fields, especially algebraic topology. The connection to topology comes from the study of classifying spaces of finite groups. These spaces can be thought of as topological versions of finite groups in that they encode all the algebraic data of a group in the homotopy theory of a space.

The $\mathfrak{p}$-local study of spaces is another area of clearly worthwhile research; in this document, the $\mathfrak{p}$-local study of spaces will be achieved by use of the Bousfield-Kan $\mathfrak{p}$-completion functor. In particular, we have arrived at our notion of the topological $\mathfrak{p}$-local study of finite groups: The examination of their $\mathfrak{p}$-completed classifying spaces.

So, given a finite group, there is both an algebraic and a topological notion of what the "$\mathfrak{p}$-part" of this group looks like. To express the obvious question: What is the relationship between these two?

Martino and Priddy conjectured [MP] that, properly understood, the algebraic and topological parts of a finite group are the same. By this we mean that each determines the other exactly, and so we might hope that the interplay of these two disciplines would lead to a greater understanding of each.

[2]This is the earliest published work of Puig I have found that details the relevant notions, though there is also an earlier unpublished manuscript that greatly influenced the development of the theory of fusion systems.

Proving the Martino-Priddy conjecture turned out to be no easy task. Martino and Priddy proved the "easy" direction—that topological information determines algebraic information—using a result of Mislin [Mis], which in turn relies on the Sullivan Conjecture proved by Miller [Mil1, Mil2]. The "hard" part was ultimately proved by Oliver [Oli1, Oli2], using the insight of Puig concerning the nature of abstract fusion systems [Pui1], the machinery of Broto-Levi-Oliver [BLO2], and the Classification Theorem of Finite Simple Groups (cf. 20^{th} century finite group theory). Both directions of the proof rely on very deep mathematics, but when the dust had settled, another strong connection between algebra and topology had been established.

And, like all good theorems, the proof of the Martino-Priddy conjecture ended up raising more questions than it answered.

In order to prove that fusion data determine the $\mathfrak{p}$-completed homotopy type of a finite group's classifying space, it was necessary to define a "classifying space" for the fusion system itself. The search for such a construction led to the development of the notion of a $\mathfrak{p}$-*local finite group*, which at its heart is an extra level of structure associated to a fusion system that allows us to create an interesting topological space.

It was observed by Solomon [Sol] that there exists 2-local fusion data that, by itself, is essentially indistinguishable from the 2-local fusion of a finite group and yet cannot be realized by any finite group. His work preceded the development of the notion of abstract fusion systems, and it ultimately turned out that he was describing an example of an *exotic* fusion system. Since the discovery of Solomon's family of exotic fusion systems, many others have been discovered for other primes, with the aid of the Classification Theorem.

These exotic fusion systems are interesting in their own right as sort of $\mathfrak{p}$-shadows of a nonexistent finite groups. Moreover, Broto-Levi-Oliver showed how to complete even an abstract fusion system to a $\mathfrak{p}$-local finite group, allowing us to study their homotopy theory.[3] So we might ask ourselves the question: Given an arbitrary abstract fusion system, is it possible to complete it to a $\mathfrak{p}$-local finite group? If so, can this

[3]Indeed, the proof the Martino-Priddy conjecture can be seen as moving from ordinary fusion systems to general abstract fusion systems in order to define their classifying spaces, and only then specializing back to ordinary ones.

be done uniquely? Oliver showed that if the fusion system is not exotic, the answer to both questions is yes, and for all exotic examples studied so far this seems to be the case. The existence and uniqueness of a classifying space for a general abstract fusion systems are two of biggest open questions in $\mathfrak{p}$-local finite group theory.

A secondary, but still extremely important, question concerns the desire for mathematical objects to play nicely with each other. For us, this means that we would like $\mathfrak{p}$-local finite groups to form a category. Finite groups certainly form a category in a natural way, and abstract fusion systems do as well. Unfortunately, when the data needed to construct a classifying space is added, there seems to be no sensible way of defining morphisms between $\mathfrak{p}$-local finite groups. This is true even if we restrict attention to the ordinary fusion systems, which is if anything even more troublesome: Any homomorphism of finite groups induces a map of their $\mathfrak{p}$-complete classifying spaces, but at this point it seems impossible to realize these data while using the intermediary of the $\mathfrak{p}$-local finite groups.

This background material is covered in more detail in Chapter 2.

1.2 Results

In this document we explore the question of what it means for a fusion system to act on a finite set. Starting with a finite group acting on a finite set, what "$\mathfrak{p}$-local" data can we extract? Can this be generalized to the notion of abstract fusion system introduced by Puig? Can we relate this to topology by defining a $\mathfrak{p}$-local finite group action?

The motivation for these questions comes from both the algebraic and topological aspects of the study of fusion systems. Algebraically, the study of group actions was an important stage in the understanding of finite groups. Indeed, historically the notion of group was introduced in terms of symmetries of some mathematical object, so perhaps we should look to fill in this gap of the development of fusion theory. On the topological side, any group action gives rise to a covering space of the classifying space of that group. Could an action of a $\mathfrak{p}$-local finite group on a finite set similarly give rise to a sort of covering space theory, or at least a notion of stabilizer subsystem whose classifying space plays an analogous role?

In our development, we give three levels of structure to the notion of "actions in the fusion context." The first is actually just a condition that must be put on the action of a $\mathfrak{p}$-group S on a finite set X. This condition is called *$\mathcal{F}$-stability* or the S-set X, and means that the S-action "respects the fusion data" in an appropriate manner. This $\mathcal{F}$-stability is a notion pleasing in its simplicity, and may well have applications to questions about the stable homotopy theory of $\mathfrak{p}$-local finite groups, but sadly it turns out to be too flabby to be useful in the unstable context we wish to investigate. $\mathcal{F}$-stable S-sets are the subject of Chapter 3.

The second level can be seen as starting with the notion of $\mathcal{F}$-stability and then adding structure relating the fusion system to the $\mathfrak{p}$-group action. The resulting algebraic objects, named *fusion action systems*, are introduced in Chapter 4 and are the central topic of study of this thesis. Fusion action systems may be interpreted in several ways—as rigidifications of the concept of $\mathcal{F}$-stability, as intermediaries between fusion systems and *transporter systems*, even as extensions of a fusion system by a finite group—but in all cases they play the role of a fusion system in a more general context.

In particular, the standard notion of fusion system can be recovered as a special case of a fusion action system acting on a one-point set. This observation begins the project of using fusion action systems to study certain associated fusion systems, tying our new notions back to their motivating origins. We go on to describe the *core* or kernel of the fusion action and the *stabilizer subsystems*, which play roles analogous to the kernel of a group action and the stabilizer subgroup of a point, respectively.

Chapter 5 concludes the content of this document with the third level of structure, the one that allows us to study the topology of our fusion actions. We call these data *$\mathfrak{p}$-local finite group actions*; the similarity with the term "$\mathfrak{p}$-local finite group" is not accidental, as once again if the underlying $\mathfrak{p}$-group action is trivial we recover the notion of [BLO2]. With this machinery set up, we are able to produce a fusion-theoretic version of the Borel construction of a finite group action, as well as defining the notion of a stabilizer subsystem and showing that it has the topological properties we would expect of it. Interestingly, we shall see that a $\mathfrak{p}$-local finite group action is not too different in its own structure from the corresponding $\mathfrak{p}$-local finite group, and that in fact the key difference comes primarily from the extra data of a map to a symmetric group.

Finally, Appendix A outlines a particular worldview that has greatly influenced the course of my research: That the study of groupoids as algebraic objects in their own right yields insight to the study of groups, and in particular we may view our fusion action systems as "fusion systems with many objects" to gain a greater understanding of fusion systems in their own right. While the content of the Appendix does not have much of a direct effect on the rest of the thesis, it suggests a general "fusion theory of finite groupoids" as an interesting future research direction.

We are still far away from such a radical generalization, so for the moment let us consider simply the fusion theory of translation groupoids, which we shall now describe.

Chapter 2

Fusion systems and $\mathfrak{p}$-local finite groups

In this chapter we review the basics of the theory of fusion systems and related concepts. We introduce the classical notion of the fusion system of a finite group and Puig's abstraction of this idea, the transporter systems of Oliver-Ventura, and the centric linking systems of Broto-Levi-Oliver. To motivate the difficult concept of a $\mathfrak{p}$-local finite group, we give a brief discussion of the Martino-Priddy Conjecture and its proof by Oliver.

Informing much of this chapter, and indeed the study of fusion systems in general, is the perspective that sometimes it is useful to consider a small category as an algebraic object in its own right: Not only can groups and rings be viewed as certain categories with a single object, but the ability to consider multiple objects gives rise to generalized notions that are difficult to describe without the language of categories. Whether categories are thought of as algebraic objects or simply a framework in which to discuss a mathematical system depends largely on one's point of view, and we will definitely make use of both perspectives in the sequel.

Throughout this document let $\mathfrak{p}$ be a prime, G a finite group, and S a finite $\mathfrak{p}$-group, which will be thought of as a Sylow subgroup of G.

2.1 Groups as categories

2.1.1 Classifying category of a group

How can we view a group as a category? The most obvious, if somewhat unenlightening, answer is simply to appeal to a definition of the notion of *group*: A category with a single object and all of whose morphisms are invertible.

Definition 2.1.1. The *classifying category* of a group G is the category $\mathcal{B}G$ with a single object $*$ and where $\mathcal{B}G(*,*) = G$ as a set with composition defined by the multiplication of G.[1] Note that, in the terminology of Appendix A, $\mathcal{B}G$ is the translation groupoid of the one-point G-set.

Let $BG := |\mathcal{B}G|$ denote the geometric realization of the nerve of $\mathcal{B}G$, also known as the *classifying space* of G.[2]

Let $\mathcal{E}G$ denote the groupoid whose objects are the elements of G and with precisely one morphism between any two objects. There is a natural G-action on $\mathcal{E}G$, which gives rise to a free G-action on the contractible G-space $EG := |\mathcal{E}G|$. Thus we have $BG \simeq G\backslash EG$. $\diamondsuit$

2.1.2 Transporter systems of finite groups

We are interested in the $\mathfrak{p}$-local structure of finite groups, so ideally our categorical version of G should pick out such structure as part of its data. The first piece of $\mathfrak{p}$-data one can associate to the finite group G is S, one of its Sylow subgroups. Moreover, the content of Sylow's theorems asserts, roughly, that not only are all Sylows equal in the eyes of G, but each contains all of G's $\mathfrak{p}$-structure. What this means will become clear shortly.

[1] As we read group multiplication from left to right but morphism composition from right to left, there are possible grounds for confusion here. Luckily, we will not actually encounter such a problematic situation in this thesis.

[2] Throughout this document, we shall refer to taking the geometric realization of a category. This is entirely a notational convenience, as every time we actually mean the geometric realization of the nerve of the category.

Notation 2.1.2. For $g \in G$, the homomorphism $G \to G : g' \mapsto gg'g^{-1}$ will be denoted by c_g. For $H \leq G$, denote by ${}^gH$ the subgroup $c_g(H)$. If ${}^gH \leq K$, conjugation by g defines a map $H \to K$, which will also be denoted c_g. ♢

We are now ready for our second categorical description of a finite group.

Definition 2.1.3. Let G be a finite group and $S \in \mathrm{Syl}_{\mathfrak{p}}(G)$. The *transporter system on S relative to G* is the category $\mathcal{T}_G = \mathcal{T}_S(G)$ whose objects are all subgroups $P \leq S$ and whose morphisms are given by

$$\mathcal{T}_G(P, Q) = N_G(P, Q) := \{g \in G | {}^gP \leq Q\}$$

$N_G(P, Q)$ is the *transporter of P to Q in G.* ♢

This definition singles out a given Sylow subgroup and plays an important role throughout this document. However, it contains too much information, especially $\mathfrak{p}'$-data. In fact, $\mathcal{T}_G$ can be easily seen to contain exactly the information of $\mathcal{B}G$, together with a choice of Sylow subgroup, by noting that $\mathcal{T}_G(1) \cong G$.[3] Indeed, the natural functor $\mathcal{B}G \to \mathcal{T}_G$ sending $*$ to 1 induces a homotopy inverse to the natural functor $\mathcal{T}_G \to \mathcal{B}G$; in the world of topology, $|\mathcal{B}G| \simeq |\mathcal{T}_G|$.

2.1.3 Fusion systems of finite groups

One way of understanding the sense in which $\mathcal{T}_G$ has too much information is to note that there may be distinct elements element $g, g' \in G$ that conjugate P to Q but are indistinguishable from the point of view of the conjugation action on P. In other words, $c_g|_P = c_{g'}|_P$ or $g^{-1}g' \in Z_G(P)$. Let us suppose that the conjugation action is the truly important $\mathfrak{p}$-local data, and define

Definition 2.1.4. For G a finite group and $S \in \mathrm{Syl}_{\mathfrak{p}}(G)$, the *fusion system on S relative to G* is the category $\mathcal{F}_G := \mathcal{F}_S(G)$ whose objects are all subgroups $P \leq S$

[3] We here make note of our notational convention: Just as for a category $\mathcal{C}$ we denote by $\mathcal{C}(a, b)$ the set of morphisms $\mathrm{Hom}_{\mathcal{C}}(a, b)$, we shall write $\mathcal{C}(a)$ for the automorphism group of the object a.

and whose morphisms are given by

$$\mathcal{F}_G(P,Q) = \mathrm{Hom}_G(P,Q) := \left\{\varphi \in \mathrm{Inj}(P,Q) \middle| \exists g \in G \text{ s.t. } \varphi = c_g|_P\right\}$$

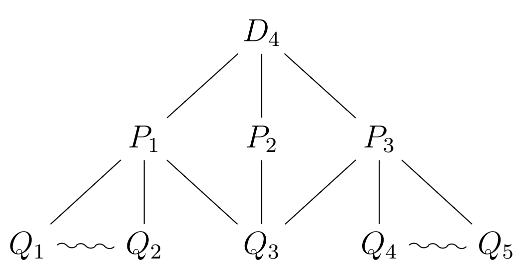
Note that we can also write $\mathcal{F}_G(P,Q) = \mathcal{T}_G(P,Q)/Z_G(P)$, so $\mathcal{F}_G$ can be thought of as a quotient of $\mathcal{T}_G$ and we have a natural projection functor $\mathcal{T}_G \to \mathcal{F}_G$. ♢

Notation 2.1.5. We shall reserve φ and ψ for morphisms in a fusion system. ♢

$\mathcal{F}_G$ is another categorical version of the finite group G—a version that extracts and focuses on the $\mathfrak{p}$-local *fusion*, or ambient conjugacy, data of the finite group. It will be the basis of our study in this document.

Example 2.1.6. The most basic example of a Sylow inclusion $S \leq G$ is the case that the supergroup G is equal to S itself. We denote the resulting fusion system by $\mathcal{F}_S$, the *minimal fusion system on* S. Minimality in this case means that if H is any finite group with $S \in \mathrm{Syl}_{\mathfrak{p}}(H)$ then $\mathcal{F}_S \subseteq \mathcal{F}_S(H) = \mathcal{F}_H$. The importance of this minimal example will become clear with the introduction of abstract fusion systems, starting in Section 2.3. ♢

Example 2.1.7. Consider D_4, the dihedral group on 4 points. This is a 2-group of order 8, which can be thought of as the group of symmetries of a square. The subgroup lattice of nonidentity subgroups of D_4, together with D_4-conjugacy relations denoted by horizontal wavy lines, is given by Figure 2-1.



$P_1, P_3 \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \qquad P_2 \cong \mathbb{Z}/4 \qquad Q_i \cong \mathbb{Z}/2 \qquad Q_3 = Z(D_4)$

Figure 2-1: Subgroup-conjugacy lattice of D_4

Every Sylow subgroup of Σ_4 is isomorphic to D_4, so we may consider the diagram of Figure 2-2, in which we record the isomorphism classes of objects of $\mathcal{F}_{\Sigma_4}$. Note in

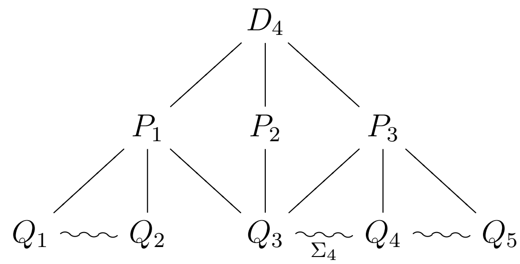


Figure 2-2: Fused 2-subgroups of Σ_4

particular the additional Σ_4-conjugacy, or *fusion*[4], relation between $Q_3 = Z(D_4)$ and both Q_4 and Q_5.

Of course, these subgroup-fusion lattices are just shadows of the structure of the fusion systems $\mathcal{F}_{D_4}$ and $\mathcal{F}_{\Sigma_4}$: The fusion system records not only which subgroups are fused, but *how* they are fused. Thus for every fusion relation, say $Q_3 \cong_{\Sigma_4} Q_4$, there is at least one explicit isomorphism $\varphi \in \mathcal{F}_{\Sigma_4}(Q_3, Q_4)$, given by conjugation by some element of Σ_4.

Moreover, each subgroup $R \leq D_4$ has an action by $N_{\Sigma_4}(R)$, which factors through the automorphism group $\mathcal{F}_{\Sigma_4}(R)$. For example, the isomorphism considered above $\varphi : Q_3 \cong_{\Sigma_4} Q_4$ can be realized as conjugation by an element of $N_{\Sigma_4}(P_3)$. Therefore φ extends to some $\widetilde{\varphi} \in \mathcal{F}_{\Sigma_4}(P)$. However, $Q_3 = Z(D_4)$ is a characteristic subgroup of D_4, so it is impossible that φ could be extended all the way to an automorphism of the entire Sylow D_4. Equivalently, φ cannot be written as conjugation by an element of Σ_4 that normalizes D_4. $\diamondsuit$

The last paragraph of Example 2.1.7 suggests that understanding of fusion systems might be accomplished through understanding of the automorphism groups of the subgroups of S. This is in fact the content of a weak form of Alperin's Fusion Theorem.

[4]Cf. [Gor] or other group theory literature, where "fusion" originally referred to conjugacy in the supergroup that was *not* realized in S itself. We do not draw this distinction, and by "fusion" simply mean conjugacy either in S or some (possibly nonexistent) supergroup.

Definition 2.1.8. For a small (preferably finite) category $\mathcal{C}$, a *system of inclusions* is a subcategory $\mathcal{I} \subseteq \mathcal{C}$ such that $\mathrm{Ob}(\mathcal{I}) = \mathrm{Ob}(\mathcal{C})$ and for any $c, c' \in \mathrm{Ob}(\mathcal{C})$ there is at most one morphism in $\mathcal{I}(c, c')$. The category $\mathcal{C}$ is an *Alperin category* (relative to $\mathcal{I}$) if every morphism of $\mathcal{C}$ can be written as a composition of morphisms of $\mathcal{I}$ and automorphisms of objects of $\mathcal{C}$. ♢

Every fusion system of a finite group naturally comes equipped with a system of inclusions, namely the actual inclusions of the subgroup lattice.

Theorem 2.1.9 (Alperin's Fusion Theorem)**.** *For G a finite group and $S \in \mathrm{Syl}_{\mathfrak{p}}(G)$, the fusion system $\mathcal{F}_G$ is an Alperin category.*

Proof. See [Alp] for a stronger version of this result. This result was strengthened further in [Gol] to the Alperin-Goldschmidt fusion theorem, which described a particular class of subgroups that the fusion system. Finally, Puig showed in [Pui2] that the class of groups identified by Goldschmidt is truly essential in order to generate the fusion system and proved in [Pui1] an abstract analogue of the Fusion Theorem that makes no reference to the finite group G (see Section 2.3 for more information on this point). □

2.1.4 Centric linking systems of finite groups

We've already seen in Definition 2.1.4 that the fusion system $\mathcal{F}_G$ can be thought of as the quotient of the transporter system $\mathcal{T}_G$ obtained by killing the action of the centralizer of the source. This quotienting process kills both $\mathfrak{p}$- and $\mathfrak{p}'$-information; if we wish to study the $\mathfrak{p}$-local structure of G, perhaps we should seek a less brutal quotient as an intermediary between $\mathcal{T}_G$ and $\mathcal{F}_G$.

To find this intermediary category, technical considerations suggest that we restrict attention to a particular collection of subgroups of S. The reasons will become clear in short order. Let us therefore introduce a seemingly ad hoc definition of the class of subgroups that will be central in the following discussion:

Definition 2.1.10. A $\mathfrak{p}$-subgroup P of G is $\mathfrak{p}$-*centric* if $Z(P) \in \mathrm{Syl}_{\mathfrak{p}}(Z_G(P))$. Equivalently, P is $\mathfrak{p}$-centric if there exists a (necessarily unique) $\mathfrak{p}'$-subgroup $Z'_G(P) \leq Z_G(P)$ such that $Z_G(P) = Z(P) \times Z'_G(P)$. ♢

This is an appropriate place to introduce some notation from finite group theory:

Definition 2.1.11. Let G be a finite group.

- $O_{\mathfrak{p}}(G)$ is the largest normal $\mathfrak{p}$-subgroup of G.
- $O_{\mathfrak{p}'}(G)$ is the largest normal $\mathfrak{p}'$-subgroup of G.
- $O^{\mathfrak{p}}(G)$ is the smallest normal $\mathfrak{p}$-power index subgroup of G.
- $O^{\mathfrak{p}'}(G)$ is the smallest normal subgroup of G with $\mathfrak{p}'$ index.

Each of these subgroups is characteristic in G. ♢

Notation 2.1.12. We shall reserve the notation $Z'_G(P)$ for $O^{\mathfrak{p}}(Z_G(P))$ in the case that P is $\mathfrak{p}$-centric in G, in which case we also have $Z'_G(P) = O_{\mathfrak{p}'}(Z_G(P))$. ♢

Notation 2.1.13. By $\mathcal{T}_G^c$ we mean the full subcategory of the transporter system $\mathcal{T}_G$ whose objects are the $\mathfrak{p}$-centric subgroups of S. We use similar notation to denote full centric subcategories of fusion systems and other related categorical versions of groups we'll encounter. ♢

We are now in the position to introduce our intermediary between $\mathcal{T}_G$ and $\mathcal{F}_G$:

Definition 2.1.14. The *centric linking system* of a finite group G with Sylow S is the category $\mathcal{L}_G^c$ whose objects are the $\mathfrak{p}$-centric subgroups of S and whose morphisms are the classes

$$\mathcal{L}_G^c(P, Q) = N_G(P, Q)/Z'_G(P)$$

♢

The quotient functors $\mathcal{T}_G^c \longrightarrow \mathcal{L}_G^c \longrightarrow \mathcal{F}_G^c$ relate our three nontrivial notions of G as a category and emphasize how some information is lost at each transition. We shall make use of the relationship between these three players in the sequel.

2.2 Groups as spaces

The goal of this section is to describe how we can use the categorical versions of a group G to construct topological spaces that will form the objects of our study.

2.2.1 Geometric realization

The classifying space functor $B : \mathcal{GRP} \to \mathcal{TOP}$ is the primary tool we use in this document for studying groups in the context of algebraic topology. We have already described B as the composition of $\mathcal{B} : \mathcal{GRP} \to \mathcal{CAT}$ with the geometric realization functor, which suggests that perhaps our alternate categorical versions of finite groups should be viewed as topological spaces via geometric realization.

For the transporter system $\mathcal{T}_G$, this works: As $|\mathcal{T}_G| \simeq BG$, essentially all the algebraic information of $\mathcal{T}_G$ is realized topologically in this manner.[5]

However, simply taking the nerve of the fusion system $\mathcal{F}_G$ will not yield an interesting space: The object 1 is initial in $\mathcal{F}_G$, so $|\mathcal{F}_G|$ is contractible.[6] We will have to be more clever about how we construct a topological space from a fusion system if we are to arrive at anything interesting; this will be the focus of Subsection 2.2.2.

This brings us to the last of our categorical versions of G introduced in 2.1, the centric linking system $\mathcal{L}_G^c$. The space $|\mathcal{L}_G^c|$ should be related to BG in some way, but as information is lost from the transition from transporter system to linking system it is unreasonable to expect that $|\mathcal{L}_G^c| \simeq BG$.

Example 2.2.1. Let G be your favorite finite group, $S \in \mathrm{Syl}_{\mathfrak{p}}(G)$, and H your favorite finite $\mathfrak{p}'$-group. Then H is a $\mathfrak{p}'$-subgroup of $Z_{G\times H}(S)$ so for any $P \leq S$ we have $H \leq Z'_{G\times H}(P)$. We conclude $\mathcal{L}_G^c \cong \mathcal{L}_{G\times H}^c$ (actual isomorphism of categories). It easily follows that the same result applies to fusion systems: $\mathcal{F}_G = \mathcal{F}_{G\times H}$ (equality of categories). ♢

[5]The problem is, roughly, that the transporter system has a "minimal" object that includes in every other one. One can see that in such a situation, all the topological data we could hope to extract from the transporter system is in some sense concentrated in the automorphisms of this minimal object; this is a situation that is explored in greater depth in [OV].

[6]As fusion systems are naturally equipped with a collection of "inclusion morphisms"—honest inclusions of subgroups in this case—this fact could be seen as a special case of the general reason why $|\mathcal{T}_G| \simeq BG$.

Since we construct the linking system by killing certain $\mathfrak{p}'$-primary data of the transporter system, we should look for an operation on topological spaces that "isolates $\mathfrak{p}$-information" in some appropriate sense.

Notation 2.2.2. Let $(-)^\wedge_\mathfrak{p} : \mathcal{TOP} \to \mathcal{TOP}$ denote the Bousfield-Kan $\mathfrak{p}$-completion functor of [BK]. There is a natural transformation $\eta : \mathrm{id}_{\mathcal{TOP}} \Rightarrow (-)^\wedge_\mathfrak{p}$; for a space $\mathcal{X}$, let $\eta_\mathcal{X} : \mathcal{X} \to \mathcal{X}^\wedge_\mathfrak{p}$ denote the resulting $\mathfrak{p}$-completion map. ♢

We shall view $(-)^\wedge_\mathfrak{p}$ largely as a black box that isolates the $\mathfrak{p}$-primary data of a space, at least in good cases:

Definition 2.2.3. A space $\mathcal{X}$ is $\mathfrak{p}$-*complete* if the $\mathfrak{p}$-completion map $\eta_\mathcal{X} : \mathcal{X} \to \mathcal{X}^\wedge_\mathfrak{p}$ is a homotopy equivalence. $\mathcal{X}$ is $\mathfrak{p}$-*good* if $\mathcal{X}^\wedge_\mathfrak{p}$ is $\mathfrak{p}$-complete. ♢

Proposition 2.2.4. $(-)^\wedge_\mathfrak{p}$ *has the following important properties:*

1. *A map of spaces $f : \mathcal{X} \to \mathcal{Y}$ induces an isomorphism of mod-$\mathfrak{p}$ cohomology if and only if $f^\wedge_\mathfrak{p} : \mathcal{X}^\wedge_\mathfrak{p} \to \mathcal{Y}^\wedge_\mathfrak{p}$ is a homotopy equivalence.*

2. *If $\pi_1(\mathcal{X})$ is finite, $\mathcal{X}$ is $\mathfrak{p}$-good.*

3. *If S if a finite $\mathfrak{p}$-group, BS is $\mathfrak{p}$-complete.*

4. *If G is a finite group, $\pi_1(BG^\wedge_\mathfrak{p}) \simeq G/O^\mathfrak{p}(G)$, the largest $\mathfrak{p}$-group quotient of G.*

5. *For spaces $\mathcal{X}$ and $\mathcal{Y}$, $(\mathcal{X} \times \mathcal{Y})^\wedge_\mathfrak{p} \simeq \mathcal{X}^\wedge_\mathfrak{p} \times \mathcal{Y}^\wedge_\mathfrak{p}$.*

6. *If $\mathcal{X}$ is not $\mathfrak{p}$-good, neither is $\mathcal{X}^\wedge_\mathfrak{p}$ (or: $\mathfrak{p}$-bad is infinitely $\mathfrak{p}$-bad).*

Proof. These results can be found in [BK]. □

Remark 2.2.5. We could use the first point of Proposition 2.2.4 to restate the notion $\mathfrak{p}$-good spaces as: $\mathcal{X}$ is $\mathfrak{p}$-good if the natural completion map $\eta_\mathcal{X} : \mathcal{X} \to \mathcal{X}^\wedge_\mathfrak{p}$ induces a mod-$\mathfrak{p}$ cohomology isomorphism. ♢

Notation 2.2.6. If the spaces $\mathcal{X}$ and $\mathcal{Y}$ satisfy the equivalent conditions of Point 2 of Proposition 2.2.4, we say that $\mathcal{X}$ and $\mathcal{Y}$ are *mod-$\mathfrak{p}$ equivalent* or *homotopy equivalent up to $\mathfrak{p}$-completion*. In this case we write $\mathcal{X} \simeq_\mathfrak{p} \mathcal{Y}$. ♢

So, even though it is unreasonable to expect that $|\mathcal{L}_G^c|$ is homotopy equivalent to BG, it is conceivable that these spaces should be equivalent up to $\mathfrak{p}$-completion. The following two theorems of [BLO1] describe the relationship of the centric linking system of G to the space $BG_{\mathfrak{p}}^{\wedge}$:

Theorem 2.2.7. *For G a finite group, the natural functors* $\mathcal{B}G \longleftarrow \mathcal{T}_G^c \longrightarrow \mathcal{L}_G^c$ *induce mod-$\mathfrak{p}$ cohomology isomorphisms on realization. In particular, $|\mathcal{L}_G^c|_{\mathfrak{p}}^{\wedge} \simeq BG_{\mathfrak{p}}^{\wedge}$.*

Theorem 2.2.8 (Weak Martino-Priddy Conjecture)**.** *For finite groups G and H, $BG_{\mathfrak{p}}^{\wedge} \simeq BH_{\mathfrak{p}}^{\wedge}$ if and only if the categories $\mathcal{L}_G^c$ and $\mathcal{L}_H^c$ are equivalent.*

2.2.2 Classifying spaces of fusion systems

In this subsection we describe how to associate a topological space to the fusion system $\mathcal{F}_G$. In doing so we introduce formal machinery, some of which will not be used until later in this document.

Homotopy colimits

The notion of a *homotopy colimit* of a functor into $\mathcal{TOP}$ is the "homotopically correct"—or invariant—notion of colimit; cf. [BK] for more of the general theory of homotopy colimits. For our purposes it is possible to chose an explicit model for the homotopy colimit that has the advantage of being combinatorial, or even algebraic, in nature. We introduce this model effectively as the definition of the homotopy colimit.

Definition 2.2.9. Let F be a functor from a small category $\mathcal{C}$ to $\mathcal{CAT}$. The *Grothendieck category* or *Grothendieck construction* of F is the category $\mathcal{G} := \mathcal{G}(F)$ whose objects are pairs (c, o) where $c \in \mathrm{Ob}(\mathcal{C})$ and $o \in \mathrm{Ob}(F(c))$. The morphism sets are defined by

$$\mathcal{G}((c,o),(c',o')) = \left\{(\alpha,\gamma)\middle|\alpha \in \mathcal{C}(c,c'), \gamma \in F(c)(F(\alpha)(o), o')\right\}$$

and with composition defined by $(\alpha', \gamma') \circ (\alpha, \gamma) = (\alpha'\alpha, \gamma' F(\gamma))$. ♢

Remark 2.2.10. As $\mathcal{GRP}$, $\mathcal{GRPD}$, and $\mathcal{SET}$ are subcategories of $\mathcal{CAT}$ (where we identify $\mathcal{SET}$ as the subcategory of categories with no nonidentity morphisms), it makes sense to speak of the Grothendieck category for a functor whose target is any of the three as well. $\diamondsuit$

Example 2.2.11. Another variant of the Grothendieck construction is the following: Let G and H be finite groups and $f : H \to \operatorname{Aut}(G)$ a group map. Then f induces a functor $\mathcal{B}f : \mathcal{B}H \to \mathcal{B}\operatorname{Aut}(G)$, and we can form the Grothendieck construction $\mathcal{G}(\mathcal{B}f)$ in the obvious way. It is not hard to see from the definitions that $\mathcal{G}(\mathcal{B}f) = \mathcal{B}(G \rtimes_f H)$, so the Grothendieck construction can be viewed as a generalization of the semidirect product of groups in this sense. $\diamondsuit$

The following theorem of Thomason relates the Grothendieck construction to the notion of homotopy colimit, and states that we can think of homotopy colimits as realizations of certain categories:

Proposition 2.2.12. *Let $\mathcal{C}$ be a small category and $F : \mathcal{C} \to \mathcal{CAT}$ a functor. Then there is a homotopy equivalence*

$$\operatorname*{hocolim}_{\mathcal{C}} |F| \simeq |\mathcal{G}(F)|$$

Proof. See [Tho]. $\square$

While we will make repeated use of the homotopy colimit in its own right, we will also need the following related construction:

Example 2.2.13 (Left homotopy Kan extensions). Let $\mathcal{C}$ and $\mathcal{D}$ be small categories and consider the following diagram:

$$\begin{array}{ccccc} & & \overset{F}{\frown\!\!\!\longrightarrow} & & \\ \mathcal{C} & \xrightarrow[\overline{F}]{} & \mathcal{CAT} & \xrightarrow[|-|]{} & \mathcal{TOP} \\ {\scriptstyle F'}\big\downarrow & & & & \\ \mathcal{D} & & & & \end{array}$$

The *left homotopy Kan extension of F over F'* is a natural, homotopically invariant functor $L = LKan_{F'}(F) : \mathcal{D} \to \mathcal{TOP}$ associated to this picture, which has the important property that $\operatorname{hocolim}_{\mathcal{D}} L \simeq \operatorname{hocolim}_{\mathcal{C}} F$.

We can describe the values of L in terms of homotopy colimits of certain functors, and our assumption that F factors through $\mathcal{CAT}$ allows us to describe L in terms of the Grothendieck construction. On objects, L is given by

$$L(d) = \operatorname*{hocolim}_{(c,\alpha)\in(F'\downarrow d)} F(c)$$

where $(F' \downarrow d)$ is the *overcategory of* F' *over* $d \in \mathcal{D}$. The objects of $(F' \downarrow d)$ are pairs (c, α), where $c \in \mathrm{Ob}(\mathcal{C})$ and $\alpha \in \mathcal{D}(F'(c), d)$. A morphism from (c, α) to (c', α') is $\gamma \in \mathcal{C}(c, c')$ such that the following diagram commutes in $\mathcal{D}$:

$$\begin{array}{ccc} c & \xrightarrow{F'(\gamma)} & c' \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\alpha'} & \\ & d & \end{array}$$

A morphism $\delta \in \mathcal{D}(d, d')$ defines a functor $(F' \downarrow d) \to (F' \downarrow d')$ given by

$$\begin{array}{ccc} (c, \alpha) & \longmapsto & (c, \delta\alpha) \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} \\ (c', \alpha') & \longmapsto & (c', \delta\alpha') \end{array}$$

This allows us to describe the left homotopy Kan extension in purely categorical terms, though it gets a bit messy. We shall see more of this in Chapter 5. $\diamondsuit$

The orbit category of a fusion system

We have seen that the passage from transporter system to linking system loses some $\mathfrak{p}'$-information, but not so much that the homotopy type of $BG^\wedge_{\mathfrak{p}}$ cannot be recovered. Following Martino and Priddy, let us conjecture an extension of this result, which at first glance may seem unreasonable in its strength: No further topological information is lost in the passage from linking system to fusion system. More explicitly,

Theorem 2.2.14. *[Martino-Priddy Conjecture] The finite groups G and H have homotopic $\mathfrak{p}$-completed classifying spaces if and only if the $\mathfrak{p}$-fusion data of G and H are the same.*

Proof. The "topology implies algebra" direction is given in [MP]. The "algebra implies topology" direction was proved by Oliver in [Oli1, Oli2], using the machinery of [BLO2] and the Classification Theorem of Finite Simple Groups. □

That "the $\mathfrak{p}$-fusion data of G and H are the same" means that there is an isomorphism of fusion systems $\mathcal{F}_G \cong \mathcal{F}_H$. The notion of isomorphism of fusion system is much stronger than saying that these categories are equivalent, or even isomorphic as categories: Such a notion would record only the shape of the fusion system as a diagram without giving due deference to the structure of the objects of the fusion system. Although we give the general notion of morphism of fusion system in Definition 2.3.7, we provide the special case here for completeness:

Definition 2.2.15. Let G and H be finite groups with respective Sylows S and T. An isomorphism $\alpha : S \to T$ is a *fusion preserving isomorphism* if for every $P, Q \leq S$ and $\beta \in \mathrm{Hom}(P, Q)$, $\beta \in \mathrm{Hom}_G(P, Q)$ if and only if $\alpha\beta\alpha^{-1} \in \mathrm{Hom}_H(\alpha P, \alpha Q)$. In this case the fusion systems $\mathcal{F}_G$ and $\mathcal{F}_H$ are *isomorphic as fusion systems.* ♢

So we claim that the data of the fusion system $\mathcal{F}_G$ determine the $\mathfrak{p}$-completed homotopy type of BG, which returns us to the question of how to associate a topological space to a fusion system. We have already seen that simply taking the nerve of $\mathcal{F}_G$ yields nothing interesting. This should not be surprising, as taking the geometric realization of the fusion system only records the shape of the category as a diagram without taking into account the fact that it is a diagram *in $\mathfrak{p}$-groups.* This is not a problem for either the transporter or linking systems of G, as for any $P \leq S$ we have $P \leq N_G(P)$ and therefore there is a natural way to identify P with a subgroup of its automorphism group. Such is not the case for fusion systems, so we must try a little harder to recover this information.

A first attempt to get an interesting space from $\mathcal{F}_G$ would look as follows: We wish to record that $\mathcal{F}_G$ is a diagram in groups, so let us simply consider the space

$\operatorname{hocolim}_{\mathcal{F}_G} BP$. If this were the right space to consider in proving the Martino-Priddy Conjecture, we should at the very least have $BG \simeq_{\mathfrak{p}} \operatorname{hocolim}_{\mathcal{F}_G} BP$. However, if S is nonabelian, it turns out to be impossible that $\operatorname{hocolim}_{\mathcal{F}_G} BP$ should have this property.[7]

Example 2.2.16. This is a heuristic argument for why $\operatorname{hocolim}_{\mathcal{F}_G} BP$ does not have the right $\mathfrak{p}$-completed homotopy type for use in the Martino-Priddy conjecture. The natural map $BS \to BG$ induced by the inclusion $S \leq G$ is Sylow, which is to say if R is any $\mathfrak{p}$-group and we have a map $BR \to BG$, there is a factorization

$$\begin{array}{ccc} BS & \longrightarrow & BG \\ & \nwarrow \quad \nearrow & \\ & BR & \end{array}$$

up to homotopy. If S is nonabelian, $\operatorname{Inn}(S)$ is nontrivial, and therefore $S \rtimes \operatorname{Inn}(S)$ contains S as a proper subgroup. As a consequence of Example 2.2.11, we have $S \rtimes \operatorname{Inn}(S) \leq \operatorname{Aut}_{\mathcal{G}(\mathcal{B}-)}(S)$, and therefore there there is an obvious map of spaces $B(S \rtimes \operatorname{Inn}(S)) \to |\mathcal{G}(\mathcal{B}-)|$ that does not factor through BS. ♢

The problem with this construction is that we are effectively double-counting some elements of S. Any noncentral $s \in S$ defines both a morphism $\check{s} \in \mathcal{B}S$ and a nonidentity morphism $c_s \in \mathcal{F}(S)$; by simply taking the homotopy colimit of $B-$ these separate morphisms both contribute even though they come from the same element of S. In other words, $\operatorname{hocolim}_{\mathcal{F}} B-$ is too big because the underlying Grothendieck construction has too many arrows.

So let's kill the offending morphisms.

Definition 2.2.17. The *orbit category*[8] of $\mathcal{F}_G$ is the category $\mathcal{O}_G := \mathcal{O}(\mathcal{F}_G)$ whose

[7] In fact, this is an "if and only if" statement, as can be inferred from the following discussion of orbit categories and the fact that if S is abelian, $\mathcal{F}_G = \mathcal{O}_G$.

[8] This notion is not to be confused with the category of orbits of G, whose objects are the transitive G-sets and where morphisms are maps of G-sets. Although there is a relationship between these two notions, we will not make use of the category of G-orbits in this document.

objects are the subgroups of S and whose morphisms are given by

$$\mathcal{O}_G(P,Q) = Q\backslash\mathcal{F}_G(P,Q)$$

In other words, the hom-set from P to Q is the set orbits of the Q-action of $\mathcal{F}_G(P,Q)$ given by postcomposition by c_q.

$\mathcal{O}_G^c$ will denote the full subcategory of $\mathcal{O}_G$ whose objects are the $\mathcal{F}$-centric subgroups of S. $\diamondsuit$

The functor $B- : \mathcal{F}_G \to \mathcal{TOP}$ does not descend to a functor $\mathcal{O}_G \to \mathcal{TOP}$, but because $\mathcal{O}_G$ is defined by quotienting out inner automorphisms, it is easy to see that there is a homotopy functor $\overline{B}- : \mathcal{O}_G \to ho\mathcal{TOP}$. If we could find a homotopy lifting

$$\begin{array}{ccc} & & \mathcal{TOP} \\ & \overset{\widetilde{B}-}{\nearrow} & \downarrow \\ \mathcal{O}_G & \xrightarrow[\overline{B}-]{} & ho\mathcal{TOP} \end{array}$$

we could consider $\text{hocolim}_{\mathcal{O}_G} \widetilde{B}-$, and relate this space to $BG^\wedge_{\mathfrak{p}}$. The following Proposition explains this relationship.

Proposition 2.2.18. *For G a finite group, consider the diagram*

$$\begin{array}{ccc} \mathcal{L}^c_G & \xrightarrow{*} & \mathcal{TOP} \\ {\scriptstyle\pi}\downarrow & \nearrow_{L} & \\ \mathcal{O}^c_G & & \end{array}$$

where π is the composite of the natural quotients $\mathcal{L}^c_G \to \mathcal{F}^c_G \to \mathcal{O}^c_G$ and L is the left homotopy Kan extension of the trivial functor $*$ over π. Then L is a homotopy lifting of $\overline{B}- : \mathcal{O}^c_G \to ho\mathcal{TOP}$, and in particular we have

$$\underset{\mathcal{O}^c_G}{\text{hocolim}}\, L \simeq \underset{\mathcal{L}^c_G}{\text{hocolim}}\, * = |\mathcal{L}^c_G| \simeq_{\mathfrak{p}} BG$$

Proof. [BLO2]. □

So far we have simply restated the original question of whether topological information is lost on the transition from linking system to fusion system: If we have a linking system in mind for $\mathcal{F}_G$, there is a homotopy lifting of $\overline{B}-$, which allows us to construct our desired space from the fusion system. But what if there is another finite group H such that $\mathcal{F}_G = \mathcal{F}_H$ and yet $\mathcal{L}^c_G \neq \mathcal{L}^c_H$: Is it possible there are two distinct homotopy liftings and thus two different spaces associated to $\mathcal{F}_G$? Or can this never happen? How do we approach this problem?

2.3 Abstraction as the answer

The basic problem introduced at the end of Section 2.2.2 is the need to think of fusion and linking systems as algebraic objects distinct from the finite groups from which they came. The work of Alperin-Broué gives further evidence that such a program should be undertaken: A consequence of their paper [AB] is that a block of a finite group gives rise to a fusion system on the defect group of that block. We shall not go into further detail as to exactly what this means beyond the observation that there may be a notion of "fusion system" that is somehow more general than that of "finite group."

Puig provided the necessary insight and abstraction to codify this generalization.[9] This section introduces his idea of abstract fusion systems (or "Frobenius categories" in the terminology of [Pui1]), though we shall use the language of Broto-Levi-Oliver. We also review the abstraction of the notion of centric linking system, due to [BLO2].

2.3.1 Fusion systems

Definition 2.3.1. Let S be a $\mathfrak{p}$-group. An *abstract fusion system on S* is a category $\mathcal{F}$ whose objects are all subgroups $P \leq S$ and whose morphisms are some collection of

[9] Cf. [Pui1] for a published version of his work, though an earlier unpublished manuscript was very influential on the development of this subject.

injective group maps: $\mathcal{F}(P,Q) \subseteq \mathrm{Inj}(P,Q)$. We require that the following conditions be satisfied:

- **(S-conjugacy)** The minimal fusion system $\mathcal{F}_S$ is a subcategory of $\mathcal{F}$.
- **(Divisibility)** Every morphism of $\mathcal{F}$ factors as an isomorphism of groups followed by an inclusion.

Composition of morphisms is composition of group maps. ♢

This is a very simple definition. In fact, it is perhaps too simple to be useful: This mimics the situation where S is a $\mathfrak{p}$-subgroup of some unnamed ambient group, but not where S is a *Sylow* $\mathfrak{p}$-subgroup. There is a great deal of additional structure that comes from such a Sylow inclusion; the question is how to codify these interesting data without reference to an ambient group. This will lead to the addition of *saturation axioms* that must be imposed on a fusion system.

Definition 2.3.2. We will need the following terms to state the saturation axioms:

- $P \leq S$ is *fully normalized in* $\mathcal{F}$ if $|N_S(P)| \geq |N_S(Q)|$ for all $Q \cong_{\mathcal{F}} P$.
- $P \leq S$ is *fully centralized in* $\mathcal{F}$ if $|Z_S(P)| \geq |Z_S(Q)|$ for all $Q \cong_{\mathcal{F}} P$.
- For any $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P,Q)$, let $N_\varphi \leq N_S(P)$ denote the group

$$\begin{aligned} N_\varphi &= \left\{ n \in N_S(P) \middle| \varphi \circ c_n \circ \varphi^{-1} \in \mathrm{Aut}_S(Q) \right\} \\ &= \left\{ n \in N_S(P) \middle| \exists s \in S \text{ s.t. } \forall p \in P, \varphi({}^n p) = {}^s\varphi(p) \right\} \end{aligned}$$

N_φ will sometimes be called the *extender* of φ. ♢

Remark 2.3.3. Perhaps some motivation for these concepts is in order. Each of these definitions comes from the idea that there are certain “global” phenomena that can be captured purely through local, fusion-theoretic data of a group. For instance, if there is an ambient Sylow G giving rise to the fusion system, then $P \leq S$ is fully normalized in $\mathcal{F}$ if and only if $N_S(P) \in \mathrm{Syl}_{\mathfrak{p}}(N_G(P))$, and similarly for the concept of full centralization.

The motivation for the extender N_φ comes from Alperin's Fusion Theorem 2.1.9. If we wish the the morphisms of a fusion system be generated by inclusions and automorphisms of subgroups, there must be some way of extending certain morphisms between different subgroups within the fusion system. The extender is the maximal subgroup of $N_S(P)$ to which we could hope to extend $\varphi \in \mathcal{F}(P, Q)$, so the question becomes when we can achieve this maximal extension. We shall examine this point in much greater detail and generality in Chapter 4. $\diamondsuit$

Definition 2.3.4 (Saturation axioms for abstract fusion systems). The fusion system $\mathcal{F}$ is *saturated* if

- Whenever P is fully $\mathcal{F}$-normalized, P is fully $\mathcal{F}$-centralized.
- Whenever P is fully $\mathcal{F}$-normalized, $\mathrm{Aut}_S(P) \in \mathrm{Syl}_{\mathfrak{p}}(\mathcal{F}(P))$.
- If Q is fully $\mathcal{F}$-centralized and $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P, Q)$, then there is some morphism $\widetilde{\varphi} \in \mathcal{F}(N_\varphi, S)$ that extends φ: $\widetilde{\varphi}|_P = \varphi$.

The first two conditions will be referred to as the *Sylow Axioms* and the third the *Extension Axiom.* $\diamondsuit$

We next define morphisms of (saturated) fusion systems, so that they may form a category:

Definition 2.3.5. For fusion systems $(S, \mathcal{F})$ and $(S', \mathcal{F}')$, a group map $\alpha : S \to S'$ is *fusion-preserving* if there exists a functor $F_\alpha : \mathcal{F} \to \mathcal{F}'$ such that $F_\alpha(P) = \alpha P$ for all $P \leq S$ and the following diagram of groups commutes for all $\varphi \in \mathcal{F}(P, Q)$:

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & \alpha P \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle F_\alpha(\varphi)} \\ Q & \xrightarrow[\alpha]{} & \alpha Q \end{array}$$

$\diamondsuit$

Remark 2.3.6. Note that there can be at most one functor F_α associated to α. In the case that α is injective, the formula $F_\alpha(\varphi) = \alpha\varphi\alpha^{-1}$ gives the unique possible choice for F_α. ♢

The functor F_α is unique if it exists, so saying that α is fusion-preserving is a condition on the group map and not extra data.

Definition 2.3.7. A *morphism of fusion systems* $(S, \mathcal{F}) \to (S', \mathcal{F}')$ is a fusion-preserving morphism $\alpha : S \to S'$. The set of all such morphisms is denoted $\mathrm{Hom}(\mathcal{F}, \mathcal{F}')$, and in the case that $(S, \mathcal{F}) = (S', \mathcal{F}')$ the resulting group of fusion preserving automorphisms of S is written $\mathrm{Aut}(\mathcal{F})$. ♢

Remark 2.3.8. Implicit in Definition 2.3.7 is the easy result that if $\alpha \in \mathrm{Hom}(\mathcal{F}, \mathcal{F}')$, $\alpha' \in \mathrm{Hom}(\mathcal{F}', \mathcal{F}'')$, and $F_\alpha, F_{\alpha'}$ are the corresponding functors, then $F_{\alpha'\alpha} = F_{\alpha'}F_\alpha$ is the functor that shows that $\alpha'\alpha \in \mathrm{Hom}(\mathcal{F}, \mathcal{F}'')$. ♢

There is an easy way to check which automorphisms of the group S actually lie in $\mathrm{Aut}(\mathcal{F})$. To express this, let us introduce a piece of terminology that is important in its own right:

Definition 2.3.9. Given a $\mathfrak{p}$-group S, subgroups $P, Q \leq R \leq S$, and an injective map $\gamma : R \to S$, the *translation along* γ *from* P *to* Q is the map

$$t_\gamma\big|_P^Q : \quad \mathrm{Hom}(P, Q) \longrightarrow \mathrm{Hom}(\gamma P, \gamma Q)$$
$$\eta \longmapsto \gamma\eta\gamma^{-1}$$

In cases where there will be no confusion, we simply write t_γ for $t_\gamma\big|_P^Q$. ♢

Proposition 2.3.10. *Given a fusion system* $\mathcal{F}$ *on the* $\mathfrak{p}$*-group* S *and* $\alpha \in \mathrm{Aut}(S)$*,* $\alpha \in \mathrm{Aut}(\mathcal{F})$ *if and only if* $t_\alpha(\mathcal{F}(P, Q)) = \mathcal{F}(\alpha P, \alpha Q)$ *for all* $P, Q \leq S$*.*

Proof. We can write out the content of Remark 2.3.6 more explicitly as follows: If there is a functor F_α making (α, F_α) a morphism of $\mathcal{F}$, for all $\varphi \in \mathcal{F}(P, Q)$ we must have $F_\alpha(\varphi) = \alpha\varphi\alpha^{-1} = t_\alpha(\varphi) \in \mathcal{F}(\alpha P, \alpha Q)$. Thus if α extends to a functor of $\mathcal{F}$, it follows that $t_\alpha(\mathcal{F}(P,Q)) \subseteq \mathcal{F}(\alpha P, \alpha Q)$ for all $P, Q \leq S$. The fact that α is an automorphism of S forces equality.

Conversely, if we have $t_\alpha(\mathcal{F}(P,Q)) = \mathcal{F}(\alpha P, \alpha Q)$, the assignment $\varphi \mapsto t_\alpha \varphi$ can easily be seen to give the action of the desired F_α on morphisms. $\square$

Example 2.3.11. $\mathcal{F}(S) \subseteq \mathrm{Aut}(\mathcal{F})$. In other words, if $\varphi \in \mathrm{Aut}(S)$ is a morphism in $\mathcal{F}$, φ is actually fusion preserving. This follows immediately from Proposition 2.3.10 together with the Divisibility Axiom of fusion systems, which implies that the restriction of a morphism in $\mathcal{F}$ to a subgroup also lies in $\mathcal{F}$.

Moreover, it is easy to see that $\mathcal{F}(S) \trianglelefteq \mathrm{Aut}(\mathcal{F})$: If $\alpha \in \mathrm{Aut}(\mathcal{F})$ and $\varphi \in \mathcal{F}(S)$, $\alpha\varphi\alpha^{-1} = t_\alpha(\varphi) \in \mathcal{F}(S)$ again by Proposition 2.3.10. $\diamondsuit$

Definition 2.3.12. The *inner automorphism group of* $\mathcal{F}$ is $\mathrm{Inn}(\mathcal{F}) := \mathcal{F}(S)$. The *outer automorphism group of* $\mathcal{F}$ is the quotient $\mathrm{Out}(\mathcal{F}) := \mathrm{Aut}(\mathcal{F})/\,\mathrm{Inn}(\mathcal{F})$. $\diamondsuit$

Let us expand our terminology slightly so as to relate fusion systems to actual finite groups.

Definition 2.3.13. If $(S, \mathcal{F})$ is a fusion system and G is a finite group, a morphism $\alpha : S \to G$ is *fusion-preserving* if for some (and hence any) $T \in \mathrm{Syl}_{\mathfrak{p}}(G)$ containing $\alpha(S)$, we have that $\alpha : (S, \mathcal{F}) \to (T, \mathcal{F}_T(G))$ is a morphism of fusion systems. $\diamondsuit$

Lemma 2.3.14. *Given a fusion system* $(S, \mathcal{F})$*, a finite group* G*, and group map* $\alpha : S \to G$*,* α *is fusion-preserving if and only if for every* $\varphi \in \mathcal{F}(P,Q)$ *there is some* $\psi \in \mathrm{Hom}_G(\alpha P, \alpha Q)$ *such that* $\alpha\varphi = \psi\alpha : P \to \alpha Q$*.*

Proof. If α is fusion-preserving, the implication is clear. Conversely, note that ψ is necessarily unique if it exists, and the composition of two such morphisms arising from G also arises from G. Therefore the assignment $\varphi \mapsto \psi$ gives a well-defined functor $\mathcal{F} \to \mathcal{F}_T(G)$, as desired. $\square$

Notation 2.3.15. Except in cases where there may be confusion, we shall simply write $\mathcal{F}$ for the pair $(S, \mathcal{F})$. Indeed, we have already done so in this section. $\diamondsuit$

2.3.2 Transporter systems and linking systems

Definition 2.3.16 ([OV])**.** Let $\mathcal{F}$ be an abstract fusion system on S. An *abstract transporter system associated to* $\mathcal{F}$ is a category $\mathcal{T}$ whose objects are some set of subgroups $P \leq S$ that is closed under $\mathcal{F}$-conjugacy and overgroups, together with functors

$$\mathcal{T}_S^{\mathrm{Ob}(\mathcal{T})}(S) \xrightarrow{\quad \delta \quad} \mathcal{T} \xrightarrow{\quad \pi \quad} \mathcal{F}$$

For any $s \in N_S(P, Q)$, set $\widehat{s} = \delta_{P,Q}(s) \in \mathcal{T}(P, Q)$. Similarly, for any $\mathfrak{g} \in \mathcal{T}(P, Q)$, set $c_{\mathfrak{g}} = \pi_{P,Q}(\mathfrak{g}) \in \mathcal{F}(P, Q)$.

The following axioms apply:

(A1) On objects, δ is the identity and π is the inclusion.

(A2) For any $P \in \mathrm{Ob}(\mathcal{T})$, define

$$E(P) = \ker\left[\pi_{P,P} : \mathcal{T}(P) \to \mathcal{F}(P)\right]$$

Then for any $P, Q \in \mathrm{Ob}(\mathcal{T})$, the group $E(P)$ acts right-freely and $E(Q)$ acts left-freely on $\mathcal{T}(P, Q)$. Moreover, the map $\pi_{P,Q} : \mathcal{T}(P, Q) \to \mathcal{F}(P, Q)$ is the orbit map of the $E(P)$-action.

(B) The functor δ is injective on morphisms, and for all $s \in N_S(P, Q)$ we have $c_{\widehat{s}} = c_s \in \mathcal{F}(P, Q)$.

(C) For all $\mathfrak{g} \in \mathcal{T}(P, Q)$ and all $p \in P$, the following diagram commutes in $\mathcal{T}$:

$$\begin{array}{ccc} P & \xrightarrow{\mathfrak{g}} & Q \\ {\scriptstyle \widehat{p}}\downarrow & & \downarrow{\scriptstyle \widehat{c_{\mathfrak{g}}(p)}} \\ Q & \xrightarrow[\mathfrak{g}]{} & Q \end{array}$$

Saturation axioms:

(I) $\delta_{S,S}(S) \in \mathrm{Syl}_{\mathfrak{p}}(\mathcal{T}(S))$.

(II) For all $\mathfrak{g} \in \mathrm{Iso}_{\mathcal{T}}(P, Q)$ and normal supergroups $\widetilde{P} \trianglerighteq P$ and $\widetilde{Q} \trianglerighteq Q$ such that $\mathfrak{g} \circ \delta_{P,P}\left(\widetilde{P}\right) \circ \mathfrak{g}^{-1} \leq \delta_{Q,Q}\left(\widetilde{Q}\right)$, there is a morphism $\widetilde{\mathfrak{g}} \in \mathcal{T}(\widetilde{P}, \widetilde{Q})$ that satisfies $\widetilde{\mathfrak{g}} \circ \widehat{1}_P^{\widetilde{P}} = \widehat{1}_Q^{\widetilde{Q}} \circ \mathfrak{g} \in \mathcal{T}(P, \widetilde{Q})$. ♢

Notation 2.3.17. As was noted in the definition of transporter systems, for any $s \in N_S(P, Q)$ we set

$$\widehat{s}\big|_P^Q = \delta_{P,Q}(s) \in \mathcal{T}(P, Q)$$

and if the source and target are obvious from the context we shall simply write $\widehat{s}$. Similarly, for any $R \leq N_S(P)$ we denote by $\widehat{R}\big|_P^P \leq \mathcal{T}(P)$ the group $\delta_{P,P}(R)$, again writing simply $\widehat{R}$ if there is no chance of confusion. ♢

Remark 2.3.18. $E(P)$ always contains $\delta_{P,P}(Z(P))$ by Axiom (B). As δ is injective, we identify $Z(P)$ with its image in $\mathcal{T}(P)$. ♢

We shall return to this point later, but for now say that $P \leq S$ is $\mathcal{F}$*-centric* if $Z_S(Q) = Z(Q)$ for all Q that are $\mathcal{F}$-conjugate to P. The collection of $\mathcal{F}$-centric subgroups is of central importance to the study of the homotopy theory of fusion system:

Definition 2.3.19 (Linking systems as minimal transporter systems)**.** Let $\mathcal{F}$ be a saturated fusion system on S. A transporter system $\mathcal{L}$ associated to $\mathcal{F}$ is an *abstract centric linking system* if:

- $\mathrm{Ob}(\mathcal{L})$ is the collection of $\mathcal{F}$-centric subgroups of S.
- For every $P \in \mathrm{Ob}(\mathcal{L})$, $E(P) = Z(P)$.

Thus an abstract linking system can be thought of as a minimal transporter system on the $\mathcal{F}$-centric subgroups of S. ♢

The following definition of centric linking system is (basically) the one given in [BLO2]. It is less compact than Definition 2.3.19, but it will be easier to work with in the sequel.

Definition 2.3.20. Let $\mathcal{F}$ be a (not necessarily saturated) fusion system on the finite $\mathfrak{p}$-group S. An *abstract centric linking system* is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups $P \leq S$, together with two functors

$$\mathcal{T}_S^c(S) \xrightarrow{\quad\delta\quad} \mathcal{L} \xrightarrow{\quad\pi\quad} \mathcal{F}^c$$

that satisfy:

(A) On objects δ is the identity and π is the inclusion. δ is injective on morphisms and π is surjective on morphisms.

(B) Identifying $Z(P)$ with a subgroup of $\mathcal{L}(P)$ via δ, the center $Z(P)$ acts right-freely on $\mathcal{L}(P,Q)$, and the map $\mathcal{L}(P,Q) \to \mathcal{F}(P,Q)$ is the orbit map of this action.

(C) For all $\mathfrak{g} \in \mathcal{L}(P,Q)$ and $p \in P$, the following diagram commutes in $\mathcal{L}$:

$$\begin{array}{ccc} P & \xrightarrow{\mathfrak{g}} & Q \\ {\scriptstyle \widehat{p}}\downarrow & & \downarrow{\scriptstyle \widehat{c_{\mathfrak{g}}(p)}} \\ Q & \xrightarrow[\mathfrak{g}]{} & Q \end{array}$$

◊

Remark 2.3.21. There are two obvious differences between Definitions 2.3.19 and 2.3.20. The first and more trivial is that in Definition 2.3.20 we do not assume that $Z(Q)$ acts left-freely on $\mathcal{L}(P,Q)$, but this turns out to follow from the other axioms and the result that every morphism in $\mathcal{L}$ is mono in the categorical sense.

The deeper difference lies in two extra Saturation Axioms of Definition 2.3.19 that come from the original definition of transporter system, and which have no analogue in 2.3.20. In [OV] it is shown that these definitions are equivalent so long as $\mathcal{F}$ is centric-saturated (i.e., if $\mathcal{F}$ obeys the saturation axioms on the centric subgroups, which by [BCG$^+$] implies that it is in fact globally saturated).

In effect, Axioms (I) and (II) of Definition 2.3.16 deserve the name "Saturation Axioms," as they imply the saturation of the underlying fusion system whenever they are in effect. ◊

Chapter 3

$\mathcal{F}$-stable S-sets

What does it mean for a fusion system to act on a finite set? In this chapter we take motivation from the ambient group case and consider the case of a G-set X restricted to a Sylow $\mathfrak{p}$-subgroup S. In this view, an action of $\mathcal{F}_G = \mathcal{F}_S(G)$ on X is an S-set that satisfies some additional, easily checked conditions. We describe how this notion can be abstracted to an arbitrary saturated fusion system $\mathcal{F}$, and record some of the properties of these $\mathcal{F}$-stable S-sets.

3.1 The ambient case

Let G be a finite group, $S \in \mathrm{Syl}_{\mathfrak{p}}(G)$, and X a finite left G-set.

Notation 3.1.1. We write ${}_G X$ whenever we wish to emphasize that X is a (left) G-set. If $\varphi : H \to G$ is an arbitrary map of groups, denote by ${}^{\varphi}_{H}X$ the H set whose action is given by twisting along φ, so that $h \cdot x := \varphi(h) \cdot x$. In particular, for $H \leq G$, we denote by ${}_H X := {}^{\iota_H^G}_{H} X$ the H-set X with action given by restriction from G. $\diamondsuit$

Let us examine what data of the G-action on X is seen by the fusion system $\mathcal{F}_S$. The most obvious piece of structure is the restricted S-set ${}_S X$ itself. It is clear, however, that not every S-set can be realized as the restriction of a G-set, and there are some obvious fusion-theoretic conditions which our ${}_S X$ must satisfy:

Notation 3.1.2. For any finite set X, let Σ_X be the group of permutations of X. A

G-set structure is the same as a group map $\rho : G \to \Sigma_X$. Let $\rho_S : S \to \Sigma_X$ be the restriction of ρ that gives the S-set structure of ${}_SX$. For any $H \leq G$, let X^H denote the H-fixed points of X, and $|X^H|$ the order of this set. ♢

Lemma 3.1.3. *In the above situation:*

(1) $\rho_S : S \to \Sigma_X$ *is* $\mathcal{F}_G$*-fusion-preserving (cf. Definition 2.3.13).*

(2) For all $g \in G$, $s \in S$, *and* $x \in X$, *we have* $g \cdot (s \cdot x) = c_g(s) \cdot (g \cdot x)$.

(3) For all $H \leq G$ *and* $g \in G$, *we have* ${}_HX \simeq {}^{c_g}_{H}X$ *as* H*-sets.*

(4) For any $P \leq S$ *and* $g \in G$, *we have* $|X^P| = |X^{{}^gP}|$.

Proof. For (1), if $c_g \in \mathcal{F}_G(P, Q)$ a direct calculation shows that defining $F_\rho(\varphi) = c_{\rho(g)}$ gives the functor F_ρ making ρ into a fusion-preserving map. Assertion (2) is obvious. The isomorphism of (3) is given by $\rho(g) : X \to X$. Lastly, (4) is a basic fact of G-sets that states that if $H, K \leq G$ are G-conjugate, the orders of the fixed points of H and K are equal. □

3.2 Definition of $\mathcal{F}$-stability

Let S be finite $\mathfrak{p}$-group, $\mathcal{F}$ a (saturated) fusion system on S, and X a finite S-set via the action map $\rho : S \to \Sigma_X$.

Notation 3.2.1. For an element $s \in S$, let $\ell_s = \rho(s) \in \Sigma_X$ be the permutation given by left translation by s. For $P \leq S$, let $\overline{P} \leq \Sigma_X$ be the image of P under the action map ρ. ♢

We wish to impose conditions on X that will mimic its being the restriction of a G-set, for G some imaginary ambient group that induces the fusion system $\mathcal{F}$.

Definition 3.2.2. The S-set X is $\mathcal{F}$*-stable* if $\rho : \mathcal{F} \to \Sigma_X$ is fusion-preserving. ♢

Thus we have simply take then first result from Lemma 3.1.3 as the defining property we wish to impose. The following result shows that we could just as easily have taken any of the other points from that Lemma as our definition:

Proposition 3.2.3. *For any S-set X, the following are equivalent:*

(1) X is $\mathcal{F}$-stable.

(2) For all $\varphi \in \mathcal{F}(P, Q)$, there is some $\sigma \in \Sigma_X$ such that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\rho} & \overline{P} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle c_\sigma} \\ Q & \xrightarrow[\rho]{} & \overline{Q} \end{array}$$

(3) For all $\varphi \in \mathcal{F}(P, Q)$, there is an abstract isomorphism of P-sets ${}_PX \cong {}^{\varphi}_{P}X$.

(4) For all $\varphi \in \mathcal{F}(P, Q)$, the orders of the fixed point sets are equal: $\left|X^P\right| = \left|X^{\varphi P}\right|$.

Proof. (1) $\Leftrightarrow$ (2): This is just an application of Proposition 2.3.14.

(2) $\Leftrightarrow$ (3): We can restate Assertion (3) to say that there exists a $\sigma \in \Sigma_X$ that *intertwines* φ. This means that for all $p \in P$ and $x \in X$, we have $\sigma(p \cdot x) = \varphi(p) \cdot \sigma(x)$. Using our notational convention of setting $\rho(p) = \ell_p$, this easily becomes the function equation $\rho \circ \varphi = c_\sigma \circ \rho$, and the equivalence is proved.

(3) $\Rightarrow$ (4): Given $\varphi \in \mathcal{F}(P, Q)$ and σ intertwining φ, by definition σ gives an isomorphism of P-sets ${}_PX \cong {}^{\varphi}_{P}X$. In particular, $\left|{}_PX^P\right| = \left|{}^{\varphi}_{P}X^P\right|$.

Now, suppose that $\varphi \in \mathcal{F}(P, Q)$ is an isomorphism; we claim that ${}^{\varphi}_{P}X^P = {}_QX^Q$. If $x \in {}^{\varphi}_{P}X^P$, then for all $p \in P$, $x = \varphi(p) \cdot x$, and as p ranges over P, $\varphi(p)$ ranges over Q, so $x \in {}_QX^Q$. Performing the same calculation with φ^{-1} yields the reverse inclusion.

Finally, the trivial observation that $\left|X^P\right| = \left|{}_PX^P\right|$ and $\left|X^Q\right| = \left|{}_QX^Q\right|$ combines with the previous two paragraphs to give the desired implication.

(4) $\Rightarrow$ (3): The statements

- ${}_PX \cong {}^{\varphi}_{P}X$ for all $\varphi \in \mathcal{F}(P, Q)$
- $\left|{}_PX^R\right| = \left|{}^{\varphi}_{P}X^R\right|$ for all $R \leq P$

are equivalent by the standard theory of G-sets, see for example [tD]. By assumption we have $\left|X^R\right| = \left|X^{\varphi R}\right|$, and the same argument in the second paragraph of (3) $\Rightarrow$ (4) shows that ${}_RX^{\varphi R} = {}^{\varphi}_{R}X^R$. The result follows. □

3.3 Basic results

In Section 3.2 we defined $\mathcal{F}$-stability, one of the fundamental concepts of this document. It was a very simple definition—too simple, it turns out, for our ultimate purpose. In this section we explore a few basic properties of this notion.

Example 3.3.1. The trivial S-set $*$ and the free S-set S are $\mathcal{F}$-stable for any fusion system $\mathcal{F}$ on S. $\diamondsuit$

Proposition 3.3.2. *For $\mathcal{F}$-stable S-sets X and Y, the disjoint union $X \amalg Y$ and cartesian product $X \times Y$ are $\mathcal{F}$-stable.*

Proof. By (4) of Proposition 3.2.3, it suffices to check that the orders of the fixed-point sets of P and Q are equal for $P \cong_{\mathcal{F}} Q$. The fact that $\left|(X \amalg Y)^P\right| = \left|X^P\right| + \left|Y^P\right|$ and $\left|(X \times Y)^P\right| = \left|X^P\right| \left|Y^P\right|$, together with the assumptions that X and Y are $\mathcal{F}$-stable, then finish the proof. $\square$

Definition 3.3.3. The $\mathcal{F}$-stable S-set X is *$\mathcal{F}$-simple* (or just *simple*) if it contains no nontrivial proper $\mathcal{F}$-stable S-sets. $\diamondsuit$

The following proposition shows that the notions of irreducibility and indecomposibility are identical for $\mathcal{F}$-stable S-sets, which justifies both conditions' being named "simplicity."

Proposition 3.3.4. *The $\mathcal{F}$-stable S-set X is simple if and only if X cannot be written as $Y \amalg Z$ for nonempty $\mathcal{F}$-stable S-sets Y and Z.*

Proof. This follows from the more general result that if X and $Y \subseteq X$ are $\mathcal{F}$-stable, then $X \backslash Y$ is as well. This follows immediately from point (4) of Proposition 3.2.3. $\square$

The following definition will be of great importance in later chapters:

Definition 3.3.5. The *core* of X is the largest subgroup $K \leq S$ that acts trivially on X. In other words, K is the kernel of the action map $\rho : S \to \Sigma_X$. $\diamondsuit$

Definition 3.3.6. A subgroup $P \leq S$ is *strongly closed* in $\mathcal{F}$ if for all $p \in P$ and $\varphi \in \mathcal{F}$, we have $\varphi(p) \leq P$ so long as this makes sense. $\diamondsuit$

Proposition 3.3.7. *The core of an $\mathcal{F}$-stable S-set X is strongly closed in $\mathcal{F}$.*

Proof. As X is $\mathcal{F}$-stable, $|X^P| = |X^Q|$ for any $\mathcal{F}$-conjugate subgroups P and Q. Let $k \in K$ be an element that acts trivially on X, so that $|X^{\langle k \rangle}| = |X|$. If $\varphi \in \mathcal{F}$ is any morphism for which $\varphi(k)$ is defined, we must then have $|X^{\langle \varphi(k) \rangle}| = |X|$, or $\varphi(k) \in K$, which is the condition for K being strongly $\mathcal{F}$-closed. □

Point (4) of Proposition 3.2.3 means that it is is not difficult to find examples of $\mathcal{F}$-stable S-sets; we will use this repeatedly in Section 3.4. Before examining the abstract case, note that we already have a large supply of examples that arise from actual finite groups:

Lemma 3.3.8. *If $\mathcal{F} = \mathcal{F}_G$ for some finite group G and X is a finite G-set, then ${}_SX$ is $\mathcal{F}$-stable.*[1]

Proof. Compare Lemma 3.1.3 with the definition of $\mathcal{F}$-stability or Proposition 3.2.3. □

Notation 3.3.9. For $P \leq S$, let $[P]$ denote the S-set S/P, up to isomorphism. Thus $[P] \cong [P']$ if and only if $P \cong_S P'$. ♢

Example 3.3.10. Consider again the case of $D_4 \in \mathrm{Syl}_2(\Sigma_4)$ introduced in Example 2.1.7. We reproduce Figure 2-2 here as Figure 3-1 to show the 2-fusion relations of Σ_4. Recall that $P_1, P_3 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ and $P_2 \cong \mathbb{Z}/4$, while each of the Q_i is isomorphic to $\mathbb{Z}/2$.

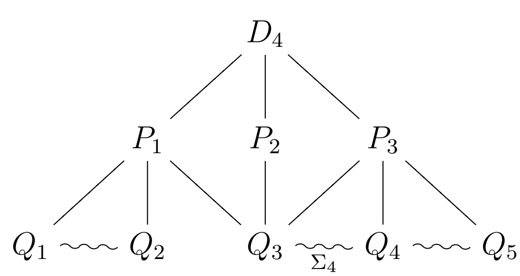


Figure 3-1: Fusion of Σ_4 at 2

[1]Note that we do not even need to require that $S \in \mathrm{Syl}_{\mathfrak{p}}(G)$ here.

We have explicitly computed the restrictions of transitive Σ_4-sets to D_4 and listed the results in Figure 3-2. Further calculations show that in fact every $\mathcal{F}_{\Sigma_4}$-stable D_4-set is a linear combination of the restrictions in this table, leading us to to the line of inquiry of Section 3.4.

$\Sigma_4 -$ set	$[1]$	$[Q_1]$	$[Q_3]$	$[P_1]$	$[P_2]$
Restriction	$3[1]$	$[Q_1]+[1]$	$[Q_3]+2[Q_4]$	$[P_1]+[Q_4]$	$[P_2]+[Q_4]$
$\Sigma_4 -$ set	$[P_3]$	$[S]$	$[\mathbb{Z}/3]$	$[\Sigma_3]$	$[A_4]$
Restriction	$3[P_3]$	$[S]+[P_3]$	$[1]$	$[Q_1]$	$[P_3]$

Figure 3-2: Restrictions of transitive Σ_4-sets to D_4

$\diamondsuit$

3.4 The Burnside ring of a fusion system

Definition 3.4.1. Recall that the *Burnside ring of the finite group* S, denoted $\mathcal{A}(S)$ is the group completion of the additive monoid of finite S-sets. The cartesian product of S-sets gives $\mathcal{A}(S)$ the structure of a unital ring.

Proposition 3.3.2 states that that the subset of $\mathcal{A}(S)$ consisting of isomorphism classes of $\mathcal{F}$-stable S-sets is closed under addition and multiplication, while the proof of 3.3.4 shows that it is closed under subtraction. Example 3.3.1 shows that both the zero element (the class of the empty S-set) and the multiplicative identity (the class of the one-point S-set) of $\mathcal{A}(S)$ are $\mathcal{F}$-stable, so the $\mathcal{F}$-stable classes form a unital subring of $\mathcal{A}(S)$. Denote this subring by $\mathcal{A}(\mathcal{F})$, the *Burnside ring of* $\mathcal{F}$. $\diamondsuit$

Notation 3.4.2. By Lemma 3.3.8, if $\mathcal{F} = \mathcal{F}_G$ then the subring of $\mathcal{A}(S)$ consisting of the restrictions of classes of G-sets is actually a subring of $\mathcal{A}(\mathcal{F})$. Denote this subring $\operatorname{res}_S^G \mathcal{A}(G)$, or just $\operatorname{res} \mathcal{A}(G)$. $\diamondsuit$

Example 3.4.3. As noted at the end of Example 3.3.10, it is easy to calculate that an additive basis for $\operatorname{res} \mathcal{A}(\Sigma_4)$ is given by

$$\{[1], \quad [Q_1], \quad [Q_3]+2[Q_4], \quad [P_1]+[Q_4], \quad [P_2]+[Q_4], \quad [P_3], \quad [S]\}$$

By direct calculation we see that in fact $\operatorname{res}\mathcal{A}(\Sigma_4) = \mathcal{A}(\mathcal{F}_{\Sigma_4})$. This is not true in general, but could be a question to examine in the future. $\diamondsuit$

3.4.1 Example: The case of an abelian $\mathfrak{p}$-group

The goal of this section is to understand the Burnside ring $\mathcal{A}(\mathcal{F})$ in the special case that the underlying $\mathfrak{p}$-group S is abelian. We derive some results about $\mathcal{F}$-stable S-sets for abelian S and note that these will in fact hold more generally, as will become apparent from the proofs. The main idea is that we can study $\mathcal{F}$-stable S-sets by understanding their fixed points and using Point (4) of Proposition 3.2.3.

The first few results do not require any conditions on S. Let S be a general $\mathfrak{p}$-group until otherwise noted, $\mathcal{F}$ a saturated fusion system on S, and X an $\mathcal{F}$-stable S-set.

If we wish to understand $|X^Q|$ for $Q \leq S$, it will be helpful to first decompose X as a disjoint union of S-orbits, so that we can examine $\left|(S/P)^Q\right|$ one at a time.

Lemma 3.4.4. *There is a bijection of finite sets $[P]^Q \cong P\backslash N_G(Q,P)$.*

Proof. Recall that $N_S(Q,P)$, the *transporter from Q to P in S*, is the set of $s \in S$ such that ${}^sQ \leq P$. Since P normalizes itself, as does Q, we have a free left action of P and a free right action of Q on $N_S(Q,P)$. The claim is that there is a bijective correspondence between Q-fixed points $[P]^Q$ and P-orbits $P\backslash N_S(Q,P)$.

Suppose that $sP \in [P]^Q$, so that for all $q \in Q$ we have $qsP = sP$, or $s^{-1}qs \in P$. Thus $s^{-1} \in N_S(Q,P)$, and conversely if $s \in N_S(Q,P)$ the same calculation shows that $s^{-1}P \in [P]^Q$. We have just seen the set-map $N_S(Q,P) \to [P]^Q$ given by sending $s \mapsto s^{-1}P$ is surjective. It is clear that two elements of $N_S(Q,P)$ have the same image if and only if one is a left P-translate of the other, so the result is proved.[2] $\square$

Remark 3.4.5. Note in particular that Lemma 3.4.4 implies that $[P]^Q \neq \emptyset$ if and only if $Q \leq_S P$. $\diamondsuit$

[2] An equivalent proof would be to note that there is a natural bijection $[P]^Q$ with the set $\operatorname{Hom}_G([Q],[P])$ of G-maps of orbits. A calculation similar to that given here then shows that we can identify $\operatorname{Hom}_G([Q],[P])$ with $P\backslash N_G(Q,P)$.

At this point we specialize to the case that S is abelian.

Observation 3.4.6. Note that in this case, transporter sets are particularly simple, as $N_S(Q,P) = S$ if $Q \leq P$ and is empty otherwise.

While this observation certainly makes life easier for us in the abelian case, it turns out to not be nearly as important a fact as the following:

Proposition 3.4.7. *If $\mathcal{F}$ is a saturated fusion system on the abelian group S, then $\mathcal{F}$ is generated by the inclusions of subgroups and $\mathcal{F}(S)$.*

Proof. This is an easy consequence of the extension axiom of saturation: For any $\varphi \in \mathcal{F}(P,Q)$ it follows immediately from the definition that $N_\varphi = S$, so φ can be written as a restriction of an $\mathcal{F}$-automorphism of S. □

Our list of examples of $\mathcal{F}$-stable S-sets currently consists of just the trivial and free S-sets. We can now add several more examples:

Definition 3.4.8. A subgroup $P \leq S$ is *weakly closed in $\mathcal{F}$* if whenever $P \leq R$ and $\varphi \in \mathcal{F}(R,S)$ we have $\varphi P = P$. ♢

Note that being strongly closed (cf. 3.3.6) implies that P is weakly closed.

Proposition 3.4.9. *If $P \leq S$ is weakly closed, then $[P]$ is $\mathcal{F}$-stable.*

Proof. By (4) of Proposition 3.2.3, it suffices to show that $\left|[P]^Q\right| = \left|[P]^{\varphi Q}\right|$ for any $Q \leq S$ and $\varphi \in \mathcal{F}(Q,S)$. By Lemma 3.4.4 and Observation 3.4.6:

$$\left|[P]^Q\right| = \begin{cases} [P:S] & \text{if } Q \leq P \\ 0 & \text{else} \end{cases} \qquad \text{and} \qquad \left|[P]^{\varphi Q}\right| = \begin{cases} [P:S] & \text{if } \varphi Q \leq P \\ 0 & \text{else} \end{cases}$$

By Proposition 3.4.7, we may assume without loss of generality that φ is defined on all of S, and in particular is defined on P. Thus $\varphi Q \leq P$ if and only if $Q \leq \varphi^{-1}P = P$ because P is weakly closed in $\mathcal{F}$. The result follows. □

To state the main result of this section, we introduce the following terminology:

Notation 3.4.10. For $P \leq S$, let $P_1 = P, P_2, \ldots, P_n$ denote the distinct $\mathcal{F}$-conjugates of P. We denote by $[P]_{\mathcal{F}}$ the disjoint union of the S-orbits $\coprod_{i=1}^{n} [P_i]$. $\diamondsuit$

Then, with slight modifications, Proposition 3.4.9 applies more generally:

Proposition 3.4.11. *For any subgroup $P \leq S$, the S-set $[P]_{\mathcal{F}}$ is $\mathcal{F}$-stable.*

Proof. Just as in the proof of Proposition 3.4.9, we must show that for any $Q \leq S$ and $\varphi \in \mathcal{F}(S)$ (here using again that all morphisms of $\mathcal{F}$ are restrictions of elements of $\mathcal{F}(S)$) we have $\left|[P]_{\mathcal{F}}^{Q}\right| = \left|[P]_{\mathcal{F}}^{\varphi Q}\right|$. Decomposing $[P]_{\mathcal{F}}$ into S-orbits, this becomes

$$\sum_{i=1}^{n} \delta_{Q \leq P_i}[P : S] \stackrel{?}{=} \sum_{i=1}^{n} \delta_{\varphi Q \leq P_i}[P : S]$$

where $\delta_{Q \leq P_i}$ is 1 for $Q \leq P_i$ and 0 otherwise. But $\varphi Q \leq P_i$ if and only if $Q \leq \varphi^{-1} P_i$, and φ^{-1} permutes the P_i, so the right hand side of the equation is simply a reordering of the left. Thus $[P]_{\mathcal{F}}$ is $\mathcal{F}$-stable. $\square$

We are almost ready to prove that the $[P]_{\mathcal{F}}$, as P ranges over the $\mathcal{F}$-conjugacy classes of subgroups of S, forms an additive basis for $\mathcal{A}(\mathcal{F})$. First, we record a basic fact that does not depend on the group S's being abelian:

Notation 3.4.12. Recall that for any $x \in X$, the S-*stablizer of* x, denoted S_x, is the maximal subgroup of S that fixed x. $\diamondsuit$

Lemma 3.4.13. *For any point $x \in X$, subgroup $P \leq S_x$, and morphism $\varphi \in \mathcal{F}(P, S)$, there is some $x' \in X$ such that $\varphi P \leq S_{x'}$.*

Proof. If $x \in X^P$ we have $|X^P| \geq 1$, so Proposition 3.2.3 implies $|X^{\varphi P}| \geq 1$. $\square$

Proposition 3.4.14. *If X is an arbitrary $\mathcal{F}$-stable S-set there exist (unique up to $\mathcal{F}$-conjugacy) subgroups $R_1, R_2, \ldots, R_m \leq S$ such that $X = \coprod_{i=1}^{m} [R_i]_{\mathcal{F}}$.*

Proof. The argument goes by induction on the order of X, being obvious in the case that X is empty. For a general $\mathcal{F}$-stable S-set, pick $x \in X$ such that the order of the stabilizer of x is maximal: $|S_x| \geq |S_{x'}|$ for all $x' \in X$. Set $R = S_x$; the claim is then that $[R]_\mathcal{F} \subseteq X$.

By 3.4.13, for each $\mathcal{F}$-conjugate of R, say $R_i = \varphi_i R$, there is some x_i such that $R_i \leq S_{x_i}$. By assumption that $|R| \geq |S_{x_i}|$, we must actually have equality. Thus the sub S-set spanned by the $\{x_i\}$ is isomorphic to $[R]_\mathcal{F}$. The proof of 3.3.4 shows that $Y = X \backslash [R]_\mathcal{F}$ is again $\mathcal{F}$-stable, so we may apply the inductive hypothesis. □

Corollary 3.4.15. *As a $\mathbb{Z}$-module, the rank of $\mathcal{A}(\mathcal{F})$ is equal to the number of $\mathcal{F}$-conjugacy classes of subgroups of S.*

Remark 3.4.16. As hinted above, we have actually proved more than we set out to: The important part of the calculation of a basis for $\mathcal{A}(\mathcal{F})$ was not that the underlying $\mathfrak{p}$-group S was abelian, but that $\mathcal{F}$ is generated by $\mathcal{F}(S)$. Indeed, whenever $\mathcal{F} = N_\mathcal{F}(S)$ the proof of 3.4.14 carries through to give the same description of the basis of $\mathcal{A}(\mathcal{F})$, with the sole modification that we must now define $[P]_\mathcal{F}$ to sum over representatives of the S-conjugacy classes of the $\mathcal{F}$-conjugacy class of P.[3] ♢

[3] Cf. [Lin] for further details on normalizer subsystems. We shall not make use of the concept of normalizer subsystems or normal subgroups of fusion systems in this document, so we omit this part of the background.

Chapter 4

Fusion action systems

The notion of $\mathcal{F}$-stability of S-sets is, it turns out, too flabby for us to work with unstably. The problem can be seen most clearly when asking the simple question: What are the morphisms of $\mathcal{F}$-stable S-sets? As $\mathcal{F}$-stability is just a condition and not additional structure, the most reasonable answer seems to be that morphisms of $\mathcal{F}$-stable S-sets are morphisms of the underlying S-sets. This cannot be "right," however, as considering the case that $S \in \mathrm{Syl}_{\mathfrak{p}}(G)$ and X a G-set will quickly show: An S-equivariant permutation of X need not be G-equivariant.

The problem stems from Part (3) of Proposition 3.2.3: For any $\mathcal{F}$-stable S-set X and $\varphi \in \mathcal{F}(P, Q)$, the P-sets ${}_P X$ and ${}^{\varphi}_{P} X$ are abstractly isomorphic, but we do not identify by *which* isomorphism. These choices of isomorphisms form an extra level of structure, of which we will make great use in the sequel. We call the totality of this structure a *fusion action system.*

4.1 The ambient case

Let us look again and more closely at the $\mathfrak{p}$-local data of a finite group acting on a finite set. Let G be a finite group, X a G-set, and $S \in \mathrm{Syl}_{\mathfrak{p}}(G)$. Recall the shorthand $\mathcal{F}_G = \mathcal{F}_S(G)$ and $\mathcal{F}_S = \mathcal{F}_S(S)$.

Definition 4.1.1. The *fusion action system on S relative to the G-action on X* is the category $\mathfrak{X}_G := \mathfrak{X}_S(G)$ whose objects are the subgroups $P \leq S$ and whose morphisms are given by

$$\mathfrak{X}_G(P,Q) = \left\{(\varphi,\sigma)\middle|\exists g \in G \text{ s.t. } \varphi = c_g|_P \text{ and } \sigma = \ell_g : {}_P X \cong {}^{\varphi}_{P} X\right\}.$$

Composition is defined coordinatewise. ♢

Example 4.1.2. The restriction of the G-action to S defines a fusion action system $\mathfrak{X}_S$. This is a minimal fusion action system, in that for all H containing S as a Sylow subgroup and acting on X with a given restricted S-action, $\mathfrak{X}_S \subseteq \mathfrak{X}_H$. Compare to to the situation $\mathcal{F}_S \subseteq \mathcal{F}_H$ for all Sylow supergroups H of S. ♢

Remark 4.1.3. The fusion action system $\mathfrak{X}_G$ has an *underlying fusion system* on S, obtained by simply ignoring the second coordinates of the morphisms of $\mathfrak{X}_G$. This is of course just $\mathcal{F}_G$. If we ignore the first coordinates of $\mathfrak{X}_G$, we obtain another interesting algebraic structure, which turns out to be, effectively, a transporter system on a quotient of S.

The G-action on X is determined by a group map $\rho : G \to \Sigma_X$. Denote by $\overline{H}$ the image of a subgroup $H \leq G$ in Σ_X. Then $\overline{S} \leq \mathrm{Syl}_{\mathfrak{p}}(\overline{G})$, and we can talk about the transporter system on $\overline{S}$ relative to this inclusion, $\mathcal{T}_{\overline{G}} := \mathcal{T}_{\overline{S}}(\overline{G})$. Then the functor $\mathfrak{X}_G \to \mathcal{T}_{\overline{G}}$ given on objects by $P \mapsto \overline{P}$ and on morphisms by projection onto the second factor is surjective, in the sense that any morphism of $\mathcal{T}_{\overline{G}}$ lies in the image of a morphism of $\mathfrak{X}_G$. ♢

Thus the fusion action system $\mathfrak{X}_G$ comes naturally equipped with natural surjective functors $\mathcal{F}_G \xleftarrow{\pi_{\mathcal{F}}} \mathfrak{X}_G \xrightarrow{\pi_{\mathcal{T}}} \mathcal{T}_{\overline{G}}$ that relate $\mathfrak{X}_G$ to two reasonably well understood algebraic objects. We shall investigate the structure of the fusion action system in terms of this pair of a fusion system and a transporter system. In particular, we shall see that the various automorphism groups of these categories can be described in terms of certain subgroups of S and G.

It turns out that working with the functor $\pi_{\mathcal{T}} : \mathfrak{X}_G \to \mathcal{T}_{\overline{G}}$ is not most convenient for our purposes. Instead, we will simply consider the projections onto the second coordinate directly.

Definition 4.1.4. For any pair of subgroups $P, Q \leq S$, let $\pi_\Sigma : \mathfrak{X}_G(P,Q) \to \Sigma_X$ be the set-map projection onto the second coordinate. When $P = Q$, the set-map π_Σ is a homomorphism of groups. In this case let $\Sigma_{\mathfrak{X}}^G(P)$ be the image of $\mathfrak{X}_G(P)$ under π_Σ and $\Sigma_{\mathfrak{X}}^S$ the image of $\mathfrak{X}_S$ under π_Σ.

In general, the notation will not record the source P or target Q. $\diamondsuit$

To emphasize the connection with fusion systems, we introduce the following notation:

Notation 4.1.5. We will denote by $\mathrm{Aut}_G(P;X)$ the group $\mathfrak{X}_G(P)$. This group is a simultaneous generalization of $\mathrm{Aut}_G(P) = \mathcal{F}_G(P)$ and of the group $\mathrm{Aut}_G(X)$ of G-set automorphisms of X. Similarly define $\mathrm{Aut}_S(P;X)$. $\diamondsuit$

There are several groups of automorphisms that arise from inspection of the category $\mathfrak{X}_G$, two of which are $\mathrm{Aut}_G(P;X)$ and $\mathrm{Aut}_S(P;X)$. We need some notation to introduce the others.

Notation 4.1.6. Let $\widehat{K}$ be the *core*, or kernel, of the G action on X. We define the *X-normalizer* and *X-centralizer in G* of a subgroup $H \leq G$ to be

$$N_G(H;X) := N_G(H) \cap \widehat{K} \qquad \text{and} \qquad Z_G(H;X) := Z_G(H) \cap \widehat{K}$$

Similarly, let K be the core of the S-action on ${}_SX$, so $K = \widehat{K} \cap S$. The *X-normalizer* and *X-centralizer in S* of $P \leq S$ are then

$$N_S(P;X) := N_S(P) \cap K \qquad \text{and} \qquad Z_S(P;X) := Z_S(P) \cap K$$

Note that $N_G(H;X)$ is just another name for $N_{\widehat{K}}(H)$. We use this notation to emphasize the idea that G is acting simultaneously on its subgroups (by conjugation) and on X (by left multiplication). $\diamondsuit$

Definition 4.1.7. For any $P \leq S$, we have the inclusions of the groups $Z_G(P;X)$, $N_G(P;X)$, $Z_G(P)$, and $N_G(P)$ as depicted in Figure 4-1. All of these inclusions are normal, so we name to the respective quotients in that Figure as well. The term G-*automizers of* P will refer to any of these quotient groups.

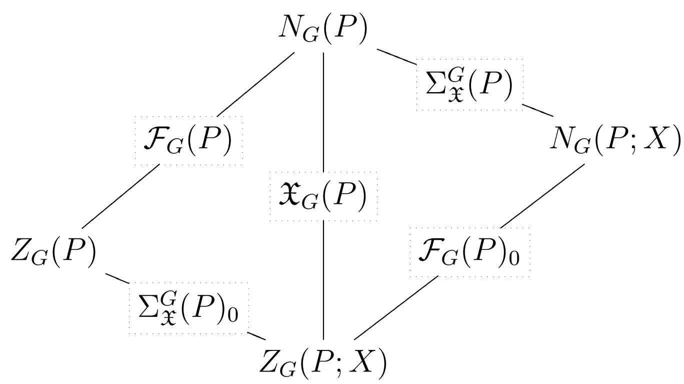


Figure 4-1: Naming the G-automizer groups

We can expand on the definitions of these G-automizers as follows:

$$\begin{array}{rcccl} \mathfrak{X}_G(P) & = & N_G(P)/Z_G(P;X) & = & \mathrm{Aut}_G(P;X) \\ \mathcal{F}_G(P) & = & N_G(P)/Z_G(P) & = & \big\{\varphi \in \mathrm{Aut}(P) \big| (\varphi,\sigma) \in \mathfrak{X}_G(P)\big\} \\ \Sigma_{\mathfrak{X}}^G(P) & = & N_G(P)/N_G(P;X) & = & \big\{\sigma \in \Sigma_X \big| (\varphi,\sigma) \in \mathfrak{X}_G(P)\big\} \\ \mathcal{F}_G(P)_0 & = & N_G(P;X)/Z_G(P;X) & = & \big\{\varphi \in \mathrm{Aut}(P) \big| (\varphi,\mathrm{id}_X) \in \mathfrak{X}_G(P)\big\} \\ \Sigma_{\mathfrak{X}}^G(P)_0 & = & Z_G(P)/Z_G(P;X) & = & \big\{\sigma \in \Sigma_X \big| (\mathrm{id}_P,\sigma) \in \mathfrak{X}_G(P)\big\} \end{array}$$

We also have the short exact sequences

$$1 \longrightarrow \Sigma_{\mathfrak{X}}^G(P)_0 \longrightarrow \mathfrak{X}_G(P) \longrightarrow \mathcal{F}_G(P) \longrightarrow 1$$

$$1 \longrightarrow \mathcal{F}_G(P)_0 \longrightarrow \mathfrak{X}_G(P) \longrightarrow \Sigma_{\mathfrak{X}}^G(P) \longrightarrow 1$$

Clearly $\Sigma_{\mathfrak{X}}^G(P)_0$ can be identified with a subgroup of $\mathrm{Aut}_P(X)$ and $\mathcal{F}_G(P)_0$ with a subgroup of $\mathcal{F}_G(P)$. If $\varphi \in \mathcal{F}_G(P)_0$, the identity map defines an isomorphism of P-sets $\mathrm{id}_X : {}_PX \cong {}_P^{\varphi}X$, or $\ell_p = \ell_{\varphi(p)}$ for all $p \in P$. Thus $\varphi(p) = p \bmod \widehat{K}$, so we have $\mathcal{F}_G(P)_0 \leq \ker\big(\mathcal{F}_G(P) \to \mathcal{F}_{\overline{G}}(\overline{P})\big)$.[1] $\diamondsuit$

[1] Here, we are using the fact that if K is an $\mathcal{F}$-strongly closed subgroup of S, there is a natural fusion system on $\overline{G}$, denoted $\mathcal{F}_{\overline{G}}$, and a fusion-preserving functor $\mathcal{F}_G \to \mathcal{F}_{\overline{G}}$. As we shall not make further use of this fact, we refer the reader to [Pui1, Lin] for more details.

Relative to S we have the same relationships amongst the groups $Z_S(P;X)$, $Z_S(P)$, $N_S(P;X)$, and $N_S(P)$. We also have the inclusions $Z_S(P;X) \leq Z_G(P;X)$, etc. These data give rise to the rather more complicated diagram of Figure 4-2.

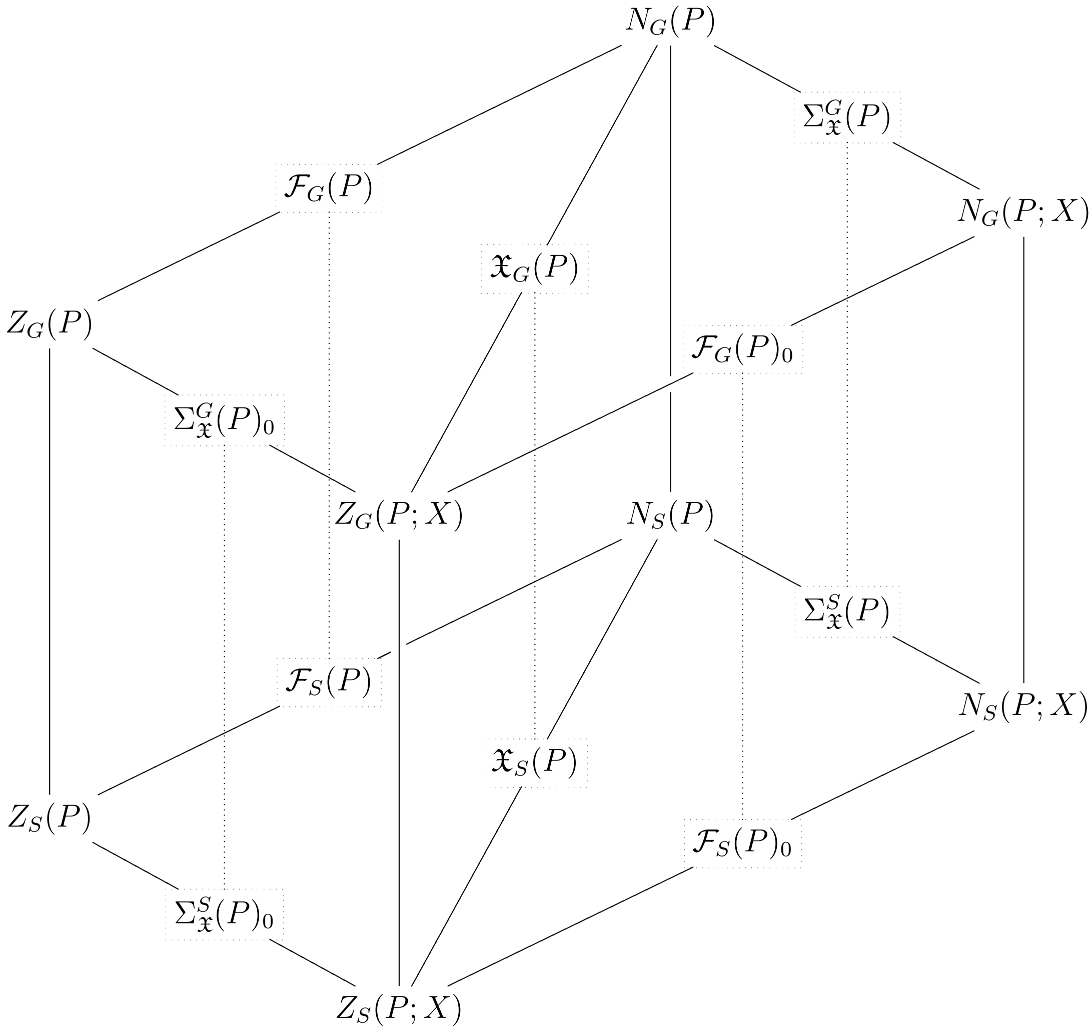


Figure 4-2: Comparing S-automizers to G-automizers

This diagram allows us to relate the "minimal" automizer groups—those that arise from the simultaneous S-action on its subgroups and X—with those that arise from G. For certain subgroups $P \leq S$, there is a particularly nice relationship; to express this, we will need the following terminology:

Definition 4.1.8. Given the fusion action system $\mathfrak{X}_G$ and $P \leq S$, we say that

- P is *fully normalized relative to* G if $N_S(P) \in \mathrm{Syl}_{\mathfrak{p}}(N_G(P))$.
- P is *fully centralized relative to* G if $Z_S(P) \in \mathrm{Syl}_{\mathfrak{p}}(Z_G(P))$.
- P is *fully* X*-normalized relative to* G if $N_S(P;X) \in \mathrm{Syl}_{\mathfrak{p}}(N_G(P;X))$.
- P is *fully* X*-centralized relative to* G if $Z_S(P;X) \in \mathrm{Syl}_{\mathfrak{p}}(Z_G(P;X))$.

The reference to G will be omitted if it is clear from the context. $\diamondsuit$

We can think of these definitions in the following manner: Instead of starting out with a chosen Sylow subgroup $S \leq G$ and a $\mathfrak{p}$-subgroup $P \leq S$, we want to investigate the $\mathfrak{p}$-subgroup P on its own terms. For instance, to understand the $\mathfrak{p}$-part of $N_G(P)$ we must pick a "right" Sylow of G: Such a Sylow must contain not just P, but also a Sylow subgroup of $N_G(P)$. Saying that P is fully normalized means that we have made this choice correctly within the G-conjugacy class of P inside S. Moreover, for a fixed Sylow S, it is always possible to do so, in the following sense:

Proposition 4.1.9. *For $P \leq S$, there is some $g \in G$ such that ${}^gP \leq S$ is fully normalized relative to G, or fully centralized relative to G, or fully X-normalized relative to G, or fully X-centralized relative to G.*

Proof. Sylow's theorems. $\square$

We now use this terminology to describe the relationships between the groups $\mathfrak{X}_G(P)$ and $\mathrm{Aut}_S(P;X)$, etc.:

Lemma 4.1.10. *Fix $\mathfrak{X}_G$ and $P \leq S$.*

(1) If P is fully normalized relative to G, then

- $\mathrm{Aut}_S(P) \in \mathrm{Syl}_{\mathfrak{p}}(\mathcal{F}_G(P))$
- $\mathrm{Aut}_S(P;X) \in \mathrm{Syl}_{\mathfrak{p}}(\mathfrak{X}_G(P))$
- $\Sigma^S_{\mathfrak{X}}(P) \in \mathrm{Syl}_{\mathfrak{p}}\left(\Sigma^G_{\mathfrak{X}}(P)\right)$

and furthermore P is fully centralized, X-normalized, and X-centralized relative to G.

(2) If P is fully X-normalized relative to G, then $\mathcal{F}_S(P)_0 \in \mathrm{Syl}_{\mathfrak{p}}(\mathcal{F}_G(P)_0)$ and P is fully X-centralized relative to G.

(3) If P is fully centralized relative to G, then $\Sigma^S_{\mathfrak{X}}(P)_0 \in \mathrm{Syl}_{\mathfrak{p}}\left(\Sigma^G_{\mathfrak{X}}(P)_0\right)$ and P is fully X-centralized relative to G.

Proof. All of the proofs are basically the same, so we just prove that if P is fully normalized then $\mathrm{Aut}_S(P;X) \in \mathrm{Syl}_{\mathfrak{p}}(\mathfrak{X}_G(P))$ and P is fully X-centralized.

We have $\mathfrak{X}_G(P) = N_G(P)/Z_G(P;X)$ and $\mathrm{Aut}_S(P;X)$ is the image in $\mathfrak{X}_G(P)$ of $N_S(P)$, which is by assumption Sylow in $N_G(P)$. The result then follows from the general statement that the image of a Sylow is Sylow in the quotient.

To see that P is fully X-centralized, consider the digram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z_G(P;X) & \longrightarrow & N_G(P;X) & \longrightarrow & \mathfrak{X}_G(P) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow \scriptstyle{\text{Sylow}} & & \uparrow \scriptstyle{\text{Sylow}} & & \\ 1 & \longrightarrow & Z_S(P;X) & \longrightarrow & N_S(P;X) & \longrightarrow & \mathrm{Aut}_S(P;X) & \longrightarrow & 1 \end{array}$$

where the unmarked inclusion is Sylow because the other two inclusions are. □

Lemma 4.1.10 tells us certain properties of $\mathfrak{X}_S(G)$ that follow from a subgroup $P \leq S$ having the property that the normalizer, X-normalizer, or centralizer of S is Sylow in the respective group of G. It gives no information, however, about the case when P is fully X-centralized relative to G, which turns out to be very important for understanding extensions of morphisms in the fusion action system. Recall that if $g \in G$ is such that ${}^gP \leq S$, then g determines the morphism $(c_g, \ell_g) \in \mathfrak{X}_G(P, {}^gP)$.

Definition 4.1.11. For $P \leq S$ and $g \in G$ such that ${}^gP \leq S$, define

$$N_{(c_g,\ell_g)} = \left\{ n \in N_S(P) \middle| \exists s \in N_S({}^gP) \text{ s.t. } (c_{gng^{-1}}, \ell_{gng^{-1}}) = (c_s, \ell_s) \right\}.$$

This is the *extender* of (c_g, ℓ_g) in the fusion action system. ♢

Remark 4.1.12. Consider the following easy observations:

- $P \cdot Z_S(P;X) \leq N_{(c_g,\ell_g)} \leq N_S(P)$.

- We can define an analogue of the notion of translation along morphisms, as in Definition 2.3.9. Given $g \in G$ and $P \leq S$ such that ${}^gP \leq S$, define

  $$t_{(c_g,\ell_g)} : \mathfrak{X}_G(P) \to \mathfrak{X}_G({}^gP) : (\varphi, \sigma) \mapsto (c_g \varphi c_g^{-1}, \ell_g \sigma \ell_g^{-1}).$$

  We could then define $N_{(c_g,\ell_g)}$ to be the subgroup of $n \in N_S(P)$ such that

  $$(c_n, \ell_n) \in t^{-1}_{(c_g,\ell_g)} \left(\mathrm{Aut}_S({}^gP;X)\right)$$

  or even more confusingly, the preimage in $N_S(P)$ of the preimage in $\mathfrak{X}_G(P)$ of $\mathrm{Aut}_S({}^gP;X)$.[2]

- The extender $N_{(c_n,\ell_n)}$ is the largest subgroup of $N_S(P)$ to which we could hope to extend (c_g, ℓ_g). Here, extension of a morphism means that we would find a $g' \in G$ such that ${}^{g'}N_{(c_g,\ell_g)} \leq S$, $\ell_{g'} = \ell_g$, and $c_{g'} = c_g$ on P,[3] and in this case we say that $(c_{g'}, \ell_{g'}) \in \mathfrak{X}_G\left(N_{(c_g,\ell_g)}, S\right)$ is an extension of $(c_g, \ell_g) \in \mathfrak{X}_G(P, S)$.

  The reason why $N_{(c_g,\ell_g)}$ is the domain of the largest possible extension is as follows: Given such an extension defined by g', pick $n \in N_S(P)$. If ${}^{g'}n \leq S$, then the fact that ${}^{g'}p = {}^gp$ for all $p \in P$ implies that $g'n(g')^{-1} \in N_S({}^gP)$. Thus $(c_{g'} c_n c_{g'}^{-1}, \ell_{g'} \ell_n \ell_{g'}^{-1}) \in \mathrm{Aut}_S({}^gP;X)$, which is just to say that $n \in N_{(c_g,\ell_g)}$.

- If g' defines an extension of (c_g, ℓ_g) as above, then in fact the extension $(c_{g'}, \ell_{g'})$ lives in $\mathfrak{X}_G\left(N_{(c_g,\ell_g)}, N_S({}^gP)\right)$. ♢

The importance of $P \leq S$ having the property that $Z_S(P;X) \in \mathrm{Syl}_{\mathfrak{p}}\left(Z_G(P;X)\right)$ is given by the following:

[2]If we followed this line of discussion further, and with greater detail, we would arrive at a notion similar to Puig's original definition of Frobenius categories in [Pui1].

[3]In other words, $g'Z_G(P;X) = gZ_G(P;X)$.

Lemma 4.1.13. *Let $P \leq S$ and $g \in G$ be such that ${}^gP \leq S$ is fully X-centralized in G. Then there is a $g' \in G$ such that ${}^{g'}N_{(c_g,\ell_g)} \leq S$ and $(c_{g'}, \ell_{g'})\big|_P = (c_g, \ell_g)$. In other words, g' defines an extension of (c_g, ℓ_g) to a morphism in $\mathfrak{X}_G\left(N_{(c_g,\ell_g)}, S\right)$.*

Proof. If ${}^gP$ is fully X-centralized, then

$$\begin{aligned}
[N_S({}^gP) : N_S({}^gP) \cdot Z_G({}^gP; X)] &= \frac{|N_S({}^gP)|\,|Z_G({}^gP;X)|\,/|Z_S({}^gP;X)|}{|N_S({}^gP)|} \\
&= [Z_S({}^gP;X) : Z_G({}^gP;X)]
\end{aligned}$$

is prime to $\mathfrak{p}$, so $N_S({}^gP) \in \mathrm{Syl}_{\mathfrak{p}}(N_S({}^gP) \cdot Z_G({}^gP;X))$. From the definition of the extender we have ${}^gN_{(c_g,\ell_g)}$ is a $\mathfrak{p}$-subgroup of $N_S({}^gP) \cdot Z_G({}^gP;X)$, so we can choose some $z \in Z_G({}^gP;X)$ so that ${}^{zg}N_{(c_g,\ell_g)} \leq N_S({}^gP)$. Then setting $g' = zg$ gives the desired extension $(c_{g'}, \ell_{g'})$. □

4.2 Abstract fusion actions of $\mathcal{F}$ on X

In this section we describe an abstraction of the fusion action systems of Section 4.1 without reference to an ambient group: We reproduce Puig's notion of abstract fusion systems in the context of fusion action systems, but we still fix the underlying fusion system $\mathcal{F}$. Let us call the resulting fusion action system an *abstract fusion action of $\mathcal{F}$ on X*.

Definition 4.2.1. For $\mathcal{F}$ a saturated fusion system on S and X an $\mathcal{F}$-stable S-set, let $\mathcal{F}^X_\Sigma$ be the category whose objects are the subgroups $P \leq S$ and whose Hom-sets are given by

$$\mathcal{F}^X_\Sigma(P,Q) = \{(\varphi,\sigma) | \varphi \in \mathcal{F}(P,Q), \sigma : {}_PX \cong {}^{\varphi}_{P}X\}.$$

There are natural functors $\mathcal{T}_S(S) \xrightarrow{\delta} \mathcal{F}^X_\Sigma \xrightarrow{\pi_{\mathcal{F}}} \mathcal{F}$, each of which is the identity on objects. On morphisms, $\pi_{\mathcal{F}}$ is the projection onto the first factor, and δ is defined by $\delta_{P,Q} : N_S(P,Q) \to \mathcal{F}^X_\Sigma(P,Q)$ is the map sending s to (c_s, ℓ_s). ◊

It is worth reemphasizing that the relationship between the coordinates of the morphism $(\varphi,\sigma) \in \mathcal{F}^X_\Sigma(P,Q)$ that follows from the condition on σ:

Definition 4.2.2. For $\varphi \in \mathrm{Hom}(P, Q)$ and $\sigma \in \Sigma_X$, we say that the pair (φ, σ) is *intertwined* if for all $p \in P$ and $x \in X$, we have $\sigma(p \cdot x) = \varphi(p) \cdot \sigma(x)$. $\diamondsuit$

We have already seen certain subcategories of $\mathcal{F}_\Sigma^X$ arise in the context of ambient fusion action systems. In particular, for G acting on X with $S \in \mathrm{Syl}_{\mathfrak{p}}(G)$, we have $\mathfrak{X}_S \subseteq \mathfrak{X}_G \subseteq \mathcal{F}_\Sigma^X$.

That $\mathfrak{X}_S$ is a subcategory of $\mathfrak{X}_G$ is really reflecting the fact that there is a natural functor $\delta : \mathcal{T}_S(S) \to \mathfrak{X}_G$. That $\mathfrak{X}_G \subseteq \mathcal{F}_\Sigma^X$ is just to say that $\mathcal{F}_\Sigma^X$ is the maximal world of discourse for fusion action systems. Therefore, we are interested in identifying those subcategories of $\mathcal{F}_\Sigma^X$ which appropriately mimic the structure of $\mathfrak{X}_G$.

Definition 4.2.3. An *abstract action* of the fusion system $\mathcal{F}$ on the $\mathcal{F}$-stable S-set X is a category $\mathfrak{X}$ such that $\mathfrak{X}_S \subseteq \mathfrak{X} \subseteq \mathcal{F}_\Sigma^X$. We also require that the *Divisibility Axiom* hold: Every morphism of $\mathfrak{X}$ factors as an isomorphism followed by an inclusion. $\diamondsuit$

In $\mathfrak{X}$, the question of whether a morphism $(\varphi, \sigma) \in \mathfrak{X}(P, Q)$ is an isomorphism is determined by whether $\varphi \in \mathcal{F}(P, Q)$ is. Also, by "inclusion" in $\mathfrak{X}$ we mean the morphism $(\iota_P^Q, \mathrm{id}_X) \in \mathfrak{X}(P, Q)$, where $\iota_P^Q : P \leq Q$ is the inclusion of subgroups.

Remark 4.2.4. As defined, the abstract action $\mathfrak{X}$ is associated to the fusion system $\mathcal{F}$, but this is not reflected in the notation. This foreshadows Section 4.3 and all that follows, where we take the view that $\mathfrak{X}$ determines $\mathcal{F}$. $\diamondsuit$

As in the case of abstract fusion systems, this definition captures the structure of a $\mathfrak{p}$-subgroup S of G acting on X, but does not reflect the more interesting situation that S is Sylow in G. We must therefore introduce saturation axioms. The following is a list of definitions that will be needed to state the saturation axioms; many of these are obvious and some are repeated from before, but all are included here for ease of reference:

Definition 4.2.5. Let $\mathcal{F}$ be a saturated fusion system, X an $\mathcal{F}$-stable S-set, and $P \leq S$.

- P is *fully normalized in* $\mathcal{F}$ if $|N_S(P)| \geq |N_S(Q)|$ for all $Q \cong_{\mathcal{F}} P$.
- P is *fully centralized in* $\mathcal{F}$ if $|Z_S(P)| \geq |Z_S(Q)|$ for all $Q \cong_{\mathcal{F}} P$.

- P is *fully X-normalized in $\mathcal{F}$* if $|N_S(P;X)| \geq |N_S(Q:X)|$ for all $Q \cong_{\mathcal{F}} P$.

- P is *fully X-centralized in $\mathcal{F}$* if $|Z_S(P;X)| \geq |Z_S(Q:X)|$ for all $Q \cong_{\mathcal{F}} P$. $\diamondsuit$

If $\mathcal{F} = \mathcal{F}_G$ for some finite group G, it is well known (cf. [BLO2], Proposition 1.3) that P is fully normalized in $\mathcal{F}$ if and only if P is fully normalized with respect to G, and similarly for both definitions of full centralization. The following Proposition shows that the same is true for the two new terms we have introduced:

Proposition 4.2.6. *If the saturated fusion system $\mathcal{F}$ on S is realized by G, $P \leq S$ is fully X-normalized if and only if*

$$N_S(P;X) \in \mathrm{Syl}_{\mathfrak{p}}(N_G(P;X))$$

Similarly, P is fully X-centralized if and only if

$$Z_S(P;X) \in \mathrm{Syl}_{\mathfrak{p}}(Z_G(P;X))$$

Proof. We prove the result for X-normalizers and note that the same argument works for X-centralizers.

The "if" implication is clear: if $N_S(P;X) = S \cap N_G(P;X)$ is Sylow in $N_G(P;X)$, then $|N_S(P;X)| \geq |N_S({}^gP;X)|$ for all $g \in G$ such that ${}^gP \leq S$.

Suppose now that $|N_S(P;X)| \geq |N_S({}^gP;X)|$ for all $g \in G$ such that ${}^gP \leq S$. Pick $T \in \mathrm{Syl}_{\mathfrak{p}}(N_G(P;X))$, and $g \in G$ such that $T \cdot P \leq {}^gS$. Then we have $P^g \leq S$, and the fact that $N_{{}^gS}(P;X) \in \mathrm{Syl}_{\mathfrak{p}}(N_G(P;X))$ implies that $N_S(P^g;X) \in \mathrm{Syl}_{\mathfrak{p}}(N_G(P^g;X))$ (this uses the fact that the core of the action is normal in G). The assumption that P is fully X-normalized now implies that the orders of the X-normalizers in S of P and ${}^gP = {}^{g^{-1}}P$ are equal, which in turn forces $N_S(P;X) \in \mathrm{Syl}_{\mathfrak{p}}(N_G(P;X))$, as desired. $\square$

The last ingredient we need to state the saturation axioms is some way of relating $\mathfrak{X}$ to some more easily understood objects. Just as the fusion action system $\mathfrak{X}_G$ came equipped with a functor to $\mathcal{F}_G$, projection onto the first coordinate of the morphisms defines a surjective functor $\mathfrak{X} \to \mathcal{F}$. In the case of ordinary fusion action systems, projection onto the second coordinate gave a surjection to a transporter system of a certain finite group. Let us determine which transporter system we should map to, without reference to an ambient G.

Definition 4.2.7. Let $\mathfrak{S}$ denote the subgroup of Σ_X generated by all permutations that appear in the second coordinate of some morphism of $\mathfrak{X}$. $\diamondsuit$

Lemma 4.2.8. *Projection onto the second coordinate identifies $\mathfrak{X}(1)$ with $\mathfrak{S}$.*

Proof. As $\mathfrak{X}(1)$ is already a group, it will suffice to show that if $(\varphi, \sigma) \in \mathfrak{X}(P, Q)$ then $(1, \sigma) \in \mathfrak{X}(1)$. As $\mathfrak{X}_S \subseteq \mathfrak{X}$ we have that the inclusion morphism $(\iota_1^P, \mathrm{id}_X) \in \mathfrak{X}(1, P)$, and therefore we also have that the composite $(\iota_1^Q, \sigma) = (\varphi, \sigma) \circ (\iota_1^P, \mathrm{id}_X)$ lies in $\mathfrak{X}(1, Q)$. Then the divisibility axiom for fusion actions forces $(\mathrm{id}_1, \sigma) \in \mathfrak{X}(1)$, and the result is proved. $\square$

As $\mathfrak{X}_S \subseteq \mathfrak{X}$, it follows that $\overline{S}$, the group of permutations of X induced by some element of S, is a $\mathfrak{p}$-subgroup of $\mathfrak{S}$. Let us ignore for moment the question of whether this is a Sylow inclusion.

Notation 4.2.9. Let $\mathcal{T}_{\mathfrak{S}}$ denote the transporter system on $\overline{S}$ relative to $\mathfrak{S}$. $\diamondsuit$

Recall that K denotes the core of the S-action on X.

Definition 4.2.10. Let $\mathfrak{X}$ be an abstract action of $\mathcal{F}$ on X and pick $P \leq S$.

- Let the functors $\mathcal{F} \xleftarrow{\pi_{\mathcal{F}}} \mathfrak{X} \xrightarrow{\pi_{\mathcal{T}}} \mathcal{T}_{\mathfrak{S}}$ be the projections onto the first and second coordinates of the morphisms of $\mathfrak{X}$, respectively. On objects, $\pi_{\mathcal{F}}$ is the identity and $\pi_{\mathcal{T}}$ is reduction mod K.

- Let π_Σ denote any of the maps $\mathfrak{X}(P, Q) \to \Sigma_X$ obtained by projection onto the second coordinate. As in the ambient case, if $P = Q$ then π_Σ is a group map.

- Let $\mathcal{F}(P)_0 := \ker[\pi_{\mathcal{T}} : \mathfrak{X}(P) \to \mathcal{T}_{\mathfrak{S}}(\overline{P})]$ denote the group of automorphisms of P that are intertwined with id_X in $\mathfrak{X}(P)$.

- Let $\Sigma_{\mathfrak{X}}(P) := \Sigma_X^{\mathcal{F}}(P; \mathfrak{X}) = \pi_\Sigma(\mathfrak{X}(P))$ denote the group of permutations of X that appear in the second coordinate of some morphism of $\mathfrak{X}(P)$.

- Let $\Sigma_{\mathfrak{X}}(P)_0 = \ker[\pi_{\mathcal{F}} : \mathfrak{X}(P) \to \mathcal{F}(P)]$ denote the set of all permutations of X that are intertwined with id_P in $\mathfrak{X}(P)$.

- For any isomorphism $(\varphi, \sigma) \in \mathfrak{X}(P, Q)$, define the group $N_{(\varphi,\sigma)} \leq N_S(P)$ to be

$$N_{(\varphi,\sigma)} = \left\{ n \in N_S(P) \middle| (\varphi c_n \varphi^{-1}, \sigma \ell_n \sigma^{-1}) \in \mathrm{Aut}_S(P; X) \right\}.$$

  In other words, for each n in the *extender* $N_{(\varphi,\sigma)}$, there is some $s \in S$ such that $(\varphi c_n \varphi^{-1}, \sigma \ell_n \sigma^{-1}) = (c_s, \ell_s) \in \mathfrak{X}(Q)$.

Similarly define $\mathcal{F}_S(P)_0 \leq \mathcal{F}(P)_0$, $\Sigma_{\mathfrak{X}}^S(P) \leq \Sigma_{\mathfrak{X}}(P)$, and $\Sigma_{\mathfrak{X}}^S(P)_0 \leq \Sigma_{\mathfrak{X}}(P)_0$ to be the respective subgroups induced by elements of S. $\diamondsuit$

We are now ready to state the saturation axioms for an abstract action of $\mathcal{F}$ on X.

Definition 4.2.11. The abstract action $\mathfrak{X}$ is *saturated* if

1. For any $P \leq S$, the following implications hold:

$$\begin{array}{ccc} P \text{ fully normalized} & \Longrightarrow & P \text{ fully } X\text{-normalized} \\ \Downarrow & & \Downarrow \\ P \text{ fully centralized} & \Longrightarrow & P \text{ fully } X\text{-centralized} \end{array}$$

2. If P is fully normalized, then

   - $\mathrm{Aut}_S(P) \in \mathrm{Syl}_{\mathfrak{p}}(\mathcal{F}(P))$
   - $\mathrm{Aut}_S(P; X) \in \mathrm{Syl}_{\mathfrak{p}}(\mathfrak{X}(P))$
   - $\Sigma_{\mathfrak{X}}^S(P) \in \mathrm{Syl}_{\mathfrak{p}}(\Sigma_{\mathfrak{X}}(P))$

3. If P is fully X-normalized, then $\mathcal{F}_S(P)_0 \in \mathrm{Syl}_{\mathfrak{p}}(\mathcal{F}(P)_0)$.

4. If P is fully centralized, then $\Sigma_{\mathfrak{X}}^S(P)_0 \in \mathrm{Syl}_{\mathfrak{p}}(\Sigma_{\mathfrak{X}}(P)_0)$.

5. If Q is fully X-centralized and $(\varphi, \sigma) \in \mathfrak{X}(P, Q)$ is an isomorphism, then there is some $(\widetilde{\varphi}, \sigma) \in \mathfrak{X}\left(N_{(\varphi,\sigma)}, S\right)$ that extends (φ, σ), i.e., such that $\varphi = \widetilde{\varphi}\big|_P$.

We shall refer to Points 2-4 collectively as the *Sylow Axioms* and Point 5 as the *Extension Axiom* for fusion actions. $\diamondsuit$

These axioms are highly redundant (e.g., the first assertion of Axiom 2 is actually an axiom for the saturation of $\mathcal{F}$); we include the entire list mainly to indicate the sort of properties we expect abstract actions of $\mathcal{F}$ to have. In particular, Axiom 1 is no requirement at all, as it follows from the assumptions that $\mathcal{F}$ is saturated and X is $\mathcal{F}$-stable (and hence its core K is strongly $\mathcal{F}$-closed):

Proposition 4.2.12. *If $P \leq S$ is fully normalized in $\mathcal{F}$ (resp. fully centralized), then P is fully X-normalized (X-centralized). Moreover, every fully X-normalized subgroup is also fully X-centralized.*

Proof. All of these facts depend on the result that if P is fully normalized in $\mathcal{F}$ and $Q \cong_{\mathcal{F}} P$, there is a morphism $\varphi \in \mathcal{F}(N_S(Q), N_S(P))$ (cf. [BLO2, Proposition A.2(b)]).

Thus if is Q is fully X-normalized, the fact that K is strongly closed in $\mathcal{F}$ forces $\varphi(c) \in K$ whenever this makes sense, so $\varphi(N_S(Q; X)) \leq N_S(P; X)$. Therefore the X-normalizer of a fully X-normalized subgroup injects into that of a fully normalized subgroup, and the first claim is proved.

The second claim follows from the facts that the image of the restriction of φ to $Z_S(Q)$ must lie in $Z_S(P)$ and that in a saturated fusion system fully normalized implies fully centralized, so for purposes of computing orders of centralizers we may as well consider P fully normalized. Intersection with K again yields the desired result.

Finally, if Q is fully X-normalized and P is fully normalized (hence fully X-centralized), the same argument shows that $|Z_S(Q; X)| = |Z_S(P; X)|$. $\square$

We close this section with a brief discussion of the motivation for the saturation axioms of Definition 4.2.11. It should be clear that we are mainly taking certain applications of Sylow's theorems from the ambient fusion action system case, especially those recorded in Lemmas 4.1.10 and 4.1.13, and simply defining them to be axioms when we do not have an actual group with which to work.

We can take this a step further. Figure 4-2 gave us a pictorial heuristic of how to view the Sylow conditions of a fusion action system. Considering the case of normalizers: If a subgroup $P \leq S$ is "good"with regard to normalizers—or fully normalized—then P is also "good" for centralizers, X-normalizers, and X-centralizers. Moreover, for any of the pairs of S- and G-automizers of P, if the S-autmoizer is Sylow in the G-autmoizer. The same is not true if P is only fully X-normalized, say, but it is true for all groups and automizers that live "beneath" the X-normalizer in Figure 4-2, and similarly if P is fully centralized.

For an abstract action of $\mathcal{F}$, we cannot reproduce all of Figure 4-2, but we still have access to the diagram of Figure 4-3.

We still have a notion of $P \leq S$ being "good" with regard to normalizers (etc.), though now that means P is fully normalized in the fusion-theoretic sense. The content of the Sylow Axioms is basically that if you're "good" at some point on Figure 4-3, all of the groups associated to P that lie beneath the good point have the property that a Sylow subgroup come from S.

4.3 Abstract fusion action systems

Now let us take the abstraction of the previous section a step further and forget not only the ambient group but also the fusion system on S itself. In other words, here we give ourselves only the data of the $\mathfrak{p}$-group S and an S-set X, and from this we examine categories that mimic $\mathfrak{X}_G$ where both G and $\mathcal{F}$ are allowed to vary.

Let us start by naming our universe of discourse:

Definition 4.3.1. Let $\mathfrak{U} := \mathfrak{U}(S; X)$ be the category whose objects are all subgroups $P \leq S$ and whose morphisms are given by

$$\mathfrak{U}(P,Q) = \big\{(\varphi,\sigma)\big|\varphi \in \mathrm{Inj}(P,Q), \sigma \in \Sigma_X, \text{ and } \varphi \text{ intertwines } \sigma\big\}$$

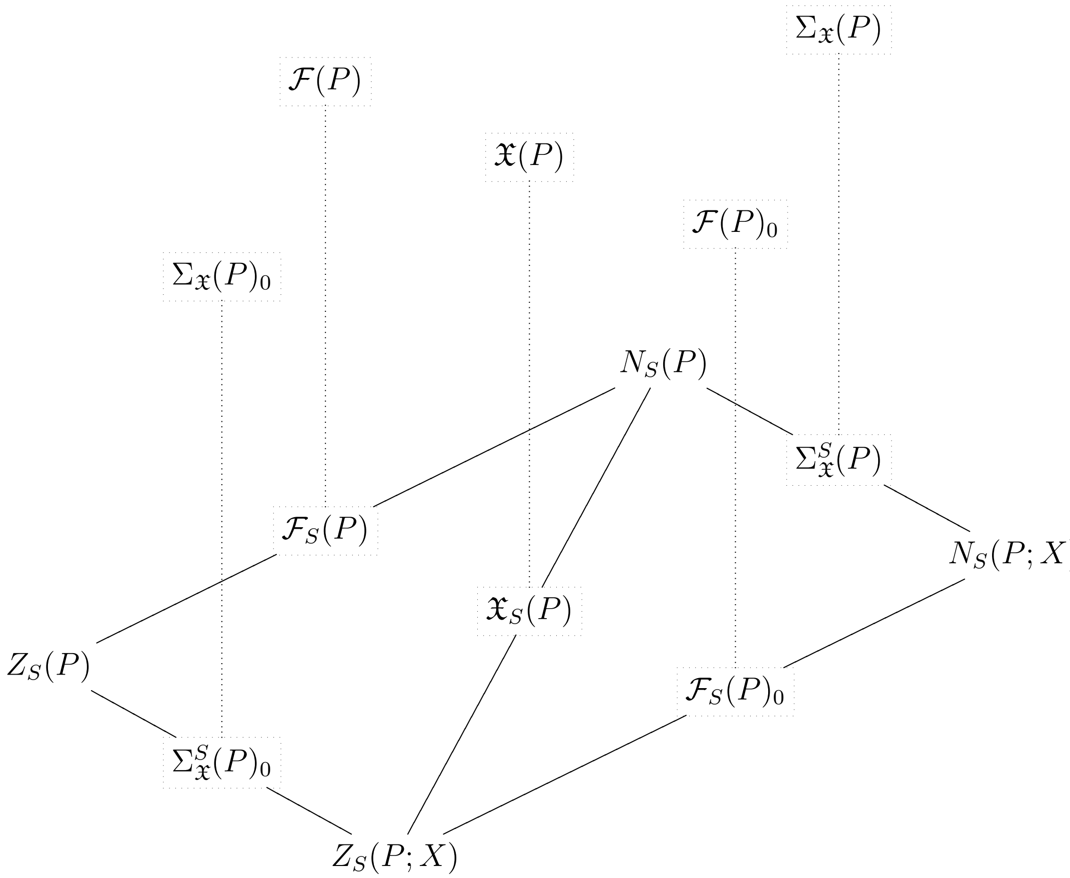


Figure 4-3: Comparing S-automorphisms to $\mathfrak{X}$-automorphisms

Composition is defined componentwise. Again, this means that $(\varphi, \sigma) \in \mathfrak{X}(P, Q)$ is an isomorphism if and only if the group map $\varphi : P \to Q$ is. $\diamondsuit$

By identifying $\mathfrak{U}$ as the "universe of discourse," we mean that this is the category in which all of our abstract fusion action systems will live.

Definition 4.3.2. An *abstract fusion action system* of S acting on X is a category $\mathfrak{X}$ such that $\mathfrak{X}_S \subseteq \mathfrak{X} \subseteq \mathfrak{U}$ and that satisfies the *Divisibility Axiom*: Every morphism of $\mathfrak{X}$ factors as an isomorphism followed by an inclusion.

The *underlying fusion system* of $\mathfrak{X}$ is the fusion system $\mathcal{F}^{\mathfrak{X}}$ on S where

$$\mathcal{F}^{\mathfrak{X}}(P, Q) = \left\{ \varphi \in \mathrm{Inj}(P, Q) \middle| (\varphi, \sigma) \in \mathfrak{X}(P, Q) \right\}.$$

In other words, $\mathcal{F}^{\mathfrak{X}}$ is the fusion system obtained by projecting the morphisms of $\mathfrak{X}$ onto the first coordinate. ♢

By referring to this underlying fusion system we are able to speak of $P \leq S$ being fully normalized or fully centralized (by definition, in $\mathcal{F}^{\mathfrak{X}}$), as well as fully X-normalized and X-centralized. The latter two are again formed by considering the normalizers or centralizers of subgroups intersected with K, the core of the action of S on X.

Lemma 4.3.3. *The S-set X is $\mathcal{F}^{\mathfrak{X}}$-stable.*

Proof. This follows from the Condition (2) of Proposition 3.2.3 and the definition of $\mathcal{F}^{\mathfrak{X}}$. □

Corollary 4.3.4. *The core K is strongly closed in $\mathcal{F}^{\mathfrak{X}}$.*

Proof. The proof of Proposition 3.3.7 carries over. This can also be seen directly by noting that if $k \in K$ and $\varphi \in \mathcal{F}^{\mathfrak{X}}(\langle k \rangle, S)$, then there is some $(\varphi, \sigma) \in \mathfrak{X}(\langle k \rangle, S)$. Then φ and σ's being intertwined implies that $\varphi(k) \cdot \sigma(x) = \sigma(k \cdot x) = \sigma(x)$ for all $x \in X$, and thus $\varphi(k) \in K$. □

Of course, we need further axioms to restrict ourselves to the interesting cases. Note that all the terms of Definition 4.2.10 still make sense in this context, so we may talk about the group $\mathfrak{S} = \mathfrak{X}(1) \leq \Sigma_X$, the functors $\pi_{\mathcal{F}}$ and $\pi_{\mathcal{T}}$, the maps π_Σ, and the extender of an isomorphism.

Definition 4.3.5. The fusion action system $\mathfrak{X}$ is *saturated* if all the axioms of Definition 4.2.11 are satisfied. ♢

Now the reason for including redundant axioms in Definition 4.2.11 should be more apparent:

Proposition 4.3.6. *If the abstract fusion action system $\mathfrak{X}$ is saturated, the underlying fusion system $\mathcal{F}^{\mathfrak{X}}$ is as well.*

Proof. All of the saturation conditions are clear except for the extension axiom. Let $\varphi \in \mathcal{F}^{\mathfrak{X}}(P,Q)$ be an isomorphism with Q fully centralized in $\mathcal{F}^{\mathfrak{X}}$. Recall that

$$N_\varphi = \left\{ n \in N_S(P) \middle| \varphi \circ c_n \circ \varphi^{-1} \in \mathrm{Aut}_S(Q) \right\}.$$

If $(\varphi, \sigma) \in \mathfrak{X}(P,Q)$ is a morphism that lies over φ, we have

$$N_{(\varphi,\sigma)} = \left\{ n \in N_S(P) \middle| \left(\varphi \circ c_n \circ \varphi - 1, \sigma \circ \ell_n \circ \sigma^{-1} \right) \in \mathrm{Aut}_S(Q;X) \right\}.$$

Clearly $N_{(\varphi,\sigma)} \leq N_\varphi$ for all possible choices of σ. If we can show that we have equality for some particular choice of σ, the extension axiom for the fusion action system $\mathfrak{X}$ will imply the desired axiom for the fusion system $\mathcal{F}^{\mathfrak{X}}$.

We can restate the problem as follows: Fix some σ such that $(\varphi,\sigma) \in \mathfrak{X}(P,Q)$, and let $N \leq \mathfrak{X}(Q)$ denote the group $(\varphi,\sigma) \circ \delta\left(N_\varphi\right) \circ (\varphi,\sigma)^{-1}$. Then N is a $\mathfrak{p}$-subgroup of $\mathfrak{X}(Q)$. If $N \leq \mathrm{Aut}_S(Q;X)$, by definition $N_\varphi = N_{(\varphi,\sigma)}$ and we are done. The goal then becomes to show that there is some $\tau \in \Sigma_{\mathfrak{X}}(Q)_0$ such that ${}^{\tau}N \leq \mathrm{Aut}_S(Q;X)$, for this will imply that $N_\varphi = N_{(\varphi,\tau\sigma)}$ and complete the proof.[4]

It is easy to see that $N \leq \Sigma_{\mathfrak{X}}(Q)_0 \cdot \mathrm{Aut}_S(Q;X)$ as follows: For $n \in N_\varphi$, we have $\varphi c_n \varphi^{-1} \in \mathrm{Aut}_S(Q)$, so $(\varphi c_n \varphi^{-1}, \sigma \ell_n \sigma^{-1})$ differs from an element of $\mathrm{Aut}_S(Q;X)$ by an element of $\Sigma_{\mathfrak{X}}(Q)_0$.

The claim is that $\mathrm{Aut}_S(Q;X)$ is Sylow in $\Sigma_{\mathfrak{X}}(Q)_0 \cdot \mathrm{Aut}_S(Q;x)$. We compute

$$\begin{aligned} [\mathrm{Aut}_S(Q;X) : \Sigma_{\mathfrak{X}}(Q)_0 \cdot \mathrm{Aut}_S(Q;X)] &= \frac{|\Sigma_{\mathfrak{X}}(Q)_0|\,|\mathrm{Aut}_S(Q;X)|}{|\Sigma_{\mathfrak{X}}(Q)_0 \cap \mathrm{Aut}_S(Q;X)|\,|\mathrm{Aut}_S(Q;X)|} \\ &= [\Sigma_{\mathfrak{X}}(Q)_0 \cap \mathrm{Aut}_S(Q) : \Sigma_{\mathfrak{X}}(Q)_0] \end{aligned}$$

By Axiom (4) for saturated fusion action systems, $\Sigma_{\mathfrak{X}}(Q)_0 \cap \mathrm{Aut}_S(Q;X) = \Sigma^S_{\mathfrak{X}}(Q)_0$ is Sylow in $\Sigma_{\mathfrak{X}}(Q)_0$. Thus there is some $\tau \in \Sigma_{\mathfrak{X}}(Q)_0$ such that ${}^{\tau}N \leq \mathrm{Aut}_S(Q;X)$, and the result is proved. □

In other words, Proposition 4.3.6 states that saturated abstract actions of a given

[4] We shall ignore the distinction between an element $\sigma \in \Sigma_S$ and the corresponding morphism (id_Q, σ) that lies in $\mathfrak{X}$.

fusion system $\mathcal{F}$ are the same as saturated abstract fusion action systems whose underlying fusion system is $\mathcal{F}$. From now on, we will conflate these two notions, using whichever is more convenient for the situation at hand. For instance, we shall simply use the symbol $\mathcal{F}$ to denote the underlying fusion system $\mathcal{F}^{\mathfrak{X}}$ of the fusion action system $\mathfrak{X}$.

We can now use the notion of abstract fusion action systems now to form a category out of our objects of study. Definition 4.3.7 will not play a major role in the rest of this thesis, but is included mostly for its relationship to Appendix A.

Definition 4.3.7. Let $(S, X, \mathfrak{X})$ and $(T, Y, \mathfrak{Y})$ be two (saturated) fusion action systems. An *equivariant map* is a pair

$$(\alpha : S \to Y, f : X \to Y) \qquad \text{such that} \qquad f(p \cdot x) = \alpha(p) \cdot f(x)$$

for all $p \in P$ and $x \in X$. An equivariant map is *fusion-action-preserving* if there is a functor $F = F_{(\alpha,f)} : \mathfrak{X} \to \mathfrak{Y}$ such that $F(P) = \alpha(P)$ and the following diagram of groupoids commutes for all $(\varphi, \sigma) \in \mathfrak{X}(P, Q)$:

$$\begin{array}{ccc}
\mathcal{B}_P X & \xrightarrow{\mathcal{B}(\alpha,f)} & \mathcal{B}_{\alpha(P)} Y \\
{\scriptstyle \mathcal{B}(\varphi,\sigma)}\downarrow & & \downarrow{\scriptstyle \mathcal{B}\big(F(\varphi,\sigma)\big)} \\
\mathcal{B}_Q X & \xrightarrow[\mathcal{B}(\alpha,f)]{} & \mathcal{B}_{\alpha(P)} Y
\end{array}$$

where $\mathcal{B}_P X$ is the translation groupoid of the P-set X and $\mathcal{B}(\alpha, f)$ is the map of groupoids defined on objects by f and on morphisms by α. That $\mathcal{B}(\alpha, f)$ is well-defined follows from the assumption that the pair (α, f) is an equivariant map. Note that if $f : X \to Y$ is surjective, the functor F is unique if it exists (and more generally, F is uniquely defined "on the image of f"). $\diamondsuit$

4.4 Properties of fusion action systems

4.4.1 General properties

This first result is a sort of converse to the Sylow conditions of the saturation axioms for fusion action systems.

Proposition 4.4.1. *Fix a saturated fusion action system $\mathfrak{X}$ and a subgroup $P \leq S$.*

(a) The following are equivalent:

1. *P is fully normalized in $\mathcal{F}$.*
2. *P is fully centralized in $\mathcal{F}$ and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_{\mathfrak{p}}(\mathcal{F}(P))$.*
3. *P is fully X-normalized in $\mathcal{F}$ and $\Sigma^S_{\mathfrak{X}}(P) \in \mathrm{Syl}_{\mathfrak{p}}(\Sigma_{\mathfrak{X}}(P))$.*
4. *P is fully X-centralized in $\mathcal{F}$ and $\mathrm{Aut}_S(P; X) \in \mathrm{Syl}_{\mathfrak{p}}(\mathfrak{X}(P))$.*

(b) P is fully centralized iff P is fully X-centralized and $\Sigma^S_{\mathfrak{X}}(P)_0 \in \mathrm{Syl}_{\mathfrak{p}}(\Sigma_{\mathfrak{X}}(P)_0)$.

(c) P is fully X-normalized iff P is fully X-centralized and $\mathcal{F}_S(P)_0 \in \mathrm{Syl}_{\mathfrak{p}}(\mathcal{F}(P)_0)$.

Proof. Half of these implications are part of the saturation axioms, so prove only the remaining ones.

(a) $P \leq S$ is fully normalized if:

- P is fully centralized and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_{\mathfrak{p}}(\mathcal{F}(P))$. In this case, the assumptions on P together with the inclusion of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Sigma_{\mathfrak{X}}(P)_0 & \longrightarrow & \mathfrak{X}(P) & \longrightarrow & \mathcal{F}(P) & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle \text{Sylow}} & & \uparrow & & \uparrow{\scriptstyle \text{Sylow}} & & \\ 1 & \longrightarrow & \Sigma^S_{\mathfrak{X}}(P)_0 & \longrightarrow & \mathrm{Aut}_S(P; X) & \longrightarrow & \mathrm{Aut}_S(P) & \longrightarrow & 1 \end{array}$$

  force $\mathrm{Aut}_S(P; X) \in \mathrm{Syl}_{\mathfrak{p}}(\mathfrak{X}(P))$. Then if Q is fully normalized and $\mathcal{F}$-conjugate to P via $(\varphi, \sigma) \in \mathfrak{X}(Q, P)$, we have that

$$(\varphi, \sigma) \circ \mathrm{Aut}_S(Q; X) \circ (\varphi, \sigma)^{-1} \leq \mathfrak{X}(P)$$

is an inclusion of a $\mathfrak{p}$-subgroup, so there is $(\psi, \tau) \in \mathfrak{X}(P)$ such that

$$(\psi\varphi, \tau\sigma) \circ \mathrm{Aut}_S(Q; X) \circ (\psi\varphi, \tau\sigma)^{-1} \leq \mathrm{Aut}_S(P; X)$$

Note that $\mathrm{Aut}_S(Q; X)$ is the image in $\mathfrak{X}(Q)$ of $N_S(Q)$, and that P being fully centralized implies that it is fully X-centralized. The extension axiom now gives the existence of $(\widetilde{\psi\varphi}, \tau\sigma) \in \mathfrak{X}(N_S(Q), N_S(P))$, from which it follows that $|N_S(Q)| = |N_S(P)|$ and P is fully normalized as well.

– P is fully X-normalized and $\Sigma^S_{\mathfrak{X}}(P) \in \mathrm{Syl}_{\mathfrak{p}}(\Sigma_{\mathfrak{X}}(P))$. The inclusion of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{F}(P)_0 & \longrightarrow & \mathfrak{X}(P) & \longrightarrow & \Sigma_{\mathfrak{X}}(P) & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle \text{Sylow}} & & \uparrow & & \uparrow{\scriptstyle \text{Sylow}} & & \\ 1 & \longrightarrow & \mathcal{F}_S(P)_0 & \longrightarrow & \mathrm{Aut}_S(P; X) & \longrightarrow & \Sigma^S_{\mathfrak{X}}(P) & \longrightarrow & 1 \end{array}$$

shows that $\mathrm{Aut}_S(P; X) \in \mathrm{Syl}_{\mathfrak{p}}(\mathfrak{X}(P))$. The rest of the proof is the same as the first point.

– P is fully X-centralized and $\mathrm{Aut}_S(P; X) \in \mathrm{Syl}_{\mathfrak{p}}(\mathfrak{X}(P))$. The argument is the same as the end of the previous two.

(b) P is fully centralized if P is fully X-centralized and $\Sigma^S_{\mathfrak{X}}(P)_0 \in \mathrm{Syl}_{\mathfrak{p}}(\Sigma_{\mathfrak{X}}(P)_0)$: Observe that $|Z_S(P)| = |Z_S(P; X)||\Sigma^S_{\mathfrak{X}}(P)_0|$. If $P \cong_{\mathcal{F}} Q$, then we have $\Sigma_{\mathfrak{X}}(P)_0 \cong \Sigma_{\mathfrak{X}}(Q)_0$, so it is easy to see that the order of (the $\mathfrak{p}$-group) $Z_S(P)$ is maximized precisely when $\Sigma^S_{\mathfrak{X}}(P)_0 \in \mathrm{Syl}_{\mathfrak{p}}(\Sigma_{\mathfrak{X}}(P)_0)$ and the order of $Z_S(P; X)$ is maximized. The result follows.

(c) P is fully X-normalized if P is fully X-centralized and $\mathcal{F}_S(P)_0 \in \mathrm{Syl}_{\mathfrak{p}}(\mathcal{F}(P)_0)$: The same argument as the previous paragraph applies, with the observation that $|N_S(P)| = |N_S(P; X)||\mathcal{F}_S(P)_0|$.

□

Next we provide a basic reality-check for our axioms: If these saturation axioms are "right," the least we would expect is for some sort of Alperin Fusion Theorem.

Proposition 4.4.2. *If $(\varphi, \sigma) \in \mathfrak{X}(P, Q)$ is an isomorphism such that Q is fully normalized, then there are morphisms $(\overline{\varphi}, \sigma') \in \mathfrak{X}(N_S(P), S)$ and $(\psi, \tau) \in \mathfrak{X}(P)$ such that $\overline{\varphi}\big|_P = \varphi \circ \psi$ and $\sigma' = \sigma \circ \tau$.*

Proof. By the assumption that Q is fully normalized, the saturation axioms state that $\mathrm{Aut}_S(Q; X) \in \mathrm{Syl}_{\mathfrak{p}}(\mathfrak{X}(Q))$, so the $\mathfrak{p}$-group $(\varphi, \sigma) \circ \mathrm{Aut}_S(P; X) \circ (\varphi^{-1}, \sigma^{-1}) \leq \mathfrak{X}(Q)$ is subconjugate to $\mathrm{Aut}_S(Q; X)$. Say

$$(\chi, \upsilon) \circ (\varphi, \sigma) \circ \mathrm{Aut}_S(P; X) \circ (\varphi, \sigma)^{-1} \circ (\chi, \upsilon)^{-1} \leq \mathrm{Aut}_S(Q; X)$$

and set $(\psi, \tau) = (\varphi^{-1}\chi\varphi, \sigma^{-1}\upsilon\sigma) \in \mathfrak{X}(P)$, so that

$$(\varphi, \sigma) \circ (\psi, \tau) \circ \mathrm{Aut}_S(P; X) \circ (\psi, \tau)^{-1} \circ (\varphi, \sigma)^{-1} \leq \mathrm{Aut}_S(Q; X).$$

Now the fact that Q is fully normalized and thus fully X-centralized implies that $(\varphi\psi, \sigma\tau) \in \mathfrak{X}(P, Q)$ has an extension $(\overline{\varphi}, \sigma') \in \mathfrak{X}\left(N_{(\varphi\chi, \sigma\tau)}(P), S\right)$ by the extension axiom of saturation. Recall that

$$N_{(\varphi\chi, \sigma\tau)}(P) = \left\{ s \in N_S(P) \middle| (\varphi\chi \circ c_s \circ \chi^{-1}\varphi^{-1}, \sigma\tau \circ \ell_s \circ \tau^{-1}\sigma^{-1} \in \mathrm{Aut}_S(Q; X) \right\}$$

so that by construction $N_S(P) \leq N_{(\varphi\chi, \sigma\tau)}(P)$. The result follows. □

One could also prove analogous statements in the cases that the target is fully centralized or X-normalized, but we will not have need for such results.

Proposition 4.4.3. *The saturated fusion action system $\mathfrak{X}$ is an Alperin category.*

Proof. The claim is that every morphism $(\varphi, \sigma) \in \mathfrak{X}(P, S)$ can be written as a composition of automorphisms of subgroups and inclusions. Proof goes by downward induction on the order of the source P. If $P = S$, then $(\varphi, \sigma) \in \mathfrak{X}(S)$, and there's nothing to prove.

Therefore suppose that $P \lneq S$. Without loss of generality, we may assume that $Q = \varphi(P)$ is fully normalized: Otherwise, pick Q' in the $\mathcal{F}$-conjugacy class of P that is fully normalized and an isomorphism $(\psi, \tau) \in \mathfrak{X}(P, Q')$. If the result is true for (ψ, τ) and $(\varphi, \sigma) \circ (\psi, \tau)^{-1}$, it clearly is for (φ, σ) as well.

Proposition 4.4.2 shows that if the target of (φ, σ) is fully normalized, then (φ, σ) can be composed with an element of $\mathfrak{X}(P)$ so that the resulting morphism extends to $\mathfrak{X}(N_S(P), S)$. But $N_S(P) \gneq P$ as P is a proper subgroup, and the inductive hypothesis gives the rest. □

4.4.2 Faithful fusion action systems

In this section we examine in closer detail a fusion action system $\mathfrak{X}$ at the opposite extreme to $\mathcal{F}$[5]: We assume that X is faithful as an S-set. We will see that in this situation, the underlying saturated fusion system $\mathcal{F}$ is always realizable by a finite group G, and moreover that G acts on X in such a way that $\mathfrak{X} = \mathfrak{X}_G$.

Let $G = \mathfrak{S}$ be the group of permutations of X that appear in some morphism of $\mathfrak{X}$. The map $S \to G : s \mapsto \ell_s$ is an injection if we assume that X is faithful as an S-set, so we will identify S with its image in G. The saturation axioms for fusion actions imply that $S \in \mathrm{Syl}_{\mathfrak{p}}(G)$, so the fusion system $\mathcal{F}_G$ is saturated.

The following proposition shows that exotic fusion systems cannot act S-faithfully on finite sets.

Proposition 4.4.4. *In this situation, $\mathcal{F} = \mathcal{F}_G$.*

Proof. First we show that $\mathcal{F} \subseteq \mathcal{F}_G$. It suffices to show that any $\varphi \in \mathcal{F}(P, S)$ is realized by conjugation by an element of G. Pick some $(\varphi, \sigma) \in \mathfrak{X}(P, S)$, so that φ and σ are intertwined, so $\sigma \circ \ell_p = \ell_{\varphi(p)} \circ \sigma \in \Sigma_X$ for all $p \in P$. This is exactly to say that σ conjugates P via φ, proving the first inclusion.

For the reverse inclusion, suppose that for some $\sigma \in G$ and $P \leq S$ we have ${}^{\sigma}P \leq S$; we must show that $c_\sigma \in \mathcal{F}(P, S)$. By the extension axiom, the morphism

[5] Note that $\mathcal{F}$ can be viewed as the unique fusion action of $\mathcal{F}$ on the one-point set.

$(\mathrm{id}, \sigma) \in \mathfrak{X}(1)$ extends to some $(\varphi, \sigma) \in \mathfrak{X}(N_{(\mathrm{id}_1,\sigma)}, S)$, where

$$N_{(\mathrm{id}_1,\sigma)} = \left\{ n \in S \middle| \sigma \circ \ell_n \circ \sigma^{-1} \in \mathrm{Aut}_S(X) \right\}.$$

In other words, for all $n \in N_{(\mathrm{id}_1,\sigma)}$, there is some (necessarily unique by faithfulness of the X-action) $s \in S$ so that $\sigma \circ \ell_n \circ \sigma^{-1} = \ell_s$. Therefore $N_{(\mathrm{id}_1,\sigma)}$ is the maximal subgroup of S that is conjugated into S by σ. Thus $P \leq N_{(\mathrm{id}_1,\sigma)}$, and if we can show that the assignment $n \mapsto s$ is equal to φ, the result will follow, as φ is by assumption a morphism of $\mathcal{F}$.

This final assertion follows again from the fact that φ and σ are intertwined: $\sigma \circ \ell_n = \ell_{\varphi(n)} \circ \sigma$ for $n \in N_{(\mathrm{id}_1,\sigma)}$ forces $s = \varphi(n)$ in the above notation. The result is proved. □

In this situation, there is a natural action of $G = \mathfrak{X}(1) \leq \Sigma_X$ on X, and the following is immediate:

Corollary 4.4.5. *The fusion action system $\mathfrak{X}$ is realized by the G-action on X: $\mathfrak{X} = \mathfrak{X}_G$.*

Proof. The key point is that if the S-action on X is faithful, then any morphism $(\varphi, \sigma) \in \mathfrak{X}(P, Q)$ is actually determined by σ alone. This follows from the assumption that φ and σ are intertwined, so that for all $p \in P$, we have $\ell_{\varphi(p)} = \sigma \ell_p \sigma^{-1}$ along with the identification $p \leftrightarrow \ell_p$. In particular, $\mathfrak{X} \subseteq \mathfrak{X}_G$, since $(\varphi, \sigma) \in \mathfrak{X}(P, Q)$ implies that ${}^{\sigma}P \leq Q$ and $c_\sigma = \varphi$.

On the other hand, for any $\sigma \in N_G(P, Q)$ we have $(c_\sigma, \sigma) \in \mathfrak{X}_G(P, Q)$, and the claim is that this is also a morphism of $\mathfrak{X}(P, Q)$. This follows from the extension axiom of saturated fusion actions as in Proposition 4.4.4 and the observation that ${}^{\sigma}P \leq S$, implies $P \leq N_{(\mathrm{id}_1,\sigma)}$. Thus $\mathfrak{X}_G \subseteq \mathfrak{X}$. □

4.5 Core subsystems

In this section let $\mathfrak{X}$ be a saturated fusion action system. Recall that $K \leq S$ denotes the core of the S action on X.

Definition 4.5.1. The *core fusion system* associated to $\mathfrak{X}$ is the fusion system $\mathcal{K}$ on K with morphisms $\mathcal{K}(P,Q) = \{\varphi \in \mathrm{Inj}(P,Q) | (\varphi, \mathrm{id}_X) \in \mathfrak{X}(P,Q)\}$. ♢

Remark 4.5.2. There is a natural functor $\mathfrak{X} \to \mathcal{B}\Sigma_X$ that sends inclusions to the identity; this is precisely what the various maps π_Σ piece together to give. $\mathcal{K}$ can be thought of the kernel fusion action system of this functor, once we note that, as the action of $\mathcal{K}$ on X is trivial, there is no distinction between a fusion system and a fusion action system. ♢

The first result of this section is that $\mathcal{K}$ is saturated as a fusion system.

Proposition 4.5.3. *$\mathcal{K}$ satisfies the following properties:*

1. *K is strongly closed in $\mathcal{F}$.*
2. *For all $P, Q \leq K$, $\varphi \in \mathcal{K}(P,Q)$ and $\psi \in \mathcal{F}(K)$, we have $\psi\varphi\psi^{-1} \in \mathcal{K}(\psi P, \psi Q)$.*
3. *For all $P, Q \leq K$ and $\chi \in \mathcal{F}(P,Q)$, there are $\psi \in \mathcal{F}(K)$ and $\varphi \in \mathcal{K}(\psi(P), Q)$ such that $\chi = \varphi \circ \psi|_P^{\psi(P)}$.*

Proof.

1. This has been observed several times already, for example in Corollary 3.3.7.
2. Pick some $(\psi, \tau) \in \mathfrak{X}(K)$ that lies over $\psi \in \mathcal{F}(K)$. Then

$$(\psi, \tau)(\varphi, \mathrm{id}_X)(\psi, \tau)^{-1} = (\psi\varphi\psi^{-1}, \mathrm{id}_X)$$

and the result follows.

3. Let $(\chi, \upsilon) \in \mathfrak{X}(P,Q)$ lie over $\chi \in \mathcal{F}(P,Q)$, and consider $(\mathrm{id}_1, \upsilon) \in \mathfrak{X}(1)$. By the extension axiom for saturated fusion action systems, $(\mathrm{id}_1, \upsilon)$ extends to a map $(\widehat{\psi}, \upsilon) \in \mathfrak{X}(N_{(\mathrm{id}_1,\upsilon)}, S)$, and it follows easily from the definition that $K \leq N_{(\mathrm{id}_1,\upsilon)}$. Let $(\psi, \upsilon) \in \mathfrak{X}(K)$ be the restriction to K (which is necessarily an automorphism of the strongly closed subgroup K), and let

$$(\varphi, \mathrm{id}_X) = (\chi, \upsilon) \circ (\psi, \upsilon)^{-1}|_{\psi(P)}^{P} \in \mathfrak{X}(\psi(P), Q).$$

Projection to $\mathcal{F}$ shows now that ψ and φ have the desired property.

□

The content of Proposition 4.5.3 is that $\mathcal{K}$ is almost a normal subsystem of $\mathcal{F}$ in the sense of [Pui1]. We say "almost" because in that definition of normality, we assume also that $\mathcal{K}$ is saturated as a fusion system, which is our current goal. We need the third point of Proposition 4.5.3 to prove saturation, but luckily the logic will not be circular.

Corollary 4.5.4. *Every $P \leq K$ is fully normalized (resp. centralized) in $\mathcal{K}$ if and only if P is fully X-normalized (resp. X-centralized) in $\mathcal{F}$.*

Proof. Note that $N_S(P;X) = N_K(P)$ and $Z_S(P;X) = Z_K(P)$ by definition, which makes the "only if" implications obvious. We prove the "if" implication for normalizers; the same argument works for centralizers.

Suppose that P is fully normalized in $\mathcal{K}$, and let $Q \leq K$ be fully X-normalized and $\mathcal{F}$-conjugate to P (such a Q must exist as K is strongly closed in $\mathcal{F}$). For any isomorphism $\chi \in \mathcal{F}(P,Q)$, we can find $\psi \in \mathcal{F}(K)$ and $\varphi \in \mathcal{K}(\psi(P),Q)$ such that $\chi = \varphi \circ \psi|_P^{\psi(P)}$ by the third point of Proposition 4.5.3. As ψ extends to an automorphism of K, it sends the X-normalizer of P to the X-normalizer of $\psi(P)$. As φ is necessarily an isomorphism in $\mathcal{K}$ by the two-out-of-three property, the assumption that $|N_K(P)| \geq |N_K(\psi(Q))|$ forces that $|N_S(P;X)| \geq |N_S(Q;X)|$. The assumption that Q is fully X-normalized forces equality, and the result is proved. □

Proposition 4.5.5. *$\mathcal{K}$ is a saturated fusion system.*

Proof. We use the equivalent saturation axioms of [Sta], in which it suffices to show that $\mathrm{Aut}_K(K) \in \mathrm{Syl}_{\mathfrak{p}}(\mathcal{K}(K))$ and that fully normalized subgroups satisfy the Extension Axiom.

To see that $\mathrm{Aut}_K(K) \in \mathrm{Syl}_{\mathfrak{p}}(\mathcal{K}(K))$: We've already noted that K is strongly closed in $\mathcal{F}$, and therefore is fully X-normalized. By the saturation axioms for fusion action systems, we have $\mathcal{F}_S(K)_0 \in \mathrm{Syl}_{\mathfrak{p}}(\mathcal{F}(K)_0)$. It is easy to see from the definitions that $\mathcal{F}(K)_0 = \mathcal{K}(K)$, and we can describe $\mathcal{F}_S(K)_0$ as the subgroup of $\mathrm{Aut}_S(K)$ consisting of those automorphisms that are induced by an element of S that acts trivially on X, so that $\mathcal{F}_S(K)_0 = \mathrm{Aut}_K(K)$.

To check the extension axiom: If $P \leq K$ is fully normalized in $\mathcal{K}$, by Corollary 4.5.4 we see that P is fully X-normalized in $\mathcal{F}$, and thus fully X-centralized. Given an isomorphism $\varphi \in \mathcal{K}(Q,P)$, we have $(\varphi, \mathrm{id}_X) \in \mathfrak{X}(Q,P)$, and the extension condition for fusion actions gives a morphism

$$(\widetilde{\varphi}, \mathrm{id}_X) \in \mathfrak{X}(N^{\mathfrak{X}}_{(\varphi,\mathrm{id}_X)}, K)$$

where the target may be assumed to be K instead of S because K is strongly closed in $\mathcal{F}$. We wish to show that

$$N^{\mathcal{K}}_{\varphi} = \left\{ n \in N_K(Q) \middle| \varphi \circ c_n \circ \varphi^{-1} \in \mathrm{Aut}_K(P) \right\}$$

is contained in $N^{\mathfrak{X}}_{(\varphi,\mathrm{id}_X)}$, as then $\widetilde{\varphi}|_{N^{\mathcal{K}}_{\varphi}}$ will give the desired extension of φ. We have

$$N^{\mathfrak{X}}_{(\varphi,\mathrm{id}_X)} = \left\{ n \in N_S(Q) \middle| (\varphi \circ c_n \circ \varphi^{-1}, \ell_n) \in \mathrm{Aut}_S(P;X) \right\}.$$

Since K acts trivially on S, for $n \in N_K(Q)$ such that $\varphi \circ c_n \circ \varphi^{-1} = c_k \in \mathrm{Aut}_K(P)$ we have

$$(\varphi \circ c_n \circ \varphi^{-1}, \ell_n) = (c_k, \mathrm{id}_X) = (c_k, \ell_k) \in \mathrm{Aut}_K(P;X) \leq \mathrm{Aut}_S(P;X)$$

and the result is proved. □

Corollary 4.5.6. *$\mathcal{K}$ is a normal subsystem of $\mathcal{F}$ in the sense of [Pui1].*

We close this section by describing how the fusion action system $\mathfrak{X}$ can be thought of as an extension of $\mathcal{K}$ by the finite group $\mathfrak{G} = \mathfrak{X}(1)$. First note that we have the functors $\mathcal{K} \overset{\iota}{\hookrightarrow} \mathfrak{X} \overset{\pi_{\mathcal{T}}}{\twoheadrightarrow} \mathcal{T}_{\mathfrak{G}}$, where ι is "injective" in the sense that $\mathcal{K}$ naturally sits as a subcategory of $\mathfrak{X}$, and $\pi_{\mathcal{T}}$ is "surjective" in the sense already described for the underlying transporter system of a fusion action system. Just as extensions of the finite group H by G determines a morphism $G \to \mathrm{Out}(H)$, we have:

Proposition 4.5.7. *The fusion action system $\mathfrak{X}$ determines a unique homomorphism $\kappa : \mathfrak{S} \to \mathrm{Out}(\mathcal{K})$.*

Proof. An easy application of the Extension Axiom for saturated fusion action systems implies that each $\sigma \in \mathfrak{S}$ appears in the second coordinate of some $(\varphi, \sigma) \in \mathfrak{X}(K)$. As $\mathcal{K}$ is normal in $\mathcal{F}$, $\varphi \in \mathrm{Aut}(\mathcal{K})$. The indeterminacy of the assignment $\sigma \mapsto \varphi$ is measured by $\mathcal{K}(K)$ by definition of $\mathcal{K}$, so the assignment $\sigma \mapsto [\varphi] \in \mathrm{Out}(\mathcal{K})$ is the desired map. $\square$

Chapter 5

Linking action systems and $\mathfrak{p}$-local finite group actions

In this chapter we discuss the homotopy theory of fusion action systems. Sections 5.1, 5.2, and 5.3 generalize the work of [BLO1, BLO2], 5.4 relates the new machinery to that introduced in [OV], and 5.5 begins the project of using fusion actions to relate a fusion system with certain subsystems.

5.1 The ambient case

Let us return for this section to the situation where we are given a finite group G, a Sylow $S \in \mathrm{Syl}_{\mathfrak{p}}(G)$, the fusion system $\mathcal{F} = \mathcal{F}_G$, and X a G-set (which is then also an $\mathcal{F}$-stable S-set). We wish to reconstruct the homotopy type of the Borel construction $B_G X := EG \times_G X$, at least up to $\mathfrak{p}$-completion, with a minimum of $\mathfrak{p}'$-data. This section reproduces some results of [BLO1] in the context of fusion action systems arising from ambient groups.

The game we want to play in reconstructing the $\mathfrak{p}$-completed homotopy type of $B_G X$ is to look for a new category that will both allow us to construct $B_G X^\wedge_{\mathfrak{p}}$ but that does not contain too much extra information. In some sense, the transporter system $\mathcal{T}_G = \mathcal{T}_S(G)$ contains all the information of G that we care about (for instance, the natural functor $\mathcal{B}G \to \mathcal{T}_G$ that sends the unique object of $\mathcal{B}G$ to the subgroup

$\{1\}$ induces a homotopy equivalence $|\mathcal{T}_G| \simeq BG$). The problem of course is that $\mathcal{T}_G$ contains too much information, especially $\mathfrak{p}'$-information. The goal then becomes figuring out the right amount of data of $\mathcal{T}_G$ to forget and still be able to understand the Borel construction.

The first way of forgetting information of $\mathcal{T}_G$ is to consider full subcategories whose sets of objects are closed under G-conjugacy and overgroups. In other words, we simply throw out all sufficiently small subgroups and the information of their Hom-sets. Exactly which subgroups we will allow must depend somehow on the fusion data of G and the action of G on X:

Definition 5.1.1. A $\mathfrak{p}$-subgroup $P \leq S$ is $\mathfrak{p}$-*centric at* X if $Z(P;X) \in \mathrm{Syl}_{\mathfrak{p}}(Z_G(P;X))$. Equivalently, P is $\mathfrak{p}$-centric at X if $Z_G(P;X) = Z(P;X) \times Z'_G(P;X)$ for some (necessarily unique) $\mathfrak{p}'$-group $Z'_G(P;X)$. $\diamondsuit$

We shall generally call such a subgroup X-*centric* and omit mentioning the prime $\mathfrak{p}$. This frees up our nomenclature so we can recall

Definition 5.1.2. A $\mathfrak{p}$-subgroup $P \leq S$ is $\mathfrak{p}$-*centric* if $Z(P) \in \mathrm{Syl}_{\mathfrak{p}}(Z_G(P))$, or equivalently if $Z_G(P) = Z(P) \times Z'_G(P)$ for some (again, unique) $\mathfrak{p}'$-group $Z'_G(P)$. This is equivalent to saying that P is X-centric for $X = *$ the trivial S-set. $\diamondsuit$

Note that the condition of being X-centric is determined purely by fusion data and makes no reference to the fusion action system $\mathcal{F}_G$. This motivates the following definition:

Definition 5.1.3. Given a saturated fusion system $\mathcal{F}$ on S and an $\mathcal{F}$-stable S-set, a subgroup $P \leq S$ is $\mathcal{F}$-*centric at* X if $Z(Q;X) = Z_S(Q:X)$ for all $Q \cong_{\mathcal{F}} P$. $\diamondsuit$

In the presence of an ambient group G, the two notions of X-centricity coincide:

Proposition 5.1.4. *If G is a finite group that acts on X, $S \in \mathrm{Syl}_{\mathfrak{p}}(G)$, and $\mathcal{F} = \mathcal{F}_G$, then a subgroup $P \leq S$ is $\mathfrak{p}$-centric at X if and only if it is $\mathcal{F}$-centric at X.*

Proof. First suppose that P is $\mathfrak{p}$-centric at X. As $Z_G(P;X) = Z(P;X) \times Z'_G(P;X)$ for some uniquely defined $\mathfrak{p}'$-subgroup $Z'_G(P;X)$, it is clear that $Z_S(P;X) = Z(P;X)$.

If $g \in G$ is such that ${}^gP \leq S$, the fact that $Z_G({}^gP; X) = {}^gZ_G(P; X)$ immediately shows that $Z_G({}^gP; X) = Z({}^gP; X) \times {}^gZ'_G(P; X)$, and we have the desired conclusion for ${}^gP$.

Conversely, suppose that P is $\mathcal{F}$-centric at X, and pick $T \in \mathrm{Syl}_{\mathfrak{p}}(Z_G(P; X))$ such that $Z(P; X) \leq T$. As $S \in \mathrm{Syl}_{\mathfrak{p}} G$, there is some $g \in G$ such that ${}^gP \leq {}^gT \leq S$. Therefore ${}^gT \leq Z_S({}^gP; X) = Z({}^gP; X)$ by the $\mathcal{F}$-centricity at X of P and we conclude that $T = Z(P; X)$, as desired. □

The main purpose of introducing the abstract fusion-centric way of thinking of X-centricity at this point is that it makes certain results cleaner to prove:

Proposition 5.1.5. *If X and Y are $\mathcal{F}$-stable S-sets and there is a surjective map of S-sets $f : X \to Y$, then every Y-centric subgroup is also X-centric.*

Proof. Let K and L be the cores of X and Y, respectively. The existence of such a surjection forces $K \leq L$. Thus $Z_S(P) \cap K \leq Z_S(P) \cap L$, so if $Z_S(P) \cap L = Z(P) \cap L$ then $Z_S(P) \cap K = Z(P) \cap K$. The result follows. □

Corollary 5.1.6. *$\mathfrak{p}$-centricity implies X-centricity for all $\mathcal{F}$-stable X.*

In particular, every $\mathfrak{p}$- or $\mathcal{F}$-centric (both henceforth "centric") subgroup of S is automatically X-centric for every $\mathcal{F}$-stable X. Moreover, the more faithful X is (so the smaller the kernel K is), the easier it is for subgroups of S to be X-centric, to the point where if X is a faithful S-set, *every* $P \leq S$ is X-centric.

Definition 5.1.7. Let $\mathcal{T}_G^{cX}$ denote the full subcategory of $\mathcal{T}_G$ whose objects are the X-centric subgroups. Similarly, for $\mathfrak{X}$ an abstract action with finite set X, let $\mathfrak{X}^{cX}$ be the full subcategory on the X-centric subgroups. ♢

The first thing we must show is that we have not lost too much information by this restriction. Let $\mathbb{X} : \mathcal{T}_G^{cX} \to \mathcal{TOP}$ be the functor

$$\begin{array}{ccc} P & \mapsto & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \ell_g} \\ Q & \mapsto & X \end{array}$$

Proposition 5.1.8. *There exists a mod-$\mathfrak{p}$ equivalence* $\operatorname{hocolim}_{\mathcal{T}_G^{cX}} \mathbb{X} \simeq_{\mathfrak{p}} B_G X$.

Proof. Let $\mathcal{E}G$ be the simplicial groupoid with objects $g \in G$, $\mathcal{K}_{cX}$ the category associated to the poset of X-centric subgroups, and K_{cX} the associated G-simplicial complex. We view the G-set X as a discrete G-category. If $\mathcal{G}(\mathbb{X})$ is the Grothendieck construction associated to the functor $\mathbb{X}$, we have $|\mathcal{G}(\mathbb{X})| \simeq \operatorname{hocolim}_{\mathcal{T}_G^{cX}} \mathbb{X}$ by [Tho], so we will use the Grothendieck construction as our model for the homotopy colimit.

There is a functor $\mathcal{E}G \times_G (X \times \mathcal{K}_{cX}) \to \mathcal{G}(\mathbb{X})$ given by

$$\begin{array}{ccc} [g,(x,P)] & \longmapsto & ({}^gP, g\cdot x) \\ {\scriptstyle [g\mapsto hg,(id_x,P\leq Q)]}\downarrow & & \downarrow {\scriptstyle h} \\ [hg,(x,Q)] & \longmapsto & ({}^{hg}Q, hg\cdot x) \end{array}$$

that is easily seen to have an inverse isomorphism of categories

$$\begin{array}{ccccc} (P,x) & \longmapsto & & & [1,(x,P)] \\ {\scriptstyle h}\downarrow & & & & \downarrow {\scriptstyle [1\mapsto h,(id_x,P\leq {}^{h^{-1}}Q)]} \\ (Q,h\cdot x) & \longmapsto & [1,(h\cdot x,Q)] & = & [h,(x,{}^{h^{-1}}Q)] \end{array}$$

Thus, on taking realizations, we get a homeomorphism of spaces

$$EG \times_G (X \times K_{cX}) \simeq \operatorname*{hocolim}_{\mathcal{T}_G^{cX}} \mathbb{X}.$$

Finally, consider the natural projection map $EG \times_G (X \times K_{cX}) \to EG \times_G X$; if we can show that this is a mod-$\mathfrak{p}$ homology isomorphism, the result will follow.

By Corollary 5.1.6, the collection of X-centric subgroups contains all centric subgroups, and it is easy to check that it is closed under $\mathfrak{p}$-overgroups. Thus [Dwy, Theorem 8.3] applies to show that K_{cX} is $\mathbb{F}_p$-acyclic, and therefore the natural map $EG \times_G (X \times K_{cX}) \to EG \times_G X$, being a fibration with fiber K_{cX}, is a mod-$\mathfrak{p}$ homology isomorphism by the Serre spectral sequence. □

There is a more drastic way of reducing information in $\mathcal{T}_G$ than simply restricting our attention to various subgroups, in which we quotient out $\mathfrak{p}'$-information directly.

Note that there is a free right action of $Z(P)$ on $\mathcal{T}_G(P,Q)$, so we can consider categories whose morphisms are orbits of subgroups of $Z(P)$ in the transporter system. The need to lose only $\mathfrak{p}'$-information is one reason why we have restricted our attention to the X-centric subgroups of S, as we shall now see.

Recall that for H a finite group, $O^{\mathfrak{p}}(H)$ is the smallest normal subgroup of H of $\mathfrak{p}$-power index. If P is X-centric, $Z_G(P;X) = Z(P;X) \times Z'_G(P;X)$ for $Z'_G(P;X)$ a $\mathfrak{p}'$-group, and we have $O^{\mathfrak{p}}(Z_G(P;X)) = Z'_G(P;X)$.

Definition 5.1.9. For a G-set X, define the $\mathfrak{p}$-*centric linking action system of* S *at* X to be the category $\mathcal{L}^{cX}_G = \mathcal{L}^{cX}_S(G)$ whose objects are X-centric subgroups of S, and where $\mathcal{L}^{cX}_G(P,Q) = N_G(P,Q)/O^{\mathfrak{p}}(Z_G(P;X))$. ♢

Just as we formed the Borel construction B_GX up to $\mathfrak{p}$-completion by considering the homotopy colimit of a functor $\mathcal{T}^{cX}_G \to \mathcal{TOP}$, there is a similarly defined functor whose homotopy colimit is of particular interest to us in the context of linking action systems. Let $\overline{\mathbb{X}} : \mathcal{L}^{cX}_G \to \mathcal{TOP}$ be the functor

$$\begin{array}{ccc} P & \mapsto & X \\ {\scriptstyle [g]}\downarrow & & \downarrow{\scriptstyle \ell_g} \\ Q & \mapsto & X \end{array}$$

By construction, the natural quotient $\pi : \mathcal{T}^{cX}_G \to \mathcal{L}^{cX}_G$ satisfies the conditions of [BLO1, Lemma 1.3]. In particular, this implies

Proposition 5.1.10. *The quotient* $\pi : \mathcal{T}^{cX}_G \to \mathcal{L}^{cX}_G$ *induces a mod-*$\mathfrak{p}$ *equivalence:*

$$\operatorname*{hocolim}_{\mathcal{L}^{cX}_G} \overline{\mathbb{X}} \simeq_{\mathfrak{p}} \operatorname*{hocolim}_{\mathcal{T}^{cX}_G} \overline{\mathbb{X}} \circ \pi.$$

Corollary 5.1.11.

$$\operatorname*{hocolim}_{\mathcal{L}^{cX}_G} \mathbb{X} \simeq_{\mathfrak{p}} B_GX$$

Proof. Combine Propositions 5.1.8 and 5.1.10 with the fact that $\mathbb{X} = \overline{\mathbb{X}} \circ \pi$. □

5.2 Classifying spaces of abstract fusion actions

5.2.1 Stating the problem

In this section let $\mathfrak{X}$ be a fixed abstract fusion action system with underlying fusion system $\mathcal{F}$. The goal is to describe what a "classifying space" for $\mathfrak{X}$ should look like. The heuristic is that $B\mathfrak{X}$ should recover the homotopy type of the Borel construction of X, up to $\mathfrak{p}$-completion, just as in the ambient case.

The problem with the heuristic is that, without an ambient group, we lack a Borel construction to aim for. Section 5.1 tells us that we could perhaps instead try to define $B\mathfrak{X}$ to be the homotopy colimit of some functor into $\mathcal{TOP}$, but again without an ambient group we do not know what the source category for such a functor should look like. We could try to rectify the situation by developing some notion of an *abstract linking action system*. This would be a category $\mathcal{L}^{\mathfrak{X}}$ associated to $\mathfrak{X}$ that would have the "right" properties so that we could construct a functor $\mathbb{X} : \mathcal{L}^{\mathfrak{X}} \to \mathcal{TOP}$, all of which would be a generalization of the ambient case of Section 5.1.

This will in fact be the plan of attack take, but first a detour: In order to properly understand what is meant by $\mathcal{L}^{\mathfrak{X}}$ having the "right" properties, we should try to understand the space we are looking for purely in terms that can be described by the fusion action system itself. The question then becomes what spaces we can make from the category $\mathfrak{X}$. The first guess of $|\mathfrak{X}|$ will not give us what we want, as this relies only on the shape of the category of $\mathfrak{X}$ and does not take into account the fact that it should be thought of as a combination of a diagram in groups together with permutations of X.

More accurately, we should think of $\mathfrak{X}$ as a diagram in groupoids via the functor $\mathcal{B}_{-}X : \mathfrak{X} \to \mathcal{GRPD}$.[1] For $P \leq S$, let $\mathcal{B}_P X$ denote the translation category of the P-set X. Recall that the objects of $\mathcal{B}_P X$ are the points of X and the morphisms are given by $\mathcal{B}_P X(x, x') = \left\{\check{p} \middle| p \in P \text{ and } p \cdot x = x'\right\}$. We then define $\mathcal{B}_{-}X$ to be the

[1]The claim that Appendix A would be irrelevant to the rest of this document is a bit of a lie; the following discussion would benefit from taking that point of view into consideration. We repeat the relevant definitions here.

functor

$$\begin{array}{ccc} P & \longmapsto & \mathcal{B}_P X \\ {\scriptstyle (\varphi,\sigma)}\downarrow & & \downarrow{\scriptstyle \mathcal{B}(\varphi,\sigma)} \\ Q & \longmapsto & \mathcal{B}_Q X \end{array}$$

where $\mathcal{B}(\varphi, \sigma)$ is the functor that acts on objects by σ and morphisms by φ.

If we then set $B_{-}X : \mathfrak{X} \to \mathcal{TOP}$ to be $|\mathcal{B}_{-}X|$, we can consider the space $\operatorname{hocolim}_{\mathfrak{X}} B_{-}X$, which is defined with information contained in $\mathfrak{X}$.

Unfortunately, this space is not what we want if S is nonabelian or acts nontrivially on X, as we show in Example 2.2.16 in the case of X a one-point set.

The problem with simply taking $\operatorname{hocolim}_{\mathfrak{X}} \mathcal{B}_{-}X$ is that this construction is in some sense double-counting the noncentral elements of S, as well as the elements not in K. In particular, any non-X-central $s \in S$ defines both a morphism in $\check{s} \in \mathcal{B}S$ and a morphism $(c_s, \ell_s) \in \mathfrak{X}(S)$; by simply taking the homotopy colimit of $B_{-}X$ these separate morphisms both contribute even though they come from the same element of S. In other words, $\operatorname{hocolim}_{\mathfrak{X}} B_{-}X$ is too big, in that it has too many arrows.

So, again, let's kill the offending morphisms.

Definition 5.2.1. The *orbit category* of $\mathfrak{X}$ is the category $\mathcal{O}^{\mathfrak{X}} := \mathcal{O}(\mathfrak{X})$ whose objects are the subgroups of S and whose morphisms are given by $\mathcal{O}^{\mathfrak{X}}(P, Q) = Q \backslash \mathfrak{X}(P, Q)$. In other words, the Hom-set from P to Q is the set orbits of the Q-action of $\mathfrak{X}(P, Q)$ given by postcomposition by (c_q, ℓ_q).

The full subcategory of $\mathcal{O}^{\mathfrak{X}}$ whose objects are the $\mathcal{F}$-centric subgroups of S will be denoted $\mathcal{O}^{c\mathfrak{X}}$. ♢

The functor $B_{-}X$ does not descend to $\mathcal{O}^{c\mathfrak{X}} \to \mathcal{TOP}$, but because $\mathcal{O}^{c\mathfrak{X}}$ is defined by quotienting out inner automorphisms, it is easy to see that there is a homotopy functor $\overline{B}_{-}X : \mathcal{O}^{c\mathfrak{X}} \to ho\mathcal{TOP}$. If we could find a homotopy lifting $\widetilde{B}$ as in Figure 5-1 we could consider the space $\operatorname{hocolim}_{\mathcal{O}^{c\mathfrak{X}}} \widetilde{B}_{-}X$ as a possibility for the classifying space of $\mathcal{F}^X$.

We shall see that the homotopy colimit of such a lifting $\widetilde{B}$ is, in fact, the solution we have been seeking.

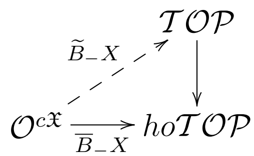


Figure 5-1: The homotopy lifting problem

Proposition 5.2.2. *In the presence of an ambient group G acting on X and giving rise to $\mathfrak{X}$,*

$$\operatorname*{hocolim}_{\mathcal{O}^{c\mathfrak{X}}} \widetilde{B}_{-}X \simeq_{\mathfrak{p}} B_G X.$$

Proof. Follows immediately from Problem 5.2.3, whose solution is detailed in Subsection 5.2.2. □

We are now in the position to fully state the problem we want to solve:

Problem 5.2.3. Given an abstract fusion action system $\mathfrak{X}$, construct an *abstract linking action system* associated to $\mathfrak{X}$. This should be a category $\mathcal{L}^{\mathfrak{X}}$ together with functors

$$\pi : \mathcal{L}^{\mathfrak{X}} \to \mathfrak{X}^{cX} \qquad \text{and} \qquad \mathbb{X} : \mathcal{L}^{\mathfrak{X}} \to \mathcal{TOP}$$

such that

- In the presence of an ambient group G, this $\mathcal{L}_G^{cX}$ and the functor $\overline{\mathbb{X}}$ described in Section 5.1 give an abstract linking action system associated to $\mathfrak{X}_G$.

- If $\overline{\pi} : \mathcal{L}^{\mathfrak{X}} \to \mathcal{O}^{c\mathfrak{X}}$ is the composite $\mathcal{L}^{\mathfrak{X}} \xrightarrow{\pi} \mathfrak{X}^{cX} \to \mathcal{O}^{c\mathfrak{X}}$, then the left homotopy Kan extension of $\mathbb{X}$ over $\overline{\pi}$ is a homotopy lifting of $\overline{B}_{-}X : \mathcal{O}^{c\mathfrak{X}} \to ho\mathcal{TOP}$, which will be denoted $\widetilde{B}_{\mathcal{L}^{\mathfrak{X}}} : \mathcal{O}^{c\mathfrak{X}} \to \mathcal{TOP}$. Consequently,

$$\operatorname*{hocolim}_{\mathcal{L}^{\mathfrak{X}}} \mathbb{X} \simeq \operatorname*{hocolim}_{\mathcal{O}^{c\mathfrak{X}}} \widetilde{B}_{\mathcal{L}^{\mathfrak{X}}}.$$

5.2.2 Solving the problem

Definition 5.2.4. An *(X-centric) linking action system associated to* $\mathfrak{X}$ is a category $\mathcal{L}^{\mathfrak{X}}$ whose objects are the X-centric subgroups of S, together with a pair of functors

$$\mathcal{T}_S^{cX} \xrightarrow{\delta} \mathcal{L}^{\mathfrak{X}} \xrightarrow{\pi} \mathfrak{X}^{cX}.$$

For each $s \in N_S(P, Q)$, let $\widehat{s}$ be the corresponding morphism in $\delta_{P,Q}(s) \in \mathcal{L}^{\mathfrak{X}}(P, Q)$. For $\mathfrak{g} \in \mathcal{L}^{\mathfrak{X}}(P, Q)$ denote the components of $\pi_{P,Q}(\mathfrak{g})$ by the pair $(c_{\mathfrak{g}}, \ell_{\mathfrak{g}})$.

The following conditions must be satisfied:

(A) The functors δ and π are the identity on objects. δ is injective and π is surjective on morphisms. Moreover, $Z(P; X)$ acts (via δ) right-freely on $\mathcal{L}^{\mathfrak{X}}(P, Q)$, and $\pi_{P,Q} : \mathcal{L}^{\mathfrak{X}}(P, Q) \to \mathfrak{X}^{cX}(P, Q)$ is the orbit map of this action.

(B) For each $p \in P$, we have $c_{\widehat{p}} = c_p : P \to P$ and $\ell_{\widehat{p}} = \ell_p : X \to X$.

(C) For each $\mathfrak{g} \in \mathcal{L}^{\mathfrak{X}}(P, Q)$ and $p \in P$, the following commutes in $\mathcal{L}^{\mathfrak{X}}$:

$$\begin{array}{ccc} P & \xrightarrow{\mathfrak{g}} & Q \\ {\scriptstyle \widehat{p}}\downarrow & & \downarrow{\scriptstyle \widehat{c_{\mathfrak{g}}(p)}} \\ P & \xrightarrow[\mathfrak{g}]{} & Q \end{array}$$

Finally, every linking action system comes naturally equipped with a functor $\mathbb{X} : \mathcal{L}^{\mathfrak{X}} \to \mathcal{TOP}$ defined by

$$\begin{array}{ccc} P & \longmapsto & X \\ {\scriptstyle \mathfrak{g}}\downarrow & & \downarrow{\scriptstyle \ell_{\mathfrak{g}}} \\ Q & \longmapsto & X \end{array}$$

where X is regarded as a discrete topological space. $\diamondsuit$

Remark 5.2.5. It is easy to see that:

- If $X = *$ is the trivial S-set and $\mathfrak{X} = \mathcal{F}$ is a saturated fusion system, the an abstract linking action system is the same as a centric linking system as defined in [BLO2]. In this case $\mathbb{X}$ is just the trivial functor $\mathcal{L}^* \to \mathcal{TOP}$ whose homotopy colimit is $|\mathcal{L}^*|$.

- In the presence of an ambient group G, the category $\mathcal{L}_G^{cX}$ of Section 5.1 is an example of an abstract linking action system associated to $\mathfrak{X}_G$. In particular, this definition satisfies the first condition of Problem 5.2.3. $\diamondsuit$

Unless it is necessary to emphasize the fact that we are looking at the X-centric subcategories, we shall omit explicit notational reference to them.

The remainder of this section is devoted to proving that these abstract linking action systems satisfy the second condition of Problem 5.2.3. We begin with some basic properties of abstract linking action systems:

Proposition 5.2.6. *Let* $P \xrightarrow{(\varphi,\sigma)} Q \xrightarrow{(\psi,\tau)} R$ *be a sequence of morphisms in* $\mathfrak{X}$*. Then for any*

$$\mathfrak{g} \in \pi_{Q,R}^{-1}((\psi,\tau)) \subseteq \mathcal{L}^{\mathfrak{X}}(Q,R) \qquad \textit{and} \qquad \widetilde{\mathfrak{g}\mathfrak{h}} \in \pi_{P,R}^{-1}((\psi\varphi,\tau\sigma)) \subseteq \mathcal{L}^{\mathfrak{X}}(P,R)$$

there is a unique $\mathfrak{h} \in \pi_{P,Q}^{-1}((\varphi,\sigma)) \subseteq \mathcal{L}^{\mathfrak{X}}(P,Q)$ *such that* $\mathfrak{g}\mathfrak{h} = \widetilde{\mathfrak{g}\mathfrak{h}}$.

Proof. Pick any $\mathfrak{h}' \in \pi_{P,Q}^{-1}((\varphi,\sigma)) \subseteq \mathcal{L}^{\mathfrak{X}}(P,Q)$. By Axiom (A), the X-center $Z(P;X)$ acts freely and transitively on $\pi_{P,R}^{-1}((\psi\varphi,\tau\sigma)) \subseteq \mathcal{L}^{\mathfrak{X}}(P,R)$, so there is a unique $z \in Z(P;X)$ such that $\widetilde{\mathfrak{g}\mathfrak{h}} = \mathfrak{g}\mathfrak{h}'\widehat{z}$. Therefore setting $\mathfrak{h} = \mathfrak{h}'\widehat{z}$ gives a morphism in $\pi_{P,Q}^{-1}((\varphi,\sigma))$ such that $\mathfrak{g}\mathfrak{h} = \widetilde{\mathfrak{g}\mathfrak{h}}$.

To prove uniqueness, suppose that we have $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathcal{L}^{\mathfrak{X}}(P,Q)$ such that $\mathfrak{g}\mathfrak{h}_1 = \mathfrak{g}\mathfrak{h}_2$, $\mathfrak{h}_i$ lifts (φ,ψ), and $\mathfrak{g}\mathfrak{h}_i = \widetilde{\mathfrak{g}\mathfrak{h}}$. Then $c_{\mathfrak{g}}c_{\mathfrak{h}_1} = c_{\mathfrak{g}}c_{\mathfrak{h}_2} \in \mathfrak{X}(P,R)$, and since every morphism of a fusion action system is mono (the first coordinate is an injective group map and the second coordinate is invertible), we conclude that $c_{\mathfrak{h}_1} = c_{\mathfrak{h}_2}$. Therefore by Axiom (A) there is a unique $z \in Z(P;X)$ so that $\mathfrak{h}_1 = \mathfrak{h}_2 \circ \widehat{z}$. Then the facts that

$$\widetilde{\mathfrak{g}\mathfrak{h}} = \mathfrak{g}\mathfrak{h}_1 = \mathfrak{g}\mathfrak{h}_2 \circ \widehat{z} = \widetilde{\mathfrak{g}\mathfrak{h}} \circ \widehat{z}$$

and $Z(P;X)$ acts right-freely on $\mathcal{L}^{\mathfrak{X}}(P,R)$ implies that $z = 1$, so $\mathfrak{h}_1 = \mathfrak{h}_2$. $\square$

Remark 5.2.7. Note that the full strength of the axioms of a linking action system is not used to prove this proposition. In particular, the fact that $\mathcal{L}^{\mathfrak{X}}$ comes equipped with a functor $\mathcal{T}_S^{cX} \to \mathcal{L}^{\mathfrak{X}}$ could have been replaced with the (apparently) weaker assumption that for every X-centric $P \le S$, there is an injection $P \to \mathcal{L}^{\mathfrak{X}}(P)$ such that Axiom (A) holds. This observation will sometimes be useful when trying to construct linking action systems. $\diamondsuit$

Proposition 5.2.6 has a number of immediate consequences:

Corollary 5.2.8. *Every morphism in $\mathcal{L}^{\mathfrak{X}}$ is categorically mono.*

Proof. Suppose that we're given morphisms $\mathfrak{g} \in \mathcal{L}^{\mathfrak{X}}(Q,R)$ and $\mathfrak{h},\mathfrak{h}' \in \mathcal{L}^{\mathfrak{X}}(P,Q)$ such that $\mathfrak{g}\mathfrak{h} = \mathfrak{g}\mathfrak{h}'$. Then $\pi(\mathfrak{g}\mathfrak{h}) = \pi(\mathfrak{g}\mathfrak{h}')$, and $\mathfrak{g}$ is a lifting of $\pi(\mathfrak{g})$, so the uniqueness statement of Proposition 5.2.6 forces $\mathfrak{h} = \mathfrak{h}'$. $\square$

Corollary 5.2.9. *If $\mathfrak{g} \in \mathcal{L}^{\mathfrak{X}}(P,Q)$ is such that $(c_{\mathfrak{g}}, \ell_{\mathfrak{g}}) \in \mathfrak{X}(P,Q)$ is an isomorphism, then $\mathfrak{g}$ is itself an isomorphism.*

Proof. Apply Proposition 5.2.6 to the sequence $Q \xrightarrow{(c_{\mathfrak{g}},\ell_{\mathfrak{g}})^{-1}} P \xrightarrow{(c_{\mathfrak{g}},\ell_{\mathfrak{g}})} Q$ with $\mathfrak{g}$ lifting $(c_{\mathfrak{g}}, \ell_{\mathfrak{g}})$ and $\mathrm{id}_Q^{\mathcal{L}^{\mathfrak{X}}}$ lifting the composite $\mathrm{id}_Q^{\mathfrak{X}}$. We obtain a unique $\mathfrak{h} \in \mathcal{L}^{\mathfrak{X}}(Q,P)$ such that $\mathfrak{g}\mathfrak{h} = \mathrm{id}_Q^{\mathcal{L}^{\mathfrak{X}}}$. It follows that $\mathfrak{g}\mathfrak{h}\mathfrak{g} = \mathfrak{g} = \mathfrak{g} \circ \mathrm{id}_P^{\mathcal{L}^{\mathfrak{X}}}$ and thus $\mathfrak{h}\mathfrak{g} = \mathrm{id}_P^{\mathcal{L}^{\mathfrak{X}}}$ as $\mathfrak{g}$ is categorically mono. Therefore $\mathfrak{h} = \mathfrak{g}^{-1}$. $\square$

Notation 5.2.10. We denote by $\mathfrak{i}_P^Q$ the morphism $\delta_{P,Q}(1) \in \mathcal{L}^{\mathfrak{X}}(P,Q)$, and call this the "inclusion" of P in Q in $\mathcal{L}^{\mathfrak{X}}$. $\diamondsuit$

Corollary 5.2.11. *For any $\mathfrak{g} \in \mathcal{L}^{\mathfrak{X}}(P,Q)$ and X-centric subgroups $P^* \le P$ and $Q^* \le Q$ such that $c_{\mathfrak{g}}(P^*) \le Q^*$, there is a unique morphism $\mathrm{res}_{P^*}^{Q^*}(\mathfrak{g}) \in \mathcal{L}^{\mathfrak{X}}(P^*,Q^*)$ such that the following diagram commutes in $\mathcal{L}^{\mathfrak{X}}$:*

$$\begin{array}{ccc} P & \xrightarrow{\ \mathfrak{g}\ } & Q \\ {\scriptstyle \mathfrak{i}_{P^*}^{P}}\uparrow & & \uparrow{\scriptstyle \mathfrak{i}_{Q^*}^{Q}} \\ P^* & \xrightarrow[\mathrm{res}_{P^*}^{Q^*}(\mathfrak{g})]{} & Q^* \end{array}$$

Proof. Apply Proposition 5.2.6 to the diagram $P^* \xrightarrow{c_{\mathfrak{g}}|_{P^*}} Q^* \xrightarrow{\iota_{Q^*}^{Q}} Q$ with $\mathfrak{i}_{Q^*}^{Q}$ lifting $\iota_{Q^*}^{Q}$ and $\mathfrak{g} \circ \mathfrak{i}_{P^*}^{P}$ lifting the composite. $\square$

Notation 5.2.12. The morphism $\mathrm{res}_{P^*}^{Q^*}(\mathfrak{g})$ is called the "restriction" of $\mathfrak{g}$, and will sometimes be denoted $\mathfrak{g}\big|_{P^*}^{Q^*}$ or just $\mathfrak{g}|_{P^*}$ if the target is clear. $\diamondsuit$

Corollary 5.2.13. *Every morphism in $\mathcal{L}^{\mathfrak{X}}$ factors uniquely as an isomorphism followed by an inclusion.*

Proof. For $\mathfrak{g} \in \mathcal{L}^{\mathfrak{X}}(P, Q)$, the morphism $(c_{\mathfrak{g}}, \ell_{\mathfrak{g}}) \in \mathfrak{X}(P, c_{\mathfrak{g}}(P))$ is an isomorphism in the underlying action system, so $\mathrm{res}_{P}^{c_{\mathfrak{g}}(P)}(\mathfrak{g})$ is an isomorphism in $\mathcal{L}^{\mathfrak{X}}$ by Corollary 5.2.9. Clearly $\mathfrak{g} = \mathfrak{i}_{c_{\mathfrak{g}}(P)}^{Q} \circ \mathrm{res}_{P}^{c_{\mathfrak{g}}(P)}(\mathfrak{g})$, and uniqueness now follows by another application of Proposition 5.2.6. $\square$

Proposition 5.2.14. *Every morphism in $\mathcal{L}^{\mathfrak{X}}$ is categorically epi.*

Proof. By Corollary 5.2.13, it suffices to show that $\mathfrak{i} := \mathfrak{i}_{P}^{Q}$ is epi for all X-centric $P \leq Q$. Moreover, it suffices to assume that $P \trianglelefteq Q$, as any inclusion of $\mathfrak{p}$-groups can be refined to a sequence of normal inclusions. Let $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathcal{L}^{\mathfrak{X}}(Q, R)$ be two morphisms such that $\mathfrak{g}_1 \circ \mathfrak{i} = \mathfrak{g}_2 \circ \mathfrak{i}$; we want to show that $\mathfrak{g}_1 = \mathfrak{g}_2$.

The image of $\mathfrak{g}_i$ in $\mathfrak{X}(Q, R)$ is $(c_{\mathfrak{g}_i}, \sigma_i)$ and the assumption that $\mathfrak{g}_1 \circ \mathfrak{i} = \mathfrak{g}_2 \circ \mathfrak{i}$ implies that $\sigma_1 = \sigma_2$. We first show that it also follows that $c_{\mathfrak{g}_1} = c_{\mathfrak{g}_2}$.

Note that the assumption that $P \trianglelefteq Q$ implies that every $q \in Q$ determines a morphism in $\widehat{q} \in \mathcal{L}^{\mathfrak{X}}(P)$, which is easily seen to be the restriction of $\widehat{q} \in \mathcal{L}^{\mathfrak{X}}(Q)$. Then the following diagram commutes in $\mathcal{L}^{\mathfrak{X}}$ by axiom (C)

$$\begin{array}{ccc} Q & \xrightarrow{\mathfrak{g}_i} & R \\ {\scriptstyle \widehat{q}}\downarrow & & \downarrow{\scriptstyle \widehat{c_{\mathfrak{g}_i}(q)}} \\ Q & \xrightarrow[\mathfrak{g}_i]{} & R \end{array}$$

for $i = 1, 2$. The assumption on the $\mathfrak{g}_i$ also implies that their restrictions to P in $\mathcal{L}^{\mathfrak{X}}$ are equal; denote this common morphism by $\mathfrak{h} \in \mathcal{L}^{\mathfrak{X}}(P, c_{\mathfrak{h}}(P))$. We can therefore form the restriction of this diagram, which gives us

$$\begin{array}{ccc} P & \xrightarrow{\mathfrak{h}} & c_{\mathfrak{h}}(P) \\ {\scriptstyle \widehat{q}}\downarrow & & \downarrow{\scriptstyle \widehat{c_{\mathfrak{g}_i}(q)}} \\ P & \xrightarrow[\mathfrak{h}]{} & c_{\mathfrak{g}}(P) \end{array}$$

for $i = 1, 2$. As all the morphisms in this restriction diagram are iso, and three of the four do not depend on choice of i, we conclude that $\widehat{c_{\mathfrak{g}_1}(q)} = \widehat{c_{\mathfrak{g}_2}(q)}$ for all $q \in Q$. The assignment $q \mapsto \widehat{q}$ is injective, so we conclude that $c_{\mathfrak{g}_1} = c_{\mathfrak{g}_2}$ on Q.

Thus we have $\pi(\mathfrak{g}_1) = \pi(\mathfrak{g}_2)$, so by Axiom (2), there is some $z \in Z(Q; X)$ such that $\mathfrak{g}_2 = \mathfrak{g}_1 \circ \widehat{z}$. We then compute $\mathfrak{g}_1 \circ \widehat{z} \circ \mathfrak{i} = \mathfrak{g}_2 \circ \mathfrak{i} = \mathfrak{g}_1 \circ \mathfrak{i}$. The fact that $\mathfrak{g}_1$ is mono implies that $\widehat{z} \circ \mathfrak{i} = \mathfrak{i}$.

Finally, the fact that $P \leq Q$ implies that $z \in Z(P; X)$, and Axiom (3) again shows that $\widehat{z} \circ \mathfrak{i} = \mathfrak{i} \circ \widehat{z}$ (for $\widehat{z}$ respectively an isomorphism in $\mathcal{L}^{\mathfrak{X}}$ of Q and P). The freeness of the right action of $Z(P; X)$ on $\mathcal{L}^{\mathfrak{X}}(P, Q)$ forces $z = 1$, and the result is proved. □

Corollary 5.2.15. *Extensions of morphisms in $\mathcal{L}^{\mathfrak{X}}$ are unique when they exist. In other words, for any $\mathfrak{g}^* \in \mathcal{L}^{\mathfrak{X}}(P^*, Q^*)$ and subgroups $P^* \leq P$ and $Q^* \leq Q$, there is at most one morphism $\mathfrak{g} \in \mathcal{L}^{\mathfrak{X}}(P, Q)$ such that the diagram*

$$\begin{array}{ccc} P & \xrightarrow{\mathfrak{g}} & Q \\ {\scriptstyle \mathfrak{i}_{P^*}^{P}}\uparrow & & \uparrow{\scriptstyle \mathfrak{i}_{Q^*}^{Q}} \\ P^* & \xrightarrow[\mathfrak{g}^*]{} & Q^* \end{array}$$

commutes in $\mathcal{L}^{\mathfrak{X}}$.

Proof. If $\mathfrak{g}'$ is another such extension, the equalities $\mathfrak{g} \circ \mathfrak{i}_{P^*}^{P} = \mathfrak{i}_{Q^*}^{Q} \circ \mathfrak{g}^* = \mathfrak{g}' \circ \mathfrak{i}_{P^*}^{P}$ and the fact that $\mathfrak{i}_{P^*}^{P}$ is epi force $\mathfrak{g} = \mathfrak{g}'$. □

Proposition 5.2.16. *If $P \leq S$ is fully normalized in $\mathcal{F}$, then $\widehat{N_S(P)}\big|_P^P \in \mathrm{Syl}_{\mathfrak{p}}\left(\mathcal{L}^{\mathfrak{X}}(P)\right)$.*

Proof. By Axiom (A) of linking action systems, $\left|\mathcal{L}^{\mathfrak{X}}(P)\right| = |\mathfrak{X}(P)| \cdot |Z(P; X)|$ and by definition of $\mathrm{Aut}_S(P;X)$,

$$|N_S(P)| = |\mathrm{Aut}_S(P;X)| \cdot |Z_S(P;X)| = |\mathrm{Aut}_S(P;X)| \cdot |Z(P;X)|\,.$$

As P is fully normalized, the saturation axioms imply that $\mathrm{Aut}_S(P;X) \in \mathrm{Syl}_{\mathfrak{p}}(\mathfrak{X}(P))$, and the result easily follows. □

Notation 5.2.17. Let $\overline{\pi}$ be the composite $\mathcal{L}^{\mathfrak{X}} \xrightarrow{\pi} \mathfrak{X} \to \mathcal{O}^{c\mathfrak{X}}$. ♢

Proposition 5.2.18. *Suppose that $\mathfrak{g}, \mathfrak{h} \in \mathcal{L}^{\mathfrak{X}}(P,Q)$ are such that $\overline{\pi}(\mathfrak{g}) = \overline{\pi}(\mathfrak{h})$. Then there is a unique element $q \in Q$ such that $\mathfrak{h} = \widehat{q} \circ \mathfrak{g}$. In other words, the map $\overline{\pi}_{P,Q} : \mathcal{L}^{\mathfrak{X}}(P,Q) \to \mathcal{O}^{c\mathfrak{X}}(P,Q)$ is the orbit map of the free left action of Q on $\mathcal{L}^{\mathfrak{X}}(P,Q)$.*

Proof. The condition on $\mathfrak{g}$ and $\mathfrak{h}$ implies that there is some $q' \in Q$ such that

$$(c_{\mathfrak{h}}, \ell_{\mathfrak{h}}) = (c_{q'}, \ell_{q'}) \circ (c_{\mathfrak{g}}, \ell_{\mathfrak{g}}).$$

Condition (A) then implies that there is a unique $z \in Z(P;X)$ such that

$$\mathfrak{h} = \widehat{q'} \circ \mathfrak{g} \circ \widehat{z} = \widehat{q'} \circ \widehat{c_{\mathfrak{g}}(z)} \circ \mathfrak{g}$$

where the second equality follows from Condition (C). Setting $q = q' \circ c_{\mathfrak{g}}(z)$ gives the desired element.

The uniqueness of q is a direct consequence of the fact that $\mathfrak{g}$ is epi and the assignment $q \mapsto \widehat{q}$ is injective. □

We now have the necessary results to show that we have solved the second point of Problem 5.2.3. Recall that $\widetilde{B}\_X : \mathcal{O}^{c\mathfrak{X}} \to \mathcal{TOP}$ is the left homotopy Kan extension of $\mathbb{X} : \mathcal{L}^{\mathfrak{X}} \to \mathcal{TOP}$ over $\overline{\pi} : \mathcal{L}^{\mathfrak{X}} \to \mathcal{O}^{c\mathfrak{X}}$, and that we are looking for a homotopy lifting of $\overline{B}\_X : \mathcal{O}^{c\mathfrak{X}} \to ho\mathcal{TOP}$, as depicted in Firgure 5-2.

For ease of notation, write $\widetilde{B}$ for $\widetilde{B}\_X$ and B for $\overline{B}\_X$.

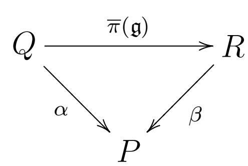
Figure 5-2: What we have vs. What we want

Proposition 5.2.19. *The functor $\widetilde{B}$ is a homotopy lifting of B.*

Proof. **On objects:** We want to show that $\widetilde{B}(P) \simeq B(P) = B_P X$. Recall that

$$\widetilde{B}(P) \simeq \underset{(Q,\alpha)\in(\overline{\pi}\downarrow P)}{\operatorname{hocolim}} \mathbb{X}(Q)$$

where $(\overline{\pi} \downarrow P)$ is the overcategory of $\overline{\pi}$ over P. The objects are pairs (Q, α), where Q is an X-centric subgroup of S, viewed as either an object of $\mathcal{L}^{\mathfrak{X}}$ or $\mathcal{O}^{c\mathfrak{X}}$ as appropriate, and α is a morphism in $\mathcal{O}^{c\mathfrak{X}}(Q, P)$. A morphism from (Q, α) to (R, β) is $\mathfrak{g} \in \mathcal{L}^{\mathfrak{X}}(Q, R)$ such that the following commutes in $\mathcal{O}^{c\mathfrak{X}}$:



Thomason's theorem [Tho] shows that this homotopy colimit expression of $\overline{B}(P)$ has the homotopy type of $|\mathcal{G}(P)|$, where $\mathcal{G}(P)$ is the Grothendieck category associated to $(\overline{\pi} \downarrow P)$ and the functor $\mathbb{X}$.

Explicitly, $\mathcal{G}(P)$ is the category whose objects are pairs $((Q, \alpha), x)$ where (Q, α) is an object of $(\overline{\pi} \downarrow P)$ and $x \in X$. A morphism $((Q, \alpha), x) \to ((R, \beta), y))$ is a morphism $\mathfrak{g} \in (\overline{\pi} \downarrow P)((Q, \alpha), (R, \beta))$—and therefore $\mathfrak{g} \in \mathcal{L}^{\mathfrak{X}}(Q, R)$—such that $\ell_{\mathfrak{g}}(x) = y$.

Let $\check{\mathcal{B}}(P) \subseteq \mathcal{G}(P)$ be the subcategory whose objects are pairs $((P, \mathrm{id}_P), x)$ for all $x \in X$, and where

$$\check{\mathcal{B}}(P)(((P, \mathrm{id}_P), x), ((P, \mathrm{id}_P), y)) = \left\{\widehat{p}\middle| p \in P, p \cdot x = y\right\}.$$

This category is isomorphic to $\mathcal{B}_P X$; the claim is that the inclusion $\check{\mathcal{B}}(P) \subseteq \mathcal{G}(P)$

induces a deformation retract after realization. This will follow if we can find a functor $\Psi : \mathcal{G}(P) \to \check{\mathcal{B}}(P)$ that is the identity on $\check{\mathcal{B}}(P)$ together with a natural transformation $F : \mathrm{Id}_{\mathcal{G}(P)} \Rightarrow \iota_{\check{\mathcal{B}}(P)}^{\mathcal{G}(P)} \circ \Psi$, where $\iota_{\check{\mathcal{B}}(P)}^{\mathcal{G}(P)} : \check{\mathcal{B}}(P) \to \mathcal{G}(P)$ is the inclusion functor.

Pick some lifting of morphisms $\xi : \mathrm{Mor}(\mathcal{O}^{c\mathfrak{X}}) \to \mathrm{Mor}(\mathcal{L}^{\mathfrak{X}})$, and assume that ξ sends identities to identities. Let $\Psi : \mathcal{G}(P) \to \check{\mathcal{B}}(P)$ be the functor

$$\begin{array}{ccc} ((Q,\alpha),x) & \longmapsto & \big((P,\mathrm{id}_P),\ell_{\xi(\alpha)}(x)\big) \\ \downarrow{\scriptstyle \mathfrak{g}} & & \downarrow{\scriptstyle \Psi(\mathfrak{g})} \\ ((R,\beta),y) & \longmapsto & \big((P,\mathrm{id}_P),\ell_{\xi(\beta)}(y)\big) \end{array}$$

where $\Psi(\mathfrak{g})$ is defined to be $\widehat{p}$ for the unique $p \in P$ such that

$$\begin{array}{ccc} Q & \xrightarrow{\mathfrak{g}} & R \\ {\scriptstyle \xi(\alpha)}\downarrow & & \downarrow{\scriptstyle \xi(\beta)} \\ P & \xrightarrow[\widehat{p}]{} & P \end{array}$$

The existence and uniqueness of p follow from Proposition 5.2.18 and the fact that $\overline{\pi}(\xi(\beta) \circ \mathfrak{g}) = \beta \circ \overline{\pi}(\mathfrak{g}) = \alpha = \overline{\pi}(\xi(\alpha))$ because $\mathfrak{g}$ is a morphism in $(\overline{\pi} \downarrow P)$.

We must check that $\widehat{p}$ indeed defines a morphism in $\mathcal{G}(P)$:

$$((P,\mathrm{id}_P),\ell_{\xi(\alpha)}(x)) \to ((P,\mathrm{id}_P),\ell_{\xi(\beta)}(y)).$$

To do this, we must show that

$$\begin{array}{ccc} P & \xrightarrow{(c_p,\ell_p)} & P \\ & {\scriptstyle \mathrm{id}_P}\searrow \quad \swarrow{\scriptstyle \mathrm{id}_P} & \\ & P & \end{array}$$

commutes in $\mathcal{O}^{c\mathfrak{X}}$ and that $\ell_p \circ \ell_{\xi(\alpha)}(x) = \ell_{\xi(\beta)}(y)$. The first is obvious from the definition of $\mathcal{O}^{c\mathfrak{X}}$, and the second follows from the fact that $\widehat{p} \circ \xi(\alpha) = \xi(\beta) \circ \mathfrak{g}$, so that $\ell_p \circ \ell_{\xi(\alpha)}(x) = \ell_{\xi(\beta)} \circ \ell_{\mathfrak{g}}(x) = \ell_{\xi(\beta)}(y)$ by the assumption on $\mathfrak{g}$.

Observe that $\Psi|_{\check{\mathcal{B}}(P)}$ is the identity functor.

Define the natural transformation $\Theta : \mathrm{Id}_{\mathcal{G}(P)} \Rightarrow \iota^{\mathcal{G}(P)}_{\check{\mathcal{B}}(P)} \circ \Psi$ by

$$\Theta(((Q,\alpha),x)) = \xi(\alpha) : ((Q,\alpha),x) \to ((P,\mathrm{id}_P),\ell_{\xi(\alpha)}(x)).$$

That this is a morphism in $\mathcal{G}(P)$ follows easily from the definition of $\mathcal{G}(P)$ above and the fact that $\xi(\alpha)$ is a lift of α to $\mathcal{L}$. If $\mathfrak{g} : ((Q,\alpha),x) \to ((R,\beta),y)$ is a morphism in $\mathcal{G}(P)$, we want the diagram

$$\begin{array}{ccc} ((Q,\alpha),x) & \xrightarrow{\mathfrak{g}} & ((R,\beta),y) \\ {\scriptstyle \xi(\alpha)}\downarrow & & \downarrow{\scriptstyle \xi(\beta)} \\ ((P,\mathrm{id}_P),\ell_{\xi(\alpha)}(x)) & \xrightarrow[\Psi(\mathfrak{g})]{} & ((P,\mathrm{id}_P),\ell_{\xi(\beta)}(y)) \end{array}$$

to commute in $\mathcal{L}^{\mathfrak{X}}$, which follows from the definition of $\Psi(\mathfrak{g})$.

Finally, note that for any object of $\check{x} \in \check{\mathcal{B}}(P)$, it is easy to see that $F(\check{x})$ is the identity. Thus the geometric realization of Θ gives a homotopy that shows that $|\Psi|$ realizes $\left|\check{\mathcal{B}}(P)\right|$ as a deformation retract of $|\mathcal{G}(P)|$, and we have the result on objects.

On morphisms: First observe that each $[\varphi,\sigma] \in \mathcal{O}^{c\mathfrak{X}}(P,P')$ induces a functor $(\overline{\pi} \downarrow P) \to (\overline{\pi} \downarrow P')$:

$$\begin{array}{ccc} (Q,\alpha) & \longmapsto & (Q,[\varphi,\sigma]\circ\alpha) \\ {\scriptstyle \mathfrak{g}}\downarrow & & \downarrow{\scriptstyle \mathfrak{g}} \\ (R,\beta) & \longmapsto & (R,[\varphi,\sigma]\circ\beta) \end{array}$$

This in turn induces a functor $[\varphi,f]_* : \mathcal{G}(P) \to \mathcal{G}(P')$, defined by

$$\begin{array}{ccc} ((Q,\alpha),x) & \longmapsto & ((Q,[\varphi,\sigma]\circ\alpha),x) \\ {\scriptstyle \mathfrak{g}}\downarrow & & \downarrow{\scriptstyle \mathfrak{g}} \\ ((R,\beta),\ell_{\mathfrak{g}}x) & \longmapsto & ((R,[\varphi,\sigma]\circ\beta),\ell_{\mathfrak{g}}x) \end{array}$$

Similarly, for $(\varphi,\sigma) \in \mathfrak{X}(P,P')$ a lift of $[\varphi,\sigma] \in \mathcal{O}^{c\mathfrak{X}}(P,P')$, there is a functor $\check{\mathcal{B}}_{(\varphi,\sigma)}$:

$\check{\mathcal{B}}(P) \to \check{\mathcal{B}}(P')$:

$$\begin{array}{ccc} ((P, \mathrm{id}_P), x) & \longmapsto & ((P', \mathrm{id}_{P'}), \sigma(x)) \\ \Big\downarrow \widehat{p} & & \Big\downarrow \widehat{\varphi p} \\ ((P, \mathrm{id}_P), p \cdot x) & \longmapsto & ((P', \mathrm{id}_{P'}), \varphi p \cdot \sigma(x)) \end{array}$$

To finish the proof that $\widetilde{B}$ is a homotopy lifting of B we must show that the diagram of functors

$$\begin{array}{ccc} \check{\mathcal{B}}(P) & \xrightarrow{\iota} & \mathcal{G}(P) \\ \Big\downarrow \check{\mathcal{B}}_{(\varphi,\sigma)} & & \Big\downarrow [\varphi,\sigma]_* \\ \check{\mathcal{B}}(P') & \xrightarrow[\iota]{} & \mathcal{G}(P') \end{array}$$

commutes up to natural transformation.

Let F_1 be the functor given by the top path of the diagram. Explicitly, F_1 is the functor

$$\begin{array}{ccc} ((P, \mathrm{id}_P), x) & \longmapsto & ((P, [\varphi, \sigma]), x) \\ \Big\downarrow \widehat{p} & & \Big\downarrow \widehat{p} \\ ((P, \mathrm{id}_P), p \cdot x) & \longmapsto & ((P, [\varphi, \sigma]), p \cdot x) \end{array}$$

Let the bottom path be the functor F_2:

$$\begin{array}{ccc} ((P, \mathrm{id}_P), x) & \longmapsto & ((P', \mathrm{id}_{P'}), \sigma(x)) \\ \Big\downarrow \widehat{p} & & \Big\downarrow \widehat{\varphi p} \\ ((P, \mathrm{id}_P), p \cdot x) & \longmapsto & ((P', \mathrm{id}_{P'}), \varphi p \cdot \sigma(x)) \end{array}$$

Choose some $\mathfrak{h} \in \mathcal{L}^{\mathfrak{X}}(P, P')$ that lifts (φ, σ), and define the natural transformation

$$\Phi : F_1 \Rightarrow F_2 : ((P, id), x) \mapsto \mathfrak{h}.$$

To see that $\mathfrak{h}$ is indeed a morphism in $\mathcal{G}(P')$ from $((P, [\varphi, \sigma]), x)$ to $((P', \mathrm{id}_{P'}), \sigma(x))$, note that

$$\begin{array}{ccc} P & \xrightarrow{\overline{\pi}(\mathfrak{h})} & P' \\ & {\scriptstyle [\varphi, f]} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_{P'}} & \\ & P' & \end{array}$$

commutes because $\mathfrak{h}$ is a lifting of $[\varphi, \sigma]$. Moreover, $\ell_{\mathfrak{h}} x = \sigma(x)$ because $\mathfrak{h}$ is a lifting of (φ, σ).

Finally, if $\widehat{p}$ is a morphism in $\check{\mathcal{B}}(P)$ from $((P, \mathrm{id}_P), x)$ to $((P, \mathrm{id}_P), y)$, the diagram

$$\begin{array}{ccc} ((P,[\varphi,\sigma]),x) & \xrightarrow{\mathfrak{h}} & ((P',\mathrm{id}_{P'}),\sigma(x)) \\ \downarrow{\scriptstyle \widehat{p}} & & \downarrow{\scriptstyle \widehat{\varphi p}} \\ ((P,[\varphi,\sigma]),y) & \xrightarrow[\mathfrak{h}]{} & ((P',\mathrm{id}_{P'}),\sigma(y)) \end{array}$$

commutes in $\mathcal{G}(P')$ because Axiom (C) and the fact that $c_{\mathfrak{h}} = \varphi$ ($\mathfrak{h}$ lifts (φ, σ)) together imply that $\widehat{\varphi p} \circ \mathfrak{h} = \mathfrak{h} \circ \widehat{p}$ in $\mathcal{L}^{\mathfrak{X}}(P, P')$.

This completes the proof that $\widetilde{B}$ is a homotopy lifting of B. $\square$

5.3 Obstruction theory

Fix an abstract fusion action system $\mathfrak{X}$. The material in this section does not depend on $\mathfrak{X}$ being saturated.

We can solve the homotopy lifting problem for $\overline{\mathcal{B}}_{-}X : \mathcal{O}^{c\mathfrak{X}} \to ho\mathcal{TOP}$, and thus create a space that we might call a "classifying space" for $\mathfrak{X}$, if we have an associated linking action system $\mathcal{L}^{\mathfrak{X}}$. In this section we describe how we can construct $\mathcal{L}^{\mathfrak{X}}$ from the data of $\mathfrak{X}$, or more accurately, we describe the difficulties in doing so. We also describe the obstructions to constructing $\mathcal{L}^{\mathfrak{X}}$ uniquely. The work of this section can basically be found mutatis mutandis in the obstruction theory of [BLO2].

Definition 5.3.1. If there is a unique $\mathcal{L}^{\mathfrak{X}}$ associated to $\mathfrak{X}$, the space $\mathrm{hocolim}_{\mathcal{L}^{\mathfrak{X}}} \mathbb{X}$ is the *classifying space* for the fusion action system. We denote this space by $B\mathfrak{X}$. $\diamondsuit$

Notation 5.3.2. We denote by $(\widetilde{\varphi}, \widetilde{\sigma}) \in \mathfrak{X}(P, Q)$ a lift of $[\varphi, \sigma] \in \mathcal{O}^{c\mathfrak{X}}(P, Q)$. $\diamondsuit$

Definition 5.3.3. Let $\mathcal{Z}^{\mathfrak{X}} : \left(\mathcal{O}^{c\mathfrak{X}}\right)^{op} \to \mathcal{A}b$ be the functor

$$\begin{array}{ccccccc} P & \mapsto & Z(P;X) & \xleftarrow{\widetilde{\varphi}^{-1}} & Z(\widetilde{\varphi}(P);X) & = & Z_S(\widetilde{\varphi}(P);X) \\ \downarrow{\scriptstyle [\varphi,\sigma]} & & \uparrow{\scriptstyle \mathcal{Z}^{\mathfrak{X}}([\varphi,\sigma])} & & & & \uparrow{\scriptstyle \mathrm{incl}} \\ Q & \mapsto & Z(Q;X) & = & = & = & Z_S(Q;X) \end{array}$$

where $\mathcal{Z}^{\mathfrak{X}}([\varphi, \sigma])$ is the unique map of groups that makes the rectangle commute. $\diamondsuit$

Proposition 5.3.4. *The functor $\mathcal{Z}^{\mathfrak{X}}$ is well-defined.*

Proof. The equalities come from the assumption that P is X-centric. We must simply show that for $(\widetilde{\varphi}', \widetilde{\sigma}')$ any another lift of $[\varphi, \sigma]$, the maps $\widetilde{\varphi}^{-1}$ and $(\widetilde{\varphi}')^{-1}$ are equal on $Z(Q; X)$. Since that $\widetilde{\varphi}$ and $\widetilde{\varphi}'$ are both lifts of φ, there is some $q \in Q$ such that $\widetilde{\varphi}' = c_q \circ \widetilde{\varphi}$. Therefore c_q^{-1} is the identity on $Z(Q; X)$, and the result follows. $\square$

The rest of this section will be devoted to proving the following fact:

Theorem 5.3.5. *The data of the abstract fusion action $\mathfrak{X}$ determines an element $u \in \lim^3_{\mathcal{O}^{c\mathfrak{X}}} \mathcal{Z}^{\mathfrak{X}}$ that vanishes precisely when there is a linking action system $\mathcal{L}^{\mathfrak{X}}$ associated to $\mathfrak{X}$. Moreover, if there is a linking action system, the group $\lim^2_{\mathcal{O}^{x}} \mathcal{Z}^{\mathfrak{X}}$ acts transitively on the set of linking action systems viewed as categories over $\mathcal{O}^{c\mathfrak{X}}$.*

Proof. **Existence:** Given $\mathfrak{X}$, let us try to construct a linking action system $\mathcal{L}$ by brute force and see where problems arise. Let $\mathcal{L}$ be a category whose objects are the X-centric subgroups of S.

If we had a linking action system $\mathcal{L}^{\mathfrak{X}}$, Proposition 5.2.18 says that the orbit of the free left action of Q on $\mathcal{L}^{\mathfrak{X}}(P, Q)$ would be $\mathcal{O}^{c\mathfrak{X}}(P, Q)$. Thus, as a set we define

$$\mathcal{L}(P, Q) = Q \times \mathcal{O}^{c\mathfrak{X}}(P, Q).$$

We now need to define composition and show that the resulting category satisfies the properties of an X-centric linking system. Let ξ be a lifting of the morphisms of $\mathcal{O}^{c\mathfrak{X}}$ to $\mathfrak{X}$ that sends identities to identities. For each $[\varphi, \sigma] \in \mathcal{O}^{c\mathfrak{X}}(P, Q)$, define $(\widetilde{\varphi}, \widetilde{\sigma})$ to be $\xi\left([\varphi, \sigma]\right) \in \mathfrak{X}(P, Q)$. The lifting ξ almost certainly does not define a functor $\mathcal{O}^{c\mathfrak{X}} \to \mathfrak{X}$, but we can measure its failure to do so as follows: For each pair of composible morphisms of $\mathcal{O}^{c\mathfrak{X}}$ $P \xrightarrow{(\varphi,\sigma)} Q \xrightarrow{(\psi,\tau)} R$ it need not be the case that

$$(\widetilde{\psi}, \widetilde{\tau}) \circ (\widetilde{\varphi}, \widetilde{\sigma}) := (\widetilde{\psi}\widetilde{\varphi}, \widetilde{\tau}\widetilde{\sigma}) \qquad \text{is equal to} \qquad (\widetilde{\psi\varphi}, \widetilde{\tau\sigma}) := \xi\left([\psi\varphi, \tau\sigma]\right)$$

in $\mathfrak{X}$. However, the image of both these morphisms in $\mathcal{O}^{c\mathfrak{X}}$ is equal, so there is some $t\left([\varphi,\sigma],[\psi,\tau]\right) \in R$ such that

$$(\widetilde{\psi}\widetilde{\varphi}, \widetilde{\tau}\widetilde{\sigma}) = (c_{t([\varphi,\sigma],[\psi,\tau])} \circ \widetilde{\psi\varphi}, \ell_{t([\varphi,\sigma],[\psi,\tau])} \circ \widetilde{\tau\sigma}).$$

In the interests of trying to reduce notational clutter, we will simply write $t(\varphi,\psi)$ for $t\left([\varphi,\sigma],[\psi,\tau]\right)$; it turns out that no additional confusion will be introduced by omitting the permutations of X from the notation. Without loss of generality, we may also assume that $t(\varphi,\psi) = 1$ if either $[\varphi,\sigma]$ or $[\psi,\tau]$ is an identity morphism.

Define for each $P, Q \leq S$ the set-map

$$\pi^{\sigma}_{P,Q} : \mathcal{L}(P,Q) \to \mathfrak{X}(P,Q) : (q, [\varphi,\sigma]) \mapsto (c_q, \ell_q) \circ (\widetilde{\varphi}, \widetilde{\sigma}) = (c_q \circ \widetilde{\varphi}, \ell_q \circ \widetilde{\sigma})$$

Ultimately we would like to patch the various maps $\pi^{\sigma}_{P,Q}$ together to make up the functor $\pi^{\xi} : \mathcal{L} \to \mathfrak{X}$ required in the definition of an X-centric linking action system. To this end, we are now ready to define what we would like composition in $\mathcal{L}$ to be. We will denote this composition by $*$.

$$* : \mathcal{L}(Q,R) \times \mathcal{L}(P,Q) \to \mathcal{L}(P,R) : (r, [\psi,\tau]) \times (q, [\varphi,\sigma]) \mapsto (r \cdot \widetilde{\psi}(q) \cdot t(\varphi,\psi), [\psi\varphi, \tau\sigma])$$

Let us check that this "composition" makes π^{ξ} functorial, in the sense that π^{ξ} sends $*$-composition to honest composition in $\mathfrak{X}$. Pick $(q,[\varphi,\sigma]) \in \mathcal{L}(P,Q)$ and $(r,[\psi,\tau]) \in \mathcal{L}(Q,R)$. Then

$$\pi^{\xi}_{Q,R}\big((r,[\psi,\tau])\big) \circ \pi^{\xi}_{P,Q}\big((q,[\varphi,\sigma])\big) = \left(c_r\widetilde{\psi}, \ell_r\widetilde{\tau}\right) \circ (c_q\widetilde{\varphi}, \ell_q\widetilde{\sigma}) = \left(c_r\widetilde{\psi}c_q\widetilde{\varphi}, \ell_r\widetilde{\tau}\ell_q\widetilde{\sigma}\right)$$

while

$$\begin{aligned} \pi^{\xi}_{P,R}((r,[\psi,\tau]) * (q,[\varphi,\sigma])) &= \pi^{\xi}_{P,R}\left((r \cdot \widetilde{\psi}(q) \cdot t(\varphi,\psi), [\psi\varphi,\tau\sigma]\right) \\ &= \left(c_{r\cdot\widetilde{\psi}(q)\cdot t(\varphi,\psi)} \circ \widetilde{\psi\varphi}, \ell_{r\cdot\widetilde{\psi}(q)\cdot t(\varphi,\psi)} \circ \widetilde{\tau\sigma}\right). \end{aligned}$$

The definition of $t(\varphi,\psi)$ then gives

$$c_{r\cdot\widetilde{\psi}(q)\cdot t(\varphi,\psi)} \circ \widetilde{\psi\varphi} = c_{r\cdot\widetilde{\psi}(q)} \circ c_{t(\varphi,\psi)} \circ \widetilde{\psi\varphi} = c_{r\cdot\widetilde{\psi}(q)}\widetilde{\psi}\widetilde{\varphi} = c_r\widetilde{\psi}c_q\widetilde{\varphi},$$

and the fact that $\widetilde{\tau}$ intertwines $\widetilde{\psi}$ gives

$$\ell_{r\cdot\widetilde{\psi}(q)\cdot t(\varphi,\psi)} \circ \widetilde{\tau\sigma} = \ell_{r\cdot\widetilde{\psi}(q)} \circ \ell_{t(\varphi,\psi)} \circ \widetilde{\tau\sigma} = \ell_{r\cdot\widetilde{\psi}(q)}\widetilde{\tau}\widetilde{\sigma} = \ell_r\widetilde{\tau}\ell_q\widetilde{\sigma}.$$

Therefore π^ξ is functorial in the sense mentioned above.

The only problem now is that the composition $*$ of $\mathcal{L}$ need not be associative. Let us see what the obstruction to associativity is. Pick a sequence of morphisms in $\mathcal{L}$:

$$P \xrightarrow{(q,[\varphi,\sigma])} Q \xrightarrow{(r,[\psi,\tau])} R \xrightarrow{(r',[\chi,\upsilon])} R'$$

and compute

$$\begin{aligned} & \Big((r',[\chi,\upsilon]) * (r,[\psi,\tau])\Big) * (q,[\varphi,\sigma]) \\ = \; & \Big(r'\cdot\widetilde{\chi}(r)\cdot t(\psi,\chi),[\chi\psi,\upsilon\tau]\Big) * (q,[\varphi,\sigma]) \\ = \; & (r'\cdot\widetilde{\chi}(r)\cdot t(\psi,\chi)\cdot\widetilde{\chi\psi}(q)\cdot t(\varphi,\chi\psi),[\chi\psi\varphi,\upsilon\tau\sigma]) \end{aligned}$$

and

$$\begin{aligned} & (r',[\chi,\upsilon]) * \Big((r,[\psi,\tau]) * (q,[\varphi,\sigma])\Big) \\ = \; & (r',[\chi,\upsilon]) * \Big(r\cdot\widetilde{\psi}(q)\cdot t(\varphi,\psi),[\psi\varphi,\psi\sigma]\Big) \\ = \; & (r'\cdot\widetilde{\chi}(r)\cdot\widetilde{\chi}\widetilde{\psi}(q)\cdot\widetilde{\chi}(t(\varphi,\psi))\cdot t(\psi\varphi,\chi),[\chi\psi\varphi,\upsilon\tau\sigma]). \end{aligned}$$

Since the second coordinates are equal, the obstruction to associativity lies in the difference between the first coordinates. Canceling the $r'\cdot\widetilde{\chi}(r)$ from both, and then using the fact that $\widetilde{\chi}\widetilde{\psi} = c_{t(\psi,\chi)} \circ \widetilde{\chi\psi}$, we are reduced to comparing

$$\begin{aligned} t(\psi,\chi)\cdot\widetilde{\chi\psi}(q)\cdot t(\varphi,\chi\psi) &= t(\psi,\chi)\cdot\widetilde{\chi\psi}(q)\cdot t(\psi,\chi)^{-1}\cdot t(\psi,\chi)\cdot t(\varphi,\chi\psi) \\ &= \widetilde{\chi}\widetilde{\psi}(q)\cdot t(\psi,\chi)\cdot t(\varphi,\chi\psi) \end{aligned}$$

to

$$\widetilde{\chi}\widetilde{\psi}(q)\cdot\widetilde{\chi}(t(\varphi,\psi))\cdot t(\psi\varphi,\chi)$$

We can then rearrange to ask the question

$$t(\varphi,\chi\psi)^{-1}\cdot t(\psi,\chi)^{-1}\cdot\widetilde{\chi}(t(\varphi,\psi))\cdot t(\psi\varphi,\chi) \stackrel{?}{=} 1 \in R'$$

To examine this question, we go back to our two ways of computing the $*$-composition of three morphisms in $\mathcal{L}$. The fact that π^{ξ} takes $*$-composition in $\mathcal{L}$ to composition in $\mathfrak{X}$ implies that, as composition in $\mathfrak{X}$ is associative,

$$c_{r'\cdot\widetilde{\chi}(r)\cdot t(\psi,\chi)\cdot\widetilde{\chi\psi}(q)\cdot t(\varphi,\chi\psi)} \circ \widetilde{\chi\psi\varphi} = c_{r'\cdot\widetilde{\chi}(r)\cdot\widetilde{\chi}\widetilde{\psi}(q)\cdot\widetilde{\chi}(t(\varphi,\psi))\cdot t(\psi\varphi,\chi)} \circ \widetilde{\chi\psi\varphi}$$

and

$$\ell_{r'\cdot\widetilde{\chi}(r)\cdot t(\psi,\chi)\cdot\widetilde{\chi\psi}(q)\cdot t(\varphi,\chi\psi)} \circ \widetilde{\upsilon\tau\sigma} = \ell_{r'\cdot\widetilde{\chi}(r)\cdot\widetilde{\chi}\widetilde{\psi}(q)\cdot\widetilde{\chi}(t(\varphi,\psi))\cdot t(\psi\varphi,\chi)} \circ \widetilde{\upsilon\tau\sigma}.$$

The second equation implies that

$$\ell_{t(\psi,\chi)\cdot t(\varphi,\chi\psi)} = \ell_{\widetilde{\chi}(t(\varphi,\psi))\cdot t(\psi\varphi,\chi)}$$

or equivalently that $t(\varphi,\chi\psi)^{-1}\cdot t(\psi,\chi)^{-1}\cdot\widetilde{\chi}(t(\varphi,\psi))\cdot t(\psi\varphi,\chi) \in K$, the core of X. The first equation implies that

$$c_{t(\psi,\chi)\cdot t(\varphi,\chi\psi)} \circ \widetilde{\chi\psi\varphi} = c_{\widetilde{\chi}(t(\varphi,\psi))\cdot t(\psi\varphi,\chi)} \circ \widetilde{\chi\psi\varphi}$$

or equivalently that $k := t(\varphi,\chi\psi)^{-1}\cdot t(\psi,\chi)^{-1}\cdot\widetilde{\chi}(t(\varphi,\psi))\cdot t(\psi\varphi,\chi) \in Z_S\left(\widetilde{\chi\psi\varphi}(P)\right)$. We have already seen that $k \in K$, so in fact $u \in Z_S\left(\widetilde{\chi\psi\varphi}(P);X\right) = Z\left(\widetilde{\chi\psi\varphi}(P);X\right)$ by the X-centricity of P.

Thus, as $\widetilde{\chi\psi\varphi} \in \mathcal{F}(P, R')$ sends $Z(P; X)$ isomorphically to $Z\left(\widetilde{\chi\psi\varphi}(P); X\right)$, there is a unique element $u = u_{\sigma,t}(\varphi, \psi, \chi) \in Z(P; X)$ such that

$$\widetilde{\chi\psi\varphi}(u) = t(\varphi, \chi\psi)^{-1} \cdot t(\psi, \chi)^{-1} \cdot \widetilde{\chi}(t(\varphi, \psi)) \cdot t(\psi\varphi, \chi).$$

If $u(\varphi, \psi, \chi) = 0$ for all morphisms, the $*$-composition on $\mathcal{L}$ is associative, so $\mathcal{L}$ is a category, and we're happy. We have no reason to believe this should be true, but we can view the failure of u to be identically zero as an obstruction to the existence of a linking action system.

It is important to remember that the data that determine u are actually morphisms of $\mathcal{O}^{c\mathfrak{X}}$, even if this is somewhat suppressed in the notation. We can view u as a normalized 3-chain in $C^3(\mathcal{O}^{c\mathfrak{X}}; \mathcal{Z}_X)$. We claim that u is a 3-cocycle. The proof from [BLO2] works mutatis mutandis. Let

$$P \xrightarrow{[\varphi,\sigma]} Q \xrightarrow{[\psi,\tau]} R \xrightarrow{[\chi,\upsilon]} R' \xrightarrow{[\omega,\varsigma]} R''$$

be a sequence of morphisms in $\mathcal{O}^{c\mathfrak{X}}$. The definition of the coboundary map gives

$$\delta u(\varphi, \psi, \chi, \omega) = \widetilde{\varphi}^{-1}\big(u(\psi, \chi, \omega)\big) \cdot u(\psi\varphi, \chi, \omega)^{-1} \cdot u(\varphi, \chi\psi, \omega) \cdot u(\varphi, \psi, \omega\chi)^{-1} \cdot u(\varphi, \psi, \chi)$$

where each term is in $Z(P; X)$. In particular we can reorder, so let us write this as

$$\delta u(\varphi, \psi, \chi, \omega) = \widetilde{\varphi}^{-1}\big(u(\psi, \chi, \omega)\big) \cdot u(\varphi, \chi\psi, \omega) \cdot u(\varphi, \psi, \chi) \cdot u(\psi\varphi, \chi, \omega)^{-1} \cdot u(\varphi, \psi, \omega\chi)^{-1}.$$

Let $\Phi := \widetilde{\omega\chi\psi\varphi}$ for short; we will show that $\Phi(u(\varphi, \psi, \chi, \omega)) = 1$, from which the fact that Φ is an injective map of groups will give the result. Using all the above we get

$$\begin{aligned}
\Phi\left(\widetilde{\varphi}^{-1}(u(\psi, \chi, \omega))\right) &= \widetilde{\omega\chi\psi\varphi}\widetilde{\varphi}^{-1}(u(\psi, \chi, \omega)) \\
&= c^{-1}_{t(\varphi, \omega\chi\psi)} \circ \widetilde{\omega\chi\psi}\widetilde{\varphi}\varphi^{-1}(u(\psi, \chi, \omega)) \\
&= t(\varphi, \omega\chi\psi)^{-1} \cdot t(\psi, \omega\chi)^{-1} \cdot t(\chi, \omega)^{-1} \\
&\qquad \cdot \widetilde{\omega}(t(\psi, \chi)) \cdot t(\chi\psi, \omega) \cdot t(\varphi, \omega\chi\psi);
\end{aligned}$$

$$\begin{aligned}
\Phi(u(\varphi,\chi\psi,\omega)) &= t(\varphi,\omega\chi\psi)^{-1}\cdot t(\chi\psi,\omega)^{-1}\cdot\widetilde{\omega}(t(\varphi,\chi\psi))\cdot t(\chi\psi\varphi,\omega);\\
\Phi(u(\varphi,\psi,\chi)) &= \widetilde{\omega\chi\psi\varphi}(u(\varphi,\psi,\chi))\\
&= c^{-1}_{t(\chi\psi\varphi,\omega)}\widetilde{\omega}\widetilde{\chi\psi\varphi}(u(\varphi,\psi,\chi))\\
&= t(\chi\psi\varphi,\omega)^{-1}\cdot\widetilde{\omega}(t(\varphi,\chi\psi))^{-1}\cdot\widetilde{\omega}(t(\psi,\chi))^{-1}\\
&\qquad\cdot\widetilde{\omega}\widetilde{\chi}(t(\varphi,\psi))\cdot\widetilde{\omega}(t(\psi\varphi,\chi))\cdot t(\chi\psi\varphi,\omega)\\
&= t(\chi\psi\varphi,\omega)^{-1}\cdot\widetilde{\omega}(t(\varphi,\chi\psi))^{-1}\cdot\widetilde{\omega}(t(\psi,\chi))^{-1}\cdot t(\chi,\omega)\\
&\qquad\cdot\widetilde{\omega\chi}(t(\varphi,\psi))\cdot t(\chi,\omega)^{-1}\cdot\widetilde{\omega}(t(\psi\varphi,\chi))\cdot t(\chi\psi\varphi,\omega);\\
\Phi(u(\psi\varphi,\chi,\omega))^{-1} &= t(\chi\psi\varphi,\omega)^{-1}\cdot\widetilde{\omega}(t(\psi\varphi,\chi))^{-1}\cdot t(\chi,\omega)\cdot t(\psi\varphi,\omega\chi);\\
\Phi(u(\varphi,\psi,\omega\chi))^{-1} &= t(\psi\varphi,\omega\chi)^{-1}\cdot\widetilde{\omega\chi}(t(\varphi,\psi))^{-1}\cdot t(\psi,\omega\chi)\cdot t(\varphi,\omega\chi\psi).
\end{aligned}$$

Writing these next to each other in this order, we find that they do in fact multiply out to give the identity. Thus $u_{\sigma,t}$ is a cocycle, as claimed.

We have made two choices in defining $u_{\sigma,t}$, namely σ and t. Let us see what happens if wef we make different choices: Let t' be another choice for t, so for each pair of morphisms $P \xrightarrow{[\varphi,\sigma]} Q \xrightarrow{[\psi,\tau]} R$ in $\mathcal{O}^{c\mathfrak{X}}$, we have

$$(\widetilde{\psi}\widetilde{\varphi},\widetilde{\tau}\widetilde{\sigma}) = (c_{t'(\varphi,\psi)}\widetilde{\psi\varphi},\ell_{t'(\varphi,\psi)}\widetilde{\tau\sigma}) = (c_{t(\varphi,\psi)}\widetilde{\psi\varphi},\ell_{t(\varphi,\psi)}\widetilde{\tau\sigma}).$$

The last equality implies that $t(\varphi,\psi)^{-1}\cdot t'(\varphi,\psi)\in C$ and $t(\varphi,\psi)^{-1}\cdot t'(\varphi,\psi)\in Z_S(\widetilde{\psi\varphi}(P)) = Z(\widetilde{\psi\varphi}(P))$. Thus we can pick some $c(\varphi,\psi)\in Z(P;X)$ so that

$$t'(\varphi,\psi) = t(\varphi,\psi)\cdot\widetilde{\psi\varphi}\big(c(\varphi,\psi)\big).$$

These $c(\varphi,\psi)$ piece together to form a 2-cochain $c\in C^2(\mathcal{O}^{c\mathfrak{X}};\mathcal{Z}_X)$.

Now, t' defines a 3-cochain u' just as t defined u; explicitly $u'(\varphi,\psi,\chi)\in Z(P;X)$ is the unique element that satisfies

$$\widetilde{\chi\psi\varphi}\big(u'(\varphi,\psi,\chi)\big) = t'(\varphi,\chi\psi)^{-1}\cdot t'(\psi,\chi)^{-1}\cdot\widetilde{\chi}\big(t'(\varphi,\psi)\big)\cdot t'(\psi\varphi,\chi).$$

The claim is that $u^{-1}\cdot u' = \delta c$, which will show that the class $[u]\in\lim^3_{\mathcal{O}^{c\mathfrak{X}}}\mathcal{Z}_X$

does not depend on the choice of t. We have

$$\delta c(\varphi,\psi,\chi)=\widetilde{\varphi}^{-1}c(\psi,\chi)\cdot c(\psi\varphi,\chi)^{-1}\cdot c(\varphi,\chi\psi)\cdot c(\varphi,\psi)^{-1}$$

and again, each of these terms lie in the center of P, so we can reorder them.

As $\widetilde{\chi\psi\varphi}:P\to R$ is an injective map of groups, we just have to show that $\widetilde{\chi\psi\varphi}(u^{-1}u')=\widetilde{\chi\psi\varphi}(\delta c)$. We compute

$$\begin{aligned}
\widetilde{\chi\psi\varphi}\big(u'(\varphi,\psi,\chi)\big) &= t'(\varphi,\chi\psi)^{-1}\cdot t'(\psi,\chi)^{-1}\cdot\widetilde{\chi}\big(t'(\varphi,\psi)\big)\cdot t'(\psi\varphi,\chi)\\
&= \widetilde{\chi\psi\varphi}\big(c(\varphi,\chi\psi)\big)^{-1}\cdot t(\varphi,\chi\psi)^{-1}\cdot\widetilde{\chi\psi}\big(c(\psi,\chi)\big)^{-1}\cdot t(\psi,\chi)^{-1}\\
&\qquad\cdot\widetilde{\chi}\big(t(\varphi,\psi)\big)\cdot\widetilde{\chi}\widetilde{\psi\varphi}\big(c(\varphi,\psi)\big)\cdot t(\psi\varphi,\chi)\cdot\widetilde{\chi\psi\varphi}\big(c(\psi\varphi,\chi)\big)\\
&= \widetilde{\chi\psi\varphi}\big(c(\varphi,\chi\psi)\big)^{-1}\cdot t(\varphi,\chi\psi)^{-1}\cdot t(\varphi,\chi\psi)\cdot\widetilde{\chi\psi\varphi}\Big(\widetilde{\varphi}^{-1}\big(c(\psi,\chi)\big)^{-1}\Big)\\
&\qquad\cdot t(\varphi,\chi\psi)^{-1}\cdot t(\psi,\chi)^{-1}\cdot\widetilde{\chi}\big(t(\varphi,\psi)\big)\cdot t(\psi\varphi,\chi)\cdot\widetilde{\chi\psi\varphi}\big(c(\varphi,\psi)\big)\\
&\qquad\cdot t(\psi\varphi,\chi)^{-1}\cdot t(\psi\varphi,\chi)\cdot\widetilde{\chi\psi\varphi}\big(c(\psi\varphi,\chi)\big)\\
&= \widetilde{\chi\psi\varphi}\big(c(\varphi,\chi\psi)\big)^{-1}\cdot\widetilde{\varphi}^{-1}\big(c(\psi,\chi)\big)^{-1})\cdot t(\varphi,\chi\psi)^{-1}\cdot t(\psi,\chi)^{-1}\\
&\qquad\cdot\widetilde{\chi}\big(t(\varphi,\psi)\big)\cdot t(\psi\varphi,\chi)\cdot\widetilde{\chi\psi\varphi}\big(c(\varphi,\psi)\cdot c(\psi\varphi,\chi)\big)
\end{aligned}$$

where the third equality uses the relations $\widetilde{\chi\psi}=c_{t(\varphi,\chi\psi)}\circ\widetilde{\chi\psi\varphi}\circ\widetilde{\varphi}^{-1}$ and $\widetilde{\chi}\circ\widetilde{\psi\varphi}=c_{t(\psi\varphi,\chi)}\circ\widetilde{\chi\psi\varphi}$. If we bring all the terms in the image of $\widetilde{\chi\psi\varphi}$ to the left side of the equation, noting that all the $c(-,-)$ commute with each other and with $u'(\varphi,\psi,\chi)$, we can rewrite the left side as $\widetilde{\chi\psi\varphi}\big(\delta c(\varphi,\psi,\chi)\cdot u'(\varphi,\psi,\chi)\big)$.

On the right hand side all that is left is what we have already calculated to be $\widetilde{\chi\psi\varphi}\big(u(\varphi,\psi,\chi)\big)$, so we are done.

Finally, suppose that we made another choice σ' of lifting of morphisms from $\mathcal{O}^{c\mathfrak{X}}$ to $\mathfrak{X}$. Since, as sets, the new "category" $\mathcal{L}'$ and the original $\mathcal{L}$ have the same hom-sets, σ' just has the effect of chosing a new t, which we have just proved not to change the class of u.

Summarizing: If $[u]=0$, then there is a choice of t giving rise to some $u'\in[u]$ which makes the composition in $\mathcal{L}$ associative. Then $\mathcal{L}$ is a category, and we claim that it becomes an X-centric linking action system once we define the functor $\mathcal{T}_S^{cX}\to\mathcal{L}$.

To do this, we first give a homomorphism

$$P \to \mathcal{L}(P) : p \mapsto (p, [\mathrm{id}_P, \mathrm{id}_X]).$$

This gives a right action of P on $\mathcal{L}(P,Q) = Q \times \mathcal{O}^{c\mathfrak{X}}(P,Q)$ by composition. For any $z \in Z(P;X)$, we have

$$(q, [\varphi, \sigma]) \cdot p = (q, [\varphi, \sigma]) * (p, [\mathrm{id}_P, \mathrm{id}_X]) = (q \cdot \widetilde{\varphi}(p), [\varphi, \sigma]),$$

using the fact that $t(\mathrm{id}_P, \varphi) = 1$. In particular, because $\widetilde{\varphi}$ is injective this shows that the action of P on $\mathcal{L}(P,Q)$ is free.

Suppose now that $(q, [\varphi, \sigma])$ and $(q', [\varphi', \sigma'])$ are morphisms in $\mathcal{L}(P,Q)$ that have the same image in $\mathfrak{X}(P,Q)$ under π^{ξ}. Then $(c_q \circ \widetilde{\varphi}, \ell_q \circ \widetilde{\sigma}) = (c_{q'} \circ \widetilde{\varphi}', \ell_{q'} \circ \widetilde{\sigma}')$. The second coordinate of the equality implies that $q^{-1}q' \in K$, and the first implies that $q^{-1}q' \in Z_S(\widetilde{\varphi}P) = (\widetilde{\varphi}P)$. Thus there is a unique $p \in Z(P;X)$ such that $\widetilde{\varphi}(p) = q^{-1}q'$, and we see that $\pi^{\xi}_{P,Q} : \mathcal{L}(P,Q) \to \mathfrak{X}(P,Q)$ is in fact the orbit map of the free right $Z(P;X)$-action.

We have enough data to apply Remark 5.2.7 to prove Proposition 5.2.6 in our situation. In particular, if for every pair of X-centric subgroups $P \leq Q$ we let $\mathfrak{i}_P^Q \in \mathcal{L}(P,Q)$ be the morphism $(1, [\iota_P^Q, \mathrm{id}_X])$, we see that Corollary 5.2.11 applies to our situation: In $\mathcal{L}$, restrictions of morphisms exist uniquely.

Any $n \in N_S(P,Q)$ is itself an element of $N_S(S,S)$, so let $\widehat{n} = (n, [\mathrm{id}_S, \mathrm{id}_X]) \in \mathcal{L}(S)$ be the corresponding morphism. By the previous paragraph there is a unique morphism $\mathrm{res}_P^Q(\widehat{n}) \in \mathcal{L}(P,Q)$ defined to be the restriction of $\widehat{n}$. This allows us to define the functor $\delta : \mathcal{T}_S^{cX} \to \mathcal{L}$ to be:

$$\begin{array}{ccc} P & \mapsto & P \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_P^Q(\widehat{n})} \\ Q & \mapsto & Q \end{array}$$

The functor $\pi = \pi^{\xi} : \mathcal{L} \to \mathfrak{X}$ has already been defined in the construction of $\mathcal{L}$. From here, the verification that $(\mathcal{L}, \delta, \pi)$ has the structure of an X-centric linking action system is straightforward.

On the other hand, suppose we have an associated X-centric linking system $\mathcal{L}^{\mathfrak{X}}$. A choice of sections $\mathrm{Mor}(\mathcal{O}^{c\mathfrak{X}}) \xrightarrow{\xi} \mathrm{Mor}(\mathfrak{X}) \xrightarrow{\xi'} \mathrm{Mor}(\mathcal{L}^{\mathfrak{X}})$ gives rise to a natural identification of $\mathcal{L}^{\mathfrak{X}}(P, Q)$ with the corresponding $\mathcal{L}(P, Q)$. Axiom (C) for $\mathcal{L}^{\mathfrak{X}}$ shows that composition in $\mathcal{L}^{\mathfrak{X}}$ is then the same as composition in $\mathcal{L}$, and then the fact that composition of morphisms is associative in $\mathcal{L}^{\mathfrak{X}}$ implies that $[u] = 0$.

Uniqueness: Suppose we have two X-centric linking action systems $\mathcal{L}_1^{\mathfrak{X}}$ and $\mathcal{L}_2^{\mathfrak{X}}$ associated to $\mathfrak{X}$, with associated projection functors $\pi_i : \mathcal{L}_i^{\mathfrak{X}} \to \mathfrak{X}$. Pick a section $\xi : \mathrm{Mor}(\mathcal{O}^{c\mathfrak{X}}) \to \mathrm{Mor}(\mathfrak{X})$, and sections $\widetilde{\xi}_i : \mathfrak{X} \to \mathcal{L}_i^{\mathfrak{X}}$ of π_i. For $i = 1, 2$, if we are given a sequence of morphisms $P \xrightarrow{[\varphi,\sigma]} Q \xrightarrow{[\psi,\tau]} R$ in $\mathcal{O}^{c\mathfrak{X}}$, then by definition the morphisms $\widetilde{\xi}_i\xi([\psi, \tau]) \circ \widetilde{\xi}_i\xi([\varphi, \sigma])$ and $\widetilde{\xi}_i\xi([\psi\varphi, \tau\sigma])$ are equal after projecting to back to $\mathcal{O}^{c\mathfrak{X}}$, so by Proposition 5.2.18 there are unique elements $t_i(\varphi, \psi) \in R$ (again using only the first coordinate for ease of notation) such that

$$\widetilde{\xi}_i\xi([\psi, \tau]) \circ \widetilde{\xi}_i\xi([\varphi, \sigma]) = t_i\widehat{(\varphi, \psi)} \circ \widetilde{\xi}_i\xi([\psi\varphi, \tau\sigma]).$$

Axiom (B) implies that these t_i each play the role of the cochain t that arises in the construction of a linking action system, so as above there exists a unique $c(\varphi, \psi) \in Z(P; X)$ such that

$$t_2(\varphi, \psi) = t_1(\varphi, \psi) \cdot [\psi\varphi]\big(c(\varphi, \psi)\big).$$

Here and for the remainder of this section, $[\psi\varphi]$ denotes the group map that appears in the first coordinate of $\xi([\psi\varphi, \tau\sigma])$, and similarly for $[\varphi]$ and $[\psi]$.

As both $\mathcal{L}_1^{\mathfrak{X}}$ and $\mathcal{L}_2^{\mathfrak{X}}$ are categories with associative composition, the associated u_1 and u_2 are both trivial, and $\delta c = u_2^{-1} u_1$ is a 2-cocycle. The claim is that the class $[c]$ measures the difference between $\mathcal{L}_1^{\mathfrak{X}}$ and $\mathcal{L}_2^{\mathfrak{X}}$, so we need to check that a different choice of $\widetilde{\xi}_i$ gives rise to the same class.

Pick a different $\widetilde{\xi}_i'$ instead of our original $\widetilde{\xi}_i : \mathrm{Mor}(\mathfrak{X}) \to \mathrm{Mor}(\mathcal{L}_i^{\mathfrak{X}})$. By construction

$$\pi_i \widetilde{\xi}_i \xi([\varphi,\sigma]) = \pi_i \widetilde{\xi}_i' \xi([\varphi,\sigma]) \in \mathrm{Mor}(\mathfrak{X}),$$

so there is a unique $w\big(\xi([\varphi,\sigma])\big) \in Z(P;X)$ such that

$$\widetilde{\xi}_i' \xi([\varphi,\sigma]) = \widetilde{\xi}_i \xi([\varphi,\sigma]) \cdot w\big(\xi([\varphi,\sigma])\big).$$

Alternatively, we could go directly from $\mathcal{O}^{c\mathfrak{X}}$ to $\mathcal{L}_i^{\mathfrak{X}}$, and then there is an associated $w \in C^1(\mathcal{O}^{c\mathfrak{X}}; \mathcal{Z}_X)$ satisfying for each $[\varphi,\sigma] \in \mathcal{O}^{c\mathfrak{X}}(P,Q)$,

$$\widetilde{\xi}' \xi([\varphi,\sigma]) = \widetilde{\xi} \xi([\varphi,\sigma]) \cdot w(\varphi).$$

Plugging this into the defining equation of t_i'

$$\widetilde{\xi}_i' \xi([\psi,\tau]) \circ \widetilde{\xi}_i' \xi([\varphi,\sigma]) = \widehat{t_i'(\varphi,\psi)} \circ \widetilde{\xi}_i' \xi([\psi\varphi,\tau\sigma])$$

we get $\widetilde{\xi}_i \xi([\psi,\tau]) \circ \widehat{w(\psi)} \circ \widetilde{\xi}_i \xi([\varphi,\sigma]) \circ \widehat{w(\varphi)} = \widehat{t_i'(\varphi,\psi)} \circ \widetilde{\xi}_i \xi([\psi\varphi,\tau\sigma]) \circ \widehat{w(\psi\varphi)}$. Repeated application of Axiom (C) turns this into

$$\begin{aligned} & \delta_R\Big([\psi]\big(w(\psi)\big) \cdot [\psi][\varphi]\big(w(\varphi)\big)\Big) \circ \widetilde{\xi}_i \xi([\psi,\tau]) \circ \widetilde{\xi}_i \xi([\varphi,\sigma]) \\ = \ & \delta_R\Big(t_i'(\varphi,\psi) \cdot [\psi\varphi]\big(w(\psi\varphi)\big)\Big) \circ \widetilde{\xi}_i \xi([\psi\varphi,\tau\sigma]). \end{aligned}$$

Now we can substitute $\widehat{t_i(\varphi,\psi)} \circ \widetilde{\xi}_i \xi([\psi\varphi,\tau\sigma])$ for the last two terms of the left hand side and use the fact that each morphism in $\mathcal{L}_i^{\mathfrak{X}}$ is categorically epi (Proposition 5.2.14) to conclude that

$$[\psi]\big(w(\psi)\big) \cdot [\psi][\varphi]\big(w(\varphi)\big) \cdot t_i(\varphi,\psi) = t_i'(\varphi,\psi) \cdot [\psi\varphi]\big(w(\psi\varphi)\big).$$

Rearrange to get

$$t_i(\varphi,\psi)^{-1} \cdot t_i'(\varphi,\psi) = c_{t_i(\varphi,\psi)}^{-1}\Big([\psi]\big(w(\psi)\big) \cdot [\psi][\varphi]\big(w(\varphi)\big)\Big) \cdot [\psi\varphi]\big(w(\psi\varphi)\big)^{-1}$$

and finally using that $c_{t_i(\varphi,\psi)} \circ [\psi\varphi] = [\psi][\varphi]$, this becomes

$$t_i(\varphi, \psi)^{-1} \cdot t'_i(\varphi, \psi) = [\psi\varphi]\Big([\varphi]^{-1}\big(w(\psi)\big) \cdot w(\varphi) \cdot w(\psi\varphi)^{-1}\Big) = [\psi\varphi]\big(\delta w(\varphi, \psi)\big).$$

In other words, c is changed by a coboundary when a different choice of lifting ξ' is made. Thus the class $[c] \in \lim^2_{\mathcal{O}^{c\mathfrak{X}}} \mathcal{Z}^{\mathfrak{X}}$ that measures the difference between $\mathcal{L}_1^{\mathfrak{X}}$ and $\mathcal{L}_2^{\mathfrak{X}}$ is uniquely determined and well defined.

If $[c] = 0$, then $t_1 = t_2$ for some choice of sections, and we get that $\mathcal{L}_1^{\mathfrak{X}}$ is isomorphic to $\mathcal{L}_2^{\mathfrak{X}}$ as categories over $\mathcal{O}^{c\mathfrak{X}}$, and conversely.

Finally, starting from $\mathcal{L}_1^{\mathfrak{X}}$, any 2-cocycle can be realized by reverse engineering the above. Therefore $\lim^2_{\mathcal{O}^{c\mathfrak{X}}} \mathcal{Z}^{\mathfrak{X}}$ acts freely transitively on the set of isomorphism classes of X-centric linking systems associated $\mathfrak{X}$ over $\mathcal{O}^{c\mathfrak{X}}$. □

5.4 Linking action systems as transporter systems

Let $\mathfrak{X}$ be a saturated fusion action system and $\mathcal{L}^{\mathfrak{X}}$ an associated linking action system.

Proposition 5.2.6 can be thought of a "unique right lifting" lemma, and we saw that it implied many useful properties for a linking action system $\mathcal{L}^{\mathfrak{X}}$. In particular, we learned from it that all morphisms are categorically mono, that restriction is a well-defined notion (given the existence of specified "inclusion morphisms"), and ultimately that all morphisms are epi. This last result "should" have been derived from a "unique left lifting" lemma, but instead we used the additional structure of the inclusion morphisms to derive it. Indeed, there is no direct left lifting analogue of Proposition 5.2.6; instead, we have to settle with the following, which turns out to have its own uses:

Proposition 5.4.1. *Let* $P \xrightarrow{(\varphi,\sigma)} Q \xrightarrow{(\psi,\tau)} R$ *be a sequence of morphisms in* $\mathfrak{X}$*. For any*

$$\mathfrak{h} \in \pi^{-1}_{P,Q}((\varphi,\sigma)) \subseteq \mathcal{L}^{\mathfrak{X}}(P,Q) \qquad \text{and} \qquad \widetilde{\mathfrak{gh}} \in \pi^{-1}_{P,R}((\psi\varphi,\tau\sigma)) \subseteq \mathcal{L}^{\mathfrak{X}}(P,R)$$

there is a unique $\mathfrak{g} \subseteq \mathcal{L}^{\mathfrak{X}}(P,Q)$ *such that* $\mathfrak{gh} = \widetilde{\mathfrak{gh}}$*. Moreover, there is a unique*

$z \in \varphi(Z(P;X))$ *such that*

$$(c_{\mathfrak{g}}, \ell_{\mathfrak{g}}) = (\varphi \circ c_z, \sigma \circ \ell_z).$$

Proof. The morphism $\mathfrak{h}$ is epi by Proposition 5.2.14, so there is at most one $\mathfrak{g}$ such that $\mathfrak{g}\mathfrak{h} = \widetilde{\mathfrak{g}\mathfrak{h}}$.

Pick any $\mathfrak{g}'$ lifting (ψ, τ). Thus $\mathfrak{g}' \circ \mathfrak{h}$ and $\widetilde{\mathfrak{g}\mathfrak{h}}$ have the same image in $\mathfrak{X}(P, R)$, and by axioms (A) and (C), there is a unique $z \in Z(P;X)$ such that

$$\widetilde{\mathfrak{g}\mathfrak{h}} = \mathfrak{g}' \circ \mathfrak{h} \circ \widehat{z} = \mathfrak{g}' \circ \widehat{\varphi(z)} \circ \mathfrak{h}.$$

Then $\mathfrak{g} = \mathfrak{g}' \circ \widehat{\varphi(z)}$ gives us the uniqueness statement. □

Remark 5.4.2. The main difference between the left "lifting" of Proposition 5.4.1 and the right lifting of Proposition 5.2.6 is that this more recent result cannot actually lift the morphism $(\psi, \tau) \in \mathfrak{X}(Q, R)$, but only some $\varphi(Z(P;X))$-translate of it. The difference between the two stems from the possibility that $Q \gneq \varphi(P)$, in which case φ need not take central elements to central elements; as Q can be bigger, it is possible that $\varphi(z)$ acts nontrivially on Q, even though z acts trivially on P. ♢

Recall that an *extension* of $\mathfrak{g} \in \mathcal{L}^{\mathfrak{X}}(P, Q)$ is $\widetilde{\mathfrak{g}} \in \mathcal{L}^{\mathfrak{X}}(\widetilde{P}, \widetilde{Q})$ for $P \leq \widetilde{P}$ and $Q \leq \widetilde{Q}$ such that

$$\begin{array}{ccc} \widetilde{P} & \xrightarrow{\widetilde{\mathfrak{g}}} & \widetilde{Q} \\ {\scriptstyle \mathfrak{i}_P^{\widetilde{P}}} \uparrow & & \uparrow {\scriptstyle \mathfrak{i}_Q^{\widetilde{Q}}} \\ P & \xrightarrow[\mathfrak{g}]{} & Q \end{array}$$

commutes in $\mathcal{L}^{\mathfrak{X}}$. We have already seen (thanks to the fact that all morphisms of $\mathcal{L}^{\mathfrak{X}}$ are both epi and mono) that extensions are unique if they exist, and we are now in the position to say when exactly they do exist:

Proposition 5.4.3. *Let* $\mathfrak{g} \in \mathcal{L}^{\mathfrak{X}}(P, Q)$ *be an isomorphism and let* $\widetilde{P}, \widetilde{Q} \leq S$ *be such that* $P \trianglelefteq \widetilde{P}$, $Q \trianglelefteq \widetilde{Q}$, *and* $\mathfrak{g} \circ \delta_{P,P}\left(\widetilde{P}\right) \circ \mathfrak{g}^{-1} \leq \delta_{Q,Q}\left(\widetilde{Q}\right)$. *Then there is a unique extension* $\widetilde{\mathfrak{g}} \in \mathcal{L}^{\mathfrak{X}}(\widetilde{P}, \widetilde{Q})$ *of* $\mathfrak{g}$.

Proof. First suppose that Q is fully X-centralized. By the extension axiom for saturated fusion actions, the morphism $(c_{\mathfrak{g}}, \ell_{\mathfrak{g}}) \in \mathfrak{X}(P, Q)$ extends to some $(\varphi, \ell_{\mathfrak{g}}) \in \mathfrak{X}\left(N_{(c_{\mathfrak{g}}, \ell_{\mathfrak{g}})}, S\right)$. The condition on $\widetilde{P}$ implies that $\widetilde{P} \leq N_{(c_{\mathfrak{g}}, \ell_{\mathfrak{g}})}$ (project the condition down to $\mathfrak{X}$) and the condition on $\widetilde{Q}$ implies that $\varphi(p') \in \widetilde{Q}$ for all $p' \in \widetilde{P}$. We can thus rename $(\varphi, \ell_{\mathfrak{g}})$ to be its restriction in $\mathfrak{X}(\widetilde{P}, \widetilde{Q})$. For the sequence $P \xrightarrow{\left(\iota_P^{\widetilde{P}}, \mathrm{id}_X\right)} \widetilde{P} \xrightarrow{(\varphi, \ell_{\mathfrak{g}})} \widetilde{Q}$ in $\mathfrak{X}$, let $\mathfrak{i}_P^{\widetilde{P}} \in \mathcal{L}^{\mathfrak{X}}(P, \widetilde{P})$ lift the first map and $\mathfrak{i}_Q^{\widetilde{Q}} \circ \mathfrak{g}$ lift the composite. Then Proposition 5.4.1 applies to give a unique extension $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ as desired (though note that it need not be the case that $\widetilde{\mathfrak{g}}$ is a lift of $(\varphi, \ell_{\mathfrak{g}})$, only a lift of a $\varphi(Z(P; X))$-translate of it).

Now consider the general case, where Q need not be fully X-centralized. Let R be fully normalized and $\mathcal{F}$-conjugate to P and Q. For any $\mathfrak{h} \in \mathcal{L}^{\mathfrak{X}}(Q, R)$,

$$\mathfrak{h} \circ \widehat{N_S(Q)}\Big|_Q^Q \circ \mathfrak{h}^{-1} \leq \mathcal{L}^{\mathfrak{X}}(R)$$

is an inclusion of a $\mathfrak{p}$-subgroup. As R is fully normalized, Proposition 5.2.16 states that $\delta_{R,R}\left(N_S(R)\right)$ is Sylow in $\mathcal{L}^{\mathfrak{X}}(R)$, so $\mathfrak{h}$ can be chosen so that

$$\mathfrak{h} \circ \widehat{N_S(Q)}\Big|_Q^Q \circ \mathfrak{h}^{-1} \leq \widehat{N_S(R)}\Big|_R^R.$$

The subgroup R is fully X-centralized, so the first part of this proof implies that there are morphisms $\widetilde{\mathfrak{h}}$ extending $\mathfrak{h}$ to $N_S(Q)$ and $\widetilde{\mathfrak{h}\mathfrak{g}}$ extending $\mathfrak{h}\mathfrak{g}$ to $\widetilde{P}$. Let $\overline{\mathfrak{h}}$ be the restricted isomorphism of $\widetilde{\mathfrak{h}}$ with source $\widetilde{Q}$, and similarly $\overline{\mathfrak{h}\mathfrak{g}}$ the restricted isomorphism of $\widetilde{\mathfrak{h}\mathfrak{g}}$ with source $\widetilde{P}$. The situation can be represented as:

$$\begin{array}{ccc} N_S(Q) & \xrightarrow{\widetilde{\mathfrak{h}}} & N_S(R) \\ \uparrow{\scriptstyle \mathfrak{i}} & & \uparrow{\scriptstyle \mathfrak{i}} \\ \widetilde{Q} & \xrightarrow[\overline{\mathfrak{h}}]{\cong} & c_{\widetilde{\mathfrak{h}}}\left(\widetilde{Q}\right) \\ \uparrow{\scriptstyle \mathfrak{i}} & & \uparrow{\scriptstyle \mathfrak{i}} \\ Q & \xrightarrow[\mathfrak{h}]{\cong} & R \end{array} \qquad\qquad \begin{array}{ccc} \overline{P} & \xrightarrow{\widetilde{\mathfrak{h}\mathfrak{g}}} & N_S(R) \\ {\scriptstyle =}\uparrow & & \uparrow{\scriptstyle \mathfrak{i}} \\ \widetilde{P} & \xrightarrow[\overline{\mathfrak{h}\mathfrak{g}}]{\cong} & c_{\widetilde{\mathfrak{h}\mathfrak{g}}}\left(\widetilde{P}\right) \\ {\scriptstyle \mathfrak{i}}\uparrow & & \uparrow{\scriptstyle \mathfrak{i}} \\ P & \xrightarrow[\mathfrak{h}\circ\mathfrak{g}]{} & R \end{array}$$

The claim is that $c_{\widetilde{\mathfrak{hg}}}\left(\widetilde{P}\right) \leq c_{\widetilde{\mathfrak{h}}}\left(\widetilde{Q}\right)$. Axiom (C) implies that for $p' \in \widetilde{P}$ and $q' \in \widetilde{Q}$,

$$\widehat{c_{\widetilde{\mathfrak{hg}}}(p')} = \overline{\mathfrak{hg}} \circ \widehat{p'} \circ \overline{\mathfrak{hg}}^{-1} \qquad \text{and} \qquad \widehat{c_{\widetilde{\mathfrak{h}}}(q')} = \overline{\mathfrak{h}} \circ \widehat{q'} \circ \overline{\mathfrak{h}}^{-1}.$$

Observing that every $p' \in \widetilde{P}$ defines a morphism in $\mathcal{L}^{\mathfrak{X}}(P)$, we can restrict this to get

$$\delta_{R,R}\left(c_{\widetilde{fg}}\left(\widetilde{P}\right)\right) = (\mathfrak{h} \circ \mathfrak{g}) \circ \delta_{P,P}\left(\widetilde{P}\right) \circ (\mathfrak{h} \circ \mathfrak{g})^{-1} \leq \mathfrak{h} \circ \delta_{Q,Q}\left(\widetilde{Q}\right) \circ \mathfrak{h}^{-1} = \delta_{R,R}\left(c_{\mathfrak{h}}\left(\widetilde{Q}\right)\right)$$

where the inequality comes from the initial assumption on $\widetilde{P}$ and $\widetilde{Q}$, and the claim is proved.

Therefore the Divisibility Axiom of fusion action systems implies that there is some $(\psi, \tau) \in \mathfrak{X}(\widetilde{P}, \widetilde{Q})$ such that

$$c_{\widetilde{\mathfrak{hg}}} = (\iota, \mathrm{id}_X) \circ c_{\widetilde{\mathfrak{h}}} \circ (\psi, \tau) \in \mathfrak{X}\left(\widetilde{P}, N_S(R)\right)$$

Now Proposition 5.2.6 implies that there is a unique $\widetilde{\mathfrak{g}} \in \mathcal{L}^{\mathfrak{X}}\left(\widetilde{P}, \widetilde{Q}\right)$ such that $\widetilde{\mathfrak{hg}} = \widetilde{\mathfrak{h}}\widetilde{\mathfrak{g}}$. Restricting this to P we get $\mathfrak{h} \circ \mathfrak{g} = \mathfrak{h} \circ \mathrm{res}_P^Q(\widetilde{\mathfrak{g}})$, and the fact that $\mathfrak{h}$ is mono implies that $\mathrm{res}_P^Q(\widetilde{\mathfrak{g}}) = \mathfrak{g}$, as desired. □

Remark 5.4.4. Note that the condition on the overgroups can be restated as follows: For every $p' \in \widetilde{P}$, there is some $q' \in \widetilde{Q}$ such that $\mathfrak{g} \circ \widehat{p'} \circ \mathfrak{g}^{-1} = \widehat{q'}$. This condition is morally the same as the definition of the extender $N_{(\varphi,\sigma)}$ used to state the extension axiom for saturated fusion actions. The key difference is that, when stated in terms of the linking action system, the extension condition on the source is sharper, which allows us to relax the assumption that the target of $\mathfrak{g}$ be fully X-centralized. ♢

Recall the notion of an *abstract transporter system* associated to a fusion system $\mathcal{F}$, as introduced in [OV] and briefly described in Subsection 2.3.2.

Corollary 5.4.5. *For a linking action system $\mathcal{L}^{\mathfrak{X}}$ associated to $\mathfrak{X}$ the composite $\mathcal{L}^{\mathfrak{X}} \to \mathfrak{X} \to \mathcal{F}$, together with $\delta : \mathcal{T}_S^{cX} \to \mathcal{L}^{\mathfrak{X}}$, give $\mathcal{L}^{\mathfrak{X}}$ the structure of a transporter system associated to $\mathcal{F}$.*

Proof. The only difficult part of the proof is the extension condition, which Proposition 5.4.3 shows to be true. □

We can interpret this result as saying that a fusion action system $\mathfrak{X}$ and an associated linking system $\mathcal{L}^{\mathfrak{X}}$ give rise to a transporter system on the underlying fusion system $\mathcal{F}$ together with a map $\mathrm{Mor}(\mathcal{L}^{\mathfrak{X}}) \to \Sigma_X$ that takes composition to multiplication and inclusions to the identity. In [OV] it is shown that this map is equivalent to the data of a group map $\pi_1(|\mathcal{L}^{\mathfrak{X}}|) \to \Sigma_X$. We now set out to reverse this process:

Fix a (saturated) fusion system $\mathcal{F}$ on the $\mathfrak{p}$-group S and a transporter system $\mathcal{T}$ associated to $\mathcal{F}$. Assume that we are given a group map $\theta : \pi_1(|\mathcal{T}|) \to \Sigma_X$, or equivalently a map $\mathrm{Mor}(\mathcal{T}) \to \Sigma_X$ that sends composition to multiplication and inclusions to the identity.

Definition 5.4.6. Let $\mathfrak{X}^\theta$ be the category with $\mathrm{Ob}(\mathfrak{X}^\theta) = \mathrm{Ob}(\mathcal{T})$ and morphisms given by

$$\mathfrak{X}^\theta(P,Q) = \left\{(\varphi,\sigma) \in \mathrm{Inj}(P,Q) \times \Sigma_X \middle| \exists \mathfrak{g} \in \mathcal{T}(P,Q) \text{ such that } (\varphi,\sigma) = (c_{\mathfrak{g}}, \theta(\mathfrak{g}))\right\}.$$

♢

This allows us to define an action of S on X as follows: the $\mathfrak{p}$-group S embeds as a subgroup of $\mathcal{T}(S)$ via the structure map $\delta : \mathcal{T}_S^{\mathrm{Ob}(\mathcal{T})} \to \mathcal{T}$. We denote by $\widehat{S}$ the image of S. θ defines a $\mathcal{T}(S)$-action on X, and thus an S-action by restriction.

Clearly $\mathfrak{X}^\theta$ is a fusion action system, or at least generates one once we allow for restrictions of morphisms to subgroups not in $\mathrm{Ob}(\mathcal{T})$. Let $\mathcal{F}^\theta$ be the underlying fusion system.

We would for $\mathfrak{X}^\theta$ to be saturated, but for now we must settle for a weaker condition.

Definition 5.4.7. For $\mathcal{C}$ a collection of subgroups of S closed under $\mathcal{F}^\theta$-conjugacy and overgroups, and $\mathfrak{X}$ a fusion action system on S, we say that $\mathfrak{X}$ is $\mathrm{Ob}(\mathcal{C})$-*saturated* if the saturation axioms hold for all $P \in \mathcal{C}$. ♢

We need a little terminology to prove object-saturation of $\mathcal{F}^\theta$:

Notation 5.4.8. In the above situation, recall that for all $P \in \mathrm{Ob}(\mathcal{T})$ we define $E(P) = \ker[\mathcal{T}(P) \to \mathcal{F}^\theta(P)]$. We also denote by $K(P)$ the kernel of the action map $\theta : \mathcal{T}(P) \to \Sigma(X)$. Finally, let C be the core of the S-action on X, so that $\widehat{C} = \widehat{S} \cap K(S)$.[2] We can therefore define the notions of X-normalizers and X-centralizers of objects of $\mathcal{T}$ in the obvious way. $\diamondsuit$

Proposition 5.4.9. *For each $P \in \mathrm{Ob}(\mathcal{T})$,*

- *P is fully normalized in $\mathcal{F}^\theta$ if and only if $\widehat{N_S(P)} \in \mathrm{Syl}_\mathfrak{p}(\mathcal{T}(P))$.*
- *P is fully centralized in $\mathcal{F}^\theta$ if and only if $\widehat{Z_S(P)} \in \mathrm{Syl}_\mathfrak{p}(E(P))$.*
- *P is fully X-normalized in $\mathcal{F}^\theta$ if and only if $\widehat{N_S(P;X)} \in \mathrm{Syl}_\mathfrak{p}(K(P))$.*
- *P is fully X-centralized in $\mathcal{F}^\theta$ if and only if $\widehat{Z_S(P;X)} \in \mathrm{Syl}_\mathfrak{p}(E(P) \cap K(P))$.*

Proof. The first two points are proved in [OV], Proposition 3.4. The proofs of the remaining two points follow basically the same argument as the second.

Proof of third point: For $P \in \mathrm{Ob}(\mathcal{T})$, let Q be $\mathcal{F}^\theta$-conjugate to P and fully normalized, so by the first point $\widehat{N_S(Q)} \in \mathrm{Syl}_\mathfrak{p}(\mathcal{T}(Q))$. Therefore

$$\widehat{N_S(Q;X)} = \widehat{N_S(Q) \cap K}(Q) \in \mathrm{Syl}_\mathfrak{p}\,\mathcal{T}(Q).$$

Now, $\mathcal{F}^\theta$ is $\mathrm{Ob}(\mathcal{T})$-saturated by [OV], so the proof of Proposition 4.2.12 applies here to give us that P is also fully X-normalized. Since $K(Q) \cong K(P)$, we have $\widehat{N_S(P;X)} \in \mathrm{Syl}_\mathfrak{p}(K(P))$ if and only if $|N_S(P;X)| = |N_S(Q;X)|$, or equivalently, if and only if P is fully X-normalized.

The proof of the fourth point is the same as that of the third, replacing every instance of $K(-)$ with $E(-) \cap K(-)$. $\square$

The notation in the following Corollary is a direct analogy with that introduced to describe fusion action systems.

[2] For this section we change the name of the core so as to free up the letter K.

Corollary 5.4.10. *For all $P \in \mathrm{Ob}(\mathcal{T})$*

- *If P is fully normalized then*
  - $\mathfrak{X}^\theta_S(P) \in \mathrm{Syl}_\mathfrak{p}(\mathfrak{X}^\theta(P))$
  - $\mathcal{F}^\theta_S(P) \in \mathrm{Syl}_\mathfrak{p}(\mathcal{F}^\theta(P))$
  - $\Sigma^S_\theta(P) \in \mathrm{Syl}_\mathfrak{p}(\Sigma_\theta(P))$
- *If P is fully X-normalized, then $\mathcal{F}^\theta_S(P)_0 \in \mathrm{Syl}_\mathfrak{p}(\mathcal{F}^\theta(P)_0)$.*
- *If P is fully centralized, then $\Sigma^S_\theta(P)_0 \in \mathrm{Syl}_\mathfrak{p}(\Sigma_\theta(P)_0)$.*

Proof. Each of these follows from the observation that the image of a Sylow is Sylow in the quotient. □

Proposition 5.4.11. *Let $Q \in \mathrm{Ob}(\mathcal{T})$ be fully X-centralized and $(\varphi, \sigma) \in \mathrm{Iso}_{\mathfrak{X}^\theta}(P, Q)$. Then there is some $(\widetilde{\varphi}, \sigma) \in \mathfrak{X}^\theta(N_{(\varphi,\sigma)}, S)$ that extends (φ, σ).*

Proof. We first claim that $\widehat{N_S(Q)} \in \mathrm{Syl}_\mathfrak{p}\left(\widehat{N_S(Q)} \cdot E(Q) \cap K(Q)\right)$:

$$\left[\widehat{N_S(Q)} : \widehat{N_S(Q)} \cdot E(Q) \cap K(Q)\right] = \frac{\left|\widehat{N_S(Q)}\right| \cdot |E(Q) \cap K(Q)|}{|N_S(Q)| \cdot \left|\widehat{N_S(Q)} \cap E(Q) \cap K(Q)\right|}$$
$$= \left[\widehat{N_S(Q)} \cap E(Q) \cap K(Q) : E(Q) \cap K(Q)\right].$$

The fact that $\widehat{N_S(Q)} \cap E(Q) \cap K(Q) = Z_S(Q; X)$ and the final point of Proposition 5.4.9 gives the claim.

Now, pick $\mathfrak{g} \in \mathcal{T}(P, Q)$ such that $(c_\mathfrak{g}, \theta(\mathfrak{g})) = (\varphi, \sigma)$. By definition of $N_{(\varphi,\sigma)}$,

$$\mathfrak{g} \circ \widehat{N_{(}\varphi, \sigma)}\Big|_P^P \circ \mathfrak{g}^{-1} \leq \widehat{N_S(Q)}\Big|_Q^Q \cdot E(Q) \cap K(Q)$$

and so by the Sylow result just proved, there is some $\mathfrak{h} \in E(Q) \cap K(Q)$ such that

$$(\mathfrak{h}\mathfrak{g}) \circ \widehat{N_{(}\varphi, \sigma)}\Big|_P^P \circ (\mathfrak{h}\mathfrak{g})^{-1} \leq \widehat{N_S(Q)}\Big|_Q^Q.$$

Then $\mathfrak{hg} \in \mathrm{Iso}_{\mathcal{T}}(P, Q)$, $N_{(\varphi,\sigma)} \trianglerighteq P$, and $N_S(Q) \triangleright Q$, so the conditions for the Extension Axiom (II) of abstract transporter systems are satisfied. Therefore there is some $\widetilde{\mathfrak{hg}} \in \mathcal{T}(N_{(\varphi,\sigma)}, N_S(Q))$ that extends $\mathfrak{hg}$. This implies that $(c_{\mathfrak{hg}}, \theta(\mathfrak{hg})) \in \mathfrak{X}^\theta(N_{(\varphi,\sigma)}, S)$ extends $(c_{\mathfrak{g}}, \theta(\mathfrak{g})) = (\varphi, \sigma)$ in $\mathfrak{X}^\theta$, and the result is proved. □

Corollary 5.4.12. *The fusion action system $\mathfrak{X}^\theta$ is $\mathrm{Ob}(\mathcal{T})$-saturated.*

Proof. All the axioms have been verified in Propositions 5.4.9 and 5.4.11. □

We have seen that a transporter system $\mathcal{T}$ together with a map $\theta : \pi_1(\mathcal{T}) \to \Sigma_X$ determine a saturated fusion action system, at least so far as the objects of $\mathcal{T}$ are aware. We are given natural functors $\mathcal{T}_S^{\mathrm{Ob}(\mathcal{T})} \longrightarrow \mathcal{T} \longrightarrow \mathfrak{X}^\theta$, and we can ask how close this is to being the data of an X-centric linking action system associated to $\mathfrak{X}^\theta$.

The following result states that, so long as all the X-centric subgroups are accounted for, $\mathcal{T}$ fails to be a linking action system "in a $\mathfrak{p}'$-way," and moreover that it contains enough data to construct a linking action system $\mathcal{L}^\theta$:

Proposition 5.4.13. *Suppose that in the above situation $\mathrm{Ob}(\mathcal{T})$ contains all X-centric subgroups of S, and let $\mathcal{T}^{cX}$ denote the fully subcategory with these as the objects. Then for any X-centric $P \leq S$, there is a unique $\mathfrak{p}'$-group $EK'(P)$ such that $E(P) \cap K(P) = \widehat{Z(P; X)} \times EK'(P)$. Furthermore, $EK'(P)$ is the subgroup of all $\mathfrak{p}'$-elements of $E(P) \cap K(P)$.*

Consequently, if $\mathcal{L}^\theta$ is the category whose objects are the X-centric subgroups of S and whose morphisms are given by

$$\mathcal{L}^\theta(P, Q) = \mathcal{T}(P, Q)/EK'(P),$$

then $\mathcal{L}^\theta$ is an X-centric linking action system associated to $\mathfrak{X}^\theta$.

Proof. Axiom (C) of transporter systems implies that $E(P)$ commutes with $\widehat{P}$, so in particular $E(P) \cap K(P)$ does as well. Therefore the fact that P is X-centric implies that

$$\widehat{S} \cap E(P) \cap K(P) = Z_S\widehat{(P; X)} = \widehat{Z(P; X)} \trianglelefteq E(P) \cap K(P),$$

and the fact that P is fully X-centralized implies that $\widehat{Z(P;X)}$ is a normal abelian Sylow subgroup of $E(P) \cap K(P)$. The Schur-Zassenhaus theorem then implies the existence and uniqueness of $EK'(P)$, from which it easily follows that $\mathcal{L}^\theta$ is an X-centric linking action system associated to $\mathfrak{X}^\theta$. $\square$

5.5 Stabilizers of $\mathfrak{p}$-local finite group actions

For $H \leq G$ finite groups, it is a basic result that

$$BH \simeq EG \times_H * \simeq EG \times_G G/H.$$

In this section we prove that the analogous statement for linking actions systems is true. Let $\mathfrak{X}$ be a saturated fusion action system, and let $\mathcal{L}^\mathfrak{X}$ be an associated linking action system. We return to the notation that K is the core of the S action on X.

The first step is to understand the most important property of the G-set G/H—that it is transitive—in the context of fusion actions:

Definition 5.5.1. The fusion action system $\mathfrak{X}$ is *transitive* if $\mathfrak{S} = \mathfrak{X}(1)$ acts transitively on X. The linking action system $\mathcal{L}^\mathfrak{X}$ is *transitive* if the underlying fusion action system is. $\diamondsuit$

Lemma 5.5.2. *If the fusion action system* $\mathfrak{X}$ *is saturated,* $\pi_\Sigma(\mathfrak{X}(1)) = \pi_\Sigma(\mathfrak{X}(K))$.

Proof. The non-obvious inclusion is $\pi_\Sigma(\mathfrak{X}(1)) \subseteq \pi_\Sigma(\mathfrak{X}(K))$, which follows from the extension axiom for fusion action systems and from the easy calculation that $K \leq N_{(\mathrm{id}_1,\sigma)}$ for any $(\mathrm{id}_1, \sigma) \in \mathfrak{X}(1)$. $\square$

Remark 5.5.3. We could therefore have defined transitivity of fusion actions in terms of the group $\pi_\Sigma(\mathfrak{X}(K)) \leq \Sigma_X$. If we want to concentrate on linking action systems, this alternate characterization has the advantage that K is always X-centric, and therefore we can define transitivity of a linking action system in terms of subgroups of S that are witnessed by $\mathcal{L}^\mathfrak{X}$. $\diamondsuit$

We wish to introduce the notion of the "stabilizer" of a point $x \in X$ from the point of view of fusion and linking actions. Recall that $S_x = \mathrm{Stab}_S(x)$ denotes the maximal subgroup of S that fixes x.

Definition 5.5.4. Given the fusion action system $\mathfrak{X}$ and $x \in X$, the *stablizer fusion action system of* x, denoted $\mathfrak{X}_x = \mathcal{S}tab_{\mathfrak{X}}(x)$, is the fusion action system on S_x acting on X with morphisms $\mathfrak{X}_x(P,Q) = \{(\varphi,\sigma) \in \mathfrak{X}(P,Q) \big| \sigma(x) = x\}$.

We denote by $\mathcal{F}_x$ the *stabilizer fusion system of* x, which is the underlying fusion system of $\mathfrak{X}_x$.

Similarly, given a linking action system $\mathcal{L}^{\mathfrak{X}}$ associated to $\mathfrak{X}$, the *stabilizer linking action system of* x, $\mathcal{L}^{\mathfrak{X}}_x = \mathcal{S}tab_{\mathcal{L}^{\mathfrak{X}}}(x)$, is the category whose objects are those X-centric subgroups that are contained in S_x and whose morphisms are given by $\mathcal{L}^{\mathfrak{X}}_x(P,Q) = \{\mathfrak{g} \in \mathcal{L}^{\mathfrak{X}}(P,Q) \big| \ell_{\mathfrak{g}}(x) = x\}$. ♢

Remark 5.5.5. We can think of $\mathfrak{X}_x$ and $\mathcal{L}^{\mathfrak{X}}_x$ as the preimages under the natural maps $\mathrm{Mor}(\mathfrak{X}) \to \Sigma_X$ and $\mathrm{Mor}(\mathcal{L}^{\mathfrak{X}}) \to \Sigma_X$, respectively, of the subgroup $\Sigma_{X-\{x\}}$. ♢

Even if the fusion action system is transitive, not all points of X are equal in the eyes of $\mathfrak{X}$ or $\mathcal{L}^{\mathfrak{X}}$: In the presence of an ambient group G, points whose stabilizers in S are Sylow in the G_x are in some sense privileged in our world. We codify this situation with the following:

Definition 5.5.6. $x \in X$ is *fully stabilized* if $|S_x| \geq |S_{x'}|$ for all $x' \in X$. ♢

Lemma 5.5.7.

(a) *The point $x \in X$ is fully stabilized if and only if $\pi_\Sigma(S_x) \in \mathrm{Syl}_{\mathfrak{p}}(\mathfrak{X}(1)_x)$.*

(b) *The point $x \in X$ is fully stabilized if and only if $\widehat{S_x}\big|_K^K \in \mathrm{Syl}_{\mathfrak{p}}\left(\mathcal{L}^{\mathfrak{X}}_x(K)\right)$.*

Proof.

(a) For any $P \leq S$, denote by $\overline{P}$ the image of P in $\mathfrak{X}(1)$. As $K \leq S_y$ for all $y \in X$ and $|\overline{S_y}| = |S_y|/|K|$, it follows immediately that x if fully stabilized if and only if $|\overline{S_x}|$ is maximal among the orders of the $\overline{S_y}$. Moreover, the Sylow axioms for saturation imply that $\overline{S} \in \mathrm{Syl}_{\mathfrak{p}}\left(\mathfrak{X}(1)\right)$, and clearly $\overline{S_y} = \overline{S} \cap (\mathfrak{X}(1)_y)$, so we find ourselves in the following situation:

Let G be a finite group that acts transitively on the finite set X. For $S \in \mathrm{Syl}_{\mathfrak{p}}(G)$ and $x \in X$, we have $|S_x| \geq |S_y|$ for all $y \in X$ if and only if $S_x \in \mathrm{Syl}_{\mathfrak{p}}(G_x)$. This result is easy to see, but we include the proof for the sake of completeness.

If $x \in X$ is such that the order of S_x is maximal, let T be a Sylow subgroup of G_x that contains S_x. As T is a $\mathfrak{p}$-subgroup of G and $S \in \mathrm{Syl}_{\mathfrak{p}}(G)$, there is some $g \in G$ such that ${}^gT \leq S$. We have ${}^gG_x = G_{g\cdot x}$, so ${}^gT \leq S_{g\cdot x}$. The assumption on the maximal order of the stabilizer of x then implies that $\left|{}^gT\right| \leq \left|S_x\right|$, from which the assumption that $S_x \leq T$ implies that $S_x = T \in \mathrm{Syl}_{\mathfrak{p}}(G_x)$.

Conversely, if $S_x \in \mathrm{Syl}_{\mathfrak{p}}(G_x)$, for any $y \in X$, pick $g \in G$ such that $g \cdot y = x$. Then the fact that ${}^gG_y = G_x$ implies that ${}^gS_y$ is a $\mathfrak{p}$-subgroup of G_x, and hence subconjugate to S_x by the Sylow assumption. Thus $\left|S_y\right| \leq \left|S_x\right|$, as desired.

(b) The group $\mathcal{L}^{\mathfrak{X}}(K)$ naturally acts on X by the composition

$$\mathcal{L}^{\mathfrak{X}}(K) \xrightarrow{\pi_{K,K}} \mathfrak{X}(K) \xrightarrow{\pi_\Sigma} \Sigma_X.$$

Axiom (B) of linking action systems implies that

$$\widehat{S_x}\big|_K^K = \left(\widehat{S}\big|_K^K\right)_x,$$

and it follows easily from the definitions that

$$\widehat{S_x}\big|_K^K = \left(\widehat{S}\big|_K^K\right) \cap \left(\mathcal{L}^{\mathfrak{X}}_x(K)\right).$$

The core K is strongly closed in $\mathcal{F}$, so in particular it is fully normalized and $N_S(K) = S$. Proposition 5.4.9 then implies that that $\widehat{S}\big|_K^K \in \mathrm{Syl}_{\mathfrak{p}}\left(\mathcal{L}^{\mathfrak{X}}(K)\right)$.

Thus we have again reduced the problem to the case of actual finite groups, as in the proof of part (a), and the result follows. □

Corollary 5.5.8. *Let $\mathfrak{X}$ be a transitive saturated fusion action system and x a fully stabilized point of X. Then for every $y \in X$, the stabilizer S_y is $\mathcal{F}$-subconjugate to S_x. In particular, the stabilizers of distinct fully stabilized points of X are isomorphic.*

Proof. First note that if (φ, σ) is a morphism of $\mathfrak{X}$ such that $\sigma(y) = x$ and $s \in S_y$, then $\varphi(s)$ lies in S_x when defined. This is simply a restatement of the fact that (φ, σ) is an intertwined pair: $\varphi(s) \cdot x = \varphi(s) \cdot \sigma(y) = \sigma(s \cdot y) = \sigma(y) = x$.

Thus the result will follow from the extension axiom if we can find some $\sigma \in \mathfrak{X}(1)$ such that $\sigma(y) = x$ and $S_y \leq N_{(1,\sigma)}$. As $\mathfrak{X}$ is assumed to be transitive, there exists a $\sigma \in \mathfrak{X}(1)$ such that $\sigma(y) = x$. Then the group $H := \left\{\sigma \circ \ell_s \circ \sigma^{-1} \middle| s \in S_y\right\}$ is a $\mathfrak{p}$-subgroup of $\mathfrak{X}(1)_x$. By Proposition 5.5.7 (a), $\overline{S_x} \in \mathrm{Syl}_{\mathfrak{p}}(\mathfrak{X}(1)_x)$, so without loss of generality we may assume that we have chosen σ such that that $H \leq \mathfrak{X}(1)_x$ and $\sigma(y) = x$. But then $S_y \leq N_{(1,\sigma)}$, as desired. $\square$

With this interpretation of the stabilizer fusion, action, and linking systems, we find ourselves in the situation examined in [OV], and we recall the following result:

Proposition 5.5.9. *Let $\mathcal{T}$ be an abstract transporter system associated to the fusion system $\mathcal{F}$ on the $\mathfrak{p}$-group S. Fix a finite group Γ and a group homomorphism $\Phi : \pi_1(|\mathcal{T}|) \to \Gamma$, or equivalently, a map $\mathrm{Mor}(\mathcal{T}) \to \Gamma$ that takes composition to multiplication and inclusions to the identity. For any subgroup $H \leq \Gamma$ let $S_H \leq S$ be the maximal subgroup whose elements (viewed as morphisms of $\mathcal{T}$) are sent to H, and assume that that $S_1 \in \mathrm{Ob}(\mathcal{T})$.*

Let $\mathcal{T}_H \subseteq \mathcal{T}$ be the subcategory whose objects are those of $\mathcal{T}$ that are contained in H and whose morphisms are given by $\mathcal{T}_H(P, Q) = \left\{\mathfrak{g} \in \mathcal{T}(P, Q) \middle| \Phi(\mathfrak{g}) \in H\right\}$.

Let $\mathcal{F}_H \subseteq \mathcal{F}$ be the fusion system on S_H generated by $\pi(\mathcal{T}_H)$, and let

$$\mathcal{T}_{S_H}^{\mathrm{Ob}(\mathcal{T}_H)}(S_H) \xrightarrow{\delta_{H,H}} \mathcal{T}_H \xrightarrow{\pi_H} \mathcal{F}_H$$

be the restrictions of the structure maps for the transporter system $\mathcal{T}$. Then:

(a) $\Phi(\mathrm{Mor}(\mathcal{T})) = \Phi(\mathcal{T}(S_1))$.

(b) $\mathcal{T}_H$ *is a transporter system associated to* $\mathcal{F}_H$ *if and only if* $\delta_{S_1,S_1}(S_H) \in \mathrm{Syl}_{\mathfrak{p}}(\mathcal{T}_H(S_1))$.

(c) *If the condition of Item (b) is satisfied and all fully centralized $P \leq S$ have the property that $Z_{S_1}(P) \leq P$ implies $P \in \mathrm{Ob}(\mathcal{T})$ then $\mathcal{F}_H$ is a saturated fusion system.*

(d) *If for all $P \in \mathrm{Ob}(\mathcal{T})$ we have $P \cap S_1 \in \mathrm{Ob}(\mathcal{T})$, then $|\mathcal{T}_H|$ has the homotopy type of the covering space of $|\mathcal{T}|$ with fundamental group $\Phi^{-1}(H)$.*

Proof. [OV, Proposition 4.1]. □

Proposition 5.5.10. *If $x \in X$ is fully stabilized, the stabilizer fusion system $\mathcal{F}_x \subseteq \mathcal{F}$ is saturated and $\mathcal{L}_x^{\mathfrak{X}}$ is a transporter system associated to it.*

Proof. From the natural map $\mathrm{Mor}(\mathcal{L}^{\mathfrak{X}}) \to \Sigma_X$ and $H = \Sigma_{X-\{x\}} \leq \Sigma_X$, in the notation of Proposition 5.5.9 we have $K = S_1$, $S_x = S_H$, $\mathcal{L}_x^{\mathfrak{X}} = \mathcal{T}_H$, and $\mathcal{F}_x = \mathcal{F}_H$. That $\mathcal{L}_x^{\mathfrak{X}}$ is a transporter system associated to $\mathcal{F}_x$ is then simply an application of Lemma 5.5.7 (b) to Proposition 5.5.9 (b).

To see that $\mathcal{F}_x$ is saturated, we appeal to Item (c) of Proposition 5.5.9 and show that if $P \leq S$ is fully centralized and $Z_K(P) \leq P$ then P is X-centric, i.e., that $Z_K(Q) = Z_S(Q; X) = Z(Q; X)$ for all Q $\mathcal{F}$-conjugate to P. If $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P, Q)$, then $\varphi(Z(P; X)) = Z(Q; X) \leq Z_S(Q; X)$. Because P is fully centralized, it is also fully X-centralized, so the assumption that $Z(P; X) = Z_S(P; X)$ and comparison of orders implies that $Z(Q; X) = Z_S(Q; X)$. Thus the conditions of Item (c) are satisfied and $\mathcal{F}_x$ is saturated. □

We now find ourselves in the following situation: Let $\mathcal{L}^{\mathfrak{X}}$ be a linking action system associated to the transitive saturated fusion action system $\mathfrak{X}$, and x a fully stabilized point of X. We would like to understand the topological information of the stabilizer linking action system $\mathcal{L}_x^{\mathfrak{X}}$ as it relates to that of $\mathcal{L}^{\mathfrak{X}}$, and indeed Proposition 5.5.9 gives us some relevant information in terms of subgroups of $\pi_1(|\mathcal{L}^{\mathfrak{X}}|)$. We can also calculate the homotopy type of $|\mathcal{L}_x^{\mathfrak{X}}|$ directly, as follows:

Let $\iota : \mathcal{L}_x^{\mathfrak{X}} \to \mathcal{L}^{\mathfrak{X}}$ be the inclusion functor, and let $F : \mathcal{L}^{\mathfrak{X}} \to \mathcal{TOP}$ be the left homotopy Kan extension of the trivial functor $* : \mathcal{L}_x^{\mathfrak{X}} \to \mathcal{TOP}$ over ι. We already have the functor $\mathbb{X} : \mathcal{L}^{\mathfrak{X}} \to \mathcal{TOP}$ defined as part of the data of $\mathcal{L}^{\mathfrak{X}}$, and we would like to relate these.

Proposition 5.5.11. *F is equivalent to $\mathbb{X}$ as functors $\mathcal{L}^{\mathfrak{X}} \to \mathcal{TOP}$.*

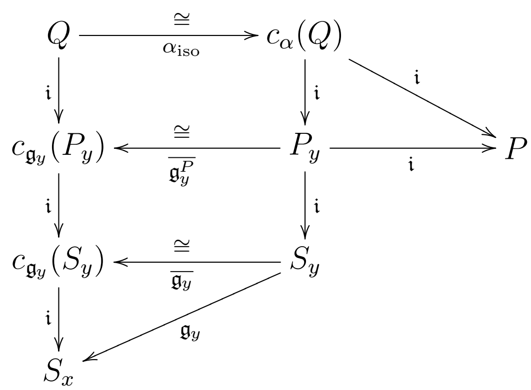
Proof. Thomason's theorem [Tho] tells us that we have a homotopy equivalence

$$F(P) \simeq \operatorname*{hocolim}_{(\iota \downarrow P)} * = |(\iota \downarrow P)|$$

for all $P \in \mathrm{Ob}(\mathcal{L}^{\mathfrak{X}})$. We first prove that every component of $|(\iota \downarrow P)|$ is contractible and that the components can be put in natural correspondence with the points of X.

The equivalence of F with $\mathbb{X}$ is not natural: For every $y \in X$, Corollary 5.5.8 implies that $\mathcal{L}^{\mathfrak{X}}(S_y, S_x)$ is nonempty as x is fully stabilized. Let $\mathfrak{g}_y \in \mathcal{L}^{\mathfrak{X}}(S_y, S_x)$ be a choice of a morphism in this hom-set for each y, and assume $\mathfrak{g}_x = \mathrm{id}_{S_x}$. Also let $\overline{\mathfrak{g}_y} = (\mathfrak{g}_y)_{\mathrm{iso}}$ be the restricted isomorphism of $\mathfrak{g}_y$ (cf. Corollary 5.2.13), $\mathfrak{g}_y^P$ the restriction of $\mathfrak{g}_y$ to $P_y \leq S_y$, and $\overline{\mathfrak{g}_y^P} = \left(\mathfrak{g}_y^P\right)_{\mathrm{iso}}$ the restricted isomorphism of $\mathfrak{g}_y^P$.

For any $(Q, \alpha) \in (\iota \downarrow P)$, so that $Q \in \mathrm{Ob}(\mathcal{L}_x^{\mathfrak{X}})$ and $\alpha \in \mathcal{L}^{\mathfrak{X}}(Q, P)$, we have $\ell_\alpha(x) = y$ for some $y \in X$. Since the pair (c_α, ℓ_α) is intertwined and $Q \leq S_x$, we have $c_\alpha(Q) \leq P_y \leq S_y$. Thus we have the following commutative diagram in $\mathcal{L}^{\mathfrak{X}}$, where all morphisms labeled $\mathfrak{i}$ are the obvious inclusions:



The top composition is α by definition. By the choices made above, we have $\ell_{\mathfrak{g}} \circ \ell_\alpha(x) = x$, and therefore there is a factorization

$$\alpha = \underbrace{\left(\mathfrak{i}_{P_y}^{P} \circ \overline{\mathfrak{g}_y^P}^{-1}\right)}_{\mathcal{L}^{\mathfrak{X}}(c_{\mathfrak{g}_y}(P_y), P)} \circ \underbrace{\left(\overline{\mathfrak{g}_y^P} \circ \mathfrak{i}_{c_\alpha(Q)}^{P_y} \circ \alpha_{\mathrm{iso}}\right)}_{\mathcal{L}_x^{\mathfrak{X}}(Q, c_{\mathfrak{g}_y}(P_y))}.$$

This is the unique (as all morphisms of $\mathcal{L}^{\mathfrak{X}}$ are categorically mono and epi) factorization of $\alpha : Q \to P$ as a composite $Q \to c_{\mathfrak{g}_y}(P_y) \to P$, so what we have really proved is the following:

For any object $(Q, \alpha) \in (\iota \downarrow P)$ such that $\ell_\alpha(x) = y$, there is a unique morphism from (Q, α) to $\left(c_{\mathfrak{g}_y}(P_y), \iota_{P_y}^{P} \circ \overline{\mathfrak{g}_y^{P}}^{-1}\right)$ in $(\iota \downarrow P)$, namely $\overline{\mathfrak{g}_y^{P}} \circ \mathfrak{i}_{c_\alpha(Q)}^{P_y} \circ \alpha_{\text{iso}}$. In other words, we have found a terminal object in the component containing (Q, α) that depends only on our choice of the $\mathfrak{g}_y$ and where α sends x. This shows that $F(P)$ is homotopically discrete, and as we have assumed that $\mathcal{L}^{\mathfrak{X}}$ is transitive, the components are naturally identified with the points of X.

All we have to do is see how F acts on morphisms, compared to the functor $\mathbb{X}$. Recall that $\mathbb{X}$ sends the morphism $\mathfrak{h} \in \mathcal{L}^{\mathfrak{X}}(P, P')$ to the map of spaces $\ell_{\mathfrak{h}}$. On the other hand, $F(\mathfrak{h})$ is induced by the functor $(\iota \downarrow P) \to (\iota \downarrow P')$ that sends (Q, α) to $(Q, \mathfrak{h} \circ \alpha)$. If (Q, α) is in the component we have identified with y we have $\ell_\alpha(x) = y$, and then $(Q, \mathfrak{h} \circ \alpha)$ is in the component corresponding to $\ell_{\mathfrak{h}} \circ \ell_\alpha(x) = \ell_{\mathfrak{h}}(y)$. This is just to say that $F(\mathfrak{h})$ permutes the space homotopy equivalent X by the permutation $\ell_{\mathfrak{h}}$, so the result is proved. □

Corollary 5.5.12. *In the above situation,*

$$\operatorname*{hocolim}_{\mathcal{L}^{\mathfrak{X}}} \mathbb{X} \simeq \left|\mathcal{L}_x^{\mathfrak{X}}\right|.$$

Remark 5.5.13. The final piece of interpretation comes from thinking of the left hand side as the $\mathfrak{p}$-local finite group action theoretic version of $EG \times_G G/H$, and the right hand side as $EH \times_H *$. ♢

Appendix A

Weltanschauung: Fusion systems of groupoids

This Appendix is a bit of an outlier, in that it does not relate directly to the content of the rest of the thesis. Instead, this could be seen as an introductory chapter to the same body of work, but presented with a very different perspective—one that was secretly lurking in the back of my mind for much of the time when the work detailed here was done. Though I have not chosen to use the perspective of this section to describe fusion action systems here, I am hopeful that this point of view may be enlightening in the future.

The material introduced here is largely informal, and will not be needed throughout the rest of this document. Much of my motivation comes from [Hig].

A.1 Groupoids

Recall that a *groupoid* is a category $\mathcal{G}$ all of whose morphisms are invertible. In the special case that $\mathcal{G}$ has a single object $*$, the composition law of the category gives $\mathrm{Hom}_{\mathcal{G}}(*, *)$ the structure of a group: Groupoids can be thought of as "groups with many objects."

If the groupoid $\mathcal{G}$ is *connected*—for any two objects $x, x' \in \mathrm{Ob}(\mathcal{G})$ we have $\mathrm{Hom}_{\mathcal{G}}(x, x')$ is nonempty—$\mathcal{G}$ is equivalent as a category to the *vertex group* $\mathcal{G}_x =$

$\mathrm{Aut}_{\mathcal{G}}(x)$ for all $x \in \mathrm{Ob}(\mathcal{G})$.[1]

A *finite groupoid* is a groupoid $\mathcal{G}$ with only finitely many objects and morphisms. For our purposes, we may as well assume that all of the groupoids we encounter here are finite.

The *classifying space* of the groupoid $\mathcal{G}$ is the geometric realization $B\mathcal{G} := |\mathcal{G}|$. If $x_1, \ldots, x_n$ are representative objects of the connected components of $\mathcal{G}$, we have $B\mathcal{G} \simeq \amalg_{i=1}^{n} B\mathcal{G}_{x_i}$ is a disjoint union of classifying spaces of discrete groups.

A groupoid is *unicursal* if every two objects have at most one morphism between them, and *simplicial* if they have exactly one. Thus the classifying space of a unicursal category is homotopically discrete, and that of a simplicial one is contractible.

A.2 Conjugation in groupoids

Let G be a finite group. Every element $g \in G$ defines an *inner automorphism* of G, $c_g : g' \mapsto gg'g^{-1}$. Viewing G as a groupoid with a single object, we might hope to generalize and describe conjugation in arbitrary groupoids.

To this end, let us define an *arrow field* of the groupoid $\mathcal{G}$ to be a collection of morphisms $\Gamma = \{\gamma_x^{x'}\}_{x \in \mathrm{Ob}(\mathcal{G})}$ with $\gamma_x^{x'} \in \mathrm{Hom}_{\mathcal{G}}(x, x')$. Note that for every object x of $\mathcal{G}$ there is a unique morphism $\gamma_x^{x'}$ with x as its source, but we do not require that the same be true for targets.

The arrow field Γ determines an endofunctor of $\mathcal{G}$ which we shall denote c_Γ. For any object $x \in \mathrm{Ob}(\mathcal{G})$, let $\gamma_x^{x'} \in \Gamma$ be the unique morphism with x as its source. c_Γ sends x to x'. For a morphism $\delta \in \mathcal{G}(x, y)$, $c_\Gamma(\delta)$ is the morphism $\gamma_y^{y'} \circ \delta \circ \left(\gamma_x^{x'}\right)^{-1} \in \mathrm{Hom}_{\mathcal{G}}(x', y')$.

If Γ and Δ are arrow fields of $\mathcal{G}$, we define the composite $\Gamma \circ \Delta$ to be the arrow field whose morphism with source x is $\gamma_{x'}^{x''} \circ \delta_x^{x'}$. The identity for this composition is the arrow field with $\gamma_x^x = \mathrm{id}_x$, so the set of arrow fields naturally has the structure of a monoid, which will be denoted $\mathrm{Inn}^+(\mathcal{G})$.

[1] One could take this basic fact to mean that the study of connected groupoids is "the same" as that of discrete groups. We take the more rigid view that it is isomorphism, not equivalence, of groupoids that is the relevant notion of sameness.

The arrow field Γ is *invertible* if the assignment $\Gamma \to \mathrm{Ob}(\mathcal{G})$ that sends a morphism to its target is bijective. If Γ is invertible, its *inverse* is the arrow field Γ^{-1} whose morphism with source x is $(\gamma_{x'}^{x})^{-1}$ for the unique $\gamma \in \Gamma$ with target x. Clearly $\Gamma \circ \Gamma^{-1} = \Gamma^{-1} \circ \Gamma$ is the identity element of $\mathrm{Inn}^{+}(\mathcal{G})$, so the set of invertible arrow fields forms the group *of inner automorphisms of* $\mathcal{G}$, denoted $\mathrm{Inn}(\mathcal{G})$.

It should be noted that $\mathrm{Inn}(\mathcal{G})$ is *not* invariant under equivalence of groupoids, only under groupoid isomorphism. In particular, if $\mathcal{G}$ is connected but has more than a single object, it is easy to see that $\mathrm{Inn}(\mathcal{G})$ is a priori much bigger than $\mathrm{Inn}(\mathcal{G}_x)$ for any $x \in \mathrm{Ob}(\mathcal{G})$. Though the groups $\mathrm{Inn}(\mathcal{G})$ and $\mathrm{Inn}(\mathcal{G}_x)$ are related, we take this evidence in support of our decision to focus on groupoid isomorphisms.

One could at this point start doing finite group theory for finite groupoids and seeing how much would carry over. There are notions of normal subgroupoid, quotient groupoid, etc., defined in [Hig], and one could formulate variations of the Sylow theorems in a few different ways.

One could even ask the question: "What do we mean by fusion in a finite groupoid?" Though this may be an interesting research direction to pursue at some point, for now we shall restrict our attention to a very particular class of groupoids.

A.3 Translation groupoids

Let G be a finite group and X a finite G-set. The *translation groupoid of* G *acting on* X is the category $\mathcal{B}_G X$ whose objects are the elements of X and whose morphisms are defined by $\mathrm{Hom}_{\mathcal{B}_G X}(x, x') = \left\{ \check{g}_x^{x'} \middle| g \cdot x = x' \right\}$. We shall often simply write $\check{g}$ for $\check{g}_x^{x'}$, when the source and target are either understood or not greatly relevant.

The important property that makes the study of translation groupoids somewhat more tractable than the general case is in some sense little more than a naming convention: From every object x of $\mathcal{B}_G X$ and every element $g \in G$, there is precisely one morphism $\check{g}_x^{g \cdot x}$ with x as a source, and similarly precisely one morphism $\check{g}_{g^{-1} \cdot x}^{x}$ with x as a target.

In particular, every element $g \in G$ determines an arrow field $\Gamma(g)$, and thus an inner automorphism c_g of $\mathcal{B}_G X$. Explicitly, the functor c_g is defined by

$$\begin{array}{ccc} x & \longmapsto & g \cdot x \\ {\scriptstyle \breve{g}'}\big\downarrow & & \big\downarrow{\scriptstyle (gg'g^{-1})\breve{}} \\ g' \cdot x & \longmapsto & gg'g^{-1} \cdot x \end{array}$$

We draw attention to the fact that the functor c_g is completely determined by two pieces of data, each of which takes place in a simpler context than that of an isomorphism of groupoids. The first is the permutation of X given by left multiplication by g, which determines the action of c_g on objects. The second is the automorphism of G given by conjugation by g, which in turn determines the action of c_g on morphisms.

Let us examine another situation in which this takes place. Suppose that X is a G-set, and we give ourselves $H, K \leq G$. Restriction of the action makes X into both an H- and a K-set, so we may talk about the groupoids $\mathcal{B}_H X$ and $\mathcal{B}_K X$. Suppose further that H is subconjugate to K via $g \in G$, so that ${}^g H \leq K$. Then c_g, defined as above, actually restricts to a morphism of groupoids $c_g : \mathcal{B}_H X \to \mathcal{B}_K X$. We still have that the complicated morphism of groupoids c_g can be described simply in terms of a permutation of X and a group map $H \to K$, even though it is possible that neither of these can be described in a manner "internal" to H or K if g lives in neither.

To take the example further, now suppose that we have the G-set X and the H-set Y, together with an *equivariant map of the pair* $(G, X) \to (H, Y)$. By this I mean that we have a group map $\alpha : G \to H$ and a set map $f : X \to Y$ that satisfy the *intertwining condition*

$$f(g \cdot x) = \alpha(g) \cdot f(x)$$

Then (α, f) determines a map of groupoids $\mathcal{B}_G X \to \mathcal{B}_H Y$, and obviously does so in a way dictated simply in terms of a set map and a group map together.

Let us call any map of translation groupoids that can be described by such a pair of maps of objects and group an *ambient map of translation groupoids.* As will hopefully become clear throughout the course of this document, ambient maps of translation groupoids are precisely the heuristic context in which we shall introduce the notion of fusion action system:

- The functors of the form $c_g : \mathcal{B}_G X \to \mathcal{B}_G X$ represent the whole group of inner automorphisms, analogous to the entire inner automorphism group of a finite group. There is too much $\mathfrak{p}'$-information here for our purposes, so we must restrict our attention.

- If S is Sylow in G and $P, Q \leq S$, ambient group maps of the form $c_g : \mathcal{B}_P X \to \mathcal{B}_Q X$ with ${}^gP \leq Q$ give rise to the morphism of an ordinary fusion action system—one induced by an actual ambient finite group.

- For S an arbitrary $\mathfrak{p}$-group and X a finite S-set, we may form a category whose morphisms are certain ambient maps of translation groupoids $\mathcal{B}_P X \to \mathcal{B}_Q X$ for all $P, Q \leq S$. Once we identify the appropriate conditions to place on the resulting category, we come to our notion of *abstract fusion action system.*

The takeaway from this entire discussion is that the theory of fusion action systems, which encompasses the classical story of abstract fusion systems, can be realized in terms of some restricted notion of "fusion system of groupoids." Perhaps this more general subject will prove to be of interest in the future, perhaps not. For now, let us be content with the fusion theory of finite translation groupoids, also known as fusion action systems.

Appendix B

Index of notation

$\mathfrak{p}$	a prime
$G, H, \ldots$	finite groups
$\mathrm{Syl}_{\mathfrak{p}}(G)$	set of Sylow subgroups of G
$S, T, \ldots$	finite $\mathfrak{p}$-groups, thought of as Sylows
$P, Q, R, \ldots$	$\mathfrak{p}$-groups, usually subgroups of S
$\mathcal{B}G, \mathcal{E}G$	classifying and simplicial categories of G
$\mathcal{C}(a,b), \mathcal{C}(a)$	the Hom-sets $\mathrm{Hom}_{\mathcal{C}}(a,b), \mathrm{Hom}_{\mathcal{C}}(a,a)$
$c_g : G \to G$	conjugation map $g' \mapsto gg'g^{-1}$
${}^gH$	$c_g(H)$, the conjugate of $H \leq G$ by g
$N_G(P,Q), \ldots$	transporter in G from P to Q
$\mathcal{T}_G := \mathcal{T}_S(G)$	transporter system on S relative to G
$\mathcal{T}_S$	minimal transporter system on S
$N_G(P), Z_G(P), \ldots$	normalizer and centralizer of P in G, etc.
$\mathrm{Aut}_G(P)$	automorphisms of P induced by conjugation from G
$\mathrm{Hom}_G(P,Q)$	group maps from P to Q induced by G
$\mathcal{F}_G := \mathcal{F}_S(G)$	fusion system on S relative to G
$\mathcal{F}_S := \mathcal{F}_S(S)$	minimal fusion system on S
$\mathcal{F}_G^c, \mathcal{T}_G^c$, etc.	full subcategories whose objects are the $\mathfrak{p}$-centric subgroups
$\mathcal{L}_G^c := \mathcal{L}_S^c(G)$	centric linking system on S relative to G
$[g], \ldots$	class of element $g \in G$, viewed as morphism in $\mathcal{L}_S^c(G)$
$(-)^\wedge_{\mathfrak{p}}$	Bousfield-Kan $\mathfrak{p}$-completion functor
$\simeq_{\mathfrak{p}}$	homotopy equivalence up to $\mathfrak{p}$-completion
$\mathcal{G}(F)$	Grothendieck construction of the functor $F : \mathcal{C} \to \mathcal{CAT}, \mathcal{SET}, \ldots$
$\mathrm{Rep}(G,H)$	$H \backslash \mathrm{Hom}(G,H)$, the representations of G in H
$\mathcal{O}_G := \mathcal{O}(\mathcal{F}_G)$	the $\mathfrak{p}$-orbit category of G
$\mathcal{F}$	abstract fusion system
$\cong_{\mathcal{F}}$	$\mathcal{F}$-conjugacy
N_φ	extender of $\varphi \in \mathrm{Mor}(\mathcal{F})$
$\mathcal{T}$	abstract transporter system

$\mathfrak{g}, \mathfrak{h}, \dots$	morphisms in abstract linking or transporter systems
$\delta : \mathcal{T}_S \to \mathcal{T}, \pi : \mathcal{T} \to \mathcal{F}$	structure maps of the transporter system
$\widehat{p}$	the group element p viewd as a morphism in $\mathcal{T}$
$\mathcal{L}$	abstract centric linking system
$X, Y, \dots$	finite sets
Σ_X	symmetric group on X
${}_GX, {}_SX, \dots$	X as a G-, S-, etc.-set
${}^{\varphi}_{H}X$	X with H-action twisted via φ
X^G	G-fixed points of X
$\overline{G}, \overline{S}$, etc.	images of G, S in Σ_X, etc.
$K, \widehat{K}$	core of S, G action on X
$[P]$	isomorphism class of S-set S/P
G_x, S_x	stabilizers of x in G, S, etc.
$\mathfrak{X}_G := \mathfrak{X}_S(G)$	fusion action system on S relative to the G-set X
ℓ_g	permutation of X given by translation by g
$\mathfrak{X}_S$	minimal fusion action system for S-set X
$\pi_{\mathcal{F}}$	natural functor $\mathfrak{X}_G \to \mathcal{F}_G$
$\pi_{\mathcal{T}}$	natural functor $\mathfrak{X}_G \to \mathcal{T}_{\overline{G}}$
π_Σ	projection onto the second coordinate $\mathfrak{X}_G(P, Q) \to \Sigma_X$
$\Sigma^G_{\mathfrak{X}}(P)$	$\pi_\Sigma(\mathfrak{X}_G(P))$
$\mathrm{Aut}_G(P; X)$, etc.	$\mathfrak{X}_G(P)$, the fusion action automorphisms of P, etc.
$N_G(H; K)$, etc.	$N_G(H) \cap \widehat{K} = N_{\widehat{K}}(H)$, the X-normalizer of H in G, etc.
$Z_G(H; K)$, etc.	$Z_G(H) \cap \widehat{K} = Z_{\widehat{K}}(H)$, the X-normalizer of H in G, etc.
$\mathcal{F}_G(P)_0$	$\ker[\pi_\Sigma : \mathfrak{X}_G(P) \to \Sigma^G_{\mathfrak{X}}(P)]$
$\Sigma^G_{\mathfrak{X}}(P)_0$	$\ker[\pi_{\mathcal{F}} : \mathfrak{X}_G(P) \to \mathcal{F}_G(P)]$
$\mathfrak{X}$	abstract fusion action system
$\mathfrak{S}$	the finite group $\mathfrak{X}(1) \leq \Sigma_X$
$N_{(\varphi,\sigma)}$	the extender of (φ, σ) in $\mathfrak{X}$
$\mathcal{F}^{\mathfrak{X}}$	fusion system on S underlying $\mathcal{F}^X$
$\mathcal{K} := \mathcal{K}^{\mathfrak{X}}$	core fusion system of $\mathfrak{X}$
$\mathcal{B}_GX, \mathcal{B}_SX, \dots$	translation categories of the G- or S-set X
$\mathcal{T}^{cX}_G, \mathfrak{X}^c, \dots$	full subcategories with the X-centric subgroups of S
$\mathcal{L}^{\mathfrak{X}}$	abstract linking action system associated to $\mathfrak{X}$
$\mathbb{X}$	the functor whose homotopy colimit gives the classifying space of $\mathcal{L}^{\mathfrak{X}}$
$\mathcal{O}^{\mathfrak{X}}$	the orbit category of $\mathfrak{X}$
$\mathcal{Z}^{\mathfrak{X}}$	the linking action obstruction functor
$\mathfrak{X}^\theta$	the fusion action system arising from the functor $\theta : \mathcal{T} \to \Sigma_X$
$\mathfrak{X}_x$	stabilizer fusion action system of $x \in X$
$\mathcal{L}^{\mathfrak{X}}_x$	stabilizer linking action system of $x \in X$
$\mathfrak{X}_x$	stabilizer fusion action system of x
$\mathcal{F}^{\mathfrak{X}}_x$	stabilizer fusion system of x
$\mathcal{L}^{c\mathfrak{X}}_G$	X-centric linking system associated to $\mathfrak{X}_G$
$\mathcal{L}^{\mathfrak{X}}$	abstract X-centric linking system associated to $\mathfrak{X}$

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