Quillen Cohomology of Pi-Algebras and Application to their Realization

by

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Abstract

We use the obstruction theory of Blanc-Dwyer-Goerss to study the realization space of certain Π-algebras with 2 non-trivial groups. The main technical tool is a result on the Quillen cohomology of truncated Π-algebras, which is an instance of comparison map induced by an adjunction. We study in more generality the behavior of Quillen (co)homology with respect to adjunctions. As a first step toward applying the obstruction theory to 3-types, we develop methods to compute Quillen cohomology of 2-truncated Π-algebras via a generalization of group cohomology.

Thesis Supervisor: Haynes R. Miller
Title: Professor of Mathematics

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Contents

Introduction

In this thesis, we study the problem of realizing a Π-algebra using the obstruction theory presented in [BDG04]. Since the obstructions to existence and uniqueness live in certain Quillen cohomology groups of Π-algebras, we will first adopt a purely algebraic point of view and try to better understand these groups. From there, we will use the obstruction theory to obtain information about realizations of Π-algebras.

In a sense, we are approaching the problem from the opposite direction of [Bla95], where the obstruction theory is set up from a topological point of view, using higher homotopy operations. Some connections between the topological and algebraic formulations of the obstruction theory are explored in [Bla99] and [BJT09].

Organization The first two chapters contain background material. Chapter 1 is about Quillen cohomology and some of its properties. In chapter 2, we review the basics of Π-algebras and discuss the truncated ones.

New material starts in the third chapter. Chapter 3 clarifies the categorical background related to Quillen cohomology and proposes a good setup where we can work with it (proposition 3.4.14). Chapter 4 studies the behavior of Quillen (co)homology with respect to adjunctions. Theorem 4.3.1 checks that the aforementioned setup is good enough to deal with adjunctions, while identifying exactly the condition required of the original adjunction. Section 4.4 works out the various comparison maps that arise from a nice adjunction. Chapter 5 uses the Postnikov truncation adjunction to describe Quillen cohomology of truncated Π-algebras. The result is theorem 5.2.2. In section 6, we use that result and the obstruction theory of [BDG04] to classify realizations of Π-algebras concentrated in dimensions 1 and n, including in particular 2-types. The main results are 6.3.5 and 6.3.6.

If one wanted to study 3-types using the obstruction theory, the primary obstructions to existence and uniqueness would lie in Quillen cohomology of 2-truncated Π-algebras. Chapter 7 proposes a program to compute the latter. First we describe Beck modules (7.1.3), abelianization (7.2.2), and pushforwards (7.3.3). Then we reduce the problem to computing the derived functors of a type of indecomposables functor (steps 7.4.1, 7.4.2, and 7.4.5).

Chapter 1

Quillen cohomology

1.1 Beck modules

Let $\mathcal{C}$ is a category with finite products, including a terminal object. Recall that a **Beck module** over an object X of $\mathcal{C}$ is an abelian group object in the slice category $\mathcal{C}/X$.

Proposition 1.1.1. *If $F : \mathcal{C} \to \mathcal{D}$ is a pullback-preserving functor , then for any object X in $\mathcal{C}$, it induces a functor on modules*

$$F_X : Ab(\mathcal{C}/X) \to Ab(\mathcal{D}/FX).$$

Moreover, F_X is additive.

Proof. F induces a functor $\mathcal{C}/X \to \mathcal{D}/FX$, which automatically preserves the terminal object (namely, the identity). This functor preserves finite products iff F preserves pullbacks over X, hence it suffices to prove the "absolute" version, i.e. if $F : \mathcal{C} \to \mathcal{D}$ preserves finite products (including the terminal object), then it induces an additive functor

$$F : Ab(\mathcal{C}) \to Ab(\mathcal{D}).$$

If A is an abelian group object in $\mathcal{C}$, then F applied to the structure maps of A yields structure maps for FA, and F applied to the structure (condition) diagrams of A yields structure diagrams for FA.

Addition in $\mathrm{Hom}_{Ab(\mathcal{D})}(FM, FN)$ is given by $(\mu_{FN})_* = (F(\mu_N))_*$, hence F is additive. In other words, for $f, g \in \mathrm{Hom}_{Ab(C)}(M, N)$, we have

$$\begin{aligned} F(f+g) &= F(\mu_N(f \times g)\Delta_N) \\ &= F(\mu_N)F(f \times g)F(\Delta_N) \\ &= \mu_{FN}(Ff \times Fg)\Delta_{FN} \\ &= Ff + Fg. \end{aligned}$$

□

Remark 1.1.2. By abuse of notation, we called the induced functor F also. The notation $Ab(F)$ or F_* might have been more appropriate, although more cumbersome.

In fact, the condition of preserving pullbacks (including finite products and the terminal object) is too strong. Recall that equalizers are a special case of pullbacks. If $f, g : X \to Y$ are two maps, their equalizer is the pullback

$$\begin{tikzcd} Eq(f,g) \arrow[r] \arrow[d] & X \arrow[d, "{(\mathrm{id},g)}"] \\ X \arrow[r, "{(\mathrm{id},f)}"'] & X \times Y \end{tikzcd}$$

which can be thought of as the "intersection of the graphs of f and g". Thus preserving pullbacks is the same as preserving finite limits (assuming there is a terminal object). In particular, the induced functor is then left exact.

In order for F to induce a functor $F : Ab(C/X) \to Ab(\mathcal{D}/FX)$ on Beck modules, it suffices that F preserve kernel pairs of split epimorphisms. By **kernel pair** of a map, we mean the pullback of the map along itself.

Non-example 1.1.3. Consider the functor $F : \mathbf{Gp} \to \mathbf{Gp}$ that associates to a group the free group on its underlying set, i.e. the comonad of the Free/Forget adjunction. Then F does NOT induce a functor on Beck modules.

To see this, use the fact (A.1.1) that a Beck module over a group G is a split extension $p : E \to G$ (plus the data of the splitting) with abelian kernel, which can be viewed as the semi-direct product $G \ltimes M \to G$. Take the trivial group 1 and the module $A \to 1$ over it, where A is any abelian group. Apply F to it. The resulting split extension

$$F(A) \to F(1)$$

does NOT have an abelian kernel. Indeed, $F(A)$ is the group of words on elements of A and their formal inverses, and the kernel of the map is the subgroup of words whose exponents add to zero, e.g. $a^2b^{-3}c$. That subgroup is highly non-abelian (as long as A is non-trivial), e.g. the elements ab^{-1} and $a^{-1}b$ do not commute.

Proposition 1.1.4. *If C has all pullbacks, then any map $f : X \to Y$ in C induces a pullback functor on Beck modules*

$$f^* : Ab(C/Y) \to Ab(C/X).$$

Moreover, f^* is additive.

Proof. First note that f induces a pullback functor on the slice categories

$$f^* : C/Y \to C/X$$

which is right adjoint to the so-called direct image functor

$$f_! : C/X \to C/Y$$

given by postcomposition. Indeed, consider the following commutative diagram:

$$\begin{tikzcd}
W \arrow[r, "\varphi"] \arrow[dr, "g"'] & f^*Z = X \times_Y Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Z \arrow[d, "p"] \\
 & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

We have the following correspondence:

$$\begin{aligned}
&\mathrm{Hom}_{C/X}\left(W \xrightarrow{g} X, f^*(Z \xrightarrow{p} Y)\right) \\
&\quad = \mathrm{Hom}_{C/X}\left(W \xrightarrow{g} X, f^*Z \xrightarrow{\pi_1} X\right) \\
&\quad = \{\varphi : W \to f^*Z \mid g = \pi_1\varphi\} \\
&\quad = \{\varphi_1 = \pi_1\varphi : W \to X, \varphi_2 = \pi_2\varphi : W \to Z \mid f\varphi_1 = p\varphi_2, \varphi_1 = g\} \\
&\quad = \{\varphi_2 : W \to Z \mid fg = p\varphi_2\} \\
&\quad = \mathrm{Hom}_{C/Y}\left(W \xrightarrow{fg} Y, Z \xrightarrow{p} Y\right) \\
&\quad = \mathrm{Hom}_{C/Y}\left(f_!(W \xrightarrow{g} X), Z \xrightarrow{p} Y\right).
\end{aligned}$$

As we've seen in the proof of 1.1.1, any limit-preserving functor induces an additive functor on the categories of abelian group objects, hence the conclusion. □

Definition 1.1.5. When f^* has a left adjoint, we call it the **pushforward**

$$f_* : Ab(C/X) \to Ab(C/Y).$$

Definition 1.1.6. When the forgetful functor $U_X : Ab(C/X) \to C/X$ has a left adjoint, we call it the **abelianization** functor $Ab_X : C/X \to Ab(C/X)$.

It is convenient to work with categories C that have all pushforwards and all abelianizations Ab_X; we will assume they exist whenever we need to. It is also convenient when the module category $Ab(C/X)$ is an abelian category for every object X, which holds for example when C is exact [Bar02, chap 2, thm 2.4].

Proposition 1.1.7. *Let $f : X \to Y$ be a morphism in C and assume f has a pushforward f_*. Then we have:*

$$Ab_Y(X \xrightarrow{f} Y) = f_* Ab_X(X \xrightarrow{\mathrm{id}} X).$$

More generally, the following diagram commutes.

$$\begin{tikzcd}
C/X \arrow[r, "f_!"] \arrow[d, "Ab_X"'] & C/Y \arrow[d, "Ab_Y"] \\
Ab(C/X) \arrow[r, "f_*"'] & Ab(C/Y).
\end{tikzcd}$$

Proof. All the functors in this diagram are left adjoints of the following functors:

$$\begin{tikzcd}
C/X & C/Y \arrow[l, "f^*"'] \\
Ab(C/X) \arrow[u, "U_X"'] & Ab(C/Y) \arrow[l, "f^*"'] \arrow[u, "U_Y"]
\end{tikzcd}$$

This diagram commutes on the nose, by construction. Since adjoint pairs compose, and adjoints are unique up to unique iso, we conclude that the diagram of left adjoints also commutes. (Technically, it only commutes up to natural iso, but the abelianization and pushforward functors can be chosen so that it commutes on the nose). □

Notation. When it's clear which category C we're working in, the category $Ab(C/X)$ of Beck modules over X will be denoted $\mathbf{Mod}_X$.

1.2 Quillen (co)homology

In this section, we recall the basics of Quillen homology and cohomology. For more details, see e.g. [Qui67, II.5] or [GS07, 4.4].

The idea is that derived functors of "global sections" are somehow a good notion of cohomology. Note that the global sections of a module M over an object X are precisely:

$$\Gamma(X, M) = \mathrm{Hom}_{C/X}(X \overset{\mathrm{id}}{\to} X, M \to X) \cong \mathrm{Hom}_{Ab(C/X)}(Ab_X X, M \to X).$$

As a first try in defining derived functors thereof, we can use homological algebra.

Definition 1.2.1. For an object X of C and a Beck module M over X, we define the **Hochschild cohomology** of X with coefficients in M as:

$$\mathrm{HH}^*(X; M) := \mathrm{Ext}^*(Ab_X X, M)$$

taken in the abelian category $Ab(C/X)$ of modules over X.

However, homotopical algebra allows us to take derived functors in a non-abelian context, where cofibrant replacements play the role of projective resolutions. We can obtain more interesting constructions that way. Quillen showed that starting from a nice category C, the simplicial categories sC and sC/X have a standard model structure, and we want the (prolonged) adjunction

$$\begin{tikzcd} sC/X \arrow[r, shift left, "Ab_X"] & sAb(C/X) \arrow[l, shift left, "U_X"] \end{tikzcd} \tag{1.2.1}$$

to be a Quillen pair. We'll study the conditions more carefully in chapter 3. There are additional assumptions on $\mathrm{Ho}\, Ab(sC/X_\bullet)$; see [Qui67, II.5] for details.

Remark 1.2.2. There is no ambiguity in the notation sC/X, by the natural isomorphism $s(C/X) \cong (sC)/X$, where X is viewed as a constant simplicial object. We use the model structure on the slice category $(sC)/X$ induced by that on sC. A map in the slice category

$$\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dr] & & Z \arrow[dl] \\
& X &
\end{tikzcd}$$

is a weak equivalence, fibration, or cofibration iff f is one. See e.g. [Qui67, II.2 prop 6] or [GS07, ex 1.7].

Let $C_\bullet \to X$ be a cofibrant replacement of X in sC, which is the same as a cofibrant replacement of $X \overset{\mathrm{id}}{\to} X$ in sC/X.

Definition 1.2.3. The **Quillen homology object** or **cotangent complex** of X is the left derived abelianization of X, given by

$$L_X := LAb_X(X \overset{\mathrm{id}}{\to} X) = Ab_X(C_\bullet) \in sAb(C/X).$$

In other words, it is a simplicial Beck module over X, defined up to weak equivalence.

Remark 1.2.4. As a derived functor, it depends (up to weak equivalence) on the choice of cofibrant replacement, and L_X is defined up to iso in $\mathrm{Ho}\, sAb(C/X)$. However, as noted in [Hov99] after definition 1.3.6, if our model structure comes with functorial factorizations, then there is no choice involved.

Definition 1.2.5. The **Quillen homology** of X is the homotopy of its Quillen homology object:

$$\mathrm{HQ}_i(X) := \pi_i L_X \in Ab(C/X).$$

Definition 1.2.6. Let $M \in Ab(C/X)$ be a Beck module over X. The **Quillen cohomology** of X with coefficients in M is given by the (simplicially) derived functors of global sections, that is

$$\begin{aligned}\mathrm{HQ}^i(X; M) &= \pi^i \operatorname{Hom}_{C/X}(C_\bullet, M) \\ &= \pi^i \operatorname{Hom}_{Ab(C/X)}(Ab_X C_\bullet, M) \in \mathbf{Ab}.\end{aligned}$$

Here π^* denotes the cohomology of a cosimplicial abelian group.

Note that Quillen cohomology is the derived functors of the composite

$$\mathrm{Hom}_{Ab(C/X)}(-, M) \circ Ab_X : sC/X \to sAb(C/X) \to s\mathbf{Ab}$$

applied to $X \overset{\mathrm{id}}{\to} X$. The first functor is a left adjoint (hence "right exact") and sends projectives to projectives. The second functor is contravariant left exact. Hence we have a Grothendieck composite spectral sequence

$$E_2^{s,t} = R^s \,\mathrm{Hom}_{Ab(C/X)}(-, M) \circ L_t Ab_X(\mathrm{id}_X) \Rightarrow R^{s+t} \left(\mathrm{Hom}_{Ab(C/X)}(-, M) \circ Ab_X\right)(\mathrm{id}_X)$$

where $L_t Ab_X$ means $\pi_t L Ab_X$, so that $L_t Ab_X(\mathrm{id}_X)$ is $\mathrm{HQ}_t(X)$. We can rewrite the spectral sequence as

$$E_2^{s,t} = \mathrm{Ext}^s_{Ab(C/X)}(\mathrm{HQ}_t(X), M) \Rightarrow \mathrm{HQ}^{s+t}(X; M) \tag{1.2.2}$$

which is as a **universal coefficients spectral sequence** (UCSS) for Quillen cohomology. For a detailed exposition of composite spectral sequences in a non-abelian setup, see [BS92].

The replacement map $C_\bullet \overset{\sim}{\to} (X \overset{\mathrm{id}}{\to} X)$ induces $Ab_X(C_\bullet) \to Ab_X X$, which is not an equivalence anymore, but the source is still cofibrant. Hence there is a map $Ab_X(C_\bullet) \to P_\bullet \overset{\sim}{\to} Ab_X X$ to a cofibrant replacement (= projective resolution) of $Ab_X X$, which upon applying $\mathrm{Hom}(-, M)$ yields a map

$$\mathrm{Hom}_{Ab(C/X)}(P_\bullet, M) \to \mathrm{Hom}_{Ab(C/X)}(Ab_X C_\bullet, M)$$

in $s\mathbf{Ab}$. Upon passing to cohomology, we obtain a well defined map

$$\mathrm{HH}^*(X; M) \to \mathrm{HQ}^*(X; M)$$

which is in fact an edge morphism in the UCSS. More precisely, the spectral sequence is cohomologically graded, and there is an edge morphism

$$\mathrm{HH}^s(X; M) = E_2^{s,0} \twoheadrightarrow E_\infty^{s,0} \hookrightarrow \mathrm{HQ}^s(X; M).$$

In that sense, Hochschild cohomology can be thought of as an abelian approximation of Quillen cohomology, and the discrepancy between the two is controlled by Quillen homology.

1.3 Simplicial base object

We focused our attention on the Quillen homology of a base object in C, but in fact one can talk about the Quillen homology of any simplicial object $X_\bullet \in sC$, and the Quillen cohomology of $X_\bullet$ with coefficients in a simplicial module $M_\bullet \in Ab(sC/X_\bullet)$. More generally, one need not work with sC, but any nice enough model category $\underline{C}$, where Quillen's assumptions hold. We first describe modules over simplicial objects.

Proposition 1.3.1. *Let C be a category with finite limits and $X_\bullet$ a simplicial object in sC. Then a Beck module $p : E_\bullet \to X_\bullet$ over $X_\bullet$ is the data of a Beck module $p_n : E_n \to X_n$ in each simplicial degree, such that the faces and degeneracies of $E_\bullet$ cover those of $X_\bullet$ and respect the abelian group object structure maps.*

Proof. The "n^{th} degree" functor $sC \to C$ preserves limits, since it is the restriction along $n \hookrightarrow \Delta^{\mathrm{op}}$. Hence it induces a functor on Beck modules and we get a Beck module $p_n : E_n \to X_n$ in each simplicial degree. This determines the constituent objects of E, the map p, and the abelian group structure maps of E. The remaining conditions are that $p : E_\bullet \to X_\bullet$ is a map in sC and the structure maps are maps in $sC/X_\bullet$. Those are exactly the conditions mentioned in the claim. □

In fact, we didn't use anything special about Δ^{op} and the statement holds for any indexing category I. Let $\mathbf{Mod}C$ denote the fibered category of Beck modules over C and $U : \mathbf{Mod}C \to C$ the "base object" forgetful functor, as in section B.1. Consider the category $C^I = \mathrm{Fun}(I, C)$ of I-diagrams in C.

Proposition 1.3.2. *Given a diagram $F : I \to C$, the category $\mathbf{Mod}_F = Ab(C^I/F)$ of Beck modules over F is the category of I-diagrams like this: the i^{th} entry is a Beck module over $F(i)$ and maps respect the structures of Beck modules (i.e. are maps in $\mathbf{Mod}C$). In other words, Mod_F is the category of I-diagrams in $\mathbf{Mod}C$ whose base diagram is F and where morphisms fix the base, i.e. the fiber over F of the forgetful functor:*

$$U^I : \mathbf{Mod}C^I \to C^I.$$

As this point it is worth noting something we swept under the rug. When our base object is a constant simplicial object, we have prolonged the abelianization functor $Ab_X : C/X \to Ab(C/X)$ to $sC/X \to sAb(C/X)$. In the general setup, we take the abelianization $\underline{C}/X \to Ab(\underline{C}/X)$. The two coincide, by the following lemma.

Lemma 1.3.3. *For any category C, we have $Ab(sC) = sAb(C)$. More generally, for any category I, we have $Ab(C^I) = Ab(C)^I$.*

Proof. Since limits are computed objectwise in a functor category, the constituent objects of an abelian group simplicial object $X_\bullet$ in $Ab(sC)$ are abelian group objects in C and together they determine the abelian group object structure of $X_\bullet$. By definition, the abelian group object structure maps commute with the simplicial structure maps. This is the same data as a simplicial abelian group object, i.e. an object in $sAb(C)$. Morphisms also match. □

After Beck modules, let's identify the abelianization.

Proposition 1.3.4. *Let $X_\bullet$ be a simplicial object in sC. Then the abelianization $Ab_{X_\bullet}X_\bullet$ is $\{Ab_{X_n}X_n\}_n$, in other words we abelianize each constituent object (over itself).*

Proof. Let $E_\bullet \overset{p}{\to} X_\bullet$ be a Beck module over $X_\bullet$. A map in $\mathrm{Hom}_{sC/X_\bullet}\left(X_\bullet \overset{id}{\to} X_\bullet, U(E_\bullet \overset{p}{\to} X_\bullet)\right)$ is a section $s : X_\bullet \to E_\bullet$ which is a map of simplicial objects, i.e. commutes with faces and degeneracies. This is the same data as a map of Beck modules in $\mathbf{Mod}_{X_\bullet}$ formed by taking the adjunct map $Ab_{X_n}X_n \to (E_n \overset{p}{\to} X_n)$ of the section $s_n : X_n \to E_n$ in each simplicial degree. Indeed, the compatibility condition with simplicial structure maps is the same in both cases. For example, the diagram:

$$\begin{tikzcd}
X_n \arrow[r, "s_n"] \arrow[d, "d_i"'] & U_{X_n}(E_n) \arrow[d, "d_i"'] \\
X_{n-1} \arrow[r, "s_{n-1}"'] & U_{X_{n-1}}(E_{n-1})
\end{tikzcd}$$

commutes iff the "adjoint" diagram (in a fibered category sense):

$$\begin{tikzcd}
Ab_{X_n}X_n \arrow[r] \arrow[d, "d_i"'] & E_n \arrow[d, "d_i"'] \\
Ab_{X_{n-1}}X_{n-1} \arrow[r] & E_{n-1}
\end{tikzcd}$$

commutes. □

There is also an "absolute" version of Quillen homology, as presented in [Qui67, II.5], examples 1 and 2, or [GS07], definition 4.24 and examples 4.25 and 4.26. It can be particularly useful when dealing with a simplicial base object which is not constant, for example the Quillen homology of simplicial sets. In this setup, $\underline{C}$ is a model category (typically of the form sC) and the abelianization

$Ab : \underline{C} \to Ab(\underline{C})$ is a left Quillen functor. The (absolute) Quillen homology object is defined as:

$$L_X^{abs} := LAb(X) = Ab(C)$$

where C is a cofibrant replacement of X. However, since $\underline{C}$ is the slice category $\underline{C}/*$ over the terminal object, we can recover the absolute version from the "relative" version presented here, in the following way.

Proposition 1.3.5. *The absolute Quillen homology object is the pushforward:*

$$L_X^{abs} = t_* L_X$$

where $\tau : X \to *$ is the unique map to the terminal object.

Proof. Let $q : C \to X$ be a cofibrant replacement of X in $\underline{C}$. Then

$$\begin{tikzcd}
C \arrow[r, "q"] \arrow[dr, "q"'] & X \arrow[d, "\mathrm{id}"] \\
 & X
\end{tikzcd}$$

is a cofibrant replacement of $X \overset{\mathrm{id}}{\to} X$ in $\underline{C}/X$ and

$$\begin{tikzcd}
C \arrow[r, "q"] \arrow[dr] & X \arrow[d, "\tau"] \\
 & *
\end{tikzcd}$$

is a cofibrant replacement of $X \overset{\tau}{\to} *$ in $\underline{C}/*$. Considering the diagram:

$$\begin{tikzcd}
C \arrow[r, "q"] \arrow[dr, "q"'] & X \arrow[r, equal] \arrow[d, "\mathrm{id}"] & X \arrow[d] \\
 & X \arrow[r, "\tau"'] & *
\end{tikzcd}$$

an easy computation gives the result:

$$\begin{aligned}
L_X &= LAb_X(X \overset{\text{id}}{\to} X) \\
&= Ab_X(C \overset{q}{\to} X) \\
&= q_* Ab_C(C \overset{\text{id}}{\to} C) \\
\tau_* L_X &= \tau_* q_* Ab_C(C \overset{\text{id}}{\to} C) \\
&= (\tau q)_* Ab_C(C \overset{\text{id}}{\to} C) \\
&= Ab_{\{*\}}(C \to *) \\
&= L_X^{abs}.
\end{aligned}$$

□

1.4 Homotopy invariance

We investigate to what extent, and in what sense, is Quillen homology a "homotopy invariant". In other words, how does it transform under a weak equivalence?

Lemma 1.4.1. *Let $f : X \to Y$ be a map in a nice model category $\underline{C}$. Then the adjunction*

$$\begin{tikzcd}
\mathbf{Mod}_X \arrow[r, shift left, "f_*"] & \mathbf{Mod}_Y \arrow[l, shift left, "f^*"]
\end{tikzcd}$$

is a Quillen pair.

Proof. Let us show that f^* preserves fibrations and acyclic fibrations. A map

$$\begin{tikzcd}
E \arrow[rr, "\varphi"] \arrow[dr] & & E' \arrow[dl] \\
& Y &
\end{tikzcd}$$

in $\mathbf{Mod}_Y = Ab(\underline{C}/Y)$ is a fibration (resp. acyclic fibration) iff it is one in $\underline{C}/Y$ [I need to check the "if" direction] iff $\varphi : E \to E'$ is one in $\underline{C}$. Consider the commutative diagram that defines f^*:

$$\begin{tikzcd}
 & & & E \arrow[r, two heads, "\varphi"] \arrow[rdd] & E' \arrow[dd] \\
f^*E \arrow[rr, "f^*\varphi"] \arrow[rrru] \arrow[rdd] & & f^*E' \arrow[rru] \arrow[ldd] & & \\
 & & & & Y \\
 & X \arrow[rrru, "f"'] & & &
\end{tikzcd}$$

The right-hand and left-hand squares are pullbacks, hence the top square is a pullback. Since fibrations and acyclic fibrations are preserved under base change, $f^*\varphi$ is also a fibration (resp. acyclic fibration). □

Question. If f is a weak equivalence, is this Quillen pair a Quillen equivalence? What if f is an acyclic fibration?

Proposition 1.4.2. *A weak equivalence $f : X \overset{\sim}{\to} Y$ in $\underline{C}$ induces a natural weak equivalence*

$$f_*L_X \overset{\sim}{\to} L_Y$$

in $Ab(\underline{C}/Y)$.

Proof. Assuming we have a functorial cofibrant replacement $q_X : QX \to X$ (notation of [Hov99], introduced right before lemma 1.1.9), we get the following commutative diagram:

$$\begin{tikzcd}
QX \arrow[r, "q_X", "\sim"'] \arrow[d, "Qf"', "\sim"] & X \arrow[d, "f", "\sim"'] \\
QY \arrow[r, "\sim", "q_Y"'] & Y
\end{tikzcd}$$

We have $L_X = Ab_X(QX \overset{q_X}{\to} X) = q_{X*}Ab_{QX}(QX \overset{\text{id}}{\to} QX)$ and likewise for Y. Now we compute:

$$\begin{aligned}
f_*L_X &= f_*q_{X*}Ab_{QX}(QX \overset{\text{id}}{\to} QX) \\
&= (fq_X)_*Ab_{QX}(QX \overset{\text{id}}{\to} QX) \\
&= (q_YQf)_*Ab_{QX}(QX \overset{\text{id}}{\to} QX) \\
&= q_{Y*}Ab_{QY}(QX \overset{Qf}{\to} QY).
\end{aligned}$$

Now the map

$$\begin{tikzcd} QX \arrow[rr, "Qf", "\sim"'] \arrow[dr, "Qf"'] & & QY \arrow[dl, "\mathrm{id}"] \\ & QY & \end{tikzcd}$$

is a weak equivalence between cofibrant objects. Upon applying the two left Quillen functors $q_{Y*}Ab_{QY}$, it induces a weak equivalence

$$f_*L_X \overset{\sim}{\to} L_Y$$

which is natural in $X \xrightarrow{f} Y$, since $q : Q \to 1$ is a natural transformation. □

Corollary 1.4.3. *In the same context, there is a first quadrant spectral sequence*

$$E^2_{s,t} = (L_s f_*)\,\mathrm{HQ}_t(X) \Rightarrow \mathrm{HQ}_{s+t}(Y).$$

Proof. Consider the composite of left adjoints

$$\underline{C}/X \xrightarrow{Ab_X} Ab(\underline{C}/X) \xrightarrow{f_*} Ab(\underline{C}/Y)$$

The first functor sends cofibrants to cofibrants (projectives), so we get a composite spectral sequence:

$$E^2_{s,t} = L_s f_* \circ L_t Ab_X \Rightarrow L_{s+t}(f_* \circ Ab_X)$$

For the object $\mathrm{id} : X \to X$, the spectral sequence becomes

$$E^2_{s,t} = (L_s f_*)\,\mathrm{HQ}_t(X) \Rightarrow \mathrm{HQ}_{s+t}(Y).$$

The natural map

$$f_*\,\mathrm{HQ}_t(X) = f_*\pi_t L_X \to \pi_t(f_*L_X) \cong \pi_t L_Y = \mathrm{HQ}_t(Y)$$

is the edge morphism

$$f_*\,\mathrm{HQ}_t(X) = E^2_{0,t} \twoheadrightarrow E^\infty_{0,t} \hookrightarrow \mathrm{HQ}_t(Y)$$

in this spectral sequence. □

Remark 1.4.4. Perhaps we can get a stronger statement. A priori, it is not obvious that the higher derived functors of f_* should vanish when f is a weak equivalence, but there might be some additional assumptions that guarantee it, or guarantee that f_* preserves all weak equivalences.

Corollary 1.4.5. *A weak equivalence $f : X \xrightarrow{\sim} Y$ induces a weak equivalence on the "absolute" Quillen homology objects $L_X^{abs} \xrightarrow{\sim} L_Y^{abs}$, and hence an isomorphism on absolute Quillen homology.*

Proof. Using proposition 1.3.5 and the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "\tau_X"'] & Y \arrow[d, "\tau_Y"] \\
& *
\end{tikzcd}$$

we obtain:

$$L_X^{abs} = \tau_{X*} L_X = \tau_{Y*} f_* L_X \xrightarrow{\sim} \tau_{Y*} L_Y = L_Y^{abs}.$$

We used the fact that τ_* preserves weak equivalences between cofibrant objects. □

Chapter 2

Π-algebras

Let $\mathbf{\Pi}$ denote the homotopy category of pointed spaces with the homotopy type of a finite (possibly empty) wedge of spheres of positive dimensions. Recall the following [BDG04, § 4].

Definition 2.0.6. A **$\mathbf{\Pi}$-algebra** is a contravariant functor $A : \mathbf{\Pi} \to \mathbf{Set}$ that sends wedges to products, i.e. a product-preserving functor $\mathbf{\Pi}^{op} \to \mathbf{Set}$.

Note that $\mathbf{\Pi}$ is a pointed category, and the definition of $\mathbf{\Pi}$-algebra is the same if the functor A takes values in $\mathbf{Set}_*$ instead, the category of pointed sets. Indeed, since A sends wedges to products, it must send the zero object $*$ to a singleton, and thus the set $A(S)$ is automatically pointed via $A(*) \to A(S)$ induced by the crushing map $S \to *$. Note also that the definition is the same if we pick a representative space in each homeomorphism class in order to make $\mathbf{\Pi}$ small, as in [Sto90, § 4].

Notation. Let $\mathbf{\Pi Alg}$ denote the category of Π-algebras, that is $\mathrm{Fun}^{\times}(\mathbf{\Pi}^{op}, \mathbf{Set})$, where $\mathrm{Fun}^{\times}$ denotes product-preserving functors.

2.1 Abelian Π-algebras

The following tidbit is essentially in [Bla93, § 3] or [BDG04, § 4.8]. We explain it here in more detail.

Proposition 2.1.1. *The category of abelian Π-algebras is the full subcategory of Π-algebras whose Whitehead products all vanish.*

Proof. The forgetful functors $\pi_1 : \mathbf{\Pi Alg} \to \mathbf{Gp}$ and $\pi_n : \mathbf{\Pi Alg} \to \mathbf{Ab}$ (for each $n \geq 2$) preserve finite products and the terminal object. Hence they induce functors on abelian group objects:

$$\pi_1 : Ab(\mathbf{\Pi Alg}) \to Ab(\mathbf{Gp}) \cong \mathbf{Ab}$$
$$\pi_n : Ab(\mathbf{\Pi Alg}) \to Ab(\mathbf{Ab}) \cong \mathbf{Ab}.$$

If we start with an object in $Ab(\mathbf{\Pi Alg})$, its structure of abelian group object must be the levelwise group multiplication (and unit and inverse). Therefore, a Π-algebra admits at most one structure of abelian group object, and it admits one iff the levelwise multiplication map $A \times A \to A$ is a map of Π-algebras. This shows that the faithful forgetful functor $Ab(\mathbf{\Pi Alg}) \to \mathbf{\Pi Alg}$ is the inclusion of a subcategory, and it is full since any map of Π-algebras is in particular a map of graded groups.

It remains to show that the levelwise multiplication map $\mu : A \times A \to A$ is a map of Π-algebras iff all Whitehead products of A vanish.

($\Rightarrow$) Since μ_1 is a map of groups, A_1 is an abelian group, i.e. its commutators vanish. (We still write its group operation multiplicatively though.) Next, μ is equivariant with respect to the π_1-action, which amounts to the following condition:

$$\begin{aligned} \mu\left((a, a') \cdot (x, x')\right) &= \mu(a, a') \cdot \mu(x, x') \\ \mu(a \cdot x, a' \cdot x') &= aa' \cdot (x + x') \\ a \cdot x + a' \cdot x' &= aa' \cdot x + aa' \cdot x' \end{aligned}$$

for all $a, a' \in A_1$ and $x, x' \in A_n$, $n \geq 2$. In particular, setting $x' = 0$ and $a = 1$, we obtain $x = a' \cdot x$, i.e. the π_1-action is trivial. Thus all Whitehead products involving elements of A_1 vanish.

For $p, q \geq 2$, let us write $W : S^{p+q-1} \to S^p \vee S^q$ for the attaching map that defines Whitehead products, via

$$A_p \times A_q = A(S^p \vee S^q) \overset{A(W)}{\to} A(S^{p+q-1}) = A_{p+q-1}.$$

Since μ is a map of Π-algebras, the following commutes:

$$\begin{tikzcd}
(A \times A)_p \times (A \times A)_q \arrow[r, "(A\times A)(W)"] \arrow[d, "\mu_p \times \mu_q"'] & (A \times A)_{p+q-1} \arrow[d, "\mu_{p+q-1}"] \\
A_p \times A_q \arrow[r, "A(W)"'] & A_{p+q-1}
\end{tikzcd}$$

which translates into the condition:

$$\begin{aligned}[x,y] + [x',y'] &= [x+x', y+y'] \\ &= [x,y] + [x,y'] + [x',y] + [x',y']\end{aligned}$$

for all $x, x' \in A_p$ and $y, y' \in A_q$, i.e. all Whitehead products must vanish.

($\Leftarrow$) If Whitehead products vanish for A, then so do they for $A \times A$. Hence $\mu : A \times A \to A$ is a map of graded groups (since A_1 is abelian) which respects the π_1-action, since it is trivial on both sides, and which commutes with Whitehead products since they all vanish.

It remains to show that μ commutes with precomposition products, i.e. for any $\alpha \in \pi_m(S^n)$ and $x, x' \in A_n$, the condition

$$\alpha^*(x + x') = \alpha^* x + \alpha^* x'$$

holds. We can use Hilton's formula to compute $\alpha^*(x + x')$; see [Bla93], remark 2.1 (c). Since all Whitehead products vanish, the formula yields exactly that condition. □

Remark 2.1.2. By C.0.7, the functor π_n preserve (small) limits, since it is the restriction along a product-preserving functor, namely the inclusion of all wedges of spheres of dimension n. It does NOT preserve colimits though.

In other words, by C.0.5 we know that limits in **ΠAlg** are computed objectwise (in the "product-preserving functor" picture), and hence levelwise (in the "graded group" picture), so the functors π_n preserve them.

Proposition 2.1.3. *If A is any Π-algebra, then ΩA is an abelian Π-algebra.*

Proof. We want to show that the map $\Omega A \times \Omega A \to \Omega A$ which corresponds to levelwise (or "objectwise") multiplication is a map of Π-algebras, i.e. that for any map $f : S \to S'$ in **Π** between wedges of spheres, the following diagram commutes:

$$\begin{tikzcd}
(\Omega A)(S \vee S) \arrow[d, equal] & (\Omega A)(S' \vee S') \arrow[d, equal] \\
(\Omega A \times \Omega A)(S) \arrow[d, "(\Omega A)(\mathrm{pinch}_S)"'] & (\Omega A \times \Omega A)(S') \arrow[l, "(\Omega A)(f \vee f)"', "=(\Omega A \times \Omega A)(f)"] \arrow[d, "(\Omega A)(\mathrm{pinch}_{S'})"] \\
(\Omega A)(S) & (\Omega A)(S') \arrow[l, "(\Omega A)(f)"]
\end{tikzcd}$$

By definition of ΩA, we can rewrite it thus:

$$\begin{tikzcd}
A(\Sigma(S \vee S)) \arrow[d, equal] & A(\Sigma(S' \vee S')) \arrow[d, equal] \\
A(\Sigma S) \times A(\Sigma S) \arrow[d, "A(\Sigma \mathrm{pinch}_S)"'] & A(\Sigma S') \times A(\Sigma S') \arrow[l, "A(\Sigma(f \vee f))"', "=A(\Sigma f) \times A(\Sigma f)"] \arrow[d, "A(\Sigma \mathrm{pinch}_{S'})"] \\
A(\Sigma S) & A(\Sigma S') \arrow[l, "A(\Sigma f)"]
\end{tikzcd}$$

Up to homotopy, we can pinch spheres along any equator we want; in particular, we can pinch along the suspension coordinate, so that $\Sigma \mathrm{pinch}_s = \mathrm{pinch}_{\Sigma S}$ holds in $\mathbf{\Pi}$. Hence the previous diagram is obtained by applying A to the following diagram in $\mathbf{\Pi}$:

$$\begin{tikzcd}
\Sigma S \vee \Sigma S \arrow[r, "\Sigma f \vee \Sigma f"] & \Sigma S' \vee \Sigma S' \\
\Sigma S \arrow[u, "\mathrm{pinch}_{\Sigma S}"] \arrow[r, "\Sigma f"'] & A(\Sigma S') \arrow[u, "\mathrm{pinch}_{\Sigma S'}"']
\end{tikzcd}$$

which is seen to commute in $\mathbf{\Pi}$, since we can pinch along the suspension coordinate on both sides.

□

Corollary 2.1.4. *The Whitehead map* $W : S^{p+q-1} \to S^p \vee S^q$ *suspends to zero, i.e.* ΣW *is null-homotopic.*

Proposition 2.1.5. *If* A *is any* Π*-algebra, then* ΩA *is not necessarily strongly abelian.*

Proof. Take $A = \pi_*(S^3)$, so that $\Omega A = \pi_*(\Omega S^3)$. Take x the canonical generator in $\pi_2(\Omega S^3) \cong \pi_3(S^3) \cong \mathbb{Z}$, in other words, the map adjoint to the identity $S^3 \to S^3$. Let $\eta : S^3 \to S^2$ denote the Hopf map. Then $x \circ \eta \in \pi_3(\Omega S^3) \cong \pi_4(S^3) \cong \mathbb{Z}/2$ is adjoint to $\Sigma\eta$, so is the non-zero element. □

2.2 Truncated Π-algebras

Thinking of $\mathbf{\Pi}$ as the category of "all spheres", it admits a nice filtration by sphere dimension. This filtration is made into a tower, by removing spheres above a certain dimension. Dualizing this filtration/tower yields the usual tower/filtration of Π-algebras by truncated Π-algebras. In this section, we make all this precise.

Definition 2.2.1. A Π-algebra A is called **n-truncated** if for all $i > n$, we have $A(S^i) = *$, the trivial pointed set.

Notation. Denote by $\mathbf{\Pi Alg}_1^n$ the full subcategory of $\mathbf{\Pi Alg}$ consisting of n-truncated Π-algebras.

Notation. Denote by $\mathbf{\Pi}_n$ the full subcategory of $\mathbf{\Pi}$ consisting of spaces with the homotopy type of a wedge of spheres of dimension at most n, and let $I_n : \mathbf{\Pi}_n \to \mathbf{\Pi}$ be the inclusion functor. We allow $n = \infty$, where $\mathbf{\Pi}_\infty$ means $\mathbf{\Pi}$. More generally, for any $i < j$, we have an inclusion functor $I_{i,j} : \mathbf{\Pi}_i \to \mathbf{\Pi}_j$, and they all commute, i.e. they satisfy $I_{i,k} = I_{j,k} I_{i,j}$.

This way we get a filtration of $\mathbf{\Pi}$:

$$\begin{tikzcd} * = \mathbf{\Pi}_0 \arrow[r, "I_{0,1}"] & \mathbf{\Pi}_1 \arrow[r, "I_{1,2}"] & \mathbf{\Pi}_2 \arrow[r, "I_{2,3}"] & \cdots \arrow[r, "I"] & \mathbf{\Pi} \end{tikzcd}$$

We can go the other way, by removing spheres above a certain dimension. Define a "truncation" functor $T_n : \mathbf{\Pi} \to \mathbf{\Pi}_n$ as follows:

$$T_n \left(\bigvee_{i=1}^{k} S^{n_i} \right) = \bigvee_{n_i \leq n} S^{n_i}$$

Remark 2.2.2. Depending on how we define the category $\mathbf{\Pi}$, this could be technically ambiguous. We can work with a version of $\mathbf{\Pi}$ where each object X is equipped with a given homotopy equivalence to (and from) a finite wedge of spheres.

T_n send a map $f : \bigvee_i S^{n_i} \to \bigvee_j S^{m_j}$ to the composite

$$\bigvee_{n_i \leq n} S^{n_i} \hookrightarrow \bigvee_i S^{n_i} \xrightarrow{f} \bigvee_j S^{m_j} \twoheadrightarrow \bigvee_{m_j \leq n} S^{m_j},$$

where the map on the left is summand inclusion and the map on the right is summand collapse. Note that we didn't really need to collapse: since the inclusion $\bigvee_{m_j \leq n} S^{m_j} \to \bigvee_j S^{m_j}$ is n-connected and admits a retraction, it induces an iso on π_i for $i \leq n$. Hence up to homotopy, there is a unique factorization of the map

$$\bigvee_{n_i \leq n} S^{n_i} \to \bigvee_i S^{n_i} \xrightarrow{f} \bigvee_j S^{m_j}$$

through $\bigvee_{m_j \leq n} S^{m_j}$.

More generally, for any $i < j$, we have an analogously defined truncation functor $T_{j,i} : \mathbf{\Pi}_j \to \mathbf{\Pi}_i$, and they all commute. Note that all functors I_{i_j} and $T_{i,j}$ preserve coproducts (wedges).

Proposition 2.2.3. *For any $i < j$, $I_{i,j}$ is left adjoint to $T_{j,i}$.*

Proof. Let S_i be an object of $\mathbf{\Pi}_i$ (a wedge of spheres of dimension at most i) and likewise for S_j. As we noted above, up to homotopy, any map $S_i \to S_j$ has a unique factorization through $T_{j,i}S_j$, hence we conclude:

$$\mathrm{Hom}_{\mathbf{\Pi}_j}(I_{i,j}S_i, S_j) \cong \mathrm{Hom}_{\mathbf{\Pi}_i}(S_i, T_{j,i}S_j)$$

□

In fact, this is a particularly nice adjunction. The unit is an equality, i.e. the composite TI is the identity functor (we omitted subscripts for T and I). The counit $IT(S) \to S$ is the inclusion of small spheres in the wedge:

$$\bigvee_{n_i \leq n} S^{n_i} \hookrightarrow \bigvee_i S^{n_i}.$$

Therefore, our filtration of $\mathbf{\Pi}$ is also a tower over $*$:

$$\begin{tikzcd} * = \mathbf{\Pi}_0 \arrow[r, shift left, "I_{0,1}"] & \mathbf{\Pi}_1 \arrow[l, shift left, "T_{1,0}"] \arrow[r, shift left, "I_{1,2}"] & \mathbf{\Pi}_2 \arrow[l, shift left, "T_{2,1}"] \arrow[r, shift left, "I_{2,3}"] & \mathbf{\Pi}_3 \arrow[l, shift left, "T_{3,2}"] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left, "I"] & \mathbf{\Pi} \arrow[l, shift left, "T"] \end{tikzcd}$$

Now, let us study what happens when we "dualize" it by applying $\mathrm{Fun}^{\times}(-, \mathbf{Set})$.

Lemma 2.2.4. *If A is an n-truncated* Π*-algebra, then $A = (IT)^*A$. In other words, A is determined by what it does on $\mathbf{\Pi}_n$.*

Proof. The counit $IT \to 1$ induces a natural transformation $A \to AIT$ (since A is contravariant). For a wedge of spheres S, we know that $IT(S) \to S$ is the inclusion:

$$\bigvee_{n_i \leq n} S^{n_i} \hookrightarrow \bigvee_i S^{n_i}.$$

Applying A yields:

$$A\left(\bigvee_i S^{n_i}\right) \to A\left(\bigvee_{n_i \leq n} S^{n_i}\right).$$

Since A sends wedges to products, this map is the projection:

$$\prod_i A(S^{n_i}) \to \prod_{n_i \leq n} A(S^{n_i}).$$

But since A is n-truncated, this map is the identity (or rather a natural iso). Thus our natural transformation $A \to AIT = (IT)^*A$ is a natural iso. □

In particular, we see that a Π-algebra A is n-truncated iff it factors through T, i.e. $A = A'T = T^*A'$ for some $A' : \mathbf{\Pi}_n \to \mathbf{Set}$. Note that such a factorization A' is unique (and sends wedges to products), since it is determined by A in the following way: $A' = A'TI = AI = I^*A$.

Proposition 2.2.5. *There is an equivalence of categories:*

$$I^* : \mathbf{\Pi Alg}_1^n \cong \mathrm{Fun}^{\times}(\mathbf{\Pi}_n^{op}, \mathbf{Set}) : T^*$$

Proof. If F is a product-preserving functor $\mathbf{\Pi}_n^{op} \to \mathbf{Set}$, then we have $I^*T^*F = (TI)^*F = F$, since TI is the identity. On the other hand, if A is an n-truncated Π-algebra, we have $T^*I^*A = (IT)^*A = A$, by lemma 2.2.4. □

Lemma 2.2.6. *For a fixed category C, the association* $\mathrm{Fun}(C, -) : \mathbf{Cat} \to \mathbf{Cat}$ *is a (strict) 2-functor. Likewise, the association* $\mathrm{Fun}(-, C) : \mathbf{Cat}^{op} \to \mathbf{Cat}$ *is a (strict) 2-functor.*

Proof. Let us prove the first assertion; the second has a similar proof. The association is clearly a 1-functor. Let us check that it respects 2-cells and all the structure, or equivalently, the enrichment over **Cat**. Given categories X and Y, we want to show that

$$\mathrm{Hom}_{\mathbf{Cat}}(X, Y) \to \mathrm{Hom}_{\mathbf{Cat}}\left(\mathrm{Fun}(C, X), \mathrm{Fun}(C, Y)\right)$$

is a map of categories. Given a morphism in the left-hand side, i.e. a natural transformation

$$\begin{tikzcd} X \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd}$$

we obtain a morphism on the right-hand side, i.e. a natural transformation

$$\begin{tikzcd} \mathrm{Fun}(C, X) \arrow[r, bend left, "F_*"] \arrow[r, bend right, "G_*"'] \arrow[r, phantom, "\Downarrow\alpha_*"] & \mathrm{Fun}(C, Y) \end{tikzcd}$$

defined as follows. Given an object φ of $\mathrm{Fun}(C, X)$, i.e. a functor $\varphi : C \to X$, we need a comparison map $\alpha_{*\varphi}$from $F_*\varphi = F\varphi$ to $G_*\varphi = G\varphi$. It is given by this diagram:

$$\begin{tikzcd}
 & F\varphi(c) \arrow[dd, "\alpha_{\varphi(c)}"] \\
c \arrow[ur, mapsto, "F\varphi"] \arrow[dr, mapsto, "G\varphi"'] & \\
 & G\varphi(c)
\end{tikzcd}$$

This respects identities and composition, i.e. if we take $F = G$ and $\alpha = \mathrm{id}_F$, then $(\mathrm{id}_F)_* = \mathrm{id}_{F_*}$ and also $(\beta\alpha)_* = \beta_*\alpha_*$. $\square$

Notation. For $i < j$, denote $\tau_{j,i} = I^*_{i,j}$, where the "dualizing" means applying $\mathrm{Fun}^\times(-, \mathbf{Set})$. Denote $\iota_{i,j} = T^*_{j,i}$. In words, the dual of "inclusion of spheres" is "truncation of *Pi*-algebras", and the dual of "truncation of spheres" is "inclusion of Π-algebras". Yes, it's awkward notation and terminology. In fact, $\tau_{j,i}$ (or $\iota_{i,j}\tau_{j,i}$) deserves to be denoted P_i, since it corresponds to the i^{th} Postnikov truncation.

Proposition 2.2.7. *For $i < j$, the functors*

$$\begin{tikzcd}
\mathbf{\Pi Alg}_1^j \arrow[r, shift left, "\tau_{j,i}"] & \mathbf{\Pi Alg}_1^i \arrow[l, shift left, "\iota_{i,j}"]
\end{tikzcd}$$

form an adjoint pair, where $\tau_{j,i}$ is the left adjoint.

Proof. Since $I_{i,j}$ and $T_{j,i}$ preserve wedges of spheres, they induce functors as in the statement upon dualizing. By lemma 2.2.6, $\tau_{j,i} = I^*_{i,j}$ is still left adjoint to $\iota i, j = T^*_{j,i}$. $\square$

Remark 2.2.8. One can easily check this adjunction directly, using the description of (truncated) Π-algebras as graded groups with extra structure. Indeed, for $A \in \mathbf{\Pi Alg}$ and $B \in \mathbf{\Pi Alg}_1^n$, we have:

$$\begin{aligned}
&\mathrm{Hom}_{\mathbf{\Pi Alg}_1^n}(P_nA, B)\\
&= \{\text{maps of graded groups } P_nA \to B \text{ respecting the extra structure}\}\\
&= \{\text{maps of graded groups } A \to \iota_nB \text{ respecting the extra structure}\}\\
&= \mathrm{Hom}_{\mathbf{\Pi Alg}}(A, \iota_nB).
\end{aligned}$$

The next to last equality holds because the additional data is trivial (maps of groups $A_m \to B_m = 0$, for $m > n$) and the additional conditions are vacuous (equations holding in groups $B_m = 0$).

Notation. Denote by **nTypes** the homotopy category of connected, pointed n-types, i.e. spaces whose homotopy groups vanish above n.

We obtain a tower/filtration a categories for connected, pointed spaces (up to weak equivalence) and one for Π-algebras, and they are related by the functor π_*. In other words, we obtain a commutative diagram of categories:

$$\begin{tikzcd}
\mathbf{Spaces} \arrow[r, "\pi_*"] \arrow[d, shift right, "P"'] & \mathbf{\Pi Alg} \arrow[d, shift right, "P"'] \\
\vdots \arrow[u, shift right, "\iota"'] \arrow[d, shift right, "P"'] & \vdots \arrow[u, shift right, "\iota"'] \arrow[d, shift right, "P"'] \\
\mathbf{3Types} \arrow[r, "\pi_*"] \arrow[u, shift right, "\iota"'] \arrow[d, shift right, "P"'] & \mathbf{\Pi Alg}_1^3 \arrow[u, shift right, "\iota"'] \arrow[d, shift right, "P"'] \\
\mathbf{2Types} \arrow[r, "\pi_*"] \arrow[u, shift right, "\iota"'] \arrow[d, shift right, "P"'] & \mathbf{\Pi Alg}_1^2 \arrow[u, shift right, "\iota"'] \arrow[d, shift right, "P"'] \\
\mathbf{1Types} \arrow[r, "\pi_*"] \arrow[u, shift right, "\iota"'] \arrow[d, shift right, "P"'] & \mathbf{\Pi Alg}_1^1 \arrow[u, shift right, "\iota"'] \arrow[d, shift right, "P"'] \\
* \arrow[r, "\pi_*"] \arrow[u, shift right, "\iota"'] & * \arrow[u, shift right, "\iota"']
\end{tikzcd}$$

where each functor P is left adjoint to the respective ι.

For small values of n, the category $\mathbf{\Pi Alg}_1^n$ has a very simple algebraic description.

Proposition 2.2.9. *1.* $\mathbf{\Pi Alg}_1^1$ *is equivalent to* **Gp**, *the category of groups.*

2. $\mathbf{\Pi Alg}_1^2$ *is equivalent to* **ModGp**, *whose objects consist of a group and a module over it (cf. section 7.0.7).*

3. $\mathbf{\Pi Alg}_1^3$ *is equivalent to the category whose objects are* $(\pi_1, \pi_2, \pi_3; q : \pi_2 \to \pi_3)$, *where* π_1 *is a group,* π_2 *and* π_3 *are (left) modules over* π_1, *and* q *is a quadratic map which is* π_1*-equivariant. Morphisms are what they should be.*

Proof. Recall that a Π-algebra can be described explicitly as a graded group, abelian above dimension 1, with a π_1-action on the higher groups, Whitehead products, and precomposition products by $\pi_m(S^n)$ (for $m > n$), satisfying certain relations.

1. Clear.

2. There is no room for Whitehead products (not involving π_1), nor for precomposition products since $\pi_m(S^1) = 0$ for all $m > 1$. Therefore, the only structure is that of the group π_1 and the

π_1-module π_2.

3. In addition to the group π_1 and the π_1-modules π_2 and π_3, there is room for the Whitehead product

$$\pi_2 \otimes \pi_2 \to \pi_3$$

as well as the precomposition product

$$\pi_3(S^2) \times \pi_2 \to \pi_3.$$

Since the latter is linear in $\pi_3(S^2) \cong \mathbb{Z}$, it suffices to know the precomposition by the Hopf map $\eta \in \pi_3(S^2)$, i.e.

$$\eta^* : \pi_2 \to \pi_3.$$

Moreover, we have the classic formula [Bau91, I.4.10] [Whi78, XI.1.16]:

$$\eta^*(x + y) - \eta^*(x) - \eta^*(y) = [x, y],$$

which is a bilinear map. Hence the Whitehead product map can be recovered from η^*. Also, there is no additional condition on η^* other than being quadratic and π_1-linear. □

Note that when we say that η^* is "quadratic", we don't mean only that $\eta^*(x + y) - \eta^*(x) - \eta^*(y)$ is a bilinear map, but also the following.

Proposition 2.2.10. *For any space X, the map* $\eta^* : \pi_2(X) \to \pi_3(X)$ *satisfies*

$$\eta^*(kx) = k^2 x$$

for any $k \in \mathbb{Z}$ *and* $x \in \pi_2(X)$*.*

Proof. Considering the universal example for Whitehead products, we have a commutative diagram

$$\begin{tikzcd}
S^3 \arrow[r, "W"] \arrow[dr, "{[\iota_2,\iota_2]}"'] & S^2 \vee S^2 \arrow[r, "{(x,x)}"] \arrow[d, "\nabla"'] & X \\
& S^2 \arrow[ur, "x"'] &
\end{tikzcd}$$

where $\iota_n \in \pi_n(S^n)$ denotes the class of the identity, and the top composite is the Whitehead product $[x, x]$. Now a Hopf invariant argument [Whi78, XI.2.5] tells us that $[\iota_2, \iota_2]$ is equal to 2η. Hence we have

$$\begin{aligned}[x, x] &= (x, x) \circ W \\ &= (2\eta)^*(x) \\ &= 2\eta^*(x)\end{aligned}$$

and therefore

$$\begin{aligned}\eta^*(2x) = \eta^*(x + x) &= \eta^*(x) + \eta^*(x) + [x, x] \\ &= 4\eta^*(x).\end{aligned}$$

By induction, the claim holds for all positive integers:

$$\begin{aligned}\eta^*\left((k+1)x\right) &= \eta^*(kx) + \eta^*(x) + [kx, x] \\ &= k^2\eta^*(x) + \eta^*(x) + k(2\eta^*(x)) \\ &= (k+1)^2\eta^*(x).\end{aligned}$$

The claim clearly holds for $k = 0$, and also for negative integers, using the equality $\eta^*(-x) = \eta^*(x)$. Indeed, we have:

$$\begin{aligned}0 = \eta^*(x - x) &= \eta^*(x) + \eta^*(-x) + [x, -x] \\ &= \eta^*(-x) - \eta^*(x).\end{aligned}$$

□

Chapter 3

Setup for Quillen cohomology

In this chapter, we study in more detail the categorical assumptions needed in order to work with Quillen cohomology. More specifically, we want the (prolonged) adjunction (1.2.1) to be a Quillen pair. For future applications, we also want four Quillen pairs in diagrams (4.4.1) and (4.4.6).

3.1 Prolonged adjunctions as Quillen pairs

Lemma 3.1.1. *The simplicial prolongation of an adjunction is still an adjunction.*

Proof. We have an adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "L"] & \mathcal{D} \arrow[l, shift left, "R"] \end{tikzcd}$$

Prolonging it to the categories of simplicial objects means applying $\mathrm{Fun}(\Delta^{\mathrm{op}}, -)$ to the diagram. By lemma 2.2.6, this preserves adjunctions. □

Proposition 3.1.2. *Assume we have an adjunction as denoted above.*

1. *If R preserves effective epis, then L preserves projectives.*
2. *If, moreover, the category $\mathcal{C}$ has finite limits and enough projectives, then the converse holds as well.*

Proof. 1. Let P be a projective in $\mathcal{C}$. We want to show LP is projective in $\mathcal{D}$. Let $f : d \to d'$ be any effective epi in $\mathcal{D}$. The we have:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(LP, d) \arrow[r, "f_*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathcal{D}}(LP, d') \arrow[d, "\cong"] \\
\mathrm{Hom}_{C}(P, Rd) \arrow[r, "(Rf)_*"'] & \mathrm{Hom}_{C}(P, Rd').
\end{tikzcd}$$

By assumption, $Rf : Rd \to Rd'$ is an effective epi in C, and P is projective, hence the bottom (and top) map is a surjection. Thus LP is projective.

2. Under the additional hypotheses, effective epis and projectives determine each other. Indeed, [Qui67, II,4 prop 2] asserts that $f : c \to c'$ is an effective epi iff the map

$$f_* : \mathrm{Hom}(P, c) \to \mathrm{Hom}(P, c')$$

is a surjection for all projective P (the “if” direction is non-trivial here). Now we start with an effective epi $f : d \to d'$ in $\mathcal{D}$ and want to show $Rf : Rd \to Rd'$ is an effective epi in C. Let P be any projective in C and consider:

$$\begin{tikzcd}
\mathrm{Hom}_{C}(P, Rd) \arrow[r, "(Rf)_*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{C}(P, Rd') \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathcal{D}}(LP, d) \arrow[r, "f_*"'] & \mathrm{Hom}_{\mathcal{D}}(LP, d').
\end{tikzcd}$$

By assumption, LP is projective and f is an effective epi, hence the bottom (and top) map is a surjection. Thus, by the criterion given above, Rf is an effective epi. □

Proposition 3.1.3. *Assume C and $\mathcal{D}$ have finite limits and enough projectives, and satisfy extra assumptions so that Quillen's theorem 4 applies (e.g. they are cocomplete and have sets of small projective generators). Assume we have an adjunction as above, and hence an induced adjunction*

$$\begin{tikzcd}
sC \arrow[r, shift left, "L"] & s\mathcal{D} \arrow[l, shift left, "R"]
\end{tikzcd} \tag{3.1.1}$$

between model categories. If L preserves projectives, or equivalently, if R preserves effective epis, then this is a Quillen pair.

Proof. We show a slightly stronger statement: R preserves fibrations and weak equivalences. Recall that a map $f : X_\bullet \to Y_\bullet$ is a fibration (resp. weak eq) if the induced map

$$f_* : \mathrm{Hom}(P, X_\bullet) \to \mathrm{Hom}(P, Y_\bullet)$$

is a fibration (resp. weak eq) of simplicial sets for all projective P. Take P a projective in C and consider:

$$\begin{tikzcd}
\mathrm{Hom}_C(P, RX_\bullet) \arrow[r, "(Rf)_*"] \arrow[d, "\cong"'] & \mathrm{Hom}_C(P, RY_\bullet) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathcal{D}}(LP, X_\bullet) \arrow[r, "f_*"'] & \mathrm{Hom}_{\mathcal{D}}(LP, Y_\bullet).
\end{tikzcd}$$

By assumption, LP is projective in $\mathcal{D}$ and f is a fibration (resp. weak eq) in $s\mathcal{D}$, hence the bottom and top maps are fibrations (resp. weak eq) of simplicial sets. Thus $Rf : RX_\bullet \to RY_\bullet$ is a fibration (resp. weak eq). □

Proposition 3.1.4. *The converse also holds: If the prolonged adjunction (3.1.1) is a Quillen pair, then R preserves effective epis.*

Proof. Take an effective epi $f : X \to Y$ in $\mathcal{D}$ and extend it to an acyclic fibration $f_\bullet : X_\bullet \to Y_\bullet$ as in the lemma below. Since R prolongs to a right Quillen functor, $Rf_\bullet$ is an acyclic fibration in sC, and hence an effective epi in each level (prop E.0.18). In particular, $Rf = Rf_0$ is an effective epi in C. □

Lemma 3.1.5. *Under the same assumptions (such that sC has a standard Quillen model structure), if $f : X \to Y$ is an effective epi in C, then there exist simplicial objects $X_\bullet$ and $Y_\bullet$ in sC with $X_0 = X$ and $Y_0 = Y$, and an acyclic fibration $f_\bullet : X_\bullet \to Y_\bullet$ with $f_0 = f$.*

Proof. Denote by $c : C \to sC$ the embedding of C as constant simplicial objects and view f as a map between constant simplicial objects $c(f) : c(X) \to c(Y)$. Factor it as a cofibration followed by an acyclic fibration:

$$\begin{tikzcd}
& Z \arrow[dr, "p", "\sim"'] & \\
c(X) \arrow[ur, hook, "i"] \arrow[rr, "c(f)"'] & & c(Y)
\end{tikzcd}$$

specifically using Quillen's construction [Qui67, II.4, prop 3]. Recall that the construction consists of attaching cells in successive dimensions to "make the map look more and more like an acyclic fibration":

$$\begin{tikzcd}
c(X) \arrow[r, "j^0"] \arrow[dr, "c(f)"'] & Z^0 \arrow[r, "j^1"] \arrow[d, "p^0"] & Z^1 \arrow[r, "j^2"] \arrow[dl, "p^1"] & \cdots \\
& c(Y) & &
\end{tikzcd}$$

where Z^n is obtained from Z^{n-1} by attaching certain n-cells. Since f is an effective epi, we don't need additional 0-cells, i.e. we can take $P_0 = \varnothing$. Then we have $Z^0 = c(X)$, $j^0 = \mathrm{id}$, $p^0 = c(f)$. Beyond that point, we don't change level 0. Indeed, the cell attachment is a pushout:

$$\begin{tikzcd}
P_{n-1} \otimes \partial\Delta[n] \arrow[r] \arrow[d] & P_{n-1} \otimes \Delta[n] \arrow[d] \\
Z^{n-1} \arrow[r, "j^n"'] & Z^n
\end{tikzcd}$$

and the inclusion of simplicial sets $\partial\Delta[n] \hookrightarrow \Delta[n]$ is the identity (or an iso) in simplicial level $\leq n-1$. In particular, for $n \geq 1$, we are pushing out along the identity in level 0, hence not doing anything; we used the fact that colimits of simplicial objects are computed levelwise. Thus we have $(Z^n)_0 = (Z^{n-1})_0 = X$ and $(p^n)_0 = (p^{n-1})_0 = f$ for all n. Then Z is obtained as $\mathrm{colim}_n Z^n$ and $p : Z \to c(Y)$ is induced by the maps p^n. Again, since colimits in simplicial objects are computed objectwise, we conclude $Z_0 = X$ and $p_0 = f$, and thus $p : Z \to c(Y)$ is a map as we were looking for. □

Remark 3.1.6. We've seen that a prolonged right Quillen functor in 3.1.3 is particularly strong, in that it preserves fibrations and ALL weak equivalences (not just between fibrant objects). However, the prolonged left Quillen functor does not enjoy this additional property in general, i.e. it need not preserve all weak equivalences, only those between cofibrant objects.

Example 3.1.7. Let R be a commutative ring, and consider the adjunction:

$$\begin{tikzcd}
\mathbf{Ab} \arrow[r, shift left, "R\otimes_{\mathbb{Z}}-"] & R\text{-}\mathbf{Mod} \arrow[l, shift left, "U"]
\end{tikzcd}$$

between categories of universal algebras. Clearly $R \otimes_{\mathbb{Z}} -$ preserves projectives (i.e. frees); equivalently, U preserves effective epis (i.e. surjections). Prolonging the adjunction and using the Dold-Kan correspondence $s\mathcal{A} \cong \mathbf{Ch}(\mathcal{A})$ for any abelian category $\mathcal{A}$ (where **Ch** denotes chain complexes bounded below at zero), we get the Quillen pair:

$$\begin{tikzcd}\mathbf{Ch}(\mathbf{Ab}) \arrow[r, shift left, "R\otimes_{\mathbb{Z}}-"] & \mathbf{Ch}(R\text{-}\mathbf{Mod}). \arrow[l, shift left, "U"]\end{tikzcd}$$

On both sides, weak equivalences are homology isos. If the left adjoint preserves all weak equivalences, in particular it preserves acyclic complexes (= exact sequences), i.e. $R \otimes_{\mathbb{Z}} -$ is an exact functor, i.e. R is flat over $\mathbb{Z}$. But there exist commutative rings that are not flat over $\mathbb{Z}$.

For more details on the model category $\mathbf{Ch}(\mathcal{A})$, see [Qui67], remark 5 at the end of II.4; see section 2.3 in [Hov99] for the unbounded version.

3.2 Slice category

Proposition 3.1.3 gives a simple criterion for when a prolonged adjunction is a Quillen pair. In our setup, let us assume the original adjunction $L : C \rightleftarrows \mathcal{D} : R$ prolongs to a Quillen pair, i.e. L preserves projectives or R preserves effective epis. Now we investigate if that is in fact enough. Let us first describe effective epis and projectives in the slice category.

Proposition 3.2.1. *If $f : Y \to Z$ is an effective epi in C, then*

$$\begin{tikzcd}Y \arrow[r, "f"] \arrow[dr] & Z \arrow[d] \\ & X\end{tikzcd}$$

is an effective epi in C/X. The converse also holds if C has coequalizers.

Proof. The map being an effective epi means that

$$\begin{tikzcd}(Y \to X) \times_{(Z\to X)} (Y \to X) \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] & (Y \to X) \arrow[r] & (Z \to X)\end{tikzcd}$$

is a coequalizer diagram. Rewrite the diagram as

$$\begin{tikzcd}Y \times_Z Y \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & Y \arrow[r] \arrow[d] & Z \arrow[dl] \\ & X &\end{tikzcd}$$

and recall that the "source" functor $C/X \to C$ creates colimits, and hence preserves those that exist in C. The colimit of

$$\begin{tikzcd}
Y \times_Z Y \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & Y \arrow[d] \\
& X
\end{tikzcd}$$

in C/X is

$$\begin{tikzcd}
& \mathrm{colim}(Y \times_Z Y \rightrightarrows Y) \arrow[dl] \\
X &
\end{tikzcd}$$

(assuming the latter exists), which is $Z \to X$ iff $f : Y \to Z$ is an effective epi. ☐

In words: A map in C/X is an effective epi iff the map of total spaces is. This is also proved in [Bar02, chap 1, prop 8.12].

Remark 3.2.2. It is NOT a general fact that $C/X \to C$ preserves colimits. For example, take C to be the poset $\{a \to b\} \amalg \{c\}$, which has no initial object. Taking $X = b$, the functor $C/X \to C$ is the inclusion $\{a \to b\} \hookrightarrow \{a \to b\} \amalg \{c\}$ and $\{a \to b\}$ has an initial object, namely a.

Proposition 3.2.3. *1. If P is projective in C, then $P \xrightarrow{p} X$ is projective in C/X.*

2. The converse also holds if C has enough projectives.

Proof. 1. Start with an effective epi

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[dr, "y"'] & Z \arrow[d, "z"] \\
& X
\end{tikzcd}$$

in C/X, which means $f : Y \to Z$ is an effective epi in C (by prop 3.2.1). We want to know if the map

$$\mathrm{Hom}_{C/X}(P \xrightarrow{p} X, Y \xrightarrow{y} X) \to \mathrm{Hom}_{C/X}(P \xrightarrow{p} X, Z \xrightarrow{z} X)$$

is surjective. Let α be a map in the right-hand side which we are trying to reach and consider the diagram:

$$\begin{tikzcd}
& Y \arrow[rr, "f"] \arrow[ddr, "y"] & & Z \arrow[ddl, "z"] \\
P \arrow[ur, dashed, "\widetilde{\alpha}"] \arrow[urrr, "\alpha"] \arrow[drr, "p"'] & & & \\
& & X &
\end{tikzcd}$$

Since P is projective in C, there is a lift $\widetilde{\alpha}$ in the top triangle, meaning $f\widetilde{\alpha} = \alpha$. If $\widetilde{\alpha}$ is in fact a map in $\mathrm{Hom}_{C/X}(P \xrightarrow{p} X, Y \xrightarrow{y} X)$, then it will be our desired lift. So it suffices to check that the triangle on the left commutes:

$$y\widetilde{\alpha} = zf\widetilde{\alpha} = z\alpha = p.$$

2. Let $E \xrightarrow{e} X$ be projective in C/X. Since C has enough projectives, pick an effective epi $P \xrightarrow{\pi} E$ from a projective P. Consider the diagram

$$\begin{tikzcd}
 & P \arrow[r, two heads, "\pi"] \arrow[dd, "e\pi"'] & E \arrow[ddl, "e"] \\
E \arrow[ur, dashed, "s"] \arrow[urr, "\mathrm{id}" near start] \arrow[dr, "e"'] & & \\
 & X &
\end{tikzcd}$$

where there exists a lift s since $E \xrightarrow{e} X$ is projective in C/X. The relation $\pi s = \mathrm{id}_E$ exhibits E as a retract of a projective in C, hence itself projective. □

In words: An object of C/X is projective iff the total space is.

Now that we know more about C/X, we can describe the standard Quillen model structure on $s(C/X) \cong sC/X$. A map

$$\begin{tikzcd}
Y_\bullet \arrow[r, "f"] \arrow[dr, "y"'] & Z_\bullet \arrow[d, "z"] \\
 & X
\end{tikzcd} \tag{3.2.1}$$

is a fibration (resp. weak equivalence) in $s(C/X)$ iff the map

$$\mathrm{Hom}_{C/X}(P \xrightarrow{p} X, Y_\bullet \xrightarrow{y} X) \xrightarrow{f_*} \mathrm{Hom}_{C/X}(P \xrightarrow{p} X, Z_\bullet \xrightarrow{z} X)$$

is a fibration (resp. weak equivalence) of simplicial sets for all projective $P \xrightarrow{p} X$ in C/X. By proposition 3.2.3, we can rephrase that as: for all projective P in C and map $p \in \mathrm{Hom}_C(P, X)$.

However, in the framework of Quillen (co)homology, we decided to work with the “slice” model structure on sC/X, where the map (3.2.1) is a fibration (resp. weak equivalence) iff the map

$$\mathrm{Hom}_C(P, Y_\bullet) \xrightarrow{f_*} \mathrm{Hom}_C(P, Z_\bullet)$$

is a fibration (resp. weak equivalence) of simplicial sets for all projective P in C. In fact, let us check that the two model structures agree.

Proposition 3.2.4. *There is a natural isomorphism of simplicial sets*

$$\coprod_{p\in\mathrm{Hom}_C(P,X)} \mathrm{Hom}_{C/X}(P \xrightarrow{p} X, Y_\bullet \xrightarrow{y} X) \xrightarrow{\cong} \mathrm{Hom}_C(P, Y_\bullet).$$

Proof. Idea : For a fixed $y : Y \to X$, the data of a map $g : P \to Y$ is the same as the data of the commutative diagram:

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[dr, "p"'] & Y \arrow[d, "y"] \\ & X \end{tikzcd}$$

and thus we can partition all maps $g : P \to Y$ according to their composite $p = yg : P \to X$. More precisely, we take the map:

$$\coprod_{p\in\mathrm{Hom}_C(P,X)} \mathrm{Hom}_{C/X}(P \xrightarrow{p} X, Y \xrightarrow{y} X) \to \mathrm{Hom}_C(P, Y)$$

which is readily seen to be surjective and injective, i.e. an iso of sets. Moreover, it is natural in $y : Y \to X$, i.e. we have a natural iso:

$$\begin{tikzcd} (Y \xrightarrow{y} X) \arrow[r, mapsto] & Y \arrow[r, mapsto] & \mathrm{Hom}_C(P, Y) \\ C/X \arrow[r] \arrow[rr, bend right, ""{name=B}] & C \arrow[r] \arrow[d, phantom, "\cong\Uparrow"] & \mathbf{Set}. \\ (Y \xrightarrow{y} X) \arrow[rr, mapsto] & & \coprod_{p\in\mathrm{Hom}_C(P,X)} \mathrm{Hom}_{C/X}(P \xrightarrow{p} X, Y \xrightarrow{y} X) \end{tikzcd}$$

By naturality, it prolongs to a natural iso of simplicial sets. Since colimits of simplicial objects are computed levelwise, we have an equality of simplicial sets:

$$\left\{ \coprod_{p\in\mathrm{Hom}_C(P,X)} \mathrm{Hom}_{C/X}(P \xrightarrow{p} X, Y_n \xrightarrow{y_n} X) \right\}_n = \coprod_{p\in\mathrm{Hom}_C(P,X)} \mathrm{Hom}_{C/X}(P \xrightarrow{p} X, Y_\bullet \xrightarrow{y} X)$$

which concludes the proof. □

Proposition 3.2.5. *The standard model structures on $s(C/X)$ and sC/X are the same.*

Proof. The top row in the diagram

$$\begin{tikzcd}
\mathrm{Hom}_C(P, Y_\bullet) \arrow[rr, "f_*"] \arrow[d, equal] & & \mathrm{Hom}_C(P, Z_\bullet) \arrow[d, equal] \\
\coprod_p \mathrm{Hom}_{C/X}(P \overset{p}{\to} X, Y_\bullet \overset{y}{\to} X) \arrow[rr, "f_*"'] & & \coprod_p \mathrm{Hom}_{C/X}(P \overset{p}{\to} X, Z_\bullet \overset{z}{\to} X)
\end{tikzcd}$$

is a fibration (resp. weak equivalence) of simplicial sets iff each summand is so. This means f is a fibration (resp. weak equivalence) in sC/X iff it is so in $s(C/X)$. Moreover, the model structures are closed i.e. cofibrations are determined by fibrations and weak equivalences (as having the LLP with respect to acyclic fibrations). Therefore the two model structures agree. □

3.3 Abelianization adjunction

In this section, we study the properties of the forgetful functor $U : Ab(C) \to C$. We usually assume it has a left adjoint $Ab : C \to Ab(C)$, called **abelianization**.

Proposition 3.3.1. *U reflects isos.*

Proof. Start with a map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ in $Ab(C)$ and assume $f = U\widetilde{f} : X \to Y$ is an iso in C, where $X = U\widetilde{X}$ and $Y = U\widetilde{Y}$. Then its inverse $f^{-1} : Y \to X$ lifts to a map $\widetilde{Y} \to \widetilde{X}$ (uniquely since U is faithful). This is the standard argument that the set-inverse of a map of groups/rings/vector spaces/etc. automatically respects the structure. For example, let's show it for the addition map. We want to show that the diagram:

$$\begin{tikzcd}
Y \times Y \arrow[r, "f^{-1}\times f^{-1}"] \arrow[d, "\mu_Y"'] & X \times X \arrow[d, "\mu_X"] \\
Y \arrow[r, "f^{-1}"'] & X
\end{tikzcd}$$

commutes, i.e. $f^{-1}\mu_Y = \mu_X(f^{-1} \times f^{-1})$. This holds iff equality holds after applying f, which it does:

$$ff^{-1}\mu_Y = f\mu_X(f^{-1} \times f^{-1})$$
$$\mu_Y = \mu_Y(f \times f)(f^{-1} \times f^{-1}) = \mu_Y.$$

The resulting map $\widetilde{f^{-1}}$ is the inverse of $\widetilde{f}$. □

Proposition 3.3.2. *Assume C has kernel pairs. Then U preserves monos.*

Proof. Let $\widetilde{f} : \widetilde{X} \hookrightarrow \widetilde{Y}$ be a mono in $Ab(C)$. We want to show that $f = U\widetilde{f}$ is a mono in C. Let $\alpha, \beta : W \to X$ be two maps coequalized by f, or equivalently, a map from W to the kernel pair of f:

$$\begin{tikzcd} X \times_Y X \arrow[rd, shift left, "pr_1"] \arrow[rd, shift right, "pr_2"'] & & \\ W \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & X \arrow[r, "f"] & Y. \end{tikzcd}$$

We want to conclude that α and β are equal. It suffices to show that the projections pr_1 and pr_2 are equal. Since U creates limits, the kernel pair diagram lifts (uniquely) to $Ab(C)$:

$$\begin{tikzcd} \widetilde{X} \times_{\widetilde{Y}} \widetilde{X} \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] & \widetilde{X} \arrow[r, hook, "\widetilde{f}"] & \widetilde{Y}. \end{tikzcd}$$

Since $\widetilde{f}$ is a mono, the projections pr_1 and pr_2 are indeed equal. □

Proposition 3.3.3. *Assume C is regular. Then U lifts the regular epi - mono factorization in C uniquely (to something that is a priori not a regular epi - mono factorization, although we will show that it is). In other words, if $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is a map in $Ab(C)$ and $X \twoheadrightarrow Z \hookrightarrow Y$ is a regular epi - mono factorization of the underlying map, then we can lift it uniquely to a factorization $\widetilde{X} \to \widetilde{Z} \to \widetilde{Y}$ in $Ab(C)$.*

Proof. [Bar02, chap 6, prop 1.7]. □

From there, Barr concludes the following proposition; we fill in the details of the proof.

Proposition 3.3.4. *If C is regular, then U preserves regular epis.*

Proof. Start with a regular epi $\widetilde{f} : \widetilde{X} \twoheadrightarrow \widetilde{Y}$ in $Ab(C)$. We want to show that $U\widetilde{f} = f : X \to Y$ is a regular epi in C. A priori it might not be, but since C is regular, we can factor it as a regular epi followed by a mono:

$$\begin{tikzcd} X \arrow[r, two heads, "e"'] \arrow[rr, bend left, "f"] & Z \arrow[r, hook, "m"'] & Y. \end{tikzcd}$$

By proposition 3.3.3, we can lift the factorization to $Ab(C)$:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{e}"'] \arrow[rr, bend left, "\widetilde{f}"] & \widetilde{Z} \arrow[r, "\widetilde{m}"'] & \widetilde{Y}. \end{tikzcd}$$

Since U is faithful, it reflects epis and monos, therefore $\widetilde{e}$ is an epi and $\widetilde{m}$ is a mono. Since $\widetilde{f} = \widetilde{me}$ is a regular epi and $\widetilde{e}$ is an epi, $\widetilde{m}$ must be a regular epi (prop D.0.13). Now $\widetilde{m}$ is a regular epi and a mono, hence an iso (prop D.0.12). Therefore $m = U\widetilde{m}$ is an iso and f is a regular epi. ◻

Now we'd like to know if U reflects regular epis.

Proposition 3.3.5. *If C is regular, then $Ab(C)$ has coequalizers of kernel pairs, created by U.*

Proof. Let $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ be any map in $Ab(C)$ and take its kernel pair:

$$\begin{tikzcd}
\widetilde{X} \times_{\widetilde{Y}} \widetilde{X} \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] & \widetilde{X} \arrow[r, "\widetilde{f}"] & \widetilde{Y}.
\end{tikzcd}$$

Since U preserves limits, the underlying diagram is still a kernel pair, and we can take its coequalizer:

$$\begin{tikzcd}
X \times_Y X \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] & X \arrow[r, "f"] \arrow[dr, two heads] & Y. \\
 & & C \arrow[u, "h"']
\end{tikzcd}$$

Since C is regular, the map $h : C \to Y$ is a mono [Bar02, chap 1, prop 8.10]. By 3.3.3, there is a unique lift $\widetilde{X} \to \widetilde{C} \to \widetilde{Y}$ of that regular epi - mono factorization. We claim that $\widetilde{C}$ is a coequalizer in $Ab(C)$. Indeed, if there is another map $\widetilde{X} \to \widetilde{W}$ coequalizing the projections:

$$\begin{tikzcd}
\widetilde{X} \times_{\widetilde{Y}} \widetilde{X} \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] & \widetilde{X} \arrow[r, "\widetilde{f}"'] \arrow[dr] \arrow[ddr] & \widetilde{Y}. \\
 & & \widetilde{C} \arrow[u] \arrow[d, dashed, "\gamma"] \\
 & & \widetilde{W}
\end{tikzcd}$$

then there is a unique underlying map $\gamma : C \to W$ since C is a coequalizer in C. It remains to check that γ respects the structure maps. Let us check it for the addition map; the argument is the same for other structure maps. In the diagram:

$$\begin{tikzcd}
X \times X \arrow[r, two heads] \arrow[d, "\mu_X"'] & C \times C \arrow[r, "\gamma\times\gamma"] \arrow[d, "\mu_Z"'] & W \times W \arrow[d, "\mu_W"'] \\
X \arrow[r, two heads] & C \arrow[r, "\gamma"'] & W
\end{tikzcd}$$

the left square and the outer rectangle commute, by assumption. We want to show the right square commutes. It does, since the top left map is an epi. □

Corollary 3.3.6. *The lifted factorization of 3.3.3 is a regular epi - mono factorization in $Ab(C)$.*

Proposition 3.3.7. *If C is regular, then U reflects regular epis.*

Proof. Let $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ be a map in $Ab(C)$ such that $f = U\widetilde{f}$ is a regular epi in C. We want to show that $\widetilde{f}$ is a regular epi.

Proof 1: Since U creates limits, the kernel pair of $\widetilde{f}$:

$$\widetilde{X} \times_{\widetilde{Y}} \widetilde{X} \rightrightarrows \widetilde{X}$$

is the unique lift of the kernel pair of f:

$$X \times_Y X \rightrightarrows X$$

and the latter has a coequalizer, namely $X \xrightarrow{f} Y$. Since U creates coequalizers of kernel pairs, there is a unique cocone lifting $X \xrightarrow{f} Y$ and it is a coequalizer of $\widetilde{X} \times_{\widetilde{Y}} \widetilde{X} \rightrightarrows \widetilde{X}$. But $\widetilde{X} \xrightarrow{\widetilde{f}} \widetilde{Y}$ is such a lift, so $\widetilde{f}$ is a regular epi.

Proof 2: Factor $\widetilde{f}$ as a regular epi followed by a mono, $\widetilde{f} = \widetilde{m}\widetilde{e}$. By 3.3.5, we can assume the underlying factorization $f = me$ is also a regular epi - mono factorization. (Alternatively, use 3.3.2 to ensure m is a mono.) Since $f = me$ is a regular epi, so is m, and hence m is an iso. Since U reflects isos, $\widetilde{m}$ is also an iso and therefore $\widetilde{f}$ is a regular epi. □

Corollary 3.3.8. *If C is regular, then $Ab(C)$ is regular (in the weaker sense).*

Proof. $Ab(C)$ has kernel pairs (or any limits that C has) and coequalizers of kernel pairs. It remains to check that the pullback of a regular epi is a regular epi:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "f^*e"'] & X \arrow[d, two heads, "e"] \\
W \arrow[r, "f"'] & Y.
\end{tikzcd}$$

Since U preserves regular epis, Ue is a regular epi. Since pullbacks are computed in C (U creates limits), we have $U(f^*e) = (Uf)^*(Ue)$, which is a regular epi since C is regular. Since U reflects regular epis, f^*e itself is a regular epi in $Ab(C)$. □

3.4 Algebraic categories

Definition 3.4.1. A category C is called **algebraic** if it has finite limits, small colimits, and a set of small projective generators (in particular enough projectives).

The interest in such categories is Quillen's theorem that if C is algebraic, then sC has a standard simplicial model structure [Qui67, II.4 thm 4]. Objects of C may not have underlying sets, but morally, the data of $\mathrm{Hom}(P, -)$ from all projectives (or from the generators) plays the role of underlying set.

Proposition 3.4.2. *Let C be an algebraic category with generator set S and let $f : X \to Y$ be a map in C.*

1. *f is a mono iff $f_* : \mathrm{Hom}(P, X) \to \mathrm{Hom}(P, Y)$ is a injective (i.e. a mono in* **Set***) for all $P \in S$.*
2. *f is a regular epi iff $f_* : \mathrm{Hom}(P, X) \to \mathrm{Hom}(P, Y)$ is surjective (i.e. a regular epi in* **Set***) for all $P \in S$.*

In particular, the family of functors $\mathrm{Hom}(P, -)$ (for all $P \in S$) collectively reflects isos, since isos are exactly maps that are monos and regular epis. In the terminology of [Bor94a, def 4.5.13], S is a strong family of generators.

Proof. (1) By definition, f is a mono iff $f_* : \mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y)$ is injective for all object A of C. Let $\pi : \amalg P_i \twoheadrightarrow A$ be a (regular) epi from a coproduct of generators $P_i \in S$. Consider the diagram:

$$\begin{tikzcd}
\mathrm{Hom}(A, X) \arrow[r, "f_*"] \arrow[d, "\pi^*"'] & \mathrm{Hom}(A, Y) \arrow[d, "\pi^*"] \\
\mathrm{Hom}(\amalg P_i, X) \arrow[r, "f_*"'] \arrow[d, equal] & \mathrm{Hom}(\amalg P_i, Y) \arrow[d, equal] \\
\prod \mathrm{Hom}(P_i, X) \arrow[r, hook, "\prod f_*"'] & \prod \mathrm{Hom}(P_i, Y)
\end{tikzcd}$$

where the maps π^* are injective since π is an epi. The bottom map is a mono since it is a product of monos. Hence $f_*\pi^* = \pi^* f_*$ is a mono, and so is the top f_*, since it is the first map of a composite which is a mono.

(2) We know f is a regular epi iff $f_* : \mathrm{Hom}(P, X) \to \mathrm{Hom}(P, Y)$ is surjective for all projective P of C. The class of objects having the lifting property with respect to effective epis is closed under

coproducts and retracts. Thus is contains all projectives, since it contains the generators, and any projective is a retract of a coproduct of generators.

More explicitly, let $g : P \to Y$ be a map we're trying to lift along f. Let $\pi : \amalg P_i \twoheadrightarrow P$ be a regular epi from a coproduct of generators $P_i \in S$, and let $s : P \to \amalg P_i$ be a section of π (since P is projective and π is a regular epi).

$$\begin{tikzcd}
 & P \arrow[dl, "s"'] \arrow[d, equal] & \\
\amalg P_i \arrow[r, two heads, "\pi"] \arrow[dr, "\widetilde{g}"'] & P \arrow[d, dashed] \arrow[dr, "g"] & \\
 & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

By assumption, $\pi^* g = g\pi$ admits an f-lift $\widetilde{g}$. Now we check that $s^*\widetilde{g} = \widetilde{g}s$ is the desired f-lift of g:

$$\begin{aligned}
\pi^* f_*(s^*\widetilde{g}) &= \pi^* s^* f_* \widetilde{g} \\
&= \pi^* s^* \pi^* g \\
&= \pi^* (\pi s)^* g \\
&= \pi^* g
\end{aligned}$$

which implies $f_*(s^*\widetilde{g}) = g$ since π^* is injective. □

Proposition 3.4.3. *In an algebraic category C, filtered colimits commute with finite limits.*

Proof. Let L be a filtered category, N a finite category, and $F : L \times N \to C$ a functor. There is a natural comparison map

$$\varphi : \operatorname*{colim}_L \lim_N F \to \lim_N \operatorname*{colim}_L F$$

which is given on each factor by

$$\operatorname*{colim}_L \lim_N F \to \operatorname*{colim}_L F(-, n)$$

i.e. the map obtained by applying colim_L to the L-diagram given by projection $\lim_N F \to F(-, n)$. We want to show the comparison map φ is an iso. By 3.4.2, it suffices to show $\operatorname{Hom}(P, \varphi)$ is an iso (of sets) for all generator P. From the definition of limit and the smallness of our generators, we obtain:

$$\begin{tikzcd}
\mathrm{Hom}(P, \mathrm{colim}_L \lim_N F) \arrow[r, "\varphi_*"] \arrow[d, equal] & \mathrm{Hom}(P, \lim_N \mathrm{colim}_L F) \arrow[d, equal] \\
\mathrm{colim}_L \mathrm{Hom}(P, \lim_N F) \arrow[d, equal] & \lim_N \mathrm{Hom}(P, \mathrm{colim}_L F) \arrow[d, equal] \\
\mathrm{colim}_L \lim_N \mathrm{Hom}(P, F) \arrow[r, "\cong"] & \lim_N \mathrm{colim}_L \mathrm{Hom}(P, F)
\end{tikzcd}$$

The bottom map (and hence φ_*) is an iso, since filtered limits commute with finite limits in **Set**. □

Proposition 3.4.4. *Let C be an algebraic category. Then $U : Ab(C) \to C$ creates filtered colimits. In particular, $Ab(C)$ has filtered colimits and U preserves them.*

Proof. Essentially the same as B.3.1. Let L be a filtered category and $\widetilde{F} : L \to Ab(C)$ a diagram whose underlying diagram $F = U\widetilde{F} : L \to C$ admits a colimit. Then there is a unique lift of the colimiting cocone in C to a cocone in $Ab(C)$. Indeed, there is at most one way to endow $\mathrm{colim}_L F$ with structure maps, since they are prescribed on each summand:

$$\begin{tikzcd}
\mathrm{colim}_L F \times \mathrm{colim}_L F \cong \mathrm{colim}_L(F \times F) \arrow[rr] & & \mathrm{colim}_L F \\
F(l) \times F(l) \arrow[u] \arrow[rr, "\mu"'] & & F(l). \arrow[u]
\end{tikzcd}$$

Applying colim_L to the structure maps of F produces those (unique) structure maps for $\mathrm{colim}_L F$. The result is the colimit of $\widetilde{F}$ in $Ab(C)$. Indeed, let $\widetilde{F} \to \Delta\widetilde{Z}$ be a cocone on $\widetilde{F}$ and $\mathrm{colim}_L F \to Z$ the corresponding unique underlying map in C. The latter map has to commute with the structure maps, since it does for each summand of the colimit. □

Proposition 3.4.5. *Let C be an algebraic category and X an object of C. Then the slice category C/X is algebraic.*

Proof. 1) C/X has finite limits, since they are computed in C (by viewing diagrams in C/X as diagrams in C).

2) C/X has small colimits, since they are created by the "source" functor $C/X \to C$.

3) Let S be a set of small projective generators for C. Then

$$\left\{ P \xrightarrow{p} X \mid P \in S, p \in \mathrm{Hom}_C(P, X) \right\}$$

is a set of small projective generators for C/X. It's a set since it's a union of sets $\mathrm{Hom}_C(P, X)$ indexed over a set S. Each object $P \to X$ is projective since the total space P is projective in C. Moreover, they form a family of generators. Indeed, let $Y \to X$ be any object of C/X, and let $\amalg P_i \twoheadrightarrow Y$ be a regular epi in C from a coproduct of generators $P_i \in S$. Then

$$\begin{tikzcd}
\amalg P_i \arrow[r, two heads] \arrow[dr] & Y \arrow[d] \\
& X
\end{tikzcd}$$

is a regular epi in C/X and $(\amalg P_i) \to X$ is the coproduct $\amalg(P_i \to X)$ in C/X. (By the same argument, if C has enough projectives, then so does C/X.)

Lastly, each $P \xrightarrow{p} X$ is small. Let I be a filtered category and $Y : I \to C/X$ an I-diagram in C/X. We have:

$$\begin{aligned}
\mathrm{Hom}\left(P \xrightarrow{p} X, \operatorname*{colim}_I (Y_i \xrightarrow{y_i} X)\right) &= \mathrm{Hom}\left(P \xrightarrow{p} X, (\operatorname*{colim}_I Y_i) \xrightarrow{y} X\right) \\
&= \left\{\varphi \in \mathrm{Hom}_C(P, \operatorname*{colim}_I Y_i) \mid y\varphi = p\right\} \\
&= \left\{\varphi \in \operatorname*{colim}_I \mathrm{Hom}_C(P, Y_i) \mid y_i\varphi_i = p \text{ for all representatives } \varphi_i : P \to Y_i\right\} \\
&= \operatorname*{colim}_I \mathrm{Hom}\left(P \xrightarrow{p} X, Y_i \xrightarrow{y_i} X\right).
\end{aligned}$$

The next to last equality comes from the fact that all the Y_i map into $\mathrm{colim}_I\, Y_i$

$$\begin{tikzcd}
P \arrow[d, "\varphi_i"] \arrow[dr, "\varphi"] \arrow[dd, bend right, "p"'] & \\
Y_i \arrow[r] \arrow[d, "y_i"] & \mathrm{colim}_I\, Y_i \arrow[dl, "y"] \\
X &
\end{tikzcd}$$

and we have the equality $p = y\varphi = y_i\varphi_i$. □

Our next goal is to show that C algebraic implies $Ab(C)$ algebraic, or rather find additional conditions on C to guarantee such a result. Cocompleteness is the most problematic issue here, so we deal with it first.

Proposition 3.4.6. *Let C be a category with finite limits and assume the forgetful functor $U : Ab(C) \to C$ has a left adjoint $Ab : C \to Ab(C)$. Then U is strictly monadic, i.e. the comparison* *functor $Ab(C) \to C^T$ is an iso of categories, where C^T denotes the category of algebras over the monad $T = UAb$.*

Proof. Same as [Mac98, VI.8 thm 1]; the proof never uses anything special about **Set**, except the availability of finite products. Let us write it here for completeness' sake.

By Beck's monadicity theorem [Mac98, VI.7 thm 1] [Bor94b, thm 4.4.4], the assertion is equivalent to saying U creates coequalizers of U-split pairs. Let $\widetilde{f}, \widetilde{g} : \widetilde{X} \rightrightarrows \widetilde{Y}$ be a U-split pair in $Ab(C)$ and let

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[l, bend right=60, "t"'] \arrow[r, "e"] & C \arrow[l, bend right=60, "s"']
\end{tikzcd}$$

be a split fork in C, i.e. $es = \mathrm{id}$, $ft = \mathrm{id}$, $gt = se$. In particular, e is a coequalizer. We want to endow C with structure maps to make it a coequalizer in $Ab(C)$. Let us do it explicitly for the addition map; the argument is the same for any operation, including of arity 0.

$$\begin{tikzcd}
X \times X \arrow[r, shift left, "f\times f"] \arrow[r, shift right, "g\times g"'] \arrow[d, "\mu_X"'] & Y \times Y \arrow[r, "e\times e"] \arrow[d, "\mu_Y"'] & C \times C \arrow[d, dashed, "\mu_C"'] \\
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "e"] & C
\end{tikzcd}$$

The top fork is still split, and in particular, $e \times e$ is still a coequalizer. For μ_C, we propose the formula $\mu_C := e\mu_Y(s \times s)$ and check that e now commutes with the structure maps:

$$\begin{aligned}
\mu_C(e \times e) &= e\mu_Y(s \times s)(e \times e) \\
&= e\mu_Y(se \times se) \\
&= e\mu_Y(gt \times gt) \\
&= e\mu_Y(g \times g)(t \times t) \\
&= eg\mu_X(t \times t) \\
&= ef\mu_X(t \times t) \\
&= e\mu_Y(f \times f)(t \times t) \\
&= e\mu_Y(ft \times ft) \\
&= e\mu_Y.
\end{aligned}$$

Moreover, since $e \times e$ (or e^n for any $n \geq 0$) is an epi, there is at most one way to endow C with structure maps. A similar argument shows that the structure maps defined above satisfy the identities they should satisfy (associativity and so on), so we get an object $\widetilde{C}$ in $Ab(C)$.

We've shown there is a unique U-lift of the cocone to C. It remains to show the lift $\widetilde{e} : \widetilde{Y} \to \widetilde{C}$ is a coequalizer in $Ab(C)$. If a map $\widetilde{Y} \to \widetilde{W}$ coequalizes $\widetilde{f}$ and $\widetilde{g}$, it induces a unique underlying map $\gamma : C \to W$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "e"] \arrow[dr] & C \arrow[d, "\gamma"] \\
& & W
\end{tikzcd}$$

and γ must commute with all structure maps, by the argument used in 3.3.5. □

Remark 3.4.7. The comparison functor $K : Ab(C) \to C^T$ sends an abelian group object $\widetilde{X}$ to the T-algebra $(U\widetilde{X}, U\epsilon_{\widetilde{X}} : UAbU\widetilde{X} \to U\widetilde{X})$. The inverse equivalence is the functor $S : C^T \to Ab(C)$ which to a T-algebra $(Z, TZ \overset{\xi_Z}{\to} Z)$ associates Z equipped with an addition map defined by:

$$Z \times Z \xrightarrow{\eta_Z \times \eta_Z} UAbZ \times UAbZ \xrightarrow{\mu_{AbZ}} UAbZ \xrightarrow{\xi_Z} Z$$

and likewise for the other structure maps. One readily checks $SK\widetilde{X} = \widetilde{X}$. The real question is to determine if KSZ is isomorphic to Z.

Proposition 3.4.8. *Assume C is algebraic and $U : Ab(C) \to C$ has a left adjoint $Ab : C \to Ab(C)$. If moreover C is complete, then $Ab(C)$ is cocomplete (and of course complete).*

Proof. By 3.4.6, $Ab(C)$ is isomorphic – "equivalent" would suffice – to the category C^T of algebras over the monad $T = UAb$. By 3.4.4, T preserves filtered colimits. Filtered colimits are just $\aleph_0$-filtered colimits, and $\aleph_0$ is a regular cardinal. Thus [Bor94b, prop 4.3.6] applies and C^T is cocomplete. □

To foster diversity of viewpoints, here is an alternate proof.

Proof. Since C is cocomplete, it suffices to show that C^T has coequalizers, by [Bor94b, prop 4.3.4]. Now the monad T preserves filtered colimits, in particular colimits along countable chains. Since C is complete, [MB05, chap 9, thm 3.9] applies and C^T has coequalizers. □

Here is yet another proof. In the case at hand, an easy argument allows us to bypass the more powerful but more complicated [Bor94b, prop 4.3.4].

Proof. $Ab(C)$ has an initial object, namely the terminal object $*$ of C with identity structure maps. Now we show that finite coproducts can be obtained from coequalizers. Let X, Y be objects in $Ab(C)$. Their coproduct is the colimit of the diagram:

$$\begin{tikzcd}
AbUX \arrow[d, two heads, "\epsilon_X"'] \arrow[dr] & & AbUY \arrow[d, two heads, "\epsilon_Y"] \arrow[dl] \\
X & AbUX \amalg AbUY & Y.
\end{tikzcd}$$

Indeed, a map from this diagram into an object Z is the data of maps $f : X \to Z$, $g : Y \to Z$, and $\varphi : AbUX \amalg AbUY \to Z$ satisfying $\varphi = (f\epsilon_X, g\epsilon_Y)$, which is exactly the data of the pair of maps f and g. Now the colimit of the diagram can be obtained by two successive pushouts, both along a regular epi, namely the counit maps ϵ_X and ϵ_Y (3.4.9). By D.0.15, we conclude that $X \amalg Y$ can be built using coequalizers.

Lastly, an arbitrary (non-empty) coproduct $\amalg_i X_i$ is the filtered colimit of its finite subcoproducts. Since $U : Ab(C) \to C$ creates filtered colimits and C is cocomplete, $Ab(C)$ has arbitrary coproducts, assuming it has coequalizers. □

Lemma 3.4.9. *If a map $f : X \twoheadrightarrow U\widetilde{Y}$ is a regular epi in C, then its adjunct map $f^\sharp : AbX \to \widetilde{Y}$ is a regular epi in $Ab(C)$. In particular, the counit $AbU\widetilde{X} \twoheadrightarrow \widetilde{X}$ is always a regular epi.*

Proof. Recall that $AbX \to \widetilde{Y}$ is a regular epi in $Ab(C)$ iff $UAbX \to U\widetilde{Y}$ is a regular epi in C. Now we have:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[rr, bend right, two heads, "f"'] & UAbX \arrow[r] & U\widetilde{Y}
\end{tikzcd}$$

where the composite is a regular epi. By D.0.13, $UAbX \to U\widetilde{Y}$ is a regular epi since C is regular. □

Remark 3.4.10. The converse is false in general. For example, take $C = \mathbf{Set}$, $X = \{*\}$, $Y = \mathbb{Z}$, and $f(*) = 1$. The map f is far from being a regular epi (i.e. surjection), but its adjunct $f^\sharp : Ab(*) = \mathbb{Z} \xrightarrow{\simeq} \mathbb{Z}$ is a regular epi, even an iso.

Lemma 3.4.11. *An object $\widetilde{X}$ of $Ab(C)$ is projective iff it is a retract of $Ab(P)$ for some projective P of C.*

Proof. (⇐) Let's try to lift a map $Ab(P) \to \widetilde{Y}$ along a regular epi $\widetilde{X} \twoheadrightarrow \widetilde{Y}$:

$$\begin{tikzcd}
 & Ab(P) \arrow[dl, dashed] \arrow[d] \\
\widetilde{X} \arrow[r, two heads] & \widetilde{Y}.
\end{tikzcd}$$

Look at the adjoint diagram

$$\begin{tikzcd}
 & P \arrow[dl, "\exists"'] \arrow[d] \\
U\widetilde{X} \arrow[r, two heads] & U\widetilde{Y}.
\end{tikzcd}$$

where the bottom map is a regular epi since U preserves them, and thus the lift exists. This proves $Ab(P)$ is projective, and we know a retract of a projective is projective.

(⇒) Let $\widetilde{X}$ be a projective in $Ab(C)$. Since C has enough projectives, there is a projective P of C with a regular epi $P \twoheadrightarrow U\widetilde{X}$. Take its adjunct map $AbP \twoheadrightarrow \widetilde{X}$, which is still a regular epi by 3.4.9. Now lift the identity of $\widetilde{X}$ along that regular epi:

$$\begin{tikzcd}
 & \widetilde{X} \arrow[dl, "\exists"'] \arrow[d, "\mathrm{id}"] \\
AbP \arrow[r, two heads] & \widetilde{X}.
\end{tikzcd}$$

and conclude $\widetilde{X}$ is a retract of AbP. □

Remark 3.4.12. In the proof of (⇒), we could not have used the counit $AbU\widetilde{X} \twoheadrightarrow \widetilde{X}$, since U of a projective $\widetilde{X}$ in $Ab(C)$ is in general not projective in C, not even if $\widetilde{X}$ is of the form AbP for some projective P of C. For example, take $C = \mathbf{Gp}$ and $Ab(C) = \mathbf{Ab}$, in which projective objects are free groups and free abelian groups, respectively. A free abelian group is far from being a free group.

Proposition 3.4.13. *Assume C is algebraic and complete, and $U : Ab(C) \to C$ has a left adjoint $Ab : C \to Ab(C)$. Then $Ab(C)$ is also algebraic.*

Proof. 1) $Ab(C)$ has finite limits since $U : Ab(C) \to C$ creates limits.

2) $Ab(C)$ has small colimits by 3.4.8.

3) Let S be a set of small projective generators for C. Then

$$\{Ab(P) \mid P \in S\}$$

is a set of small projective generators for $Ab(C)$. It's clearly a set, and each $Ab(P)$ is projective. Moreover, they form a family of generators. For any object $\widetilde{X}$ of $Ab(C)$, take an effective epi $\amalg P_i \twoheadrightarrow U\widetilde{X}$ from a coproduct of generators in S. Then the adjunct map

$$\amalg Ab(P_i) = Ab\left(\amalg P_i\right) \twoheadrightarrow \widetilde{X}$$

is a regular epi. (By the same argument, if C has enough projectives, then so does $Ab(C)$.)

Lastly, each $Ab(P)$ is small. Let I be a filtered category and $\widetilde{X}_i$ be an I-diagram in $Ab(C)$. We have:

$$\begin{aligned}
&\operatorname{Hom}_{Ab(C)}\left(Ab(P), \operatorname*{colim}_I \widetilde{X}_i\right)\\
&\quad= \operatorname{Hom}_C\left(P, U \operatorname*{colim}_I \widetilde{X}_i\right)\\
&\quad= \operatorname{Hom}_C\left(P, \operatorname*{colim}_I U\widetilde{X}_i\right) \quad \text{by 3.4.4}\\
&\quad= \operatorname*{colim}_I \operatorname{Hom}_C\left(P, U\widetilde{X}_i\right)\\
&\quad= \operatorname*{colim}_I \operatorname{Hom}_{Ab(C)}\left(Ab(P), \widetilde{X}_i\right).
\end{aligned}$$

□

Putting all the ingredients together, we obtain a good setup for Quillen cohomology. It is essentially an observation of Quillen [Qui67, II.5 (4) before thm 5], which we state and prove in more detail.

Proposition 3.4.14. *Let C be a complete algebraic category with all abelianizations, and let X be an object of C. Then C/X and $Ab(C/X)$ are algebraic and the prolonged adjunction*

$$\begin{tikzcd} sC/X \arrow[r, shift left, "Ab_X"] & s\mathbf{Ab}(C/X) \arrow[l, shift left, "U_X"] \end{tikzcd}$$

is a Quillen pair.

Proof. Both C/X and $Ab(C/X)$ are algebraic, by 3.4.5 and 3.4.13. Moreover, C is regular [Qui67, II.4, cor of prop 2], and therefore C/X is also regular [Bar02, chap 1, prop 8.12]. By proposition 3.3.4, the right adjoint $U_X : Ab(C/X) \to C/X$ preserves regular epis, hence the prolonged adjunction is a Quillen pair, by 3.1.3. □

A category of universal algebras, i.e. monadic over (possibly graded) sets, satisfies the assumptions of 3.4.14. However, the proposed setup avoids requiring underlying sets, in the spirit of

Quillen's original work [Qui67, II.4 thm 4].

Remark 3.4.15. The setup above is not quite enough to do Quillen cohomology. There are additional assumptions on the homotopy category Ho $Ab(sC/X_\bullet)$: conditions (A) and (B) at the beginning of [Qui67, II.5].

Chapter 4

Behavior with respect to adjunctions

In this chapter, we study the effect of an adjunction on Quillen cohomology.

4.1 Effect on Beck modules

Under mild assumptions, an adjunction passes to the categories of abelian group objects.

Proposition 4.1.1. *Assume $L : \mathcal{C} \to \mathcal{D}$ preserves finite products and has a right adjoint $R : \mathcal{D} \to \mathcal{C}$. Then the induced functors $L : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ and $R : Ab(\mathcal{D}) \to Ab(\mathcal{C})$ still form an adjoint pair.*

Proof. $\mathrm{Hom}_{Ab(\mathcal{C})}(\widetilde{c}, \widetilde{Rd})$ is the subset of $\mathrm{Hom}_{\mathcal{C}}(c, Rd) \cong \mathrm{Hom}_{\mathcal{D}}(Lc, d)$ consisting of maps $c \to Rd$ which commute with the structure maps. So we need to show that this holds iff the adjoint map $Lc \to d$ commutes with structure maps. This is true by the naturality of the adjunction, and the fact that $\widetilde{Lc}$ and $\widetilde{Rd}$ have structure maps induced by those of $\widetilde{c}$ and $\widetilde{d}$, respectively. For example, the diagram of multiplication maps:

$$\begin{tikzcd}
c \times c \arrow[r, "\mu_c"] \arrow[d] & c \arrow[d] \\
R(d \times d) \cong Rd \times Rd \arrow[r, "R\mu_d"'] & Rd
\end{tikzcd}$$

commutes iff the adjoint diagram:

$$\begin{tikzcd}
L(c \times c) \cong Lc \times Lc \arrow[r, "L\mu_c"] \arrow[d] & Lc \arrow[d] \\
d \times d \arrow[r, "\mu_d"'] & d
\end{tikzcd}$$

commutes. ☐

The point of that proposition was to identify the induced left adjoint. The right adjoint $R : \mathcal{D} \to C$ always passes to abelian group objects $R : Ab(\mathcal{D}) \to Ab(C)$, since it preserves limits. A priori, we don't know what its left adjoint $\widetilde{L} : Ab(C) \to Ab(\mathcal{D})$ will look like.

Now we'll see how an adjunction passes to slice categories. There are two versions, depending if one starts with a ground object in C or in $\mathcal{D}$. Choose a ground object c in C and consider:

$$C/c \xrightarrow{L} \mathcal{D}/Lc \xrightarrow{R} C/RLc \xrightarrow{\eta_c^*} C/c$$

where $\eta_c : c \to RLc$ is the unit map.

Proposition 4.1.2. *We get an induced adjoint pair:*

$$\begin{tikzcd} C/c \arrow[r, shift left, "L"] & \mathcal{D}/Lc \arrow[l, shift left, "\eta_c^* R"] \end{tikzcd} \tag{4.1.1}$$

Proof.

$$\begin{aligned} &\mathrm{Hom}_{C/c}(c' \to c, \eta_c^* R(d' \to Lc)) \\ &= \mathrm{Hom}_{C/c}(c' \to c, \eta_c^*(Rd' \to RLc)) \\ &= \mathrm{Hom}_{C/RLc}(\eta_{c!}(c' \to c), Rd' \to RLc) \end{aligned}$$

These consist of maps $c' \to Rd'$ in C making the following diagram commute:

$$\begin{tikzcd} c' \arrow[r] \arrow[d] & Rd' \arrow[d] \\ c \arrow[r, "\eta_c"'] & RLc \end{tikzcd}$$

which is equivalent to the commutativity of the adjoint diagram:

$$\begin{tikzcd} Lc' \arrow[r] \arrow[d] & d' \arrow[d] \\ Lc \arrow[r, "\mathrm{id}"'] & Lc. \end{tikzcd}$$

But the data of a map $Lc' \to d'$ in $\mathcal{D}$ making this diagram commute is precisely a map in

$$\mathrm{Hom}_{\mathcal{D}/Lc}\left(L(c' \to c), d' \to Lc\right). \qquad \square$$

For the second way, choose a ground object d in $\mathcal{D}$ and consider:

$$\mathcal{D}/d \overset{R}{\to} C/Rd \overset{L}{\to} \mathcal{D}/LRd \overset{\epsilon_{d!}}{\to} \mathcal{D}/d$$

where $\epsilon_d : LRd \to d$ is the counit map.

Proposition 4.1.3. *We get an induced adjoint pair:*

$$\begin{tikzcd} C/Rd \arrow[r, shift left, "\epsilon_{d!}L"] & \mathcal{D}/d \arrow[l, shift left, "R"] \end{tikzcd} \tag{4.1.2}$$

Proof. Essentially the same. A map in $\mathrm{Hom}_{\mathcal{D}/d}\left(\epsilon_{d!}L(c' \to Rd), d' \to d\right)$ is a map $Lc' \to d'$ making the following diagram commute:

$$\begin{tikzcd} Lc' \arrow[r] \arrow[d] & d' \arrow[d] \\ LRd \arrow[r, "\epsilon_d"'] & d \end{tikzcd}$$

which is equivalent to the commutativity of the adjoint diagram:

$$\begin{tikzcd} c' \arrow[r] \arrow[d] & Rd' \arrow[d] \\ Rd \arrow[r, "\mathrm{id}"'] & Rd. \end{tikzcd}$$

The top map $c' \to Rd'$ in such a commutative diagram is precisely a map in

$$\mathrm{Hom}_{C/Rd}\left(c' \to Rd, R(d' \to d)\right). \qquad \square$$

These adjunctions pass to categories of abelian group objects if the left adjoint preserves finite products. In the case 4.1.2, this holds iff $L : C \to \mathcal{D}$ preserves pullbacks over c (proposition 1.1.1). In 4.1.3, the left adjoint $\epsilon_{d!}L$ preserves finite products (in particular the terminal object) iff $L : C \to \mathcal{D}$ preserves pullbacks over Rd and $\epsilon_d : LRd \to d$ is an iso, in which case $\epsilon_{d!}$ is an iso of categories. Of course, the latter condition is unreasonably strong and happens rarely. In general, we'll need to use the pushforward ϵ_{d*}.

Proposition 4.1.4. *If $L : \mathcal{C} \to \mathcal{D}$ preserves kernel pairs of split epis over Rd, we get an induced adjoint pair:*

$$Ab(\mathcal{C}/Rd) \underset{R}{\overset{\epsilon_{d*}L}{\rightleftarrows}} Ab(\mathcal{D}/d).$$

Proof.

$$\begin{aligned}&\mathrm{Hom}_{Ab(\mathcal{D}/d)}\left(\epsilon_{d*}L(c' \to Rd), d' \to d\right)\\ &= \mathrm{Hom}_{Ab(\mathcal{D}/LRd)}\Big(Lc' \to LRd, \epsilon_d^*(d' \to d)\Big)\end{aligned}$$

These consist of maps $Lc' \to d'$ that make the diagram

$$\begin{tikzcd}
Lc' \arrow[r] \arrow[d] & d' \arrow[d] \\
LRd \arrow[r, "\epsilon_d"'] & d
\end{tikzcd}$$

commute AND respect the structure maps of the columns. This is equivalent to maps $c' \to Rd'$ that make the adjoint diagram

$$\begin{tikzcd}
c' \arrow[r] \arrow[d] & Rd' \arrow[d] \\
Rd \arrow[r, "\mathrm{id}"'] & Rd.
\end{tikzcd}$$

commute AND respect the structure maps (by definition of the structure maps of $Lc' \to LRd$ and $Rd' \to Rd$). These are precisely maps in

$$\mathrm{Hom}_{Ab(\mathcal{C}/Rd)}\left(c' \to Rd, R(d' \to d)\right). \qquad \square$$

Corollary 4.1.5. *Assume $L : \mathcal{C} \to \mathcal{D}$ preserves kernel pairs of split epis.*

1. *For any object c in $\mathcal{C}$, there is an induced adjunction on Beck modules:*

$$Ab(\mathcal{C}/c) \underset{\eta_c^*R}{\overset{L}{\rightleftarrows}} Ab(\mathcal{D}/Lc).$$

2. *For any object d in $\mathcal{D}$, there is an induced adjunction on Beck modules:*

$$Ab(\mathcal{C}/Rd) \underset{R}{\overset{\epsilon_{d*}L}{\rightleftarrows}} Ab(\mathcal{D}/d).$$

Remark 4.1.6. We don't really care whether or not the left adjoint L passes to Beck modules. The right adjoint certainly does, and we can work with the left adjoint of the induced functor $\eta_c^* R : Ab(\mathcal{D}/Lc) \to Ab(\mathcal{C}/c)$, assuming it exists. However, the situation simplifies when L passes to Beck modules, since the induced left adjoint is essentially L itself; we obtain it for free.

4.2 Effect on Hochschild cohomology

When two categories are related by an adjunction, what relationships do we get on Hochschild cohomology? Assume we have an adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "L"] & \mathcal{D} \arrow[l, shift left, "R"] \end{tikzcd}$$

where the left adjoint L passes to Beck modules. As we have seen in corollary 4.1.5, we get two induced adjunctions, depending if one starts with a base object in $\mathcal{C}$ or $\mathcal{D}$.

4.2.1 Base object in $\mathcal{C}$

Pick a base object c in $\mathcal{C}$. The induced adjunction on Beck modules fits into the diagram

$$\begin{tikzcd} \mathcal{C}/c \arrow[r, shift left, "Ab_c"] \arrow[d, shift right, "L"'] & Ab(\mathcal{C}/c) \arrow[l, shift left, "U_c"] \arrow[d, shift right, "L"'] \\ \mathcal{D}/Lc \arrow[r, shift left, "Ab_{Lc}"] \arrow[u, shift right, "\eta_c^* R"'] & Ab(\mathcal{D}/Lc). \arrow[l, shift left, "U_{Lc}"] \arrow[u, shift right, "\eta_c^* R"'] \end{tikzcd} \tag{4.2.1}$$

where the diagram of right adjoints commutes (on the nose), and thus the diagram of left adjoints commutes as well. In particular, applying that to id_c, we obtain:

$$LAb_c c = Ab_{Lc} Lc.$$

Take a module N over Lc. On the one hand, we have:

$$\begin{aligned} \mathrm{HH}^*(c; \eta_c^* RN) &= \mathrm{Ext}^*(Ab_c c, \eta_c^* RN) \\ &= \mathrm{H}^* \operatorname{Hom}_{\mathbf{Mod}_c}(P_\bullet, \eta_c^* RN) \\ &= \mathrm{H}^* \operatorname{Hom}_{\mathbf{Mod}_{Lc}}(LP_\bullet, N) \end{aligned}$$

where $P_\bullet \to Ab_c c$ is a projective resolution. We want to compare this to:

$$\begin{aligned}\mathrm{HH}^*(Lc; N) &= \mathrm{Ext}^*(Ab_{Lc}Lc, N)\\ &= \mathrm{H}^* \operatorname{Hom}_{\mathbf{Mod}_{Lc}}(Q_\bullet, N)\end{aligned}$$

where $Q_\bullet \to Ab_{Lc}Lc$ is a projective resolution. Assume the induced left adjoint $L : \mathbf{Mod}_c \to \mathbf{Mod}_{Lc}$ preserves projectives (which is the case for example when its right adjoint $\eta_c^* R$ preserves epis, i.e. is exact). Then $LP_\bullet$ is projective but is not a resolution of $LAb_c c$. However, the map factors as:

$$LP_\bullet \hookrightarrow Q_\bullet \xrightarrow{\sim} LAb_c c = Ab_{Lc}Lc$$

and that first map induces:

$$\operatorname{Hom}_{\mathbf{Mod}_{Lc}}(Q_\bullet, N) \to \operatorname{Hom}_{\mathbf{Mod}_{Lc}}(LP_\bullet, N)$$

which, upon passing to cohomology, induces a well defined map. We sum up the argument in the following proposition.

Proposition 4.2.1. *If the left adjoint L induces a functor on Beck modules which preserves projectives, then we get a comparison map in Hochschild cohomology:*

$$\mathrm{HH}^*(Lc; N) \to \mathrm{HH}^*(c; \eta_c^* RN)$$

The computation above exhibits $\mathrm{HH}^*(c; \eta_c^* RN)$ as the derived functors of $\operatorname{Hom}_{\mathbf{Mod}_{Lc}}(-, N) \circ L$ applied to $Ab_c c$. Since L sends projectives to projectives, we obtain a Grothendieck composite spectral sequence:

$$E_2^{s,t} = \mathrm{Ext}^s(L_t L(Ab_c c), N) \Rightarrow \mathrm{HH}^{s+t}(c; \eta_c^* RN)$$

which is first quadrant, cohomologically graded. The comparison map is the edge morphism

$$\mathrm{HH}^s(Lc; N) = \mathrm{Ext}^s(LAb_c c, N) = E_2^{s,0} \twoheadrightarrow E_\infty^{s,0} \hookrightarrow \mathrm{HH}^s(c; \eta_c^* RN).$$

If $L : \mathbf{Mod}_c \to \mathbf{Mod}_{Lc}$ happens to be exact, then $LP_\bullet$ is a projective resolution of $LAb_c c = Ab_{Lc}Lc$, and the comparison map is an iso.

Now take a module M over c. Since there is a map $\mathrm{Hom}_{\mathbf{Mod}_c}(Ab_c c, M) \to \mathrm{Hom}_{\mathbf{Mod}_{Lc}}(LAb_c c, LM)$ given by applying L, one might want to compare $\mathrm{HH}^*(c; M)$ with $\mathrm{HH}^*(Lc; LM)$, but it's not clear that we can. One thing we can do is use the unit of the induced adjunction

$$M \to \eta_c^* RLM$$

to obtain the diagram

$$\begin{tikzcd}
\mathrm{HH}^*(c; M) \arrow[r, "\mathrm{unit}_*"] & \mathrm{HH}^*(c; \eta_c^* RLM) \\
& \mathrm{HH}^*(Lc; LM) \arrow[u, "\mathrm{comparison}"']
\end{tikzcd}$$

which, in degree $* = 0$, becomes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Mod}_c}(Ab_c c, M) \arrow[r, "\mathrm{unit}_*"] \arrow[dr, "L"'] & \mathrm{Hom}_{\mathbf{Mod}_c}(Ab_c c, \eta_c^* RLM) \arrow[d, equal] \\
& \mathrm{Hom}_{\mathbf{Mod}_{Lc}}(LAb_c c, LM)
\end{tikzcd}$$

4.2.2 Base object in $\mathcal{D}$

Pick a base object d in $\mathcal{D}$. The induced adjunction on Beck modules fits into the diagram

$$\begin{tikzcd}
C/Rd \arrow[r, shift left, "Ab_{Rd}"] \arrow[d, shift right, "\epsilon_! L"'] & Ab(C/Rd) \arrow[l, shift left, "U_{Rd}"] \arrow[d, shift right, "\epsilon_{d*} L"'] \\
\mathcal{D}/d \arrow[r, shift left, "Ab_d"] \arrow[u, shift right, "R"'] & Ab(\mathcal{D}/d) \arrow[l, shift left, "U_d"] \arrow[u, shift right, "R"']
\end{tikzcd}$$

where the diagram of right adjoints commutes (on the nose), and thus the diagram of left adjoints commutes as well.

Take a module N over d. On the one hand, we have:

$$\begin{aligned} \mathrm{HH}^*(d; N) &= \mathrm{Ext}^*(Ab_d d, N) \\ &= \mathrm{H}^* \mathrm{Hom}_{\mathbf{Mod}_d}(P_\bullet, N) \end{aligned}$$

where $P_\bullet \to Ab_d d$ is a projective resolution. We want to compare this to:

$$\begin{aligned} \mathrm{HH}^*(Rd; RN) &= \mathrm{Ext}^*(Ab_{Rd}Rd, RN) \\ &= \mathrm{H}^* \operatorname{Hom}_{\mathbf{Mod}_{Rd}}(Q_\bullet, RN) \\ &= \mathrm{H}^* \operatorname{Hom}_{\mathbf{Mod}_d}(\epsilon_{d*}LQ_\bullet, N) \end{aligned}$$

where $Q_\bullet \to Ab_{Rd}Rd$ is a projective resolution. Here again we assume the induced left adjoint $\epsilon_{d*}L : \mathbf{Mod}_{Rd} \to \mathbf{Mod}_d$ preserves projectives. Then $\epsilon_{d*}LQ_\bullet$ is projective and we have a map:

$$\epsilon_{d*}LQ_\bullet \to \epsilon_{d*}LAb_{Rd}Rd = \epsilon_{d*}Ab_{LRd}LRd = Ab_d(Lrd \xrightarrow{\epsilon_d} d) \xrightarrow{Ad_d(\epsilon_d)} Ab_d d.$$

It has a factorization:

$$\epsilon_{d*}LQ_\bullet \hookrightarrow P_\bullet \xrightarrow{\sim} Ab_d d$$

and this first map induces:

$$\operatorname{Hom}_{\mathbf{Mod}_d}(P_\bullet, N) \to \operatorname{Hom}_{\mathbf{Mod}_d}(\epsilon_{d*}LQ_\bullet, N)$$

which, upon passing to cohomology, induces a well defined map:

$$\mathrm{HH}^*(d; N) \to \mathrm{HH}^*(Rd; RN)$$

which is the comparison map. The computation above exhibits $\mathrm{HH}^*(Rd; RN)$ as the derived functors of $\operatorname{Hom}_{\mathbf{Mod}_d}(-, N) \circ \epsilon_{d*}L$ applied to $Ab_{Rd}Rd$. Since $\epsilon_{d*}L$ sends projectives to projectives, we obtain a Grothendieck composite spectral sequence:

$$E_2^{s,t} \, \mathrm{Ext}^s \, (L_t(\epsilon_{d*}L)(Ab_{Rd}Rd), N) \Rightarrow \mathrm{HH}^{s+t}(Rd; RN)$$

which is first quadrant, cohomologically graded. The comparison map is $Ab_d(\epsilon_d)^*$ followed by an edge morphism:

$$\begin{aligned} \mathrm{HH}^s(d; N) = \mathrm{Ext}^s(Ab_d d, N) &\xrightarrow{Ab_d(\epsilon_d)^*} \mathrm{Ext}^s(\epsilon_{d*}LAb_{Rd}Rd, N) \\ &= E_2^{s,0} \twoheadrightarrow E_\infty^{s,0} \hookrightarrow \mathrm{HH}^s(Rd; RN). \end{aligned}$$

If $\epsilon_{d*}L : \mathbf{Mod}_{Rd} \to \mathbf{Mod}_d$ happens to be exact, then $\epsilon_{d*}LQ_{\bullet}$ is a projective resolution of $\epsilon_{d*}LAb_{Rd}Rd$, and we get an iso:

$$\mathrm{Ext}^*(\epsilon_{d*}LAb_{Rd}Rd, N) \cong \mathrm{HH}^*(Rd; RN).$$

Now take a module M over Rd. Is there a way to compare $\mathrm{HH}^*(d; \epsilon_{d*}LM)$ and $\mathrm{HH}^*(Rd; M)$? Again, the only obvious thing one can do is use the unit of the induced adjunction

$$M \to R\epsilon_{d*}LM$$

to obtain the diagram

$$\begin{tikzcd}
\mathrm{HH}^*(Rd; M) \arrow[r, "\mathrm{unit}_*"] & \mathrm{HH}^*(Rd; R\epsilon_{d*}LM) \\
& \mathrm{HH}^*(d; \epsilon_{d*}LM). \arrow[u, "\mathrm{comparison}"']
\end{tikzcd}$$

Note that in degree $* = 0$, the comparison map becomes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Mod}_d}(Ab_d d, N) \arrow[r, "Ab_d(\epsilon_d)^*"] \arrow[d, equal] & \mathrm{Hom}_{\mathbf{Mod}_d}(\epsilon_{d*}LAb_{Rd}Rd, N) \arrow[r, equal, "\sim"] & \mathrm{Hom}_{\mathbf{Mod}_{Rd}}(Ab_{Rd}Rd, RN) \arrow[d, equal] \\
\mathrm{HH}^0(d; N) & & \mathrm{HH}^0(Rd; RN).
\end{tikzcd}$$

4.3 Good setup for adjunctions

Let us check that a nice adjunction between ground categories behaves well at the level of Quillen (co)homology.

Theorem 4.3.1. *Let C and $\mathcal{D}$ be complete algebraic categories with abelianizations. Let $L : C \rightleftarrows \mathcal{D} : R$ be an adjunction that prolongs to a Quillen pair ($\Leftrightarrow$ R preserves regular epis $\Leftrightarrow$ L preserves projectives). Then diagrams (4.4.1) and (4.4.6) consist of four Quillen pairs.*

Proof. **Case 1:** Ground object c in C (4.4.1). The induced right adjoint on slice categories is $\eta_c^* R : \mathcal{D}/Lc \to C/c$ and it preserves regular epis. Indeed, $R : \mathcal{D}/Lc \to C/RLc$ preserves regular epis by assumption and 3.2.1. The pullback η_c^* also preserves regular epis since C is regular and again by 3.2.1.

The induced right adjoint on Beck modules $\eta_c^* R : Ab(\mathcal{D}/Lc) \to Ab(C/c)$ preserves regular epis. It follows from the same argument, and the fact that regular epis in $Ab(-)$ are preserved and reflected by the forgetful functor U, by 3.3.4 and 3.3.7.

Case 2: Ground object d in $\mathcal{D}$ (4.4.6). The induced right adjoint on slice categories is just $R : \mathcal{D}/d \to C/Rd$, which preserves regular epis. The induced right adjoint on Beck modules $R : Ab(\mathcal{D}/d) \to Ab(C/Rd)$ also preserves regular epis. □

Remark 4.3.2. The result holds whether or not the left adjoint L passes to Beck modules, since the proof only relies on properties of the induced right adjoints.

4.4 Effect on Quillen (co)homology

We look at the behavior of Quillen homology with respect to adjunctions. As in section 4.2, start with an adjunction

$$\begin{tikzcd} C \arrow[r, shift left, "L"] & \mathcal{D} \arrow[l, shift left, "R"] \end{tikzcd}$$

where the left adjoint L passes to Beck modules, and the prolonged adjunction is a Quillen pair (3.1.3 and 3.1.4).

4.4.1 Base object in C

Pick a base object c in C. The induced adjunction on Beck modules fits into the diagram

$$\begin{tikzcd} sC/c \arrow[r, shift left, "Ab_c"] \arrow[d, shift right, "L"'] & sAb(C/c) \arrow[l, shift left, "U_c"] \arrow[d, shift right, "L"'] \\ s\mathcal{D}/Lc \arrow[r, shift left, "Ab_{Lc}"] \arrow[u, shift right, "\eta_c^* R"'] & sAb(\mathcal{D}/Lc). \arrow[l, shift left, "U_{Lc}"] \arrow[u, shift right, "\eta_c^* R"'] \end{tikzcd} \tag{4.4.1}$$

where everything has been simplicially prolonged, and we have four Quillen pairs (by 4.3.1). The diagram of right adjoints commutes and so does diagram of left adjoints. Starting with a cofibrant replacement $q_c : Qc \xrightarrow{\sim} c$ of id_c, we can apply L to obtain $LQc \to Lc$, where the source is still cofibrant but the map is not a weak equivalence anymore. We can factor this map as

$$LQc \overset{\psi}{\rightarrow} QLc \overset{\sim}{\rightarrow} Lc \tag{4.4.2}$$

and we obtain

$$\begin{aligned}
LAb_c(Qc \to c) &= Ab_{Lc}L(Qc \to c) \\
L(L_c) &= Ab_{Lc}(LQc \to Lc) \\
&\to Ab_{Lc}(QLc \to Lc) = L_{Lc}.
\end{aligned}$$

So we get a map

$$L(L_c) \to L_{Lc} \tag{4.4.3}$$

which is in fact $Ab_{Lc}(\psi)$. The composite spectral sequence (of left derived functors) for $L \circ Ab_c$ applied to id_c is

$$E^2_{s,t} = L_s L\left(\mathrm{HQ}_t(c)\right) \Rightarrow \pi_{s+t}L(L_c) = \pi_{s+t}Ab_{Lc}L(Qc \to c). \tag{4.4.4}$$

There is an edge morphism

$$E^2_{0,t} = L\left(\mathrm{HQ}_t(c)\right) \twoheadrightarrow E^\infty_{0,t} \hookrightarrow \pi_t L(L_c)$$

which is simply the homology comparison map of the right exact functor L, applied to the chain complex L_c (using implicitly the Dold-Kan correspondence):

$$L\left(\mathrm{HQ}_t(c)\right) = LH_t(L_c) \to H_t L(L_c) = \pi_t L(L_c).$$

For a detailed study of homology comparison, see [Bar06, thm 2.2 and 2.6]. Following this homology comparison by the effect of the comparison map (4.4.3) on π_*, we obtain the Quillen homology comparison:

$$L\left(\mathrm{HQ}_t(c)\right) \to \pi_t L(L_c) \to \mathrm{HQ}_t(Lc).$$

If the original functor L preserves pullbacks then the induced L on Beck modules also preserves finite limits, hence is left exact (and thus exact). In that case, the homology comparison is an iso and the (silly) spectral sequence (4.4.4) is just that iso. So really, the Quillen homology comparison measures the failure of L to preserve cofibrant replacements.

In the special case where L preserves all weak equivalences (or equivalently, all weak equivalences with cofibrant source), then the map ψ is a weak equivalence. Since Ab_{Lc} is a left Quillen functor, the comparison map (4.4.3) is also a weak equivalence, and we obtain an iso in Quillen homology:

$$L\,(\mathrm{HQ}_*(c)) \xrightarrow{\sim} \mathrm{HQ}_*(Lc).$$

Effect on Quillen cohomology Given a module N over Lc, we can apply the functor

$$\mathrm{Hom}_{\mathbf{Mod}_{Lc}}(-, N)$$

to the comparison map (4.4.3):

$$\mathrm{Hom}_{\mathbf{Mod}_{Lc}}(L_{Lc}, N) \to \mathrm{Hom}_{\mathbf{Mod}_{Lc}}(L(L_c), N) \cong \mathrm{Hom}_{\mathbf{Mod}_c}(L_c, \eta_c^* RN)$$

and upon passing to cohomology, we obtain a comparison map in Quillen cohomology:

$$\mathrm{HQ}^*(Lc; N) \to \mathrm{HQ}^*(c; \eta_c^* RN). \tag{4.4.5}$$

In the special case where L preserves all weak equivalences, $LQc \to Lc$ is a cofibrant replacement (i.e. the map ψ in (4.4.2) is a weak equivalence) so the comparison in Quillen cohomology (4.4.5) is an iso. In fact, if the comparison of cotangent complexes (4.4.3) is a weak equivalence, then the Quillen cohomology comparison is an iso, since $L(L_c)$ and L_{Lc} are cofibrant. This fact can also be expressed using the UCSS (1.2.2).

4.4.2 Base object in $\mathcal{D}$

The reasoning is very similar when we pick a base object d in $\mathcal{D}$. The induced adjunction on Beck modules fits into the diagram

$$\begin{tikzcd}
sC/Rd \arrow[r, shift left, "Ab_{Rd}"] \arrow[d, shift right, "\epsilon_! L"'] & sAb(C/Rd) \arrow[l, shift left, "U_{Rd}"] \arrow[d, shift right, "\epsilon_{d*} L"'] \\
s\mathcal{D}/d \arrow[r, shift left, "Ab_d"] \arrow[u, shift right, "R"'] & sAb(\mathcal{D}/d) \arrow[l, shift left, "U_d"] \arrow[u, shift right, "R"']
\end{tikzcd} \tag{4.4.6}$$

and again, everything has been simplicially prolonged, we have four Quillen pairs, and the diagrams of right and left adjoints commute. Starting with a cofibrant replacement $q_{Rd} : Q(Rd) \xrightarrow{\sim} Rd$ of id_{Rd}, we can apply $\epsilon_{d!} L$ to obtain $LQ(Rd) \to d$, where the source is still cofibrant but the map is not a weak equivalence anymore. We can factor this map as

$$LQ(Rd) \xrightarrow{\psi} Qd \xrightarrow{\sim} d$$

and we obtain

$$\begin{aligned} \epsilon_{d*}LAb_{Rd}(Q(Rd) \to d) &= Ab_d(LQ(Rd) \to LRd \xrightarrow{\epsilon_d} d) \\ \epsilon_{d*}L(L_{Rd}) &= Ab_d(LQ(Rd) \to d) \\ &\to Ab_d(Qd \xrightarrow{\sim} d) = L_d. \end{aligned}$$

So we get a map

$$\epsilon_{d*}L(L_{Rd}) \to L_d \tag{4.4.7}$$

which is in fact $Ab_d(\psi)$. The composite spectral sequence (of left derived functors) for $\epsilon_{d*}L \circ Ab_{Rd}$ applied to id_{Rd} is

$$E^2_{s,t} = L_s(\epsilon_{d*}L)\left(\mathrm{HQ}_t(Rd)\right) \Rightarrow \pi_{s+t}\epsilon_{d*}L(L_{Rd}) = \pi_{s+t}Ab_d(LQ(Rd) \to d). \tag{4.4.8}$$

Again, there is an edge morphism

$$E^2_{0,t} = \epsilon_{d*}L\left(\mathrm{HQ}_t(Rd)\right) \twoheadrightarrow E^\infty_{0,t} \hookrightarrow \pi_t\epsilon_{d*}L(L_{Rd})$$

which is the homology comparison map of the right exact functor $\epsilon_{d*}L$, applied to the chain complex L_{Rd}. Following this homology comparison by the effect of the comparison map 4.4.7 on π_*, we obtain the Quillen homology comparison:

$$\epsilon_{d*}L\left(\mathrm{HQ}_t(Rd)\right) \to \pi_t\epsilon_{d*}L(L_{Rd}) \to \mathrm{HQ}_t(d).$$

If the original functor L preserves pullbacks, then the induced L on Beck modules is exact and the homology comparison for $\epsilon_{d*}L$ becomes that of ϵ_{d*} only. If additionally ϵ_{d*} is exact as well, then the homology comparison is an iso and the spectral sequence (4.4.8) is just that iso.

In the special case where $\epsilon_{d!}L$ preserves all weak equivalences, then the map ψ is a weak equivalence. Since Ab_d is a left Quillen functor, the comparison map (4.4.7) is also a weak equivalence, and we obtain an iso in Quillen homology:

$$\epsilon_{d*}L\left(\mathrm{HQ}_*(Rd)\right) \xrightarrow{\sim} \mathrm{HQ}_*(d).$$

Effect on Quillen cohomology Given a module N over d, we can apply the functor $\mathrm{Hom}_{\mathbf{Mod}_d}(-, N)$ to the comparison map (4.4.7):

$$\mathrm{Hom}_{\mathbf{Mod}_d}(L_d, N) \to \mathrm{Hom}_{\mathbf{Mod}_d}(\epsilon_{d*}L(L_{Rd}), N) \cong \mathrm{Hom}_{\mathbf{Mod}_{Rd}}(L_{Rd}, RN)$$

and upon passing to cohomology, we obtain a comparison map in Quillen cohomology:

$$\mathrm{HQ}^*(d; N) \to \mathrm{HQ}^*(Rd; RN). \tag{4.4.9}$$

As above, if the comparison of cotangent complexes (4.4.7) is a weak equivalence, then the Quillen cohomology comparison (4.4.9) is an iso.

4.5 Example: Commutativization of groups

Consider the "commutativization" functor $Com : \mathbf{Gp} \to \mathbf{Ab}$ that kills commutators, i.e.

$$Com(G) = G/[G, G].$$

Note that Com is left adjoint to the inclusion functor $\iota : \mathbf{Ab} \to \mathbf{Gp}$. We will check that Com passes to Beck modules even though it is not limit-preserving.

Proposition 4.5.1. *Com does NOT preserve pullbacks in general.*

Proof. Denote by $F(S)$ be the free group on the set S. Consider the "sink" diagram

$$\begin{tikzcd}
 & F(v) \simeq \mathbb{Z} \arrow[d] & v \arrow[d, mapsto] \\
\mathbb{Z} \simeq F(u) \arrow[r] & F(x, y) & {[x, y] = xyx^{-1}y^{-1}} \\
u \arrow[r, mapsto] & 1 &
\end{tikzcd}$$

whose pullback in **Gp** is $F(u) \times 1 \cong F(u) \simeq \mathbb{Z}$. Once we apply Com to the diagram, we get

$$\begin{tikzcd}
 & FAb(v) \simeq \mathbb{Z} \arrow[d] & v \arrow[d, mapsto] \\
\mathbb{Z} \simeq FAb(u) \arrow[r] & FAb(x,y) \sim \mathbb{Z}^2 & 0 \\
u \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

whose pullback in **Ab** is $FAb(u) \times FAb(v) \simeq \mathbb{Z}^2$. The natural map

$$\mathbb{Z} \simeq Com\Big(F(u) \times_{F(x,y)} F(v)\Big) \to Com(F(u)) \times_{Com(F(x,y))} Com(F(v)) \simeq \mathbb{Z}^2$$

is NOT an iso. □

Proposition 4.5.2. *Com does NOT preserve kernel pairs in general.*

Proof. Take a non-trivial group G whose commutator subgroup $[G,G]$ is equal to the whole group G, e.g. take $G = A_5$. Now take as map the inclusion of a non-trivial abelian subgroup $C \hookrightarrow G$, e.g. take C to be the cyclic subgroup generated by a non-identity element. The kernel pair diagram

$$\begin{tikzcd}
 & C \arrow[d, hook] \\
C \arrow[r, hook] & G
\end{tikzcd}$$

has pullback C, viewed as the diagonal subgroup of $C \times C$. Once we apply Com to the diagram, we get

$$\begin{tikzcd}
 & C \arrow[d] \\
C \arrow[r] & 0
\end{tikzcd}$$

whose pullback in **Ab** is $C \oplus C$. Therefore Com does NOT preserve this pullback. □

Nevertheless, let us show that Com does pass to Beck modules. Recall that for a (left) G-module M, the semidirect product $G \ltimes M$ is the group with underlying set $G \times M$ and multiplication

$$(g,m)(g',m') = (gg', m + gm').$$

Proposition 4.5.3. *$Com(G \ltimes M) \cong Com(G) \oplus M_G$.*

Proof. Commutators in $G \ltimes M$ are given by

$$[(g_1, m_1), (g_2, m_2)] = \left([g_1, g_2], m_1 - g_1 g_2 g_1^{-1} m_1 + g_1 m_2 - g_1 g_2 g_1^{-1} g_2^{-1} m_2\right).$$

Applying Com to the split extension $G \ltimes M \to G$ yields a split extension $Com(G \ltimes M) \to Com(G)$ in **Ab** whose kernel is M modulo the subgroup

$$\begin{aligned}&\left\langle m_1 - g_1 g_2 g_1^{-1} m_1 + g_1 m_2 - g_1 g_2 g_1^{-1} g_2^{-1} m_2 \mid g_i \in G, m_i \in M\right\rangle \\ &= \langle m - gm \mid g \in G, m \in M\rangle\end{aligned}$$

so it is M_G, the abelian group of coinvariant of M. □

Corollary 4.5.4. *Com passes to Beck modules, on which it induces the coinvariants functor $(-)_G$: $\mathbf{Mod}_G \to \mathbf{Ab}$.*

Proof. Com preserves the pullback that defines the multiplication structure map:

$$\begin{aligned}Com\left((G \ltimes M) \times_G (G \ltimes M)\right) &= Com\left(G \ltimes (M \times M)\right) \\ &= Com(G) \oplus (M \times M)_G \\ &= Com(G) \oplus (M_G \oplus M_G) \\ &= (Com(G) \oplus M_G) \times_{Com(G)} (Com(G) \oplus M_G) \\ &= Com(G \ltimes M) \times_{Com(G)} Com(G \ltimes M).\end{aligned}$$

In **Gp** as well as in **Ab**, we think of the module as the kernel of the split extension, and in this case, we see that a G-module M is sent to the abelian group M_G. □

Remark 4.5.5. In **Ab**, a Beck module consists only of a split extension with the data of the splitting (cf. A.2.1). Therefore, ANY functor $F : C \to$ **Ab** passes to Beck modules. We've shown it explicitly for Com and identified the induced functor.

Now let us see what the adjunction

$$\begin{tikzcd}\mathbf{Gp} \arrow[r, shift left, "Com"] & \mathbf{Ab} \arrow[l, shift left, "\iota"]\end{tikzcd}$$

does on Quillen homology. First, note that the right adjoint ι preserves regular epis, which are just surjections. Hence the prolonged adjunctions are Quillen pairs. Moreover, the left adjoint Com passes to Beck modules and induces the coinvariants functor $(-)_G$.

Note also that the unit of the adjunction is $\eta_G : G \twoheadrightarrow G/[G,G]$ and the counit is the identity $\epsilon_A : A \overset{\text{id}}{\to} A$. We'll work with a base object G in **Gp**, since we get nothing new from a base object in **Ab**. Diagram (4.4.1) becomes:

$$\begin{tikzcd}
s\mathbf{Gp}/G \arrow[r, shift left, "Ab_G"] \arrow[d, shift right, "Com"'] & s\mathbf{Mod}_G \arrow[l, shift left, "G\ltimes-"] \arrow[d, shift right, "(-)_G"'] \\
s\mathbf{Ab}/Com(G) \arrow[r, shift left, "Src"] \arrow[u, shift right, "\eta_G^*\iota"'] & s\mathbf{Ab}. \arrow[l, shift left, "Com(G)\oplus-"] \arrow[u, shift right, "Triv"']
\end{tikzcd} \tag{4.5.1}$$

where Src is the "source" functor, as in proposition A.2.4, and $Triv$ is the functor assigning to an abelian group the trivial G-action. Indeed, we know that the right adjoint on Beck modules is $\eta_G^*\iota$. From a Beck module $Com(G) \oplus A$, we first view it as a split extension of groups, which means A has a trivial $Com(G)$ action, and then pull it back along $\eta_G : G \to G/[G,G]$, which endows A with the trivial G-action.

Remark 4.5.6. In 4.5.4, we've checked explicitly that Com passes to Beck modules and shown that the induced functor is coinvariants $(-)_G$. Per remark 4.1.6, we could also look at the induced right adjoint $\eta_G^*\iota = Triv$ and use its left adjoint to complete the diagram above. The left adjoint of $Triv$ is indeed $(-)_G$, which confirms that our computation was correct.

We can now formulate the result about Quillen homology.

Proposition 4.5.7. *Let $C_\bullet \to G$ be a cofibrant replacement of G in groups and let L_G denote the cotangent complex of G. Then we have*

$$\pi_* \left(C_\bullet/[C_\bullet, C_\bullet]\right) = \pi_* \left((L_G)_G\right).$$

Proof. Starting from a cofibrant replacement of G in **Gp** (or equivalently, of id_G in **Gp**$/G$) in the upper left corner of the commutative diagram (4.5.1), going down then right yields:

$$\begin{aligned}
Src \circ Com(C_\bullet \to G) &= Src\,(Com(C_\bullet) \to Com(G)) \\
&= Com(C_\bullet) = C_\bullet/[C_\bullet, C_\bullet]
\end{aligned}$$

whereas going right then down yields:

$$(Ab_G(C_\bullet \to G))_G = (L_G)_G$$

by definition of the cotangent complex. The two simplicial abelian groups are weakly equivalent, whence the claim. □

In fact, one can compute both sides explicitly and show that they coincide. By proposition A.1.6, we know that $L_G \to I_G$ is a cofibrant replacement, in particular a flat resolution, so taking coinvariants yields the derived functors thereof, also called group homology:

$$\pi_* \left((L_G)_G\right) = L_*(-)_G(I_G) = \mathrm{H}_*(G; I_G).$$

From the long exact sequence associated to the short exact sequence of G-modules

$$0 \to I_G \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

we conclude that the connecting morphism $\mathrm{H}_{i+1}(G;\mathbb{Z}) \to \mathrm{H}_i(G; I_G)$ is an iso for all $i \geq 0$. For $i \geq 1$, it follows from the flatness of $\mathbb{Z}G$, and for $i = 0$, it follows from the iso

$$\mathbb{Z} \cong (\mathbb{Z}G)_G = \mathrm{H}_0(G;\mathbb{Z}G) \xrightarrow{\cong} \mathrm{H}_0(G;\mathbb{Z}) = (\mathbb{Z})_G = \mathbb{Z}.$$

We conclude the following:

$$\pi_i \left((L_G)_G\right) = \mathrm{H}_{i+1}(G;\mathbb{Z})$$

for all $i \geq 0$. On the other hand, [GS07, ex 4.26] uses a different argument to show:

$$\pi_i \left(C_\bullet/[C_\bullet, C_\bullet]\right) = \mathrm{H}_{i+1}(G;\mathbb{Z})$$

for all $i \geq 0$. Thus our proposition is consistent with these computations.

In particular, for $i = 0$, we obtain $\mathrm{H}_1(G;\mathbb{Z}) = G/[G, G]$. This is the usual Hurewicz theorem since $\pi_1(BG) = G$. Here is a fun way to prove it using only simple methods from the present document.

Proposition 4.5.8. *Indeed, we have* $\mathrm{H}_1(G;\mathbb{Z}) = G/[G, G]$.

Proof. First note that over the terminal object in **Gp**, i.e. the trivial group $\{1\}$, the forgetful functor

$$U_{\{1\}} : Ab(\mathbf{Gp}/\{1\}) \to \mathbf{Gp}/\{1\}$$

is just the embedding $\mathbf{Ab} \to \mathbf{Gp}$. Therefore, its left adjoint $Ab_{\{1\}}$ is *Com*. Using this, we compute:

$$\begin{aligned}
\mathrm{H}_1(G;\mathbb{Z}) &\cong \mathrm{H}_0(G; I_G) \\
&= (I_G)_G \\
&= \mathbb{Z} \otimes_{\mathbb{Z}G} I_G \\
&= \tau_*(I_G), \text{ where } \tau : G \to \{1\} \\
&= \tau_*(Ab_G G) \\
&= Ab_{\{1\}}(G \to \{1\}) \\
&= Com(G) = G/[G,G].
\end{aligned}$$

□

4.6 Example: Commutativization of algebras

Using the notation of A.3 and A.4, consider the "commutativization" functor

$$\begin{aligned}
Com : \mathbf{Alg}_R &\to \mathbf{Com}_R \\
A &\mapsto A/[A,A]
\end{aligned}$$

which kills the 2-sided ideal generated by commutators. It is left adjoint to the inclusion functor $\mathbf{Com}_R \to \mathbf{Alg}_R$.

Proposition 4.6.1. *1. The functor Com* : $\mathbf{Alg}_R \to \mathbf{Com}_R$ *passes to Beck modules.*

2. It induces the "central quotient" functor

$$\mathrm{HH}_0 : A - \mathbf{Bimod}_R \to Com(A) - \mathbf{Mod}$$

which coequalizes the two actions.

Proof. We use the "split extension" picture of Beck modules. Start with a Beck module over A in $\mathbf{Alg}_R$, i.e. a split extension $A \oplus M \to A$ satisfying $M^2 = 0$. Applying *Com* to it yields a split extension

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & Com(A \oplus M) \arrow[r, shift left, "Com(p)"] & A \arrow[l, shift left, "Com(s)"] \arrow[r] & 0 \end{tikzcd}$$

in $\mathbf{Com}_R$. It remains to show that its kernel has square zero.

Commutators in $A \oplus M$ Using the decomposition $(a, m) = (a, 0) + (0, m)$, commutators will be generated by those of the forms

$$\begin{aligned} [(a,0),(a',0)] &= ([a,a'],0) \\ [(a,0),(0,m')] &= (a,0)(0,m') - (0,m')(a,0) \\ &= (0, a \cdot m') - (0, m' \cdot a) \\ &= (0, a \cdot m' - m' \cdot a) \\ [(0,m),(a',0)] &= -[(a',0),(0,m)] \ \text{ nothing new} \\ [(0,m),(0,m')] &= 0 \end{aligned}$$

and hence the kernel is

$$K \simeq M/ \langle a \cdot m - m \cdot a \rangle \tag{4.6.1}$$

where we kill the sub-A-bimodule generated by all elements of that form.

K **has square zero** Take two elements $x, x' \in K = \ker Com(p) \subset Com(A \oplus M)$ and choose representatives (c, m) and (c', m') in $A \oplus M$, where $c, c' \in [A, A]$. Then xx' is represented by

$$\begin{aligned} (c,m)(c',m') &= (cc', c \cdot m' + m \cdot c') \\ &\sim (0, c \cdot m' + m \cdot c'). \end{aligned}$$

Thus it suffices to check that all elements of the form $c \cdot m$ and $m \cdot c$ are zero in $Com(A \oplus M)$, for any $m \in M$ and $c \in [A, A]$. Suffices to check it for $c = [a, a'] = aa' - a'a$.

$$\begin{aligned}
(aa' - a'a) \cdot m &= aa' \cdot m - a'a \cdot m \\
&= a \cdot (a' \cdot m) - (a' \cdot m) \cdot a + (a' \cdot m) \cdot a - a' \cdot (a \cdot m) \\
&= a \cdot (a' \cdot m) - (a' \cdot m) \cdot a + a' \cdot (m \cdot a - a \cdot m) \in [A \oplus M, A \oplus M] \\
m \cdot (aa' - a'a) &= m \cdot aa' - m \cdot a'a \\
&= (m \cdot a) \cdot a' - a' \cdot (m \cdot a) + a' \cdot (m \cdot a) - (m \cdot a') \cdot a \\
&= (m \cdot a) \cdot a' - a' \cdot (m \cdot a) + (a' \cdot m - m \cdot a') \cdot a \in [A \oplus M, A \oplus M].
\end{aligned}$$

This proves the first assertion, and the computation (4.6.1) proves the second. □

The adjunction

$$\begin{tikzcd} \mathbf{Alg}_R \arrow[r, shift left, "Com"] & \mathbf{Com}_R \arrow[l, shift left, "\iota"] \end{tikzcd}$$

allows us to compare the two categories. According to 4.6.1, the comparison diagram 4.2.1 becomes

$$\begin{tikzcd}
\mathbf{Alg}_R/A \arrow[r, shift left, "A\otimes I_{(-)}\otimes A"] \arrow[d, shift right, "Com"'] & A - \mathbf{Bimod}_R \arrow[l, shift left, "A\oplus -"] \arrow[d, shift right, "\mathrm{HH}_0"'] \\
\mathbf{Com}_R/Com(A) \arrow[r, shift left, "Com(A)\otimes\Omega_{(-)/R}"] \arrow[u, shift right, "\eta_A^*\iota"'] & Com(A) - \mathbf{Mod} \arrow[l, shift left, "Com(A)\oplus -"] \arrow[u, shift right, "\text{same action}"']
\end{tikzcd} \tag{4.6.2}$$

where "same action", the right adjoint on the right, means that we view a $Com(A)$-module as an A-bimodule by acting via the counit $A \to Com(A) = A/[A, A]$ both on the left and the right. The abelianizations are described in A.3.4 and A.4.4.

Two special "extreme" cases are of particular interest.

1. $A = R$ When the R-algebra A is just R itself – and is in particular commutative – the comparison diagram becomes

$$\begin{tikzcd}
\mathbf{Alg}_R/R \arrow[r, shift left, "R\otimes I_{(-)}\otimes R"] \arrow[d, shift right, "Com"'] & R - \mathbf{Bimod}_R \arrow[l, shift left, "R\oplus -"] \arrow[d, shift right, "\mathrm{id}"'] \\
\mathbf{Com}_R/R \arrow[r, shift left, "R\otimes\Omega_{(-)/R}"] \arrow[u, shift right, "\iota"'] & R - \mathbf{Mod}. \arrow[l, shift left, "R\oplus -"] \arrow[u, shift right, "\mathrm{id}"']
\end{tikzcd} \tag{4.6.3}$$

Essentially, the diagram says that killing all products can be done in two steps, by killing all commutators first. One could try to use the Grothendieck composite spectral sequence for the non-abelian setting [BS92, thm 4.4] to relate the Quillen homology in $\mathbf{Alg}_R$ to the Quillen homology in $\mathbf{Com}_R$, i.e. André-Quillen cohomology.

2. $R = \mathbb{Z}$ When R is $\mathbb{Z}$, $\mathbf{Alg}_R$ and $\mathbf{Com}_R$ are just rings and commutative rings, respectively, and $A - \mathbf{Bimod}_R$ becomes $A - \mathbf{Bimod}$.

Chapter 5

Quillen cohomology of truncated Π-algebras

In this chapter, we describe explicitly the standard model structure on $s\mathbf{\Pi Alg}_1^n$, the category of simplicial n-truncated Π-algebras. Then we use the truncation adjunction to describe Quillen cohomology of truncated Π-algebras.

5.1 Standard model structure

Recall Quillen's construction [Qui67, II.4 thm 4] to makes sC into a simplicial model category, where C is an algebraic category and sC denotes the category of simplicial objects in C. The classic reference is [Qui67, II.4]. For a more recent reference, see [GJ91, II.4].

There is an explicit description of the model structure on $s\mathbf{\Pi Alg}$ in [BDG04, § 4.5]. Now we do the same with the category $\mathbf{\Pi Alg}_1^n$ of n-truncated Π-algebras. Most of the arguments will only be manifestations of the fact that $\mathbf{\Pi Alg}_1^n$ is a category of universal algebras.

We first describe the ingredients involved in the construction: effective epis, projective objects, and free objects. First, recall that $\mathbf{\Pi Alg}$ has **free** objects. The forgetful functor

$$U : \mathbf{\Pi Alg} \to \mathbf{GrSet}$$

to graded sets (with grading $1, 2, 3, \ldots$) has a left adjoint

$$F : \mathbf{GrSet} \to \mathbf{\Pi Alg}$$
$$F(\{X_i\}) = \pi_* \left(\bigvee_{i=1}^{\infty} \bigvee_{X_i} S^i \right).$$

This Π-algebra is called **free** on the graded set $\{X_i\}$. For a somewhat less drastic adjunction, one can forget down to graded pointed sets:

$$U_* : \mathbf{\Pi Alg} \to \mathbf{GrSet}_*$$

and this functor has a left adjoint

$$F_* : \mathbf{GrSet}_* \to \mathbf{\Pi Alg}$$
$$F_*(\{(X_i, x_i)\}) = \pi_* \left(\bigvee_{i=1}^{\infty} \bigvee_{X_i - \{x_i\}} S^i \right).$$

The category $\mathbf{\Pi Alg}_1^n$ also has free objects.

Proposition 5.1.1. *The left adjoint of the forgetful functor*

$$U : \mathbf{\Pi Alg}_1^n \to \mathbf{GrSet}$$

is the functor

$$F_n : \mathbf{GrSet} \to \mathbf{\Pi Alg}_1^n$$
$$F_n(\{X_i\}) = P_n \pi_* \left(\bigvee_{i=1}^{\infty} \bigvee_{X_i} S^i \right) = \pi_* \left(P_n \bigvee_{i=1}^{n} \bigvee_{X_i} S^i \right).$$

In other words, we have $F_n = P_n F$.

Proof. Combine two adjunctions:

$$\begin{tikzcd} \mathbf{GrSet} \arrow[r, shift left, "F"] & \mathbf{\Pi Alg} \arrow[l, shift left, "U"] \arrow[r, shift left, "P_n"] & \mathbf{\Pi Alg}_1^n \arrow[l, shift left, "\iota_n"] \end{tikzcd}$$

□

Remark 5.1.2. In the case $n = 2$, we know that $\mathbf{\Pi Alg}_1^2$ is equivalent to the category **ModGp**. The previous proposition tells us that a free 2-truncated Π-algebra on a pair of sets (S, T) is $P_2\pi_*(\bigvee_S S^1 \vee \bigvee_T S^2)$. This wedge of spheres has $\pi_1 = F(S)$, the free group generated by the circles, and $\pi_2 = \mathbb{Z}\pi_1[T]$, the free π_1-module generated by the 2-spheres. Thus we recover exactly the free objects as in proposition 7.1.2.

Now that we have free objects, we'd like to identify projective objects. For this we need to identify effective epimorphisms. Recall a few definitions.

Definition 5.1.3. In a category with kernel pairs, a map $f : X \to Y$ is an **effective epimorphism** [Qui67, II.4] or **regular epimorphism** [Bor94b, def 4.3.1] if the following diagram is a coequalizer:

$$\begin{tikzcd} X \times_Y X \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] & X \arrow[r, "f"] & Y. \end{tikzcd} \tag{5.1.1}$$

We will prefer the term "regular".

Definition 5.1.4. An object P is **projective** if for any regular epi $f : X \to Y$, the map

$$f_* : \mathrm{Hom}(P, X) \to \mathrm{Hom}(P, Y)$$

is a surjection of sets. In other words, maps from P lift through any regular epi:

$$\begin{tikzcd} & P \arrow[ld, dashed] \arrow[d] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Definition 5.1.5. A category has **enough projectives** if for any object X, there is a regular epi from a projective $P \to X$.

Recall that in the category of groups, a regular epi is just a surjection of underlying sets. The reason is that any map that coequalizes the diagram (5.1.1) will have a kernel containing $\ker f$. We can generalize this argument to our case.

Proposition 5.1.6. *In* $\mathbf{\Pi Alg}$ *or* $\mathbf{\Pi Alg}_1^n$, *a map* $f : X \to Y$ *is a regular epi iff it is a surjection of underlying graded sets.*

Proof. (⇐) f is a surjection and we have the following commutative diagram in **ΠAlg**:

$$\begin{tikzcd}
X \times_Y X \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] & X \arrow[r, two heads, "f"] \arrow[dr, "\alpha"'] & Y \arrow[d, dashed, "\exists ? \beta"] \\
 & & Z
\end{tikzcd} \tag{5.1.2}$$

Since each functor π_i preserves limits, applying it to the diagram yields a diagram of groups:

$$\begin{tikzcd}
X_i \times_{Y_i} X_i \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] & X_i \arrow[r, two heads, "f_i"] \arrow[dr, "\alpha_i"'] & Y_i \arrow[d, dashed, "\exists ! \beta_i"] \\
 & & Z_i
\end{tikzcd}$$

where there exists a unique map of groups β_i. It can be defined as choosing a preimage by f_i and then applying α_i. Hence in the diagram (5.1.2), there is a unique map β of graded groups. It remains to check that this β is a map of Π-algebras. This follows from the fact that α is, and so is f, hence preimages by f can be chosen so as to respect the extra structure. To be very explicit, write x's for f-preimages of the corresponding y's in Y. Then we have:

$$\begin{aligned}
\beta(y_1 \cdot y_k) &= \alpha(x_1 \cdot x_k) \\
&= \alpha(x_1) \cdot \alpha(x_k) \\
&= \beta(y_1) \cdot \beta(y_k) \\
\beta([y, y']) &= \alpha([x, x']) \\
&= [\alpha(x), \alpha(x')] \\
&= [\beta(y), \beta(y')] \\
\beta\left(\gamma^*(y_k)\right) &= \alpha\left(\gamma^*(x_k)\right) \qquad \text{where } \gamma \in \pi_m(S^k) \\
&= \gamma^* \alpha(x_k) \\
&= \gamma^* \beta(y_k).
\end{aligned}$$

(⇒) Note that $\operatorname{im} f$ is a Π-algebra, and the inclusion $\iota : \operatorname{im} f \to Y$ is a map of Π-algebras. Now consider the diagram

$$\begin{tikzcd}
X \times_Y X \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] & X \arrow[r, "f"] \arrow[dr, "f"'] & Y \arrow[d, dashed, "\exists ! \beta"] \\
 & & \operatorname{im} f.
\end{tikzcd}$$

From $\beta f = f$, we get $\beta|_{\operatorname{im} f} = \mathrm{id}_{\operatorname{im} f} = \beta\iota$. Now consider the commutative diagram

$$\begin{tikzcd}
X \times_Y X \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] & X \arrow[r, "f"] \arrow[dr, "f"'] & Y \arrow[d, shift right, "\mathrm{id}_Y"'] \arrow[d, shift left, "\iota\beta"] \\
 & & Y.
\end{tikzcd}$$

By the uniqueness of the map on the right, we conclude $\iota\beta = \mathrm{id}_Y$, so $\operatorname{im} f$ equals Y, i.e. f is surjective. □

Remark 5.1.7. Quillen notes that this proposition always holds in a category of universal algebras. See [Qui67, II.4], remark 1 after proposition 1.

Corollary 5.1.8. *In* $\mathbf{\Pi Alg}$ *or* $\mathbf{\Pi Alg}_1^n$*, an object is projective iff it is a retract of a free.*

Proof. ($\Rightarrow$) Note that the counit of the adjunction $F \dashv U$ maps surjectively onto any object, i.e. $FUX \twoheadrightarrow X$ is a regular epi. If P is projective, then we have the dotted arrow in

$$\begin{tikzcd}
 & P \arrow[dl, dashed] \arrow[d, "\mathrm{id}_P"] \\
FUP \arrow[r, two heads] & P
\end{tikzcd}$$

which exhibits P as a retract of the free FP.

($\Leftarrow$) Frees are projective and retracts of projectives are projective (standard argument, see e.g. [Bor94a, prop 4.6.4]). Explicitly, let $P \xrightarrow{s} F \xrightarrow{p} P$ exhibit P as a retract of a free, and consider the following diagram:

$$\begin{tikzcd}
 & F \arrow[d, shift right, "p"'] \arrow[ddl, dashed, two heads, "g''"'] \\
 & P \arrow[u, shift right, "s"'] \arrow[d, "g"] \arrow[dl, dashed, "g'"] \\
X \arrow[r, two heads, "f"'] & Y.
\end{tikzcd}$$

Since F is free, lift $gp : F \to Y$ to $g'' : F \to X$ by choosing f-preimages of the gp-image of the generators. (For this reason, a free is projective.) Now define $g' := g''s$, which is a lift of g, as we wanted:

$$fg' = fg''s = gps = g. \qquad \square$$

Remark 5.1.9. Again, this holds in any category of universal algebras, for the same reason.

Corollary 5.1.10. *The Postnikov truncation functor* $P_n : \mathbf{\Pi Alg} \to \mathbf{\Pi Alg}_1^n$ *sends projectives to projectives.*

Proof. A projective P in $\mathbf{\Pi Alg}$ is a retract of a free:

$$\begin{tikzcd} P \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & F \arrow[r] & P. \end{tikzcd}$$

Applying truncation P_n, we obtain a retract:

$$\begin{tikzcd} P_n(P) \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & P_n(F) \arrow[r] & P_n(P). \end{tikzcd}$$

By proposition 5.1.1, $P_n(F)$ is free in $\mathbf{\Pi Alg}_1^n$. By corollary 5.1.8, $P_n(P)$ is projective in $\mathbf{\Pi Alg}_1^n$. □

Note that $\mathbf{\Pi Alg}$ and $\mathbf{\Pi Alg}_1^n$ have enough projectives. Moreover, the regular epi $FUX \twoheadrightarrow X$ shows that the set of free objects $\pi_*(P_nS^1), \pi_*(P_nS^2), \ldots, \pi_*(P_nS^n)$ forms a set of generators for $\mathbf{\Pi Alg}_1^n$, i.e. any object receives a regular epi from a coproduct of generators. Recall the following definition.

Definition 5.1.11. An object P is **small** if $\mathrm{Hom}(P, -)$ preserves filtered colimits.

The next fact holds in any category of universal algebras C.

Proposition 5.1.12. *Any free object on a finite (graded) set of generators is small.*

Proof. Let J be a filtered category, S a finite (graded) set and $X = \mathrm{colim}_j X_j$. We have:

$$\begin{aligned} \mathrm{Hom}_C(F(S), \operatorname*{colim}_j X_j) &= \mathrm{Hom}_{\mathbf{Set}}\left(S, U(\operatorname*{colim}_j X_j)\right) \\ &= \mathrm{Hom}_{\mathbf{Set}}\left(S, \operatorname*{colim}_j U(X_j)\right) \\ &= \operatorname*{colim}_j \mathrm{Hom}_{\mathbf{Set}}\left(S, U(X_j)\right) \\ &= \operatorname*{colim}_j \mathrm{Hom}_C(F(S), X_j). \end{aligned}$$

In the second equality, we have used the fact that the "underlying set" functor preserves filtered colimits; see [Mac98, IX.1], proposition 2 and the remark after it. For the third equality, we have used the fact that every finite set is small. Indeed, $\mathrm{Hom}_{\mathbf{Set}}(S, -)$ is the product $\prod_S$, i.e. the limit over

S viewed as a discrete category, and finite limits commute with (small) filtered colimits [Mac98, IX.2, thm 1]. □

In particular, our category $\mathbf{\Pi Alg}_1^n$ has a set of small projective generators: $\pi_*(P_n S^1)$, $\pi_*(P_n S^2)$, $\ldots$, $\pi_*(P_n S^n)$.

Proposition 5.1.13. *The category $s\mathbf{\Pi Alg}_1^n$ of simplicial truncated Π-algebras has a standard model structure.*

Proof. $\mathbf{\Pi Alg}_1^n$ has finite limits (in fact all small limits), small colimits, and a set of small projective generators. Hence Quillen's theorem 4 applies. □

Recall that in the standard model structure, a map $f : X_\bullet \to Y_\bullet$ is a weak equivalence (resp. fibration) if the map

$$f_* : \mathrm{Hom}_C(P, X_\bullet) \to \mathrm{Hom}_C(P, Y_\bullet)$$

is a weak equivalence (resp. fibration) of simplicial sets. Cofibrations are the maps with the left lifting property with respect to acyclic fibrations. We'd like a more explicit description. In the case of simplicial Π-algebras, a map is a weak equivalence (resp. fibration) iff it is levelwise a weak equivalence (resp. fibration) of simplicial groups, and a cofibration iff it is a retract of a free map [BDG04, § 4.5]. We'll see that the description is essentially the same for truncated Π-algebras.

Let us use the adjunction

$$\begin{tikzcd} \mathbf{\Pi Alg} \arrow[r, shift left, "P_n"] & \mathbf{\Pi Alg}_1^n \arrow[l, shift left, "\iota_n"] \end{tikzcd}$$

to compare the two model structures.

Corollary 5.1.14. *The adjunction*

$$\begin{tikzcd} s\mathbf{\Pi Alg} \arrow[r, shift left, "P_n"] & s\mathbf{\Pi Alg}_1^n \arrow[l, shift left, "\iota_n"] \end{tikzcd}$$

is a Quillen pair.

Proof. The left adjoint $P_n : \mathbf{\Pi Alg} \to \mathbf{\Pi Alg}_1^n$ preserves projectives (by 5.1.10). □

In particular, P_n preserves cofibrations. In fact, more is true.

Proposition 5.1.15. *$P_n : s\mathbf{\Pi Alg} \to s\mathbf{\Pi Alg}_1^n$ preserves weak equivalences and fibrations.*

Proof. Let $f : X_\bullet \to Y_\bullet$ be a fibration (resp. weak eq) in $s\mathbf{\Pi Alg}$. Let P be a projective of $\mathbf{\Pi Alg}_1^n$, exhibited as a retract of a free by $P \overset{s}{\to} F \overset{p}{\to} P$. The following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{\Pi Alg}_1^n}(P, P_n X_\bullet) \arrow[r, "(P_n f)_*"] \arrow[d, "p^*"'] & \mathrm{Hom}_{\mathbf{\Pi Alg}_1^n}(P, P_n Y_\bullet) \arrow[d, "p^*"] \\
\mathrm{Hom}_{\mathbf{\Pi Alg}_1^n}(F, P_n X_\bullet) \arrow[r, "(P_n f)_*"] \arrow[d, "s^*"'] & \mathrm{Hom}_{\mathbf{\Pi Alg}_1^n}(F, P_n Y_\bullet) \arrow[d, "s^*"] \\
\mathrm{Hom}_{\mathbf{\Pi Alg}_1^n}(P, P_n X_\bullet) \arrow[r, "(P_n f)_*"] & \mathrm{Hom}_{\mathbf{\Pi Alg}_1^n}(P, P_n Y_\bullet)
\end{tikzcd}$$

exhibits $P_n f$ (in the top and bottom rows) as a retract of the middle row, so it suffices that this row be a fibration (resp. weak eq) of simplicial sets.

Note that $F = F_n(S)$ is free on a graded set S empty above dimension n, so we have:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{\Pi Alg}_1^n}(F, P_n X_\bullet) &= \mathrm{Hom}_{\mathbf{GrSet}}(S, UP_n X_\bullet) \\
&= \mathrm{Hom}_{\mathbf{GrSet}}(S, UX_\bullet) \\
&= \mathrm{Hom}_{\mathbf{\Pi Alg}}(F(S), X_\bullet).
\end{aligned}$$

Using this, we obtain:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{\Pi Alg}_1^n}(F, P_n X_\bullet) \arrow[r, "(P_n f)_*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathbf{\Pi Alg}_1^n}(F, P_n Y_\bullet) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathbf{\Pi Alg}}(F(S), X_\bullet) \arrow[r, "f_*"'] & \mathrm{Hom}_{\mathbf{\Pi Alg}}(F(S), Y_\bullet).
\end{tikzcd}$$

Since f is a fibration (resp. weak eq) in $s\mathbf{\Pi Alg}$, the bottom and top rows are fibrations (resp. weak eq) of simplicial sets. □

Now we can describe explicitly the model structure on $s\mathbf{\Pi Alg}_1^n$.

Proposition 5.1.16. *Let $f : X_\bullet \to Y_\bullet$ be a map in $s\mathbf{\Pi Alg}_1^n$. Then f is:*

1. *a fibration (resp. weak eq) iff $\iota_n f$ is one in $s\mathbf{\Pi Alg}$;*

2. *a cofibration iff it is a retract of a free map.*

Proof. 1. (⇒) As we have seen in corollary 5.1.14, ι_n preserves fibrations and weak equivalences.

(⇐) We have $f = P_n \iota_n f$, and we have seen in 5.1.15 that P_n preserves fibrations and weak equivalences.

2. This is Quillen's remark 4 at the end of section II.4 in [Qui67]. □

Remark 5.1.17. This proposition also holds in any category of universal algebras, which has underlying (possibly graded) sets. A map between simplicial objects is then a fibration (resp. weak equivalence) iff it is so at the level of underlying sets.

Remark 5.1.18. Note that the right Quillen functor ι_n does not preserve cofibrations, not even free maps or even free objects for that matter. Well, except $P_n\pi_*(S^1)$.

5.2 Truncation isomorphism

In this section, we use the machinery of 4.4 and the previous section to show that the Quillen cohomology of truncated Π-algebras can be computed "within the world of truncated Π-algebras". We use the adjunction

$$\begin{tikzcd}\mathbf{\Pi Alg} \arrow[r, shift left, "P_n"] & \mathbf{\Pi Alg}_1^n \arrow[l, shift left, "\iota_n"]\end{tikzcd}$$

which is extremely nice. The left adjoint P_n preserves limits (by C.0.7) and as we have seen in 5.1.15, it preserves all weak equivalences. Moreover, the counit $\epsilon : P_n\iota_n \to 1$ of the adjunction is the identity. Taking as base object a Π-algebra A in **ΠAlg**, apply (4.4.3) to obtain an equivalence of cotangent complexes:

$$P_n(L_A) \xrightarrow{\sim} L_{P_nA} \tag{5.2.1}$$

i.e. truncating the cotangent complex of a Π-algebra yields the cotangent complex of the truncation. Now for Quillen cohomology, start with a module N over P_nA. Applying (4.4.5), we get a natural iso:

$$\mathrm{HQ}^*_{\mathbf{\Pi Alg}_1^n}(P_nA; N) \xrightarrow{\cong} \mathrm{HQ}^*_{\mathbf{\Pi Alg}}(A; \eta_A^*\iota_nN) \tag{5.2.2}$$

where $\eta_A : A \to \iota_nP_nA$ is the Postnikov truncation map. Note that starting with a base object T in $\mathbf{\Pi Alg}_1^n$ yields nothing new, only the same results with the particular case $A = \iota_nT$.

We'd like a nicer description of $\eta_A^*\iota_nN$. Think of a module over A as an abelian Π-algebra on which A acts (cf. [BDG04, § 4.11]), i.e. the kernel of the split extension as opposed to its "total space".

Lemma 5.2.1. *The category* $\mathbf{Mod}_{P_nA}$ *of modules over* P_nA *is isomorphic to the full subcategory* $\mathbf{Mod}_A^{n\text{-}tr}$ *of* $\mathbf{Mod}_A$ *of modules that happen to be n-truncated.*

Proof. Look at the adjunction on modules:

$$\begin{tikzcd}\mathbf{Mod}_A \arrow[r, shift left, "P_n"] & \mathbf{Mod}_{P_nA} \arrow[l, shift left, "\eta_A^*\iota_n"]\end{tikzcd}$$

from 4.1.5. We already know the composite $P_n\eta_A^*\iota_n$ is the identity. Moreover, $\eta_A^*\iota_n$ lands in $\mathbf{Mod}_A^{n\text{-tr}}$. By restricting to the latter, we obtain the adjunction

$$\begin{tikzcd}\mathbf{Mod}_A^{n\text{-tr}} \arrow[r, shift left, "P_n"] & \mathbf{Mod}_{P_nA} \arrow[l, shift left, "\eta_A^*\iota_n"]\end{tikzcd}$$

where both composites $P_n\eta_A^*\iota_n$ and $\eta_A^*\iota_nP_n$ are the identity, i.e. an iso of categories. □

The lemma justifies the abuse of notation in the following repackaged statement.

Theorem 5.2.2. *Let A be a Π-algebra and N a module over A that is n-truncated. Then there is a natural iso*

$$\mathrm{HQ}^*_{\mathbf{\Pi Alg}_1^n}(P_nA; N) \xrightarrow{\cong} \mathrm{HQ}^*_{\mathbf{\Pi Alg}}(A; N).$$

Consequently, the Quillen cohomology of a Π-algebra with coefficients in a truncated module can be computed within the world of truncated Π-algebras. The following example is of interest to us in light of theorems 1.3 and 9.6 in [BDG04].

Example 5.2.3. Let A be an n-truncated Π-algebra. For k a positive integer, the k-fold loops Ω^kA form a module over A (which is zero if $k \geq n$) and we are interested in the cohomology groups $\mathrm{HQ}^*(A; \Omega^kA)$. Since Ω^kA is $(n-k)$-truncated, proposition 5.2.2 says:

$$\mathrm{HQ}^*_{\mathbf{\Pi Alg}_1^{n-k}}(P_{n-k}A; \Omega^kA) \simeq \mathrm{HQ}^*_{\mathbf{\Pi Alg}}(A; \Omega^kA).$$

Chapter 6

Realizations of 2-stages

In this chapter, we investigate some special cases of the realization problem for Π-algebras using the obstruction theory of [BDG04] (theorems 1.3 and 9.6). Let us recall their main results.

6.1 Blanc-Dwyer-Goerss obstruction theory

Start with a Π-algebra A and consider the moduli space $\mathcal{TM}(A)$ of its topological realizations. Try to build it using the moduli spaces $\mathcal{TM}_n(A)$ of "potential n-stages", which are simplicial spaces that look more and more like realizations of A.

- Geometric realization induces a weak equivalence $\mathcal{TM}_\infty(A) \xrightarrow{\sim} \mathcal{TM}(A)$.
- There is a weak equivalence $\mathcal{TM}_\infty(A) \xrightarrow{\sim} \operatorname{holim}_n \mathcal{TM}_n(A)$.
- Successive stages $\mathcal{TM}_n(A) \to \mathcal{TM}_{n-1}(A)$ are related in a certain fiber square.
- $\mathcal{TM}_0(A)$ is weakly equivalent to $B\operatorname{Aut}(A)$.

The interpretation in terms of π_0 is the following.

- A potential 0-stage exists and is unique (up to weak equivalence).
- Given a potential $(n-1)$-stage Y, there is an obstruction class

  $$o_Y \in \mathrm{HQ}^{n+2}(A; \Omega^n A) / \operatorname{Aut}(A, \Omega^n A)$$

  to lifting it to a potential n-stage.

- If Y is liftable, different lifts (up to weak equivalence) are classified by $\mathrm{HQ}^{n+1}(A;\Omega^n A)$, meaning that the group acts transitively on the set of lifts.
- Realizations correspond to successive lifts all the way up to infinity.

We can be more precise on the indeterminacy of classifying lifts. There is a fiber sequence [with correct indexing!]

$$\mathcal{H}_A^{n+1}(A;\Omega^n A) \to \mathcal{TM}_n(A)_Y \to \mathcal{M}(Y) \tag{6.1.1}$$

where $\mathcal{M}(Y)$ is the moduli space of Y (i.e. the component of Y in $\mathcal{TM}_{n-1}(A)$), $\mathcal{TM}_n(A)_Y$ consists of components of $\mathcal{TM}_n(A)$ sitting over it, and $\mathcal{H}_A^{n+1}(A;\Omega^n A)$ is a Quillen cohomology space, whose π_0 is the corresponding Quillen cohomology. In particular, the set of (weak equivalence classes of) lifts of Y is in bijection with $\mathrm{HQ}^{n+1}(A;\Omega^n A)/\pi_1\mathcal{M}(Y)$. The Quillen cohomology space satisfies more generally:

$$\pi_i\mathcal{H}_A^k(A;M) = \mathrm{HQ}^{k-i}(A;M). \tag{6.1.2}$$

as explained in [BDG04, 6.7].

6.2 Toy example: Eilenberg-Maclane spaces

As a warmup, take a Π-algebra A of Eilenberg-Maclane type $K(G,n)$, i.e. $A_n = G$ and all other groups are trivial. Note that this data determines A as a Π-algebra. More precisely, there exists a unique Π-algebra with this data, as long as G is an abelian group if $n \geq 2$.

Proposition 6.2.1. *Let k be a positive integer. In the example above, the Quillen cohomology groups $\mathrm{HQ}^i(A;\Omega^k A)$ are zero for all i.*

Proof. Since A is m-truncated, we have:

$$\mathrm{HQ}^i(A;\Omega^k A) \cong \mathrm{HQ}^i(P_{n-k}A;\Omega^k A) = \mathrm{HQ}^i(*;\Omega^k A)$$

where $*$ is the trivial Π-algebra. Since the latter is free, $\mathrm{HQ}^i(*;M)$ is zero for $i > 0$ and since it is the zero object in Π-algebras, $\mathrm{HQ}^0(*;M)$ is zero, for any module M over $*$ (i.e. abelian Π-algebra). □

Remark 6.2.2. There is another, more direct proof which doesn't use 5.2.2. Since A is $(n-1)$-connected, it admits a cofibrant replacement $C_\bullet$ whose constituent Π-algebras are all $(n-1)$-connected; take for example the simplicial resolution obtained by applying repeatedly the comonad

Free Forget (the pointed version). Now $\Omega^k A$ is $(n-1)$-truncated, so once we apply the "derivations" functor

$$\mathrm{Hom}_{\mathbf{\Pi Alg}/A}\left(-,\Omega^k A\right)$$

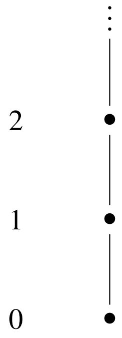
to $C_\bullet$, the resulting cosimplicial abelian group is identically zero.

Corollary 6.2.3. *A is realizable, in a unique way (up to weak equivalence).*

Proof. The obstructions to existence live in the trivial groups $\mathrm{HQ}^{k+2}(A;\Omega^k A)$, $k \geq 1$; obstructions to uniqueness live in the trivial groups $\mathrm{HQ}^{k+1}(A;\Omega^k A)$. □

We've recovered the fact that Eilenberg-Maclane spaces exist and are unique up to weak equivalence. Well, there are many simpler and nicer proofs of that fact, but it was only a warmup to use the machinery.

Let us represent the process schematically in a **realization tree** for A:



The bottom node represents the unique potential 0-stage, and the nodes above represent all possible successive lifts. According to theorems 9.3 and 9.4 of [BDG04], successive lifts all the way to infinity correspond to a topological realization of A.

We've only presented the π_0 point of view. However, a nice feature of the obstruction theory is that it gives information on the whole moduli space of realizations $\mathcal{TM}(A)$. Let us study the whole tower instead of only π_0 of each stage.

Proposition 6.2.4. *The topological monoid of self homotopy equivalences* $\mathrm{Aut}^h(K(G,m))$ *is discrete, with* $\pi_0 = \mathrm{Aut}(G)$*, the group of group automorphisms of* G*.*

Proof. By (6.1.2) and 6.2.1, the fiber in the fiber sequence (6.1.1) is weakly contractible for all $n \geq 1$. Therefore, the tower consists of weak equivalences $\mathcal{TM}_n \xrightarrow{\sim} \mathcal{TM}_{n-1}$ at all stages and we obtain $\mathcal{TM}_\infty \xrightarrow{\sim} \mathcal{TM}_0 \simeq B\,\mathrm{Aut}(A)$. By the structure theorem for moduli spaces, we know

$\mathcal{TM}(A) \simeq \coprod_{\langle X \rangle} B\,\mathrm{Aut}^h(X)$, where the coproduct runs over (weak) homotopy types of realizations of A, of which there is only one in this case. In short, we have:

$$B\,\mathrm{Aut}^h(K(G,m)) \simeq \mathcal{TM}(A) \xleftarrow{\sim} \mathcal{TM}_\infty \xrightarrow{\sim} \mathcal{TM}_0 \simeq B\,\mathrm{Aut}(A)$$

so that the 0^{th} Postnikov truncation

$$\mathrm{Aut}^h(K(G,m)) \to \pi_0\,\mathrm{Aut}^h(K(G,m))$$

is a weak equivalence and we have $\pi_0\,\mathrm{Aut}^h(K(G,m)) \simeq \mathrm{Aut}(A) = \mathrm{Aut}(G)$. □

Of course, it is not hard to show this fact using classical methods.

Proof. The topological monoid of pointed self-maps $\mathrm{Map}_*\left(K(G,m), K(G,m)\right)$ is discrete:

$$\begin{aligned}
&\pi_i\,\mathrm{Map}_*\left(K(G,m), K(G,m)\right) \\
&\quad = \pi_0\,\mathrm{Map}_*\left(S^i, \mathrm{Map}_*\left(K(G,m), K(G,m)\right)\right) \\
&\quad = \pi_0\,\mathrm{Map}_*\left(S^i \wedge K(G,m), K(G,m)\right) \\
&\quad = \left[\Sigma^i K(G,m), K(G,m)\right]_* \\
&\quad = \widetilde{\mathrm{H}}^m\left(\Sigma^i K(G,m); G\right) \\
&\quad = \widetilde{\mathrm{H}}^{m-i}\left(K(G,m); G\right) \\
&\quad = \begin{cases} \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{H}_m\,K(G,m), G\right) = \mathrm{Hom}_{\mathbb{Z}}(G,G) & \text{for } i = 0 \\ 0 & \text{for } i \geq 1. \end{cases}
\end{aligned}$$

Here we used the Hurewicz theorem and the universal coefficient theorem. Note that the result still holds for $m = 1$ and G a non-abelian group, in which case we need to say π_0 is $\mathrm{Hom}_{\mathbf{Gp}}(G,G)$, a formula that covers all cases.

Now $\mathrm{Aut}^h(K(G,m))$ is the submonoid of $\mathrm{Map}_*\left(K(G,m), K(G,m)\right)$ consisting of (pointed) self homotopy equivalences, which means it is the union of path components corresponding to invertible elements. We conclude:

$$\pi_i\,\mathrm{Aut}^h(K(G,m)) = \begin{cases} \mathrm{Aut}(G) & \text{for } i = 0 \\ 0 & \text{for } i \geq 1. \end{cases}$$

□

As mentioned above, it was only a toy example to see the machinery at work.

6.3 2-types and a variant

Now for a more serious application, take A to be a 2-truncated Π-algebra, i.e. A_1 is a group, A_2 is a module over it, and all the groups above dimension 2 are trivial. As we saw in 2.2.9, this is precisely the data of a 2-truncated Π-algebra.

6.3.1 π_0 point of view

We first seek to describe the set of realizations of A.

Proposition 6.3.1. *Any* 2*-truncated* Π*-algebra is realizable.*

Proof. The action of A_1 on A_2 induces a (pointed) action of A_1 on the Eilenberg-Maclane space $K(A_2, 2)$. Use this action to form the Borel construction

$$\begin{tikzcd} X = EA_1 \times_{A_1} K(A_2, 2) \arrow[d] \\ BA_1 \end{tikzcd}$$

By the long exact sequence of the fibration $K(A_2, 2) \to X \to BA_1$, we have $\pi_i(X) \simeq A_i$ for all i, and by construction, the action of $\pi_1(X)$ on $\pi_2(X)$ is the prescribed one. In other words, $\pi_*(X)$ is isomorphic to A as a Π-algebra. □

Proposition 6.3.2. *Realizations of A are classified by group cohomology* $\mathrm{H}^3(A_1; A_2)$*. More precisely, their weak equivalence classes are in bijection with*

$$\mathrm{H}^3(A_1; A_2)/\operatorname{Aut}(A).$$

Proof. Notice that the module $\Omega^k A$ is zero for $k \geq 2$. This means once we reach a potential 1-stage, it lifts uniquely up to infinity. The only possible obstruction that could happen is in lifting from the potential 0-stage to a potential 1-stage. Since we know A is realizable, the obstruction to existence vanishes. Lifts to potential 1-stages are parametrized by

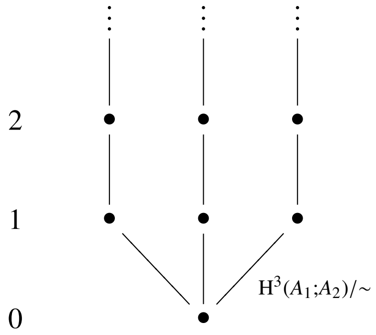
$$\mathrm{HQ}^2(A; \Omega A) \cong \mathrm{HQ}^2_{\mathbf{\Pi Alg}^1_1}(P_1 A; \Omega A) = \mathrm{HQ}^2_{\mathbf{Gp}}(A_1; A_2)$$

and as we have seen in A.1.7, this Quillen cohomology in groups is equal to group cohomology (shifted by one):

$$\mathrm{H}^3(A_1; A_2).$$

The indeterminacy is the action of $\pi_1 \mathcal{T}\mathcal{M}_0(A) \cong \pi_1 B\,\mathrm{Aut}(A) \cong \mathrm{Aut}(A)$. □

The realization tree for A looks like this:



Remark 6.3.3. The topological interpretation of the cohomology class in $\mathrm{H}^3(A_1; A_2)$ is the 1st k-invariant of the realizing space X, which is the obstruction to a cross-section of the Postnikov truncation $X \cong P_2X \to P_1X \simeq K(A_1, 1)$.

Here is one way of establishing the interpretation. Use the correspondence of obstruction theories between the Quillen cohomology of Π-algebras and $(\mathcal{S}, \mathcal{O})$-cohomology [BB09, thm 4.5] of a suitable diagram that encodes the Π-algebra we're trying to realize [BB09, 4.3]. The obstruction theory with $(\mathcal{S}, \mathcal{O})$-cohomology consists of lifting hom-sets to mapping spaces in a coherent way, one Postnikov level at a time. When lifting the "mapping n-types" to $(n + 1)$-types, the cohomology classes parametrizing different lifts are essentially the n^{th} k-invariants of the mapping spaces [BB09, 5.5].

Hence we recover a classic result of Whitehead and Maclane on the classification of pointed homotopy 2-types [MW50, thm 1, 2].

We can extend the argument to a more general case. Let A be a Π-algebra with a group A_1 and an A_1-module A_n in dimension n, and zero elsewhere. Again, there is a unique Π-algebra with such data. The Borel construction in 6.3.1 also works to realize this A (just replace 2 by n).

Proposition 6.3.4. *Realizations of A are classified by group cohomology* $\mathrm{H}^{n+1}(A_1; A_n)$. *More pre-*

cisely, their weak equivalence classes are in bijection with

$$\mathrm{H}^{n+1}(A_1; A_n)/\operatorname{Aut}(A).$$

Proof. The module $\Omega^k A$ is zero for $k \geq n$, so once we reach a potential $(n-1)$-stage, it lifts uniquely up to infinity. To get there from the potential 0-stage, we encounter some obstruction groups $\mathrm{HQ}^*(A; \Omega^k A)$. For $1 \leq k \leq n-2$, these groups are in fact all trivial. Indeed, we have:

$$\mathrm{HQ}^i(A; \Omega^k A) \cong \mathrm{HQ}^i(P_{n-k}A; \Omega^k A) = \mathrm{HQ}^i(A_1; \Omega^k A)$$

where the abuse of notation A_1 means the Π-algebra with the group A_1 in dimension 1 and trivial elsewhere. Such a Π-algebra admits a cofibrant replacement $C_\bullet$ whose constituent Π-algebras are also concentrated in dimension 1; take for example the simplicial resolution obtained from Free Forget. In other words, a cofibrant replacement of such a Π-algebra is just a cofibrant replacement of the group A_1. (This is a very special feature of the free Π-algebra $\pi_*(S^1)$.) Since $\Omega^k A$ is concentrated in dimension $n-k > 1$, applying the "derivations" functor

$$\mathrm{Hom}_{\mathbf{\Pi Alg}/A}\left(-, \Omega^k A\right)$$

to $C_\bullet$ yields a cosimplicial abelian group which is identically zero. It follows that $\mathcal{H}^i(A; \Omega^k A)$ is in fact contractible for $1 \leq k \leq n-2$, so that the maps

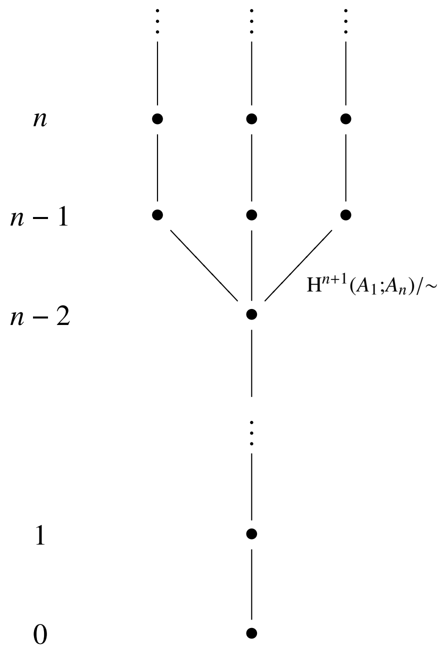
$$\mathcal{TM}_{n-2}(A) \to \cdots \to \mathcal{TM}_1(A) \to \mathcal{TM}_0(A) \simeq B\operatorname{Aut}(A)$$

in the tower are all weak equivalences. The only possible obstruction is in lifting from the potential $(n-2)$-stage to a potential $(n-1)$-stage. Since we know A is realizable, the obstruction to existence vanishes. Lifts to potential $(n-1)$-stages are parametrized by

$$\begin{aligned} \mathrm{HQ}^n(A; \Omega^{n-1}A) &\cong \mathrm{HQ}^n_{\mathbf{\Pi Alg}_1^1}(P_1 A; \Omega^{n-1}A) \\ &= \mathrm{HQ}^n_{\mathbf{Gp}}(A_1; A_n) \\ &\cong \mathrm{H}^{n+1}(A_1; A_n) \end{aligned}$$

and the indeterminacy is the action of $\pi_1(\mathcal{TM}_{n-2}(A)) \cong \operatorname{Aut}(A)$. □

The realization tree in this case looks like this:



6.3.2 Moduli point of view

Now we want to describe the moduli space of realizations $\mathcal{TM}(A)$.

Theorem 6.3.5.

$$\pi_i\mathcal{TM}(A) \simeq \begin{cases} \mathrm{H}^{n+1}(A_1;A_n)/\operatorname{Aut}(A) & \textit{for } i = 0 \\ \mathrm{H}^{n+1-i}(A_1;A_n) & \textit{for } 2 \le i < n \\ \operatorname{Der}(A_1, A_n) & \textit{for } i = n \\ 0 & \textit{for } i > n \end{cases}$$

and $\pi_1\mathcal{TM}(A)$ *is an extension by* $\mathrm{H}^n(A_1;A_n)$ *of a subgroup of* $\operatorname{Aut}(A)$ *corresponding to realizable automorphisms.*

Proof. We know that $\mathcal{TM}_\infty \overset{\sim}{\to} \mathcal{TM}_{n-1}$ is a weak equivalence. Using the identifications $\mathcal{TM}_\infty \overset{\sim}{\to} \mathcal{TM}$ and $\mathcal{TM}_{n-2} \overset{\sim}{\to} \mathcal{TM}_0 \simeq B\operatorname{Aut}(A)$, we can exhibit the moduli space $\mathcal{TM}$ as the total space of a fiber sequence:

$$\mathcal{H}^n(A;\Omega^{n-1}A) \to \mathcal{TM} \to B\operatorname{Aut}(A). \tag{6.3.1}$$

By the long exact sequence of homotopy groups, we obtain:

$$\pi_i \mathcal{H}^n(A; \Omega^{n-1}A) \xrightarrow{\simeq} \pi_i \mathcal{TM}$$

for all $i \geq 2$, which proves the claim in that range. The bottom part of the long exact sequence is:

$$\begin{aligned} \pi_2 B\operatorname{Aut}(A) \to & \pi_1 \mathcal{H}^n(A; \Omega^{n-1}A) \to \pi_1 \mathcal{TM} \to \pi_1 B\operatorname{Aut}(A) \to \\ & \to \pi_0 \mathcal{H}^n(A; \Omega^{n-1}A) \to \pi_0 \mathcal{TM} \to \pi_0 B\operatorname{Aut}(A) \end{aligned}$$

which we can write as

$$0 \to \mathrm{H}^n(A_1; A_n) \to \pi_1 \mathcal{TM} \to \operatorname{Aut}(A) \to \mathrm{H}^{n+1}(A_1; A_n) \to \pi_0 \mathcal{TM} \to *$$

remembering that $\mathrm{H}^{n+1}(A_1; A_n)$ is really a pointed set, on which $\operatorname{Aut}(A)$ acts. The sequence gives us $\pi_0 \mathcal{TM} \simeq \mathrm{H}^{n+1}(A_1; A_n)/\operatorname{Aut}(A)$. Now the kernel of $\operatorname{Aut}(A) \to \mathrm{H}^{n+1}(A_1; A_n)$ is precisely the stabilizer of the basepoint $\kappa \in \mathrm{H}^{n+1}(A_1; A_n)$, so we get a short exact sequence of groups:

$$\mathrm{H}^n(A_1; A_n) \hookrightarrow \pi_1 \mathcal{TM} \twoheadrightarrow \operatorname{Stab}(\kappa)$$

which proves the claim on $\pi_1 \mathcal{TM}$. □

Interpretation Working with pointed spaces instead of simplicial groups, one can see that the space $\mathcal{H}^n(A; \Omega^{n-1}A)$ above is equivalent to the derived mapping space $\operatorname{Map}_{BA_1}(BA_1, BA_1(A_n, n+1))$ of pointed maps over BA_1 [BDG04, 3.6]. That is why they have the same homotopy groups, as computed previously. Here we used the notation of [BDG04, 3.1], where $BG(M, i)$ denotes the extended Eilenberg-Maclane object $EG \times_G K(M, i)$.

The fiber sequence 6.3.1 exhibits $\mathcal{TM}$ as the homotopy quotient $(\mathcal{H}^n)_{h\operatorname{Aut}(A)}$. In fact, there is a general phenomenon at work. As explained in [BDG04, 1.1], the moduli space $\mathcal{TM}(A)$ always fibers over $B\operatorname{Aut}(A)$, with fiber the relative moduli space $\mathcal{TM}'(A)$ consisting of realizations X equipped with an identification $f : \pi_* X \simeq A$. In our case, the map $\mathcal{TM}(A) \to B\operatorname{Aut}(A)$ can be identified with $\mathcal{TM}_{n-1} \to \mathcal{TM}_{n-2}$, whose fiber $\mathcal{H}^n(A; \Omega^{n-1}A)$ is therefore equivalent to the relative moduli space $\mathcal{TM}'(A)$.

More explicitly: Picking an actual k-invariant $\kappa \in \mathrm{H}^{n+1}(A_1; A_n)$ is the same as choosing an (isomorphism class of) identification $f : \pi_* X \simeq A$ for the realization X corresponding to κ. The action of $\varphi \in \mathrm{Aut}(A)$ on $\mathrm{H}^{n+1}(A_1; A_n)$ corresponds to the postcomposition action $\varphi \cdot (X, f) = (X, \varphi f)$. The stabilizer of (X, f) is the subgroup of all $\varphi \in \mathrm{Aut}(A)$ satisfying

$$(X, f) = \varphi \cdot (X, f) = (X, \varphi f)$$

in that set of isomorphism classes. In other words, there is a self weak equivalence $h : X \simeq X$ realizing the automorphism:

$$\begin{tikzcd}
\pi_* X \arrow[r, "f"] \arrow[d, "\pi_* h"'] & A \arrow[d, "\varphi"] \\
\pi_* X \arrow[r, "f"'] & A.
\end{tikzcd}$$

That is why $\mathrm{Stab}(\kappa)$ corresponds to the subgroup of realizable automorphisms of A, once all basepoints have been chosen.

Conclusion 6.3.6. *Realizations of A correspond to* $\mathrm{H}^{n+1}(A_1; A_n)/\,\mathrm{Aut}(A)$*, viewed as the k-invariant up to its indeterminacy. This π_0 statement can be promoted to a moduli statement: The moduli space of realizations is weakly equivalent to the mapping space where the k-invariant lives, up to the action of* $\mathrm{Aut}(A)$*:*

$$\mathcal{TM}(A) \simeq \mathrm{Map}_{BA_1}\left(BA_1, BA_1(A_n, n+1)\right)_{h\,\mathrm{Aut}(A)}.$$

Taking the relative moduli space removes the indeterminacy:

$$\mathcal{TM}'(A) \simeq \mathrm{Map}_{BA_1}\left(BA_1, BA_1(A_n, n+1)\right).$$

There are other approaches to describing the moduli space of 2-stage spaces in the literature. See [MS09, § 2] for an overview. However, some of the approaches are less convenient when dealing with the non simply-connected case.

Chapter 7

Computations with 2-truncated Π-algebras

In chapter 6, we have seen that the obstruction theory works out nicely for 2-types. Can we use it to study realizations of 3-types? The primary obstructions to realizing a 3-truncated Π-algebra A lie in $\mathrm{HQ}^*(A; \Omega A) = \mathrm{HQ}^*(P_2 A; \Omega A)$, that is, Quillen cohomology of a 2-truncated Π-algebra. In this chapter, we develop tools to compute such cohomology groups.

Definition 7.0.7. Consider the category **ModGp** (fibred over **Gp**) whose objects are pairs (G, A), where G is a group and A is a (left) module over G. Morphisms are pairs of maps $(f_1 : G \to G', f_2 : A \to A')$ such that f_2 is f_1-equivariant, i.e. $f_2(g \cdot a) = f_1(g) \cdot f_2(a)$.

Remark 7.0.8. Baues denotes this category $\mathbf{Mod}_{\mathbb{Z}}^{\wedge}$ [Bau91, def I.1.7].

We are interested in this category because it is the category $\mathbf{\Pi Alg}_1^2$ of 2-truncated Π-algebras. However, the notation **ModGp** is used to suggest a more general construction which we explore in appendix B.

7.1 Beck modules

Our first task is to describe Beck modules in the category **ModGp**.

Lemma 7.1.1. *The forgetful functor* $\pi_2 : \mathbf{ModGp} \to \mathbf{Ab}$ *which sends* (G, A) *to the abelian group* A *has a left adjoint, which sends an abelian group* A *to* $(1, A)$*.*

Proof. A map in $\mathrm{Hom}_{\mathbf{ModGp}}((1,A),(G,B))$ is the data of a group map $1 \to G$ (no data) and an equivariant map $\varphi : A \to B$. But here the equivariance condition $\varphi(1 \cdot a) = 1 \cdot \varphi(a)$ is automatic, hence we conclude:

$$\mathrm{Hom}_{\mathbf{ModGp}}((1,A),(G,B)) \cong \mathrm{Hom}_{\mathbf{Ab}}(A,B).$$

□

In fact, **ModGp** has free objects.

Proposition 7.1.2. *The "underlying set" functor* $U : \mathbf{ModGp} \to \mathbf{Set}^2$ *that sends* (G,A) *to* (UG,UA) *has a left adjoint ("free" functor)* $F : \mathbf{Set}^2 \to \mathbf{ModGp}$ *which sends a pair of sets* (S,T) *to* $(FS, \mathbb{Z}(FS)[T])$*, i.e.* π_1 *is the free group on* S *and* π_2 *is the free* π_1*-module.*

Proof. Let's forget in two steps:

$$\begin{tikzcd}
\mathbf{ModGp} \arrow[r, "U_2"] & \mathbf{Gp} \times \mathbf{Set} \arrow[r, "U_1 \times \mathrm{id}"] & \mathbf{Set} \times \mathbf{Set} \\
(G,A) \arrow[r, mapsto] & (G,UA) \arrow[r, mapsto] & (UG,UA)
\end{tikzcd}$$

so we can free up in two corresponding steps. The left adjoint of $U_1 \times \mathrm{id}$ is $F \times \mathrm{id}$, where F means the free group functor. To complete the proof, we need to show that the left adjoint of U_2 is the "free π_1-module" functor defined by $F(G,T) = (G, \mathbb{Z}G[T])$.

$$\begin{aligned}
&\mathrm{Hom}_{\mathbf{ModGp}}(F(G,T),(G',A')) \\
&= \mathrm{Hom}_{\mathbf{ModGp}}((G,\mathbb{Z}G[T]),(G',A')) \\
&= \{\text{group map } \varphi_1 : G \to G' \text{ and } \varphi_1\text{-equivariant map } \varphi_2 : \mathbb{Z}G[T] \to A'\} \\
&= \left\{\varphi_1 : G \to G' \text{ and map of } G\text{-modules } \varphi_2 : \mathbb{Z}G[T] \to \varphi_1^* A'\right\} \\
&= \left\{\varphi_1 : G \to G' \text{ and map of sets } T \to U(\varphi_1^* A') = UA'\right\} \\
&= \mathrm{Hom}_{\mathbf{Gp}}(G,G') \times \mathrm{Hom}_{\mathbf{Set}}(T,UA') \\
&= \mathrm{Hom}_{\mathbf{Gp}\times\mathbf{Set}}((G,T),(G',UA')) \\
&= \mathrm{Hom}_{\mathbf{Gp}\times\mathbf{Set}}((G,T),U_2(G',A')).
\end{aligned}$$

□

Proposition 7.1.3. *A Beck module over an object* (G,A) *consists of the following data:*

- *G-modules M and B;*

- *A G-map* $\varphi : M \otimes_{\mathbb{Z}} A \to B$*, where the tensor product has the diagonal G-action.*

The correspondence is as follows: Given this data one constructs

$$\begin{pmatrix} A \oplus B \\ G \ltimes M \end{pmatrix} \xrightarrow{p} \begin{pmatrix} A \\ G \end{pmatrix},$$

where the action of $G \ltimes M$ *on* $A \oplus B$ *is given by*

$$(g, m) \cdot (a, b) = (g \cdot a, \varphi(m, g \cdot a) + g \cdot b).$$

Conversely, given a Beck module

$$\begin{tikzcd} \begin{pmatrix} A' \\ G' \end{pmatrix} \arrow[r, shift left, "p"] & \begin{pmatrix} A \\ G \end{pmatrix} \arrow[l, shift left, "e"] \end{tikzcd}$$

one extracts the data as follows.

- $M := \ker(\pi_1 p : G' \to G)$*, endowed with the usual G-action (cf. remark after A.1.1);*

- $B := \ker(\pi_2 p : A' \to A)$*, with the G-action* $g \cdot b = proj_B(e(g) \cdot b)$*;*

- $\varphi(m, a) := proj_B(m \cdot e(a))$*.*

Proof. Since the functors π_1 : **ModGp** → **Gp** and π_2 : **ModGp** → **Ab** preserve limits, by proposition 1.1.1 and the structure of Beck modules in **Gp** and **Ab** (cf. A.1.1 and A.2.1), we know that a Beck module in **ModGp** is of the form

$$\begin{pmatrix} A \oplus B \\ G \ltimes M \end{pmatrix} \xrightarrow{p} \begin{pmatrix} A \\ G \end{pmatrix},$$

where B is an abelian group and M is a (left) G-module. This determines the projection map p and the three structure maps; moreover, they automatically satisfy the abelian group object conditions. The only additional data is that the "total space" $(G \ltimes M, A \oplus B)$ is an object of **ModGp**, i.e. $A \oplus B$ has a $(G \ltimes M)$-action. The additional conditions are that the projection map is a map in **ModGp** and the three structure maps are maps in **ModGp**$/(G, A)$. Let us describe those conditions.

1. The projection map p is a map in ModGp.

$$\begin{aligned} p_2((g,m)\cdot(a,b)) &= p_1(g,m)\cdot p_2(a,b) \\ &= g\cdot a. \end{aligned}$$

In other words, the A-component of $(g,m)\cdot(a,b)$ is just $g\cdot a$. Let us denote its B-component by $\psi(g,m,a,b)$, so we have

$$(g,m)\cdot(a,b) = (g\cdot a, \psi(g,m,a,b)).$$

2. The multiplication map μ is a map in ModGp$/(G,A)$. First, we have the following pullback in **ModGp**:

$$\begin{pmatrix} A\oplus B \\ G\ltimes M \end{pmatrix} \times_{\begin{pmatrix} A \\ G \end{pmatrix}} \begin{pmatrix} A\oplus B \\ G\ltimes M \end{pmatrix} = \begin{pmatrix} A\oplus B\oplus B \\ G\ltimes(M\times M) \end{pmatrix},$$

where G acts diagonally on $M\times M$ and $G\ltimes(M\times M)$ acts on $A\oplus B\oplus B$ as follows:

$$(g,m_1,m_2)\cdot(a,b_1,b_2) = (g\cdot a, \psi(g,m_1,a,b_1), \psi(g,m_2,a,b_2)).$$

The multiplication map μ is given by the formula

$$\mu\begin{pmatrix} a,b_1,b_2 \\ g,m_1,m_2 \end{pmatrix} = \begin{pmatrix} a,b_1+b_2 \\ g,m_1+m_2 \end{pmatrix},$$

which commutes with the projection down to (G,A). Applying μ to the left-hand side of the previous equality, we obtain

$$\begin{aligned} \mu\left((g,m_1,m_2)\cdot(a,b_1,b_2)\right) &= \mu(g,m_1,m_2)\cdot\mu(a,b_1,b_2) \\ &= (g,m_1+m_2)\cdot(a,b_1+b_2) \\ &= (g\cdot a, \psi(g,m_1+m_2,a,b_1+b_2). \end{aligned}$$

Applying μ to the right-hand side yields

$$\mu\left(g\cdot a, \psi(g,m_1,a,b_1), \psi(g,m_2,a,b_2)\right) = (g\cdot a, \psi(g,m_1,a,b_1)+\psi(g,m_2,a,b_2)).$$

Equating the two sides yields the condition

$$\psi(g, m_1 + m_2, a, b_1 + b_2) = \psi(g, m_1, a, b_1) + \psi(g, m_2, a, b_2). \tag{7.1.1}$$

In particular, setting b_1 and m_2 to 0, we obtain

$$\psi(g, m, a, b) = \psi(g, m, a, 0) + \psi(g, 0, a, b).$$

3. The unit map e is a map in ModGp$/(G, A)$.

$$\begin{aligned} e(g \cdot a) &= e(g) \cdot e(a) \\ (g \cdot a, 0) &= (g, 0) \cdot (a, 0) \\ &= (g \cdot a, \psi(g, 0, a, 0) \end{aligned}$$

We obtain the condition $\psi(g, 0, a, 0) = 0$, which we already knew from (7.1.1) by setting m_2 and b_2 to 0.

4. The inverse map ι is a map in ModGp$/(G, A)$.

$$\begin{aligned} \iota\left((g, m) \cdot (a, b)\right) &= \iota(g, m) \cdot \iota(a, b) \\ (g \cdot a, -\psi(g, m, a, b)) &= (g, -m) \cdot (a, -b) \\ &= (g \cdot a, \psi(g, -m, a, -b) \end{aligned}$$

We obtain the condition $\psi(g, -m, a, -b) = -\psi(g, m, a, b)$, which we already knew from (7.1.1) by setting $m_2 = -m_1$ and $b_2 = -b_1$.

Now let us describe the action of $G \ltimes M$ on $A \oplus B$. Applying the action to $(a, b) = (a, 0) + (0, b)$ yields

$$\psi(g, m, a, b) = \psi(g, m, a, 0) + \psi(g, m, 0, b).$$

Combined with condition (7.1.1), we obtain:

$$\psi(g, 0, a, b) = \psi(g, m, 0, b) \text{ for all } g, m, a, b,$$

so these are all equal to $\psi(g, 0, 0, b)$. This defines an action of G on B. Indeed, for a fixed $g \in G$, the formula $g * b := \psi(g, 0, 0, b)$ gives a linear endomorphism of B, and they compose correctly:

$$\begin{aligned}
(gg', 0) \cdot (0, b) &= (g, 0) \cdot ((g', 0) \cdot (0, b)) \\
&= (g, 0) \cdot (0, \psi(g', 0, 0, b)) \\
&= (0, \psi\,(g, 0, 0, \psi(g', 0, 0, b)))\,; \\
gg' * b &= \psi(gg', 0, 0, b) \\
&= B - \text{component of } (gg', 0) \cdot (0, b) \\
&= \psi\,(g, 0, 0, \psi(g', 0, 0, b)) \\
&= g * (g' * b).
\end{aligned}$$

We will denote this action $g \cdot b$ from now on. Let us now describe the other piece of data, coming from $\psi(g, m, a, 0)$:

$$\begin{aligned}
(g, m) \cdot (a, 0) &= (1, m)(g, 0) \cdot (a, 0) \\
&= (1, m) \cdot ((g, 0) \cdot (a, 0)) \\
&= (1, m) \cdot (g \cdot a, 0) \\
&= (g \cdot a, \psi(1, m, g \cdot a, 0).
\end{aligned}$$

Taking B-components, we obtain:

$$\psi(g, m, a, 0) = \psi(1, m, g \cdot a, 0).$$

In light of this, let us define:

$$\varphi(m, a) := \psi(1, m, a, 0),$$

which is linear in m and in a, thus can be viewed as a map $\varphi : M \otimes_{\mathbb{Z}} A \to B$. To see how this map interacts with the G-actions, let's use the factorization $(g, 0)(1, m) = (g, g \cdot m)$:

$$\begin{aligned}
(g, g \cdot m) \cdot (a, 0) &= (g, 0) \cdot ((1, m) \cdot (a, 0)) \\
(g \cdot a, \psi(g, g \cdot m, a, 0)) &= (g, 0) \cdot (a, \psi(1, m, a, 0)) \\
&= (g \cdot a, \psi\,(g, 0, a, \psi(1, m, a, 0)))
\end{aligned}$$

Taking B-components, we obtain:

$$\begin{aligned}
\psi(g, g\cdot m, a, 0) &= \psi\,(g, 0, a, \psi(1, m, a, 0)) \\
\psi(1, g\cdot m, g\cdot a, 0) &= \psi\,(g, 0, 0, \psi(1, m, a, 0)) \\
\varphi(g\cdot m, g\cdot a) &= g\cdot \psi(1, m, a, 0) \\
&= g\cdot \varphi(m, a).
\end{aligned}$$

Hence the map φ is G-linear, where G acts diagonally on $M \otimes_{\mathbb{Z}} A$. The map φ and the G-action on B determine the action of $G \ltimes M$ on $A \oplus B$, since we have $\psi(g, m, a, b) = \varphi(m, g\cdot a) + g\cdot b$ and thus:

$$(g, m)\cdot(a, b) = (g\cdot a, \varphi(m, g\cdot a) + g\cdot b)\,. \tag{7.1.2}$$

This proves one direction of the correspondence.

Conversely, given G-modules M and B, and a G-linear map $\varphi : M \otimes_{\mathbb{Z}} A \to B$, we want to show that the formulas above define a Beck module over (G, A). First, let us check that the formula (7.1.2) defines an actions of $G \ltimes M$ on $A \oplus B$. For fixed (g, m) it's clearly an endomorphism of $A \oplus B$. Let us check that they compose correctly:

$$\begin{aligned}
(g, m)\cdot((g', m')\cdot(a, b)) &= (g, m)\cdot(g'\cdot a, \varphi(m', g'\cdot a) + g'\cdot b) \\
&= (gg'\cdot a, \varphi(m, gg'\cdot a) + g\cdot\varphi(m', g'\cdot a) + gg'\cdot b)\,; \\
((g, m)(g', m'))\cdot(a, b) &= (gg', m + g\cdot m')\cdot(a, b) \\
&= (gg'\cdot a, \varphi(m + g\cdot m', gg'\cdot a) + gg'\cdot b) \\
&= (gg'\cdot a, \varphi(m, gg'\cdot a) + \varphi(g\cdot m', gg'\cdot a) + gg'\cdot b) \\
&= (gg'\cdot a, \varphi(m, gg'\cdot a) + g\cdot\varphi(m', g'\cdot a) + gg'\cdot b)\,.
\end{aligned}$$

By our discussion of Beck modules in **ModGp**, the only thing that remains to check is that the multiplication map μ is in **ModGp**$/(G, A)$, which is equivalent to condition (7.1.1) above. In our construction, the condition becomes

$$\varphi(m + m', g\cdot a) + g\cdot(b + b') = \varphi(m, g\cdot a) + g\cdot b + \varphi(m', g\cdot a) + g\cdot b'$$

which holds indeed. □

Proposition 7.1.4. *In the correspondence of proposition 7.1.3, a morphism in* $Mod_{(G,A)}$ *from* (M, B, φ) *to* (M', B', φ') *corresponds to the data of* G*-maps* $\alpha : M \to M'$ *and* $\beta : B \to B'$ *such that the following diagram commutes:*

$$\begin{tikzcd}
M \otimes A \arrow[r, "\varphi"] \arrow[d, "\alpha\otimes 1"'] & B \arrow[d, "\beta"] \\
M' \otimes A \arrow[r, "\varphi'"'] & B'.
\end{tikzcd}$$

Proof. Easy verification. □

The category of modules over (G, A) has a good notion of tensor product.

Definition 7.1.5. The **tensor product** of two (G, A)-modules (M, B, φ) and (M', B', φ') is

$$(M, B, \varphi) \otimes (M', B', \varphi') =$$
$$= (M \otimes M', M \otimes B' \oplus B \otimes M', m \otimes m' \otimes a \mapsto (m, \varphi'(m', a), \varphi(m, a) \otimes m')$$

where by default, $\otimes$ applied to G-modules means tensored over $\mathbb{Z}$, with the diagonal G-action. This tensor product should be thought of as "over the action of the object $(\{1\}, 0)$ of **ModGp**", or as "over the module $(\mathbb{Z}, 0, 0)$", by analogy with the tensor product of modules over groups.

Proposition 7.1.6.

1. $\mathbf{Mod}_{(G,A)}$ *equipped with the direct sum* $\oplus$ *and tensor product* $\otimes$ *is a bimonoidal category, where the zero module plays the role of zero object, and* $(\mathbb{Z}, 0, 0)$ *is the unit object (here* $\mathbb{Z}$ *has the trivial* G*-action).*
2. *The full embedding* $\mathrm{Zero} : \mathbf{Mod}_G \to \mathbf{Mod}_{(G,A)}$ *and the forgetful functor* $\pi_1 : \mathbf{Mod}_{(G,A)} \to \mathbf{Mod}_G$ *are bimonoidal functors, i.e. compatible with the usual* $\oplus$ *and* $\otimes$ *in* $\mathbf{Mod}_G$*.*

Proof. Routine verification. □

7.2 Abelianization

Now we set out to find the abelianization of $(G, A) \overset{\mathrm{id}}{\to} (G, A)$.

Lemma 7.2.1. *If* A *is a* G*-module, then we have naturally isomorphic* G*-modules:*

$$\mathbb{Z}G \otimes^{l}_{\mathbb{Z}} A \cong \mathbb{Z}G \otimes^{d}_{\mathbb{Z}} A$$

where the left-hand side has a G-action by left multiplication on the first factor and the right-hand side has the diagonal G-action.

Proof. Consider the map:

$$\theta : \mathbb{Z}G \otimes^{l}_{\mathbb{Z}} A \to \mathbb{Z}G \otimes^{d}_{\mathbb{Z}} A$$
$$g \otimes a \mapsto g \otimes ga$$

It is a map of abelian groups: linear in g by definition and clearly linear in a. Moreover, it is a G-map:

$$\begin{aligned} \theta(h \cdot (g \otimes a)) &= \theta(hg \otimes a)) \\ &= hg \otimes hga \\ &= h \cdot (g \otimes ga) \\ &= h \cdot \theta(g \otimes a). \end{aligned}$$

In fact, θ is an isomorphism of abelian groups (and hence of G-modules) whose inverse is:

$$\theta^{-1}(g \otimes a) = g \otimes g^{-1}a.$$

Finally, note that the formula defines an isomorphism that is natural in G and in A. □

Proposition 7.2.2. *For (G, A) in* **ModGp***, the abelianization $Ab_{(G,A)}(G, A)$ of the identity is the following module:*

$$\left(I_G, \mathbb{Z}G \otimes_{\mathbb{Z}} A, I_G \otimes_{\mathbb{Z}} A \overset{\text{incl}\otimes 1}{\longrightarrow} \mathbb{Z}G \otimes_{\mathbb{Z}} A\right),$$

where the second G-module has the diagonal action.

Proof. It is straightforward to check that this works. For a more enlightening discussion, let's see how we obtain the result. First, by categorical nonsense (proposition B.2.3), the first G-module has to be I_G. Indeed we have:

$$\pi_1 Ab_{(G,A)}(G, A) = Ab_{\pi_1(G,A)}\pi_1(G, A) = Ab_G G = I_G$$

where π_1 is the "ground level" forgetful functor, denoted U in appendix B.

Now, let us identify global sections in **ModGp**, i.e. maps in

$$\mathrm{Hom}_{\mathbf{ModGp}/(G,A)}\left(\begin{pmatrix}A\\G\end{pmatrix}\overset{\mathrm{id}}{\to}\begin{pmatrix}A\\G\end{pmatrix},\begin{pmatrix}A\oplus B\\G\ltimes M\end{pmatrix}\overset{p}{\to}\begin{pmatrix}A\\G\end{pmatrix}\right).$$

Such a section $s:\begin{pmatrix}A\\G\end{pmatrix}\to\begin{pmatrix}A\oplus B\\G\ltimes M\end{pmatrix}$ consists of the following data:

- A section s_1 of $G\ltimes M\overset{p_1}{\to}G$, where by abuse of notation we also denote the corresponding crossed homomorphism $G\to M$ by s_1.
- A section s_2 of $A\oplus B\to A$ that is s_1-equivariant (to ensure that s is a map in **ModGp**). By the same abuse of notation, we also denote the relevant component $A\to B$ by s_2.

The condition on s_2 is the following:

$$\begin{aligned}s_2(g\cdot a)&=s_1(g)\cdot s_2(a)\\(g\cdot a,s_2(g\cdot a))&=(g,s_1(g))\cdot(a,s_2(a))\\&=(g\cdot a,\varphi(s_1(g),g\cdot a)+g\cdot s_2(a)).\end{aligned}$$

In other words, s_2 satisfies the following cocycle condition in B:

$$s_2(g\cdot a)=\varphi(s_1(g),g\cdot a)+g\cdot s_2(a). \tag{7.2.1}$$

The map $s_2\in\mathrm{Hom}_{\mathbf{Ab}}(A,B)\cong\mathrm{Hom}_G(\mathbb{Z}G\otimes_{\mathbb{Z}}A,B)$ corresponds to an adjoint map of G-modules:

$$\begin{aligned}\sigma_2:\mathbb{Z}G\otimes_{\mathbb{Z}}A&\to B\\g\otimes a&\mapsto g\cdot s_2(a)\end{aligned}$$

where G acts on the left-hand side by left multiplication. Note the following:

$$\begin{aligned}s_2(ga)=1\cdot s_2(ga)&=\sigma_2(1\otimes ga)\\\varphi(s_1(g),ga)&=\varphi(\sigma_1(1-g),ga)\\g\cdot s_2(a)&=\sigma_2(g\otimes a).\end{aligned}$$

So the map s_2 satisfies the cocycle condition (7.2.1) iff its adjoint map σ_2 satisfies the "adjoint" cocycle condition:

$$\sigma_2(1 \otimes ga) = \varphi(\sigma_1(1-g), ga) + \sigma_2(g \otimes a). \tag{7.2.2}$$

We're looking for a G-module B' and a map $\psi : I_G \otimes_{\mathbb{Z}} A \to B'$ such that the above data is the same as a map in $\mathrm{Hom}_{\mathbf{Mod}_{(G,A)}}((I_G, B', \psi), (M, B, \varphi))$. Such a map consists of the data of two G-maps $\sigma_1 : I_G \to M$ and $\beta : B' \to B$ such that the following diagram commutes:

$$\begin{tikzcd}
I_G \otimes_{\mathbb{Z}} A \arrow[r, "\psi"] \arrow[d, "\sigma_1 \otimes 1"'] & B' \arrow[d, "\beta"] \\
M \otimes A \arrow[r, "\varphi"'] & B
\end{tikzcd}$$

or in other words, the following condition holds:

$$\varphi(\sigma_1(1-g), a) = \beta\psi(1-g, a).$$

Comparing this to condition (7.2.2), one good candidate is to take:

$$\begin{aligned}
&B' = \mathbb{Z}G \otimes_{\mathbb{Z}} A \\
&\psi(1-g, a) = 1 \otimes a - g \otimes g^{-1}a \\
&\beta = \sigma_2
\end{aligned}$$

We only need to check that this $\psi : I_G \otimes_{\mathbb{Z}} A \to \mathbb{Z}G \otimes_{\mathbb{Z}} A$ is indeed a G-map, where the left-hand side has the diagonal action and the right-hand side has the action by multiplication on the left factor. It is clearly a map of abelian groups; let us check the G-equivariance:

$$\begin{aligned}
\psi(h \cdot ((1-g) \otimes a)) &= \psi(h(1-g) \otimes ha) \\
&= \psi((1-hg) \otimes ha - (1-h) \otimes ha) \\
&= 1 \otimes h \cdot a - hg \otimes g^{-1}h^{-1}ha - 1 \otimes h \cdot a + h \otimes h^{-1}ha \\
&= h \otimes a - hg \otimes g^{-1}a \\
&= h \cdot \left(1 \otimes a - g \otimes g^{-1}a\right) \\
&= h \cdot \psi((1-g) \otimes a).
\end{aligned}$$

Now it's annoying to have the left multiplication action on $\mathbb{Z}G \otimes_{\mathbb{Z}} A$, since A is itself a G-module and in that case we usually take the diagonal action. Let us use lemma 7.2.1 to fix this:

$$\begin{tikzcd}
I_G \otimes_{\mathbb{Z}} A \arrow[r, "\psi"] & \mathbb{Z}G \otimes^l_{\mathbb{Z}} A \arrow[r, "\cong"] & \mathbb{Z}G \otimes^d_{\mathbb{Z}} A \\
(1-g) \otimes a \arrow[r, mapsto] & 1 \otimes a - g \otimes g^{-1}a \arrow[r, mapsto] & 1 \otimes a - g \otimes gg^{-1}a = (1-g) \otimes a.
\end{tikzcd}$$

This is precisely the statement. □

7.3 Pushforward

In order to compute any abelianization, let us describe pushforwards in **ModGp** (cf. proposition 1.1.7). First, let us describe free modules over (G, A).

Lemma 7.3.1. *The forgetful functor* $\mathbf{Mod}_{(G,A)} \to G\text{-}\mathbf{Mod}^2$ *defined by* $(M, B, \varphi) \mapsto (M, B)$ *has a left adjoint that sends* (M, B) *to*

$$(M, (M \otimes_{\mathbb{Z}} A) \oplus B, M \otimes_{\mathbb{Z}} A \hookrightarrow (M \otimes_{\mathbb{Z}} A) \oplus B).$$

Proof. A map in

$$\mathrm{Hom}_{\mathbf{Mod}_{(G,A)}}\left((M, (M \otimes_{\mathbb{Z}} A) \oplus B, M \otimes_{\mathbb{Z}} A \hookrightarrow (M \otimes_{\mathbb{Z}} A) \oplus B), (M', B', \varphi')\right)$$

consists of the data of two G-maps:

$$\alpha : M \to M'$$
$$\beta : (M \otimes_{\mathbb{Z}} A) \oplus B \to B'$$

making the following diagram commute:

$$\begin{tikzcd}
M \otimes_{\mathbb{Z}} A \arrow[r, hook] \arrow[d, "\alpha\otimes 1"'] & M \otimes_{\mathbb{Z}} A \oplus B \arrow[d, "\beta"] \\
M' \otimes_{\mathbb{Z}} A \arrow[r, "\varphi'"'] & B'.
\end{tikzcd}$$

The condition says that β is $\varphi' \circ (\alpha \otimes 1)$ on the first summand, and there is no constraint for β on the second summand. Hence this is the data of any two G-maps

$$\alpha : M \to M'$$
$$\beta_2 : B \to B'$$

i.e. a map in $\mathrm{Hom}_{\mathbf{Mod}^2_G}((M, B), (M', B'))$. $\square$

In particular, the free (G, A)-module on sets (S, T), is given by

$$(\mathbb{Z}G[S], (\mathbb{Z}G[S] \otimes_{\mathbb{Z}} A) \oplus \mathbb{Z}G[T], \mathbb{Z}G[S] \otimes_{\mathbb{Z}} A \hookrightarrow (\mathbb{Z}G[S] \otimes_{\mathbb{Z}} A) \oplus \mathbb{Z}G[T]).$$

Notation. Let us denote the latter free module $F(S, T)$. The arguments could also be abelian groups or G-modules.

Lemma 7.3.2. *For a map* $f : \begin{pmatrix} A \\ G \end{pmatrix} \to \begin{pmatrix} A' \\ G' \end{pmatrix}$, *there is a natural* G'*-map*

$$\psi : f_{1*}(M \otimes_{\mathbb{Z}} A) \to f_{1*}M \otimes_{\mathbb{Z}} A'$$
$$g' \otimes (m \otimes a) \mapsto (g' \otimes m) \otimes g' f_2(a).$$

Proof. The map is well-defined: in $f_{1*}(M \otimes_{\mathbb{Z}} A)$ we have

$$g' f_1(g) \otimes (m \otimes a) = g' \otimes (gm \otimes ga)$$

and the first expression is sent to

$$(g' f_1(g) \otimes m) \otimes g' f_1(g) f_2(a)$$

whereas the second expression is sent to

$$\begin{aligned} &(g' \otimes gm) \otimes g' f_2(ga) \\ &= (g' f_1(g) \otimes m) \otimes g' f_1(g) f_2(a). \end{aligned}$$

Moreover, the map is a G'-map:

$$\begin{aligned} g'' \cdot (g' \otimes (m \otimes a)) &= g''g' \otimes (m \otimes a) \\ &\mapsto (g''g' \otimes m) \otimes g''g' f_2(a) \\ &= g'' \cdot ((g' \otimes m) \otimes g' f_2(a)) \\ &= g'' \cdot \psi (g' \otimes (m \otimes a)) . \end{aligned}$$

That the map is natural (in f) is easy to check. □

Proposition 7.3.3. *For a map* $f : \begin{pmatrix} A \\ G \end{pmatrix} \to \begin{pmatrix} A' \\ G' \end{pmatrix}$, *the pushforward functor* f_* *associates to a module* (M, B, φ) *over* $\begin{pmatrix} A \\ G \end{pmatrix}$ *the module:*

$$\left(f_{1*}M, f_{1*}M \otimes_{\mathbb{Z}} A' \amalg_{f_{1*}(M \otimes_{\mathbb{Z}} A)} f_{1*}B, f_{1*}M \otimes_{\mathbb{Z}} A' \xrightarrow{i_1} f_{1*}M \otimes_{\mathbb{Z}} A' \amalg_{f_{1*}(M \otimes_{\mathbb{Z}} A)} f_{1*}B \right)$$

where the pushout is defined using the map ψ *in the previous lemma.*

Proof. Recall what the pullback does. Given a module (M', B', φ') over (G', A'), its pullback $f^*(M', B', \varphi')$ is the module

$$\left(f_1^* M', f_1^* B', f_2^* \varphi' \right)$$

where the latter map is given by

$$\begin{aligned} f_2^* \varphi' : f_1^* M' \otimes_{\mathbb{Z}} A &\to f_1^* B' \\ m' \otimes a &\mapsto \varphi'(m', f_2(a)). \end{aligned}$$

A map in $\mathrm{Hom}_{\mathbf{Mod}_{(G,A)}} ((M, B, \varphi), f^*(M', B', \varphi'))$ consists of the data of G-maps

$$\begin{aligned} \alpha &: M \to f_1^* M' \\ \beta &: B \to f_1^* B' \end{aligned}$$

satisfying the condition

$$\beta\varphi(m \otimes a) = \varphi' \left(\alpha(m) \otimes f_2(a) \right)$$

for all $m \in M$, $a \in A$. By adjunction, this is the same data as two G'-maps

$$\underline{\alpha} : f_{1*}M \to M'$$
$$\underline{\beta} : f_{1*}B \to B'$$

satisfying the condition

$$\underline{\beta}(g' \otimes \varphi(m \otimes a)) = \varphi'\left(\underline{\alpha}(g' \otimes m) \otimes g' f_2(a)\right) \tag{7.3.1}$$

for all $m \in M$, $a \in A$, and $g' \in G'$ (or only for $g' = 1$, by equivariance). Using free modules over (G', A') , this is the same data as a map

$$F(f_{1*}M, f_{1*}B) \to (M', B', \varphi')$$

satisfying (7.3.1). By lemma 7.3.1, the free module is

$$(f_{1*}M, (f_{1*}M \otimes_{\mathbb{Z}} A') \oplus f_{1*}B, \mathrm{incl}_1)$$

so we're looking for G'-maps $\underline{\alpha}, \underline{\beta}$

$$\begin{tikzcd}
f_{1*}M \otimes_{\mathbb{Z}} A' \arrow[r, hook, "incl_1"] \arrow[d, "\underline{\alpha}\otimes 1"'] & (f_{1*}M \otimes_{\mathbb{Z}} A') \oplus f_{1*}B \arrow[d, "\varphi' \circ(\underline{\alpha}\otimes 1)+\underline{\beta}"] \\
M' \otimes_{\mathbb{Z}} A' \arrow[r, "\varphi'"'] & B'
\end{tikzcd}$$

such that the diagram

$$\begin{tikzcd}
f_{1*}(M \otimes_{\mathbb{Z}} A) \arrow[r, "f_{1*}\varphi"] \arrow[d, "\psi"'] & f_{1*}B \arrow[d, hook, "incl_2"] \\
f_{1*}M \otimes_{\mathbb{Z}} A' \arrow[d, "\underline{\alpha}\otimes 1"'] & (f_{1*}M \otimes_{\mathbb{Z}} A') \oplus f_{1*}B \arrow[d, "\varphi' \circ(\underline{\alpha}\otimes 1)+\underline{\beta}"] \\
M' \otimes_{\mathbb{Z}} A' \arrow[r, "\varphi'"'] & B'
\end{tikzcd}$$

commutes. Taking the pushout P of the top left corner, we see this is the same data as $\underline{\alpha}$ and the bottom right map $P \to B'$ in the commutative diagram

$$\begin{tikzcd}
f_{1*}(M \otimes_{\mathbb{Z}} A) \arrow[r, "f_{1*}\varphi"] \arrow[d, "\psi"'] & f_{1*}B \arrow[d] \arrow[dd, bend left=60, "\underline{\beta}"] \\
f_{1*}M \otimes_{\mathbb{Z}} A' \arrow[r] \arrow[d, "\underline{\alpha}\otimes 1"'] & P \arrow[d, dashed] \\
M' \otimes_{\mathbb{Z}} A' \arrow[r, "\varphi'"'] & B'
\end{tikzcd}$$

This proves the assertion. □

7.4 Hochschild cohomology

We start with an object (G, A) of **ModGp** and a module (M, B, φ) over it, cf. 7.1.3. Our goal is to compute the Hochschild cohomology

$$\begin{aligned}
&\mathrm{HH}^*\left((G, A); (M, B, \varphi)\right) \\
&= \mathrm{Ext}^*\left(Ab_{(G,A)}(G, A), (M, B, \varphi)\right) \\
&= \mathrm{Ext}^*\left((I_G, \mathbb{Z}G \otimes_{\mathbb{Z}} A, \mathrm{incl}), (M, B, \varphi)\right)
\end{aligned}$$

where the last equality comes from 7.2.2.

7.4.1 Extended group cohomology

As in the case of groups, the module of differentials embeds in a projective (even free) module. Indeed, there is a short exact sequence of (G, A)-modules:

$$0 \longrightarrow (I_G, \mathbb{Z}G \otimes_{\mathbb{Z}} A, \mathrm{incl}) \longrightarrow (\mathbb{Z}G, \mathbb{Z}G \otimes_{\mathbb{Z}} A, \mathrm{id}) \longrightarrow (\mathbb{Z}, 0, 0) \longrightarrow 0 \qquad (7.4.1)$$

where $\mathbb{Z}$ in the rightmost module has the trivial G-action. The middle module is the free module $F(1, 0)$, cf. 7.3.1.

Notation. In a manner analogous to group cohomology, let us write

$$\mathrm{H}^*\left((G, A); (M, B, \varphi)\right) := \mathrm{Ext}^*\left((\mathbb{Z}, 0, 0), (M, B, \varphi)\right)$$

in the category $\mathbf{Mod}_{(G,A)}$. They are the derived functors of what could be called the **orthogonal invariants** functor:

$$\begin{aligned} M^{G,\perp} &:= \operatorname{Hom}((\mathbb{Z},0,0),(M,B,\varphi)) = \mathrm{H}^0((G,A);(M,B,\varphi)) \\ &= \{m \in M \mid gm = m \ \forall g \in G \text{ and } \varphi(m,a) = 0 \ \forall a \in A\}. \end{aligned}$$

Applying $\operatorname{Hom}(-,(M,B,\varphi))$ to (7.4.1), we obtain the long exact sequence:

$$\begin{aligned} \cdots &\to \operatorname{Ext}^i((\mathbb{Z},0,0),(M,B,\varphi)) \to \operatorname{Ext}^i(F(1,0),(M,B,\varphi)) \to \\ &\to \operatorname{Ext}^i((I_G,\mathbb{Z}G \otimes_{\mathbb{Z}} A,\text{incl}),(M,B,\varphi)) \to \operatorname{Ext}^{i+1}((\mathbb{Z},0,0),(M,B,\varphi)) \to \cdots \end{aligned} \tag{7.4.2}$$

and since $F(1,0)$ is projective, the long exact sequence breaks into the exact sequence

$$0 \to M^{G,\perp} \to M \to \Gamma((G,A),(M,B,\varphi)) \to \mathrm{H}^1((G,A);(M,B,\varphi)) \to 0 \tag{7.4.3}$$

for $i = 0$ and the isomorphisms

$$\mathrm{HH}^i((G,A);(M,B,\varphi)) \cong \mathrm{H}^{i+1}((G,A);(M,B,\varphi))$$

for $i \geq 1$. In (7.4.3), Γ refers to "global sections" or "derivations", and as for group cohomology, the sequence exhibits H^1 as derivations modulo principal derivations.

Conclusion 7.4.1. *Computing* HH^* *is essentially the same as computing* H^*, *which is our new goal.*

7.4.2 Reduction: $M = 0$

Our notion of H^* is indeed a generalization of group cohomology. The idea is that we can forget what's happening on the "second floor" and only look at the "ground floor", the case of groups. More precisely, we have the forgetful functor π_1 which sits in the adjunction

$$\begin{tikzcd} \mathbf{ModGp} \arrow[r, shift left, "\pi_1"] & \mathbf{Gp} \arrow[l, shift left, "\text{Zero}"] \end{tikzcd}$$

and as we know by B.2.1, Zero is in fact a left and right adjoint of π_1, so π_1 preserves all limits. The unit of the adjunction is

$$\eta : \begin{pmatrix} A \\ G \end{pmatrix} \to \begin{pmatrix} 0 \\ G \end{pmatrix}$$

and the counit is the identity functor on **Gp**. The induced adjunction on Beck modules (cf. 4.1.5) is

$$\begin{tikzcd} \mathbf{Mod}_{(G,A)} \arrow[r, shift left, "\pi_1"] & \mathbf{Mod}_G \arrow[l, shift left, "\text{Zero}"] \\ (M,B,\varphi) \arrow[r, mapsto] & M \\ (N,0,0) & N \arrow[l, mapsto] \end{tikzcd}$$

Take a base object (G,A) in **ModGp** and a G-module N. The induced π_1 preserves projectives (since Zero is exact), so we have a comparison map (cf. 4.2.1)

$$\mathrm{HH}^*(G;N) \overset{\cong}{\to} \mathrm{HH}^*\left((G,A);(N,0,0)\right)$$

which is an iso since π_1 is exact. Similarly, π_1 induces an iso

$$\mathrm{H}^*(G;N) \overset{\cong}{\to} \mathrm{H}^*\left((G,A);(N,0,0)\right)$$

This can be checked directly, or by the 5-lemma applied to the map that π_1 induces from the long exact sequence (7.4.2) to the analogous one for groups.

We have a short exact sequence of (G,A)-modules:

$$0 \to (0,B,0) \to (M,B,\varphi) \to (M,0,0) \to 0$$

where in fact, the map on the right is the unit map of the adjunction on Beck modules, $1 \to \text{Zero} \circ \pi_1$. Applying $\mathrm{Hom}\left((\mathbb{Z},0,0),-\right)$, we obtain a long exact sequence

$$\begin{gathered} \cdots \to \mathrm{Ext}^i\left((\mathbb{Z},0,0),(0,B,0)\right) \to \mathrm{Ext}^i\left((\mathbb{Z},0,0),(M,B,\varphi)\right) \to \\ \to \mathrm{Ext}^i\left((\mathbb{Z},0,0),(M,0,0)\right) \to \mathrm{Ext}^{i+1}\left((\mathbb{Z},0,0),(0,B,0)\right) \to \cdots \end{gathered} \tag{7.4.4}$$

which we can rewrite as

$$\begin{gathered} \cdots \to \mathrm{H}^i\left((G,A);(0,B,0)\right) \to \mathrm{H}^i\left((G,A);(M,B,\varphi)\right) \to \\ \to \mathrm{H}^i(G;M) \to \mathrm{H}^{i+1}\left((G,A);(0,B,0)\right) \to \cdots \end{gathered} \tag{7.4.5}$$

In that sense, the groups $\mathrm{H}^i\left((G,A);(0,B,0)\right)$ (and the maps around them) measure the loss of information when forgetting from extended group cohomology $\mathrm{H}^i\left(((G,A);(M,B,\varphi)\right)$ to usual group cohomology $\mathrm{H}^i(G;M)$.

Conclusion 7.4.2. *Assuming we know how to compute group cohomology and assuming we can understand the long exact sequence (7.4.5), the important step is to study the cohomology groups* $\mathrm{H}^*\left((G,A);(0,B,0)\right)$*. This is our new goal.*

7.4.3 Reduction: Derived functors of indecomposables

We're trying to compute $\mathrm{Ext}^*\left((\mathbb{Z},0,0),(0,B,0)\right)$ in the category of (G,A)-modules. Consider B fixed and let us vary the source.

Proposition 7.4.3.

$$\mathrm{Hom}_{\mathbf{Mod}_{(G,A)}}\left((M',B',\varphi'),(0,B,0)\right) \cong \mathrm{Hom}_{\mathbf{Mod}_G}(B'/\operatorname{im}\varphi',B)$$

Proof. A map from (M',B',φ') to $(0,B,0)$ consists of the data of a G-map $\beta : B' \to B$ such that the diagram

$$\begin{tikzcd}
M' \otimes_{\mathbb{Z}} A \arrow[r, "\varphi'"] \arrow[d] & B' \arrow[d, "\beta"] \\
0 \arrow[r] & B
\end{tikzcd}$$

commutes, i.e. such that β vanishes on $\operatorname{im}\varphi'$. This is exactly the data of a G-map $\beta : B'/\operatorname{im}\varphi' \to B$. □

We have factored our functor as follows:

$$\begin{tikzcd}
\mathbf{Mod}_{(G,A)} \arrow[r, "Q"] \arrow[rr, bend right, "\mathrm{Hom}(-,(0,B,0))"'] & \mathbf{Mod}_G \arrow[r, "\mathrm{Hom}(-,B)"] & \mathbf{Ab}
\end{tikzcd} \tag{7.4.6}$$

where the functor $Q : (M,B,\varphi) \mapsto B/\operatorname{im}\varphi$ can be thought as some kind of "indecomposables" functor.

Proposition 7.4.4. *The functor Q preserves projectives.*

Proof. In both categories, projectives are retract of frees, so it suffices to show that Q preserves frees (or at least sends them to projectives). A free (G,A)-module on sets S and T is

$$F(S,T) = (\mathbb{Z}G[S], \mathbb{Z}G \otimes_{\mathbb{Z}} A \oplus \mathbb{Z}G[T], \mathrm{incl}_1)$$

and the functor Q sends it to $\mathbb{Z}G[T]$, which is a free G-module. $\square$

Therefore, the factorization (7.4.6) applied to the module $(Z, 0, 0)$ yields a Grothendieck spectral sequence (first quadrant, cohomologically graded):

$$E_2^{s,t} = \operatorname{Ext}^s\left(L_t Q(\mathbb{Z}, 0, 0), B\right) \Rightarrow \operatorname{Ext}^{s+t}\left((\mathbb{Z}, 0, 0), (0, B, 0)\right) \tag{7.4.7}$$

Note that $E_2^{s,0}$ is zero, since $Q(\mathbb{Z}, 0, 0)$ is zero. Moreover, the edge morphism

$$\operatorname{Ext}^t\left((\mathbb{Z}, 0, 0), (0, B, 0)\right) \twoheadrightarrow E_\infty^{0,t} \hookrightarrow E_2^{0,t} = \operatorname{Hom}\left(L_t Q(\mathbb{Z}, 0, 0), B\right)$$

is the homology preservation map [Bar06, 2.2, 2.6] of the left exact functor $\operatorname{Hom}(-, B)$, applied to Q of a projective resolution of $(\mathbb{Z}, 0, 0)$.

Conclusion 7.4.5. *Assuming we can understand the composite spectral sequence (7.4.7), an important step is to compute* $L_* Q(\mathbb{Z}, 0, 0)$*, the left derived functors of* Q *applied to* $(\mathbb{Z}, 0, 0)$*. It is now our goal.*

7.4.4 Bar resolution of $(\mathbb{Z}, 0, 0)$

In order to compute $L_* Q(\mathbb{Z}, 0, 0)$, we will construct a bar resolution of $(\mathbb{Z}, 0, 0)$, which is the simplicial resolution associated to the comonad "Free of Forget", in the adjunction:

$$\begin{tikzcd}
\mathbf{Ab}^2 \arrow[r, shift left, "F"] & \mathbf{Mod}_{(G,A)} \arrow[l, shift left, "U"]
\end{tikzcd}$$

$$\begin{tikzcd}
(S, T) \arrow[r, mapsto] & (\mathbb{Z}G \otimes_{\mathbb{Z}} S, (\mathbb{Z}G \otimes_{\mathbb{Z}} S) \otimes_{\mathbb{Z}} A \oplus \mathbb{Z}G \otimes_{\mathbb{Z}} T, \mathrm{incl}_1)
\end{tikzcd}$$

$$\begin{tikzcd}
(M, B) & (M, B, \varphi) \arrow[l, mapsto]
\end{tikzcd}$$

The G-actions are: left multiplication on $\mathbb{Z}G \otimes S$, diagonal action on $(\mathbb{Z}G \otimes S) \otimes A$ (meaning G acts on the $\mathbb{Z}G$ factor and on A), and left multiplication on $\mathbb{Z}G \otimes T$. Here, tensor products without subscript mean over $\mathbb{Z}$. Let us denote by $X_\bullet$ the simplicial resolution of $(\mathbb{Z}, 0, 0)$, so that $X_n = (FU)^{n+1}(\mathbb{Z}, 0, 0)$. A straighforward computation yields the following.

Proposition 7.4.6. *The constituent* (G, A)*-modules* X_n *are*

$$\left(\mathbb{Z}G^{\otimes n+1}, \oplus^{n+1}(\mathbb{Z}G^{\otimes n+1} \otimes A), \mathrm{incl}_1\right).$$

On $\mathbb{Z}G^{\otimes n+1}$*, G acts by left multiplication i.e. on the left-most tensor factor* $\mathbb{Z}G$*. On* $\oplus^{n+1}(\mathbb{Z}G^{\otimes n+1} \otimes A)$*, the action is more subtle: On the first summand, G acts by multiplication on the left-most* $\mathbb{Z}G$ *factor and on the factor A, whereas on the subsequent summands, G only acts by multiplication on the left-most factor* $\mathbb{Z}G$*.*

Corollary 7.4.7. $Q(X_\bullet)$ *is the simplicial G-module whose* n^{th} *constituent is* $\oplus^n(\mathbb{Z}G^{\otimes n+1} \otimes A)$*, on which G acts by left multiplication, i.e. on the left-most* $\mathbb{Z}G$ *factor of each summand.*

Since $X_\bullet \to (\mathbb{Z}, 0, 0)$ is a free resolution of $(\mathbb{Z}, 0, 0)$ in $\mathbf{Mod}_{(G,A)}$, the derived functors of Q on $(\mathbb{Z}, 0, 0)$ can be computed as $\pi_* Q(X_\bullet)$, i.e. the homology of the associated chain complex.

Appendix A

Examples related to Quillen cohomology

In this appendix, we review some standard examples related to Quillen cohomology. We discuss in particular the category of Beck modules, the abelianization functor, and the pushforward.

A.1 Groups

A.1.1 Beck modules

Proposition A.1.1. *A Beck module over a group G is a split extension of G (with the data of a splitting) with abelian kernel:*

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] & E \arrow[r, shift left, "p"] & G \arrow[l, shift left, "s"] \arrow[r] & 1. \end{tikzcd}$$

Proof. Standard computation; see [Bar02, section 6.1]. □

Remark A.1.2. This category is equivalent to the standard category of (say, left) modules over G. To a split extension, one associates the kernel K with induced action from G, given by $g \cdot k = s(g)k$. To a usual module K, one associates the semidirect product $G \ltimes K \to G$. Note that this is the same as a module over the group ring $\mathbb{Z}G$. It's also the same as (covariant) functors from the one-object category G to Ab.

Proposition A.1.3. *For a map of groups $f : H \to G$, the pushforward functor f_* associates to an H-module M the G-module:*

$$\mathbb{Z}G \otimes_{\mathbb{Z}H} M.$$

Proof. We know this is the pushforward from $\mathbb{Z}H$-modules to $\mathbb{Z}G$-modules. □

Proposition A.1.4. *1. The module $Ab_X X$ of a group G is the augmentation ideal $I_G = \ker(\mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z})$.*

2. The abelianization functor $Ab_G : \mathbf{Gp}/G \to Ab(\mathbf{Gp}/G)$ associates to $H \to G$ the G-module:

$$\mathbb{Z}G \otimes_{\mathbb{Z}H} I_H.$$

Proof. 1. Standard computation in homological algebra. Let us describe the correspondence more explicitly. Recall that for groups, "global sections" correspond to crossed homomorphisms:

$$\begin{aligned}\Gamma(G,M) &= \mathrm{Hom}_{\mathbf{Gp}/G}(G \xrightarrow{\mathrm{id}} G, G \ltimes M \to G)\\ &\cong \{\varphi : G \to M \mid \varphi(gg') = \varphi(g) + g \cdot \varphi(g')\}.\end{aligned}$$

Such a crossed homomorphism corresponds, via the adjunction, to the G-module map $\alpha : I_G \to M$ defined by:

$$\alpha(1-g) = \varphi(g).$$

2. Follows from propositions 1.1.7 and A.1.3. □

A.1.2 Hochschild cohomology

Proposition A.1.5. *For a module $G \ltimes M \to G$ over a group G, Hochschild cohomology is essentially group cohomology, with grading shifted by one:*

$$\mathrm{HH}^i(G;M) \cong \begin{cases} \Gamma(G,M) \text{ if } i = 0, \\ \mathrm{H}^{i+1}(G;M) \text{ if } i > 0. \end{cases}$$

Proof. We have:

$$\mathrm{HH}^i(G;M) = \mathrm{Ext}^i(Ab_G G, M) = \mathrm{Ext}^i(I_G, M).$$

There is also a short exact sequence of G-modules:

$$0 \to I_G \to \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Since $\mathbb{Z}G$ is a projective (free!) G-module, we have $\mathrm{Ext}^i(\mathbb{Z}G, M) = 0$ for all $i > 0$ and the associated long exact sequence of right derived functors of $\mathrm{Hom}(-, M)$ yields:

$$\mathrm{Ext}^i(I_G, M) \cong \mathrm{Ext}^{i+1}(\mathbb{Z}, M) = \mathrm{H}^{i+i}(G; M)$$

for $i > 0$ and

$$0 \to M^G \hookrightarrow M \to \Gamma(G, M) \twoheadrightarrow \mathrm{H}^1(G; M) \to 0$$

around $i = 0$. □

A.1.3 Quillen cohomology

Proposition A.1.6. *Let G be a group. The Quillen homology object of G is weakly equivalent to a constant object:*

$$L_G \xrightarrow{\sim} Ab_G G.$$

Proof. Standard homological / homotopical algebra. One way to prove it relies on Quillen's machinery of simplicial modules over a simplicial ring (since G-modules are $\mathbb{Z}G$-modules) and the fact that a weak equivalence of simplicial sets induces an iso on homology. □

Corollary A.1.7. *The Quillen homology of a group G is:*

$$\mathrm{HQ}_*(G) = \begin{cases} I_G \text{ if } * = 0, \\ 0 \text{ if } * \neq 0. \end{cases}$$

The Quillen cohomology of G with coefficients in a module M is:

$$\mathrm{HQ}^*(G; M) = \mathrm{HH}^*(G; M)$$

as given in A.1.5.

In other words: For groups, it all reduces to homological algebra. The homotopical algebra doesn't bring anything new.

A.2 Abelian groups

Proposition A.2.1. *A Beck module over an abelian group A is a split extension of A in abelian groups:*

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & E \cong A \oplus K \arrow[r, shift left, "p"] & A \arrow[l, shift left, "e"] \arrow[r] & 0. \end{tikzcd}$$

Proof. Not hard. Basically the same as for groups, except the "total space" has to be abelian. □

Remark A.2.2. This category is equivalent to **Ab**. To a split extension, one associates the kernel K. To an abelian group K, one associates the projection $A \oplus K \to A$. It's also the same as functors from the trivial category $*$ to Ab.

Proposition A.2.3. *For a map of abelian groups $f : A \to B$, the pushforward functor f_* is the identify functor on Ab, i.e. sends $A \oplus K$ to $B \oplus K$.*

Proof. The pullback functor f^* is the identity on **Ab**, under the above identification. □

Proposition A.2.4. *The abelianization functor $Ab_A : \mathbf{Ab}/A \to Ab(\mathbf{Ab}/A) \cong \mathbf{Ab}$ is the "source" functor, which sends $B \to A$ to B.*

Proof. Under the above identification, the forgetful functor $U_A : \mathbf{Ab} \to \mathbf{Ab}/A$ sends an abelian group K to $A \oplus K \xrightarrow{\pi_1} A$. Thus we have:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Ab}/A}(B \to A, U_A(K)) &= \mathrm{Hom}_{\mathbf{Ab}/A}(B \to A, A \oplus K \to A) \\ &= \mathrm{Hom}_{\mathbf{Ab}}(B, K). \end{aligned}$$

□

A.2.1 Hochschild cohomology

Proposition A.2.5. *For a module B over an abelian group A, i.e. just an abelian group B, Hochschild cohomology is given by:*

$$\mathrm{HH}^i(A; B) = \mathrm{Ext}^i_{\mathbf{Ab}}(A, B).$$

Proof. By proposition A.2.4, we have:

$$\begin{aligned} \mathrm{HH}^i(A;B) &= \mathrm{Ext}^i(Ab_A A, B) \\ &= \mathrm{Ext}^i_{\mathbf{Ab}}(A,B) \end{aligned}$$ □

A.3 Associative algebras

By **ring**, we will mean by default an associative, unital ring. Fix a ground ring R which is commutative, and consider the usual notion of **associative R-algebra**, i.e. a ring A which is also an R-module in a compatible way. Equivalently, it is a ring A with a ring map $R \to A$ which lands in the center of A.

Notation. Let $\mathbf{Alg}_R$ denote the category of R-algebras.

A.3.1 Beck modules

Proposition A.3.1. *A Beck module over an associative R-algebra A is a split extension of A (with the data of a splitting) with square zero kernel:*

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & E \cong A \oplus M \arrow[r, shift left, "p"] & A \arrow[l, shift left, "s"] \arrow[r] & 0. \end{tikzcd}$$

Equivalently, it is the data of an A-bimodule M over R, i.e. the two actions coincide for scalars (elements of R).

Proof. Standard; see e.g. [Bar02, § 6.1]. Since it is a formative exercise, let us prove it anyway.

As abelian groups, we have a split extension

$$\begin{tikzcd} A \oplus M \arrow[r, shift left, "p"] & A \arrow[l, shift left, "s"] \end{tikzcd}$$

which determines the abelian group object structure maps. The latter must be maps in $\mathbf{Alg}_R/A$, and the projection map p must be a map in $\mathbf{Alg}_R$. Those are all the conditions.

1. The projection map p is a map in $\mathbf{Alg}_R$.

$$p\left((a,m)(a',m')\right) = p(a,m)p(a',m') = aa'$$

In other words, we have

$$(a,m)(a',m') = (aa', \varphi(a,m,a',m'))$$

for some function φ.

2. The multiplication map μ is a map in $\mathbf{Alg}_R/A$. Note that the product inside $A \oplus M \oplus M = (A \oplus M) \times_A (A \oplus M)$ is given by:

$$(a,m_1,m_2)(a',m_1',m_2') = \left(aa', \varphi(a,m_1,a',m_1'), \varphi(a,m_2,a',m_2')\right)$$

and μ is a ring map. Applying it to both sides, we obtain:

$$\begin{aligned} \mu(\text{LHS}) &= \mu(a,m_1,m_2)\mu(a',m_1',m_2') \\ &= (a,m_1+m_2)(a',m_1'+m_2') \\ &= \left(aa', \varphi(a,m_1+m_2,a',m_1'+m_2')\right) \\ \mu(\text{RHS}) &= \left(aa', \varphi(a,m_1,a',m_1') + \varphi(a,m_2,a',m_2')\right) \end{aligned}$$

and hence the condition:

$$\varphi(a,m_1+m_2,a',m_1'+m_2') = \varphi(a,m_1,a',m_1') + \varphi(a,m_2,a',m_2').$$

In particular, we get:

$$\varphi(a,m,a',m') = \varphi(a,m,a',0) + \varphi(a,0,a',m')$$

as well as $\varphi(a,0,a',0) = 0$, which is the condition of the unit map s being a map in $\mathbf{Alg}_R/A$.

Using the decomposition $(a,m) = (a,0) + (0,m)$, we also obtain:

$$\begin{aligned} (a,m)(a',m') &= (a,0)(a',0) + (a,0)(0,m') + (0,m)(a',0) + (0,m)(0,m') \\ &= (aa',0) + (0,\varphi(a,0,0,m')) + (0,\varphi(0,m,a',0)) + (0,\varphi(0,m,0,m')) \\ &= (aa', \varphi(a,0,0,m') + \varphi(0,m,a',0) + \varphi(0,m,0,m')). \end{aligned}$$

However, we know:

$$\varphi(0,m,0,m') = \varphi(0,m,0,0) + \varphi(0,0,0,m') = 0 + 0 = 0$$

so the previous condition becomes:

$$\begin{aligned}\varphi(a, m, a', m') &= \varphi(a, 0, 0, m') + \varphi(0, m, a', 0)\\ &=: a \cdot m' + m \cdot a'.\end{aligned}$$

By the associativity of the product in $A \oplus M$, these formulas for $a \cdot m$ and $m \cdot a$ define a left and right action of A on M, respectively. We can rewrite the formulas as:

$$\begin{aligned}a \cdot m &= s(a)\, m \in M \subset A \oplus M\\ m \cdot a &= m\, s(a) \in M \subset A \oplus M.\end{aligned}$$

It remains to describe the behavior as R-algebras.

3. R-action. For a scalar $r \in R$, also denote by r its image in A (or any R-algebra). We know the section (unit map) is a map in $\mathbf{Alg}_R$:

$$\begin{tikzcd} R \arrow[d] \arrow[dr] & \\ A \oplus M \arrow[r, shift left, "p"] & A \arrow[l, shift left, "s"] \end{tikzcd}$$

so that r is $(r, 0)$ in $A \oplus M$. In particular, $(r, 0)$ is in the center of $A \oplus M$ and the two actions of A on M agree over R.

In summary, we have shown that a Beck module over A is exactly the data of a split extension $A \oplus M \to A$ of R-algebras with square zero kernel (and the data of the splitting), or equivalently, of an A-bimodule M over R. The three categories are isomorphic, via the constructions above. ☐

A.3.2 Abelianization

Notation. For an R-algebra A, denote by $m : A \otimes_R A \to A$ the multiplication map, and let $I_A := \ker(A \otimes_R A \xrightarrow{m} A)$ be its kernel.

Proposition A.3.2. *The abelianization of an R-algebra A over itself is $Ab_A A = I_A$.*

Proof. Standard computation in homological algebra. We will check the result and establish the correspondence more precisely. Let us first identify the "global sections" $\mathrm{Hom}_{\mathbf{Alg}_R/A}(A \xrightarrow{\mathrm{id}} A, A \oplus M \xrightarrow{p} A)$. Such a map is the data of a function $\varphi : A \to M$ making $a \mapsto (a, \varphi(a))$ into an R-algebra map $A \to A \oplus M$. Respecting multiplication means:

$$\begin{aligned} aa' \mapsto (aa', \varphi(aa')) &= (a, \varphi(a))\,(a', \varphi(a')) \\ &= (aa', a \cdot \varphi(a') + \varphi(a) \cdot a') \end{aligned}$$

so the condition is the Leibniz rule $\varphi(aa') = a \cdot \varphi(a') + \varphi(a) \cdot a'$. The condition of R-linearity means

$$\begin{tikzcd} R \arrow[d] \arrow[dr] & \\ A \arrow[r, "(\mathrm{id},\varphi)"'] & A \oplus M \end{tikzcd}$$

commutes, in other words $(r, \varphi(r)) = (r, 0)$, so the condition is $\varphi(r) = 0$. Thus global sections for associative R-algebras are just R-linear derivations $\mathrm{Der}_R(A, M)$.

Now we exhibit a natural equivalence of R-modules

$$\begin{aligned} \alpha : \mathrm{Hom}_{A-\mathbf{Bimod}_R}(I_A, M) &\cong \mathrm{Der}_R(A, M) : \beta \\ f &\mapsto \varphi(a) = f(1 \otimes a - a \otimes 1) \\ f\left(\sum a_i \otimes b_i\right) = \sum a_i \varphi(b_i) &\mapsfrom \varphi \end{aligned}$$

Verifications for α. The formula for α does define an R-derivation:

$$\begin{aligned} a\varphi(a') + \varphi(a)a' &= af(1 \otimes a' - a' \otimes 1) + f(1 \otimes a - a \otimes 1)a' \\ &= f(a \otimes a' - aa' \otimes 1) + f(1 \otimes aa' - a \otimes a') \\ &= f(1 \otimes aa' - aa' \otimes 1) \\ &= \varphi(aa') \\ \varphi(r) &= f(1 \otimes r - r \otimes 1) \\ &= f\left((1 \otimes 1)r - r(1 \otimes 1)\right) \\ &= f(0) = 0. \end{aligned}$$

Moreover, the definition of α is clearly additive (in fact R-linear) in f.

Verifications for β. First of all, the formula for $\beta(\varphi)$ makes sense since it is R-bilinear. Let us check that it is a map of A-bimodules:

$$\begin{aligned} f\left(a \sum a_i \otimes b_i\right) &= f\left(\sum aa_i \otimes b_i\right) \\ &= \sum (aa_i)\varphi(b_i) \\ &= a \sum a_i\varphi(b_i) \\ &= af\left(\sum a_i \otimes b_i\right) \end{aligned}$$

and likewise for the right action. Moreover, the definition of β is clearly R-linear in φ.

$\alpha \circ \beta$ is the identity. Let's compute the composite $\alpha \circ \beta$ (and denote by a tilde something modified by the composite):

$$\begin{aligned} \varphi \mapsto f\left(\sum a_i \otimes b_i\right) = \sum a_i\varphi(b_i) & \\ \mapsto \widetilde{\varphi}(a) &= f(1 \otimes a - a \otimes 1) \\ &= 1\varphi(a) - a\varphi(1) \\ &= \varphi(a) \end{aligned}$$

which means $\widetilde{\varphi} = \varphi$, i.e. $\alpha \circ \beta$ is the identity.

$\beta \circ \alpha$ is the identity.

$$\begin{aligned} f \mapsto \varphi(a) &= f(1 \otimes a - a \otimes 1) \\ \mapsto \widetilde{f}\left(\sum a_i \otimes b_i\right) &= \sum a_i\varphi(b_i) \\ &= \sum a_i f(1 \otimes b_i - b_i \otimes 1) \\ &= \sum f(a_i \otimes b_i - a_i b_i \otimes 1) \\ &= f\left(\sum a_i \otimes b_i - \sum a_i b_i \otimes 1\right) \quad \text{, but recall } \sum a_i b_i = 0 \\ &= f\left(\sum a_i \otimes b_i\right) \end{aligned}$$

which means $\widetilde{f} = f$, i.e. $\beta \circ \alpha$ is the identity. □

A more elegant way of saying this is that there is a universal R-derivation

$$\begin{aligned} d : A &\to I_A \\ a &\mapsto 1 \otimes a - a \otimes 1 \end{aligned}$$

such that the natural map

$$\begin{aligned} \mathrm{Hom}_{A-\mathbf{Bimod}_R}(I_A, M) &\to \mathrm{Der}_R(A, M) \\ f &\mapsto f \circ d \end{aligned}$$

is an iso, which we called α earlier.

A.3.3 Pushforward

For a map $f : A \to B$ of R-algebras, basic homological algebra tells us the following fact.

Proposition A.3.3. *The pushforward functor* $f_* : A - \mathbf{Bimod}_R \to B - \mathbf{Bimod}_R$ *is given by*

$$f_*(M) = B \otimes_A M \otimes_A B$$

equipped with B-multiplication on the left and right.

Corollary A.3.4. *The abelianization functor is* $Ab_B(A \to B) = B \otimes_A I_A \otimes_A B$.

A.4 Commutative algebras

Notation. Let $\mathbf{Com}_R$ denote the category of commutative R-algebras.

A.4.1 Beck modules

Proposition A.4.1. *A Beck module over a commutative R-algebra A is a split extension of A (with the data of a splitting) with square zero kernel:*

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & E \cong A \oplus M \arrow[r, shift left, "p"] & A \arrow[l, shift left, "s"] \arrow[r] & 0. \end{tikzcd}$$

Equivalently, it is the data of an A-module M in the usual sense (i.e. an abelian group with an action of A).

Proof. Same as for associative algebras A.3.1, except the total space E must be commutative, which means:

$$a \cdot m = s(a)m = ms(a) = m \cdot a. \qquad \square$$

A.4.2 Abelianization

We use the adjunction $Com : \mathbf{Alg}_R \rightleftarrows \mathbf{Com}_R : \iota$ to identify the abelianization in $\mathbf{Com}_R$.

Proposition A.4.2. *For a commutative R-algebra A, the abelianization functor $Ab_A : \mathbf{Com}_R/A \to A - \mathbf{Mod}$ satisfies*

$$Ab_A A = I_A / I_A^2.$$

Proof. Starting from id_A in the upper left corner of 4.6.2, we obtain

$$\begin{tikzcd}
\mathrm{id}_A \arrow[r, mapsto] \arrow[d, mapsto] & I_A \arrow[d, mapsto] \\
\mathrm{id}_A \arrow[r, mapsto] & Ab_A A = \mathrm{HH}_0(I_A).
\end{tikzcd}$$

Now $\mathrm{HH}_0(I_A)$ is obtained from I_A by modding out the sub-bimodule generated by elements of the form

$$\begin{aligned}
a \cdot m - m \cdot a &= a \cdot (1 \otimes a' - a' \otimes 1) - (1 \otimes a' - a' \otimes 1) \cdot a \\
&= a \otimes a' - aa' \otimes 1 - 1 \otimes a'a + a' \otimes a \\
&= -(1 \otimes aa' - a' \otimes a - a \otimes a' + aa' \otimes 1) \quad \text{since } A \text{ is commutative} \\
&= -(1 \otimes a - a \otimes 1)(1 \otimes a' - a' \otimes 1)
\end{aligned}$$

which is exactly the sub-bimodule I_A^2. Thus we have $Ab_A A = \mathrm{HH}_0(I_A) = I_A / I_A^2$. $\square$

As for associative R-algebras, the "global sections" of a Beck module $A \oplus M \to A$ over the commutative R-algebra A are precisely the R-derivations $\mathrm{Der}_R(A, M)$.

Notation. The module $Ab_A A = I_A / I_A^2$ representing R-derivations is called the **module of differentials** and denoted $\Omega_{A/R}$.

Again, there is a universal R-derivation

$$\begin{aligned} d : A &\to \Omega_{A/R} \\ a &\mapsto 1 \otimes a - a \otimes 1 \end{aligned}$$

such that the natural map

$$\begin{aligned} \mathrm{Hom}_{A-\mathbf{Mod}}(\Omega_{A/R}, M) &\to \mathrm{Der}_R(A, M) \\ f &\mapsto f \circ d \end{aligned}$$

is an iso. Compare [GS07], the setup before proposition 4.27.

A.4.3 Pushforward

For a map $f : A \to B$ of commutative R-algebras, basic homological algebra tells us the following fact.

Proposition A.4.3. *The pushforward functor* $f_* : A - \mathbf{Mod} \to B - \mathbf{Mod}$ *is given by*

$$f_*(M) = B \otimes_A M$$

equipped with B-multiplication on the left.

Corollary A.4.4. *The abelianization functor is* $Ab_B(A \to B) = B \otimes_A \Omega_{A/R}$.

A.4.4 Quillen cohomology

In the category $\mathbf{Com}_R$ of commutative R-algebras, Quillen cohomology is given by

$$\begin{aligned} \mathrm{HQ}^*(A; M) &= \mathrm{H}^* \mathrm{Hom}_{\mathbf{Mod}_A}(Ab_A C_\bullet, M) \\ &= \mathrm{H}^* \mathrm{Der}_R(A \otimes_{C_\bullet} \Omega_{C_\bullet/R}, M) \end{aligned}$$

which is the celebrated André-Quillen cohomology of commutative R-algebras [Qui70].

Appendix B

Fibered category of Beck modules

B.1 Construction

The category **ModGp** encountered in chapter 7 is an example of a construction of independent interest. For any nice category C, instead of looking only at Beck modules over a single object X, we want to know what happens when we change the base object X. For this, we assemble all the module categories $\mathbf{Mod}_X$ together.

Definition B.1.1. The (fibered) **category of Beck modules** over C, denoted $\mathbf{Mod}C$ is the category whose objects are of pairs (X, E), where X is an object of C and $E \to X$ is a Beck module over X. A morphism from (X, E) to (Y, E') consists of maps $f : X \to Y$ and $\varphi : E \to E'$ in C making the obvious diagram commute:

$$\begin{tikzcd}
E \arrow[r, "\varphi"] \arrow[d] & E' \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

and such that the horizontal arrows respect the group structure maps of $E \to X$ and $E' \to Y$.

Consider the forgetful functor $U : \mathbf{Mod}C \to C$ taking the pair (X, E) to the base object X. Its fiber over an object X (i.e. subcategory of $\mathbf{Mod}C$ of objects sent to X and morphisms sent to id_X) is exactly the category $\mathbf{Mod}_X$ of Beck modules over X. Now let's make sure that $\mathbf{Mod}C$ is indeed fibered over C.

Proposition B.1.2. *A morphism from (X, E) to (Y, E') as defined above is the same data as a map $f : X \to Y$ in C and a map $\varphi : E \to f^*E'$ in $\mathbf{Mod}_X$.*

Proof. Consider the commutative diagram:

$$\begin{tikzcd}
E \arrow[rr, "\varphi"] \arrow[dr] \arrow[dd, "p"'] & & E' \arrow[dd, "p'"] \\
& f^*E' \arrow[ur] \arrow[dl] \arrow[dr] & \\
X \arrow[rr, "f"'] & & Y.
\end{tikzcd}$$

Viewing it as a diagram in C, the map φ is in $\mathrm{Hom}_{C/Y}(E \overset{fp}{\to} Y, E' \overset{p'}{\to} Y) = \mathrm{Hom}_{C/X}(E \overset{p}{\to} X, f^*E' \overset{f^*p'}{\to} X)$. By construction of the pullback f^*E' and its structure maps, this adjoint map φ is actually in the subset $\mathrm{Hom}_{Ab(C/X)}(E \overset{p}{\to} X, f^*E' \overset{f^*p'}{\to} X)$ iff the original φ respects the structure maps of $E \to X$ and $E' \to Y$. □

Corollary B.1.3. *Pullback squares*

$$\begin{tikzcd}
f^*E' \arrow[r, "\varphi"] \arrow[d] & E' \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

are Cartesian morphisms in $\mathbf{Mod}C$*. The forgetful functor* $U : \mathbf{Mod}C \to C$ *makes* $\mathbf{Mod}C$ *into a fibered category over* C *in the sense of [Vis07, § 3.1.1]. The system of pullbacks makes it into a cleaved category.*

B.2 Relationship to the ground category

Proposition B.2.1. *The "zero section" functor* $Z : C \to \mathbf{Mod}C$ *which sends* X *to* $(X, 0_X)$ *is a both a left and right adjoint of* U*. In particular,* U *preserves all limits and colimits.*

Proof. It follows from the fact that 0_X is a zero object in the additive category $\mathbf{Mod}_X$ and the pullback of zero is again zero, i.e. $f^*0_Y = 0_X$.

$$\begin{aligned}
\mathrm{Hom}_C(X, U(Y, E')) &= \mathrm{Hom}_C(X, Y) \\
&= \mathrm{Hom}_{\mathbf{Mod}C}((X, 0_X), (Y, E')) \\
&= \mathrm{Hom}_{\mathbf{Mod}C}(Z(X), (Y, E')).
\end{aligned}$$

$$\begin{aligned}\mathrm{Hom}_C(U(X,E),Y) &= \mathrm{Hom}_C(X,Y)\\ &= \mathrm{Hom}_{\mathbf{Mod}C}((X,E),(Y,0_Y))\\ &= \mathrm{Hom}_{\mathbf{Mod}C}((X,E),Z(Y)).\end{aligned}$$

□

In fact, there is a more general relative version of this proposition, where we work over fixed objects. The previous case is over the terminal object.

Proposition B.2.2. *Consider the forgetful functor* $U : \mathbf{Mod}C/(X,E) \to C/X$.

1. *The left adjoint of* U *is the "zero section" functor* Z*, which sends* $Y \xrightarrow{f} X$ *to* $(Y, 0_Y) \to (X,E)$.
2. *The right adjoint of* U *is the "pullback" functor* $Pull$*, which sends* $Y \xrightarrow{f} X$ *to* $(Y, f^*E) \to (X,E)$.

Proof. 1) Similar to the absolute version:

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{Mod}C/(X,E)}\left(Z(Y \xrightarrow{f} X), (X',E') \xrightarrow{(f',\varphi')} (X,E)\right)\\ &= \mathrm{Hom}_{\mathbf{Mod}C/(X,E)}\left((Y,0_Y) \xrightarrow{(f,0)} (X,E), (X',E') \xrightarrow{(f',\varphi')} (X,E)\right)\\ &= \{(g,\varphi) : (Y,0_Y) \to (X',E') \mid (f',\varphi') \circ (g,\varphi) = (f,0)\}\\ &= \{g : Y \to X' \mid f' \circ g = f\} \qquad \text{since } \varphi \text{ must be 0, and 0 works}\\ &= \mathrm{Hom}_{C/X}\left(Y \xrightarrow{f} X, X' \xrightarrow{f'} X\right)\\ &= \mathrm{Hom}_{C/X}\left(Y \xrightarrow{f} X, U\left((X',E') \xrightarrow{(f',\varphi')} (X,E)\right)\right).\end{aligned}$$

2) We have the following:

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{Mod}C/(X,E)}\left((X',E') \xrightarrow{(f',\varphi')} (X,E), Pull(Y \xrightarrow{f} X)\right)\\ &= \mathrm{Hom}_{\mathbf{Mod}C/(X,E)}\left((X',E') \xrightarrow{(f',\varphi')} (X,E), (Y,f^*E) \xrightarrow{f} (X,E)\right)\\ &= \{(g,\varphi) : (X',E') \to (Y,f^*E) \mid f \circ (g,\varphi) = (f',\varphi')\}.\end{aligned}$$

This consists of the data of $g : X' \to Y$ and a map $E' \to g^*(f^*E) \cong (fg)^*E = (f')^*E$ in $\mathbf{Mod}_{X'}$

making the diagram commute. But since

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

is a pullback square, our map φ must "be" (under the usual identification) the map $\varphi' : E' \to (f')^*E$ in $\mathbf{Mod}_{X'}$. Hence it provides no additional data and no constraint, and the above set of morphisms is:

$$\begin{aligned} &\{g : X' \to Y \mid f \circ g = f'\} \\ &= \mathrm{Hom}_{C/X}\left(X' \xrightarrow{f'} X, Y \xrightarrow{f} X\right) \\ &= \mathrm{Hom}_{C/X}\left(U\left((X', E') \xrightarrow{(f',\varphi')} (X, E)\right), Y \xrightarrow{f} X\right). \end{aligned}$$ □

We can use this proposition to study the abelianization in $\mathbf{Mod}C$. The forgetful functor $U : \mathbf{Mod}C \to C$ sending (X, E) to X takes the "ground level" part of the data. With this in mind, we'll show that the ground level part of the abelianization is the abelianization of the ground level part.

Proposition B.2.3. *The following diagram commutes:*

$$\begin{tikzcd} \mathbf{Mod}C/(X,E) \arrow[r, "Ab_{(X,E)}"] \arrow[d, "U"'] & Ab(\mathbf{Mod}C/(X,E)) \arrow[d, "U"'] \\ C/X \arrow[r, "Ab_X"] & Ab(C/X). \end{tikzcd}$$

Proof. By propositions B.2.2 and 4.1.1, the diagram above consists of left adjoints. Let us write all four adjunctions:

$$\begin{tikzcd} \mathbf{Mod}C/(X,E) \arrow[r, shift left, "Ab_{(X,E)}"] \arrow[d, shift right, "U"'] & Ab(\mathbf{Mod}C/(X,E)) \arrow[l, shift left, "U_{(X,E)}"] \arrow[d, shift right, "U"'] \\ C/X \arrow[r, shift left, "Ab_X"] \arrow[u, shift right, "Pull"'] & Ab(C/X). \arrow[l, shift left, "U_X"] \arrow[u, shift right, "Pull"'] \end{tikzcd}$$

The diagram of right adjoints commutes on the nose:

$$U_{(X,E)} \circ Pull = Pull \circ U_X$$

by definition of induced functor on category of abelian group objects. Thus their left adjoints are naturally isomorphic:

$$U \circ Ab_{(X,E)} \cong Ab_X \circ U.$$

As in proposition 1.1.7, the abelianizations can be chosen so that this is an equality on the nose. □

Proposition B.2.4. *The "total space" functor $Src : \mathbf{Mod}C \to C$ which sends $E \xrightarrow{p} Y$ to E (viewed as object in C) has a left adjoint, which sends an object X of C to $Ab_X X \to X$. The notation Src stands for "source".*

Proof. A map in $\mathrm{Hom}_{\mathbf{Mod}C}\left(Ab_X X \to X, E \xrightarrow{p} Y\right)$ consists of a map $f : X \to Y$ and a map $Ab_X X \to f^*E$ in $\mathbf{Mod}_X$.

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Mod}_X}(Ab_X X \to X, f^*E \to X) &= \mathrm{Hom}_{C/X}(X \xrightarrow{\mathrm{id}} X, f^*E \to X) \\ &= \mathrm{Hom}_{C/Y}(f_!(X \xrightarrow{\mathrm{id}} X), E \to Y) \\ &= \mathrm{Hom}_{C/Y}(X \xrightarrow{f} Y, E \to Y)\end{aligned}$$

Therefore, a map in $\mathrm{Hom}_{\mathbf{Mod}C}\left(Ab_X X \to X, E \xrightarrow{p} Y\right)$ consists of the data of f and φ in such a diagram:

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

which is the data of φ only, since f must be $p\varphi$. Since there is no constraint on φ, we conclude:

$$\mathrm{Hom}_{\mathbf{Mod}C}\left(Ab_X X \to X, E \xrightarrow{p} Y\right) = \mathrm{Hom}_C(X, E).$$ □

B.3 Limits and completeness

In this section, we study limits in $\mathbf{Mod}C$ and show that $\mathbf{Mod}C$ is complete if C is. Let's proceed in steps.

Proposition B.3.1. *The forgetful functor $U : Ab(C) \to C$ creates limits, in the sense of [Mac98, § 5.1].*

Proof. 1) Start with a diagram $\widetilde{F} : I \to Ab(C)$ whose underlying diagram $F := U\widetilde{F}$ in C has a limit. We will endow $\lim F$ with structure maps to produce a limit of $\widetilde{F}$. The structure maps of the objects in the diagram $\widetilde{F}$ can be expressed as natural transformations

$$\begin{aligned} F \times F &\xrightarrow{\mu} F \\ * &\xrightarrow{e} F \\ F &\xrightarrow{\iota} F, \end{aligned}$$

where $*$ is the terminal diagram (i.e. constant on the terminal object), and $F \times F$ is the composite

$$I \xrightarrow{(F,F)} C \times C \xrightarrow{\times} C.$$

In other words, we take the objectwise product, unit, and inverse structure maps. Applying the functor lim yields maps

$$\begin{aligned} \lim(F \times F) \cong \lim F \times \lim F &\xrightarrow{\lim \mu} \lim F \\ \lim * \cong * &\xrightarrow{\lim e} \lim F \\ \lim F &\xrightarrow{\lim \iota} \lim F. \end{aligned}$$

(Detail: We haven't assumed that C is complete, so technically there is no functor $\lim : C^I \to C$. We could work around this by restricting to the full subcategory of C^I of diagrams admitting a limit, on which the functor lim is defined. More explicitly, we can unwind the construction: A natural transformation $h : F \to G$ always induces $\lim h : \lim F \to \lim G$ whose associated cone on G is given by $\lim F \xrightarrow{\pi_i} F_i \xrightarrow{h_i} G_i$ for any index i in I.)

These form structure maps of an abelian group object, since applying lim to the condition diagrams of F yields condition diagrams for $\lim F$. Let us denote $\widetilde{\lim F} \in Ab(C)$ the object $\lim F$ equipped with these structure maps. By construction, $\widetilde{\lim F}$ comes with a cone on $\widetilde{F}$ which is a U-lift of $\lim F$ and its limiting cone on F.

2) Let us check that $\widetilde{\lim F}$ is the limit of $\widetilde{F}$ in $Ab(C)$. Given a cone $\widetilde{\psi} : \Delta\widetilde{Z} \to \widetilde{F}$, look at the underlying cone ψ; it has a unique map $\varphi Z \to \lim F$ associated to it. We need to check that φ is a map in $Ab(C)$. But by construction it is, since all the maps in the natural transformation $\widetilde{\psi}$ are in $Ab(C)$.

For example, consider the diagram

$$\begin{tikzcd}
Z \times Z \arrow[r, "(\varphi{,}\varphi)"] \arrow[d, "\mu_Z"'] & \lim F \times \lim F \arrow[r, "\cong"] \arrow[d, "\mu_{\lim F}"] & \lim(F \times F) \arrow[d, "\lim \mu_F"] \\
Z \arrow[r, "\varphi"'] & \lim F \arrow[r, equal] & \lim F.
\end{tikzcd}$$

It commutes iff the adjoint diagram commutes:

$$\begin{tikzcd}
\Delta Z \times \Delta Z = \Delta(Z \times Z) \arrow[r, "(\psi{,}\psi)"] \arrow[d, "\Delta\mu_Z"'] & F \times F \arrow[d, "\mu_F"] \\
\Delta Z \arrow[r, "\psi"'] & F.
\end{tikzcd}$$

This diagram does commute, since $\widetilde{\psi}$ was a natural transformation between I-diagrams in $Ab(C)$. Likewise for the unit and inverse structure maps. Hence we have $\widetilde{\lim F} = \lim \widetilde{F}$ in $Ab(C)$, as desired.

3) So far we've shown that U lifts limits, but there is more to creating limits. We need to show that there is a *unique* cone lifting $\lim F$ and its limiting cone $\pi : \Delta \lim F \to F$. Let L be such a lift, i.e. L is the underlying object $\lim F$ equipped with (possibly exotic) structure maps, and we have a lift $\widetilde{\pi} : \Delta L \to \widetilde{F}$ of the cone π. Saying that this cone $\widetilde{\pi}$ is a lift in $Ab(C)$ means that all the projection maps $\pi_i : \lim F \to F_i$ respect the structure maps. For example, the following diagram commutes:

$$\begin{tikzcd}
\lim F \times \lim F \arrow[r, "\mu"] \arrow[d, "\pi_i \times \pi_i"'] & \lim F \arrow[d, "\pi_i"] \\
F_i \times F_i \arrow[r, "\mu_i"'] & F_i.
\end{tikzcd}$$

Hence the i^{th} component of μ is $\mu_i(\pi_i \times \pi_i)$. But a map to a limit is uniquely determined by its components, hence μ is unique, and by the same argument, so are the unit and inverse structure maps. These structure maps are precisely the ones we used in part (1), i.e. L is $\widetilde{\lim F}$. Since U is faithful, the lift of the cone π is unique, and as we've seen in part (2), it is a limiting cone for $\widetilde{F}$. □

Corollary B.3.2. *If C is complete, then so is $Ab(C)$, and $U : Ab(C) \to C$ preserves limits.*

Proof. Let $\widetilde{F} : I \to \mathbf{Mod}C$ be a diagram indexed by some small category I. Since C is complete, the underlying diagram $F := U\widetilde{F}$ has a limit, with limiting cone $\pi : \Delta \lim F \to F$. Since U creates limits, there is a unique U-lift $(L, \widetilde{\pi})$ of $(\lim F, \pi)$, and it's a limit in $\mathbf{Mod}C$. Thus U preserves limits, since they are essentially unique. □

Remark B.3.3. Note that creating limits and preserving limits are distinct notions, neither one implying the other. The previous argument only shows that a limit-creating functor U must preserve limits of diagrams whose underlying diagram has a limit.

Example B.3.4. (Preserving limits $\nRightarrow$ Creating limits) Consider the projection functor $C \times \mathcal{D} \to C$. Clearly it preserves limits, but (for general $\mathcal{D}$) it does not create them, or even lift them uniquely.

Example B.3.5. (Creating limits $\nRightarrow$ Preserving limits) Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ be the category of natural numbers viewed as a totally ordered set, i.e. with a unique morphism from i to j iff $i \leq j$. Let $[n]$ be its full subcategory with objects $\{0, 1, \cdots, n\}$.

Limits in $\mathbb{N}$ are as follows: the minimum number appearing in the diagram if the diagram is non-empty, and non-existent if the diagram is empty, i.e. there is no terminal object. Limits in $[n]$ are the same except there is a terminal object, namely n. Therefore, the inclusion functor $[n] \to \mathbb{N}$ does NOT preserve limits, but it does create them.

Corollary B.3.6. *If C is complete, then so is each category of Beck modules $\mathbf{Mod}_X$, for X an object in C.*

Proof. C being complete implies the slice category C/X is also complete [Bor94a, prop 2.16.3]. Hence $\mathbf{Mod}_X = Ab(C/X)$ is also complete. □

Corollary B.3.7. *For any map $f : X \to Y$, the pullback functor $f^* : \mathbf{Mod}_Y \to \mathbf{Mod}_X$ preserves limits.*

Proof. We have the commutative diagram:

$$\begin{tikzcd}
Ab(C/Y) \arrow[r, "f^*"] \arrow[d, "U_Y"'] & Ab(C/X) \arrow[d, "U_X"] \\
C/Y \arrow[r, "f^*"] & C/X.
\end{tikzcd}$$

As seen above, U_Y preserves limits, and so does f^* (downstairs) since it's a right adjoint. Hence, if we start with a limit and its cone in $\mathbf{Mod}_Y$ we obtain:

$$\begin{aligned}
U_X f^*(\lim F) &= f^* U_Y(\lim F) \\
&= \lim(f^* U_Y F).
\end{aligned}$$

Since U_X creates limits, $f^*(\lim F)$ with its cone is the unique U_X-lift of $\lim(f^* U_Y F)$ with its cone, and it is itself a limit. In other words, we have $f^*(\lim F) = \lim f^* F$. $\square$

Proposition B.3.8. *If C is complete, then so is* $\mathbf{Mod}C$.

Proof. 1) Let $\widetilde{F} : I \to \mathbf{Mod}C$ be a diagram and denote $F := U\widetilde{F}$ the diagram of underlying objects in C. Since C is complete, it admits a limit $X := \lim F$. We're looking for a "limit module" over X; let's just pull back all the modules in $\widetilde{F}$ via the cone π. Having a diagram $\widetilde{F}$ in $\mathbf{Mod}C$ means that for every index $i \in I$, we have and object $\widetilde{F_i} \to F_i$ and for every map $\alpha : i \to j$ we have a map

$$\begin{tikzcd}
\widetilde{F_i} \arrow[r] \arrow[d] & \widetilde{F_j} \arrow[d] \\
F_i \arrow[r, "F_\alpha"'] & F_j
\end{tikzcd}$$

in $\mathbf{Mod}C$, which is the same as F_α and the associated map $\widetilde{F_i} \to F_\alpha^* \widetilde{F_j}$ in $\mathbf{Mod}_{F_i}$. Now, X has a limiting cone on F, i.e.

$$\begin{tikzcd}
 & X \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] & \\
F_i \arrow[rr, "F_\alpha"'] & & F_j
\end{tikzcd}$$

Pulling back the modules over X via π, we obtain:

$$\begin{tikzcd}
\pi_i^* \widetilde{F_i} \arrow[rr] \arrow[dr] & & \pi_i^* F_\alpha^* \widetilde{F_j} \cong (F_\alpha \pi_i)^* \widetilde{F_j} = \pi_j^* \widetilde{F_j} \arrow[dl] \\
 & X &
\end{tikzcd}$$

which defines an I-diagram $\pi^* \widetilde{F}$ in $\mathbf{Mod}_X$ (One needs to be careful with the natural iso $f^* g^* \cong (gf)^*$, but it works). By B.3.6, $\mathbf{Mod}_X$ is complete, so we can take its limit $M := \lim \pi^* \widetilde{F}$.

2) Let us check that $M \to X$ and its cone $\widetilde{\pi}$ over $\widetilde{F}$ is a limit in $\mathbf{Mod}C$. Given a cone $\widetilde{\psi} : \Delta(\widetilde{Z} \to Z) \to \widetilde{F}$, look at the cone ψ of underlying objects and take its associated map $\varphi : Z \to X = \lim F$. We're looking for a map

$$\begin{tikzcd}
\widetilde{Z} \arrow[r, "\widetilde{\varphi}"] \arrow[d] & M \arrow[d] \\
Z \arrow[r, "\varphi"'] & X
\end{tikzcd}$$

that commutes with the cones, i.e. such that $\widetilde{\pi} \circ \widetilde{\varphi}$ is $\widetilde{\psi}$. This is the same as a map $\widetilde{Z} \to \varphi^* M$ in $\mathbf{Mod}_Z$ such that the cones $\widetilde{\pi} \circ \widetilde{\varphi}$ and $\widetilde{\psi}$ in $\mathbf{Mod}_Z$ over $\psi^* \widetilde{F}$ agree. There is a unique such map, because of the following:

$$\begin{aligned} \varphi^* M &= \varphi^* \lim \pi^* \widetilde{F} \\ &= \lim \varphi^* \pi^* \widetilde{F} \quad \text{by B.3.7} \\ &\cong \lim (\pi\varphi)^* \widetilde{F} \\ &= \lim \psi^* \widetilde{F}. \end{aligned} \qquad \square$$

Question. Does $\mathbf{Mod}C$ inherit other nice properties from C? Here are some properties that would be of particular interest.

- Having all pushforwards;
- Having all abelianizations;
- Beck modules over any object form an abelian category;
- Cocompleteness;
- Regularity;
- Exactness.

The answer is probably yes for most of them, maybe with additional assumptions. In light of proposition 3.4.8, we probably need additional assumptions for cocompleteness.

B.4 Representability

Following [Vis07, § 3.1.2], one can think of the fibered category $\mathbf{Mod}C$ as a (contravariant) pseudofunctor $\mathbf{Mod}_{(-)}$ associating to each object X of C its category of Beck modules $\mathbf{Mod}_X$ and to each map $f : X \to Y$ the pullback functor $f^* : \mathbf{Mod}_Y \to \mathbf{Mod}_X$,

$$\mathbf{Mod}_{(-)} : C \to \mathbf{AbCat}.$$

In the examples of appendix A, we have seen that the category of Beck modules over an object X is sometimes naturally equivalent to a category of functors into $\mathbf{Ab}$, i.e. presheaves of abelian groups.

This raises the question: Is there a (covariant) pseudofunctor $I_{(-)}$ making the following diagram commute up to natural isomorphism?

$$\begin{tikzcd} & \mathbf{Cat} \arrow[dr, "\mathrm{Fun}(-,\mathbf{Ab})"] & \\ C \arrow[ur, dashed, "I_{(-)}"] \arrow[rr, "\mathbf{Mod}_{(-)}"'] & & \mathbf{AbCat} \end{tikzcd}$$

Definition B.4.1. The category C has **representable Beck modules** if there exists such a pseudofunctor $I_{(-)}$, called an **indexing functor**, and the category I_X, satisfying $\mathbf{Mod}_X \cong \mathbf{Ab}^{I_X}$ is called an **indexing category** for the object X of C.

Example B.4.2. $C = \mathbf{Set}$ has representable Beck modules, taking the indexing category I_X of a set X to be the discrete small category X.

Example B.4.3. $C = \mathbf{Gp}$ has representable Beck modules, taking the indexing category I_G of a group G to be the one-object groupoid G.

Example B.4.4. $C = \mathbf{Ab}$ has representable Beck modules, taking the indexing category I_A of an abelian group A to be the trivial category $\{*\}$ with one object and its identity map.

Non-example B.4.5. $C = \mathbf{ComRing}$, the category of commutative (associative, unital) rings, does NOT have representable Beck modules. For a commutative ring R, the category $\mathbf{Mod}_R$ of Beck modules over it is the usual category of R-modules (see Haynes's notes). Taking $R = \mathbb{F}_p$ the field with p elements, then $\mathbf{Mod}_R$ is the category of $\mathbb{F}_p$-vector spaces, which itself is not of the form $\mathbf{Ab}^I$ for any category I. Indeed, fixing an index (object) i of I, consider the (covariant) functor:

$$\begin{tikzcd} I \arrow[r, "\mathrm{Hom}_I(i,-)"] & \mathbf{Set} \arrow[r, "Free"] & \mathbf{Ab} \end{tikzcd}$$

This is an object of $\mathbf{Ab}^I$ whose identify endomorphism has infinite additive order. This can't happen in $\mathbb{F}_p$-vector spaces, where every map is p-torsion.

Question. When does C have representable Beck modules? The question can be broken down into two.

1. When is an abelian category $\mathcal{A}$ equivalent to a functor category $\mathbf{Ab}^I$?
2. Assuming C is such that for every object X, the category $\mathbf{Mod}_X$ is equivalent to a functor category $\mathbf{Ab}^{I_X}$ for some (small) category I_X, when does C have representable Beck modules?

> In other words, can we make the indexing categories I_X and equivalences $\mathbf{Mod}_X \cong \mathbf{Ab}^{I_X}$ natural in X?

Question. Does C have some nice features if it has representable Beck modules? In other words, is it an interesting property?

It is more common to look for a ring R (or ringoid, a.k.a. preadditive category) such that the abelian category $\mathcal{A}$ is the category of modules over R (additive functors $R \to \mathbf{Ab}$).

B.5 Analogy with the tangent bundle

Here's a far-fetched idea: Does the fibered category $\mathbf{Mod}C$ deserve to be thought of as the "tangent bundle" of C? The tangent bundle of a smooth manifold M gives over each point x the tangent space T_xM, which can be thought of as the best linear approximation of M around x. Heuristically, let's think of C/X as a neighborhood of X, in other words an object $Y \xrightarrow{f} X$ is a point "close" to X. Now let's think of abelian group objects as providing the best approximation by an additive category. Then $\mathbf{Mod}_X$ is a best linear approximation of C around the object X.

Here's one way in which the analogy is not completely silly. Notice that if V is a smooth manifold that happens to be a vector space (i.e. Euclidean space), then for every point $x \in V$, we have $T_xV \cong V$. The same holds for Beck modules.

Lemma B.5.1. *Let I be any category and let $\mathcal{A}$ denote the abelian category $\mathbf{Ab}^I$. Then for every object F of $\mathcal{A}$, we have an equivalence of abelian categories:*

$$\mathbf{Mod}_F = Ab(\mathcal{A}/M) \cong \mathcal{A}.$$

The equivalence associates to a module $E \xrightarrow{p} F$ the object $\ker p$, and to an object K of $\mathcal{A}$, the module $F \oplus K \to F$, with structure maps given by the "objectwise" addition in K.

Proof. Let's look at one node of the diagram at a time. For an index i in I, consider the functor $i^* : \mathbf{Ab}^I \to \mathbf{Ab}$ that extracts the abelian group at index i, i.e. evaluates a functor in $\mathbf{Ab}^I$ at i. This is the restriction functor along the inclusion of the point category $i : \{*\} \to I$ that selects the object i. By proposition C.0.3 i^* preserves limits, and thus induces a functor on Beck modules. By our knowledge of Beck modules in $\mathbf{Ab}$ (proposition A.2.1), we know that the Beck module $E \xrightarrow{p} F$, which looks like

$$\begin{tikzcd}
E_i \arrow[r, "E_\alpha"] \arrow[d, "p_i"'] & E_j \arrow[d, "p_j"] \\
F_i \arrow[r, "F_\alpha"] & F_j
\end{tikzcd}$$

(where $\alpha : i \to j$ is a typical map in I) is actually of the form

$$\begin{tikzcd}
K_i \arrow[r, "K_\alpha"] \arrow[d] & K_j \arrow[d] \\
F_i \oplus K_i \arrow[r, "E_\alpha"] \arrow[d, "p_i"'] & F_j \oplus K_j \arrow[d, "p_j"] \\
F_i \arrow[r, "F_\alpha"] & F_j
\end{tikzcd}$$

where K_i is $\ker p_i$ and K_α is the restriction of E_α to K_i. Moreover, since $p : F \oplus K \to F$ is a map in $\mathcal{A}$, i.e. a natural transformation, and so is the zero section $e : F \to F \oplus K$, we know that the map E_α is actually $F_\alpha \oplus K_\alpha$, or in matrix form $\begin{bmatrix} F_\alpha & 0 \\ 0 & K_\alpha \end{bmatrix}$. The upper row comes from p, and the lower-left corner comes from e. This determines completely the structure maps, which must be the objectwise addition, zero, and negative in each abelian group K_i. The only additional information is that these structure maps are maps in $\mathcal{A}/F$. Writing down these conditions explicitly, they say that K_α must be a group map, which is automatic. In other words, there is no constraint on K, any object of $\mathcal{A}$ will do.

Now look at the correspondence described in the statement. The composite $\mathcal{A} \to \mathbf{Mod}_F \to \mathcal{A}$ is the identify functor. The composite $\mathbf{Mod}_F \to \mathcal{A} \to \mathbf{Mod}_F$ sends $E \xrightarrow{p} F$ to $F \oplus \ker p \to F$, which is a natural iso in $\mathcal{A}/F$ since the Beck module comes equipped with the data of the splitting (the unit map). By the argument above, it is a natural iso in $\mathbf{Mod}_F$, i.e. it recovers the Beck module structure. Finally, note that both directions of the correspondence are additive functors, so we obtain an equivalence of abelian categories $\mathbf{Mod}_F \cong \mathcal{A}$. □

Proposition B.5.2. *Let $\mathcal{A}$ be an abelian category. Then for every object M of $\mathcal{A}$, we have an equivalence of abelian categories:*

$$\mathbf{Mod}_M = Ab(\mathcal{A}/M) \cong \mathcal{A}.$$

The equivalence associates to a module $E \xrightarrow{p} M$ the object $\ker p$*, and to an object K of $\mathcal{A}$, the* *module $M \oplus K \to M$, with the following structure maps:*

- *Multiplication $\mu : M \oplus K \oplus K \to M \oplus K$ is given by* $\begin{bmatrix} \mathrm{id}_M & 0 & 0 \\ 0 & \mathrm{id}_K & \mathrm{id}_K \end{bmatrix}$;
- *Unit $e : M \to M \oplus K$ is given by inclusion, that is* $\begin{bmatrix} \mathrm{id}_M \\ 0 \end{bmatrix}$;
- *Inverse $\iota : M \oplus K \to M \oplus K$ is given by* $\begin{bmatrix} \mathrm{id}_M & 0 \\ 0 & -\mathrm{id}_K \end{bmatrix}$.

Proof. Essentially follows from the splitting lemma, the Yoneda embedding and the lemma above. Recall the Yoneda embedding:

$$Y : \mathcal{A} \to \mathbf{Ab}^{\mathcal{A}^{op}}$$
$$B \mapsto \mathrm{Hom}_{\mathcal{A}}(-, B)$$

which is full, faithful, and left exact. Since it's left exact, it preserves small limits and hence induces a functor on Beck modules. Starting with a Beck module $E \xrightarrow{p} M$ in $\mathcal{A}$, we get a Beck module $Y(E) \xrightarrow{Y(p)} Y(M)$ in $\mathbf{Ab}^{\mathcal{A}^{op}}$. By the lemma above, its structure maps must be the obvious ones on $\ker Y(p)$ (objectwise addition, zero, and inverse). Since Y is faithful, this determines the structure maps for $E \xrightarrow{p} M$. Moreover, we have $\ker Y(p) = Y(\ker p)$, and by the splitting lemma, $E \xrightarrow{p} M$ is canonically isomorphic to $M \oplus \ker p \to M$ in $\mathcal{A}$. Put structure maps on the latter by the formulas in the statement. Notice that Y sends all those maps to the structure maps of $Y(M) \oplus \ker Y(p) \to Y(M)$ (we used the fact that Y is additive for the inverse structure map ι). Hence those candidate structure maps ARE those of $E \xrightarrow{p} M$. (Incidentally, this shows that the formulas in the statement do define a Beck module, although one can easily check it directly.)

We can conclude $\mathbf{Mod}_M \cong \mathcal{A}$ exactly as in the lemma. □

Question.

1. Can we push the analogy further? More precisely, we're looking for a universal property satisfied by the tangent bundle of a manifold, and a universal property satisfied by the fibered category of Beck modules.
2. Is $Ab(C) \to C$ terminal among additive categories equipped with a faithful, limit-creating functor to C?

Appendix C

Kan extension arguments

Whenever we have a functor $F : I \to J$, it induces a restriction functor $F^* : C^J \to C^I$. Kan extensions allow us to go the other way. Let's recall a few things about them, which can be found in [Mac98, X.3].

The right Kan extension of a functor $X : I \to C$ along F is the "closest" (i.e. terminal) functor as in this diagram:

$$\begin{tikzcd}
I \arrow[rr, "X"] \arrow[dr, "F"'] & {\scriptstyle \eta \Uparrow} & C \\
& J \arrow[ur, "\operatorname{Ran}_F X"'] &
\end{tikzcd}$$

which means the following:

$$\operatorname{Hom}_{\mathrm{Fun}(I,C)}(F^*Z, X) \cong \operatorname{Hom}_{\mathrm{Fun}(J,C)}(Z, \operatorname{Ran}_F X)$$

i.e. the right Kan extension Ran_F (if it exists) is right adjoint to restriction F^*.

Dually, the left Kan extension is the closest (i.e. initial) functor as in this diagram:

$$\begin{tikzcd}
I \arrow[rr, "X"] \arrow[dr, "F"'] & {\scriptstyle \Downarrow \epsilon} & C \\
& J \arrow[ur, "\operatorname{Lan}_F X"'] &
\end{tikzcd}$$

which means:

$$\operatorname{Hom}_{\mathrm{Fun}(I,C)}(X, F^*Z) \cong \operatorname{Hom}_{\mathrm{Fun}(J,C)}(\operatorname{Lan}_F X, Z)$$

i.e. the left Kan extension Lan_F is left adjoint to restriction F^*.

The right Kan extension can be computed as a limit:

$$\mathrm{Ran}_F X(j) = \lim_{j \to Fi} X(i)$$

where the limit is taken over the comma category $(j \downarrow F)$. In particular, if I is small and C is complete, then $\mathrm{Ran}_F X$ exists for any X and F, and F^* has a right adjoint Ran_F.

Dually, the left Kan extension can be computed as a colimit:

$$\mathrm{Lan}_F X(j) = \operatorname*{colim}_{Fi \to j} X(i)$$

where the limit is taken over the comma category $(F \downarrow j)$. In particular, if I is small and C is cocomplete, then $\mathrm{Lan}_F X$ exists for any X and F, and F^* has a left adjoint Lan_F.

As we have seen several times, we're interested in knowing if F^* preserves limits and, to a lesser extent, colimits. Of course, if C is cocomplete, then F^* is a right adjoint so it preserves limits. But frankly, I don't like to assume (co)completeness properties when we don't need to.

Proposition C.0.3. *Let I, J, C, be categories (we may want I and J to be small, though I don't see why), and $F : I \to J$ a functor. Then the restriction functor $F^* : C^J \to C^I$ preserves limits.*

Proof. Recall that limits in functor categories are computed objectwise or "pointwise" [Bor94a, prop 2.15.1]; it follows basically from the fact that natural transformations are defined objectwise. Let $X : K \to C^J$ be a diagram of functors which has a limit. Trying to compute $\lim_K F^*X$ objectwise, we get:

$$\begin{aligned}
(\lim_K F^*X)(i) &= \lim_K (F^*X)(i) \\
&= \lim_K X(F(i)) \\
&= (\lim_K X)(F(i)) \\
&= (F^* \lim_K X)(i)
\end{aligned}$$

and thus $F^* \lim_K X$ with its cone is indeed $\lim_K F^*X$. □

In fact, colimits in functor categories are also computed objectwise, hence we obtain the following.

Proposition C.0.4. *Let I, J, C, be categories and $F : I \to J$ a functor. Then the restriction functor $F^* : C^J \to C^I$ preserves colimits.*

Proof. Same as above. □

Sometimes, as in Lawvere's notion of algebraic theories [Law63], we are interested in the full subcategory $\mathrm{Fun}^\times(I, C)$ of product-preserving functors.

Proposition C.0.5. *If a diagram in $\mathrm{Fun}^\times(I, C)$ has a limit when viewed in $\mathrm{Fun}(I, C)$ (i.e. objectwise limit), then this is also the limit in $\mathrm{Fun}^\times(I, C)$. In other words, the inclusion $\mathrm{Fun}^\times(I, C) \subset \mathrm{Fun}(I, C)$ creates limits.*

Proof. It follows from the fact that products are limits and hence commute with any other limits. More precisely, let $X : K \to \mathrm{Fun}^\times(I, C)$ be a diagram of functors that has a limit L in $\mathrm{Fun}(I, C)$. We know this is an objectwise limit:

$$L(i) = \lim_K X(i).$$

Thus L is itself product-preserving:

$$\begin{aligned} L(i \times j) &= \lim_K X(i \times j) \\ &= \lim_K (X(i) \times X(j)) \\ &= \left(\lim_K X(i)\right) \times \left(\lim_K X(j)\right) \\ &= L(i) \times L(j). \end{aligned}$$

(We wrote a finite product for ease of reading, but it should be understood as an arbitrary product.) Since $\mathrm{Fun}^\times(I, C)$ is a full subcategory of $\mathrm{Fun}(I, C)$, L is actually the limit of X in $\mathrm{Fun}^\times(I, C)$. □

In particular, this inclusion also preserves limits if C is complete. Warning: it need NOT preserve limits in general.

Example C.0.6. Let I be a discrete category, and C be the "disjoint union" of the category of sets of cardinality not one with a discrete one-object category (i.e. a "disjoint basepoint"). Note that C doesn't have a terminal object since it has more than one component, and it has exactly one idempotent object, namely the added basepoint $*$. We call an object c **idempotent** if any (non-empty) product of copies of c is (naturally) isomorphic to c. Since I is discrete, we have:

$$\begin{aligned}\mathrm{Fun}^{\times}(I,C) &= \mathrm{Fun}(I, Idem(C))\\ &= \mathrm{Fun}(I,\{*\}) = \{*\}.\end{aligned}$$

This has a terminal object, namely the constant functor on the added basepoint. However, this object is not terminal when viewed in $\mathrm{Fun}(I,C)$. In fact, that category doesn't have a terminal object since C doesn't have one. Hence in this example, the inclusion $\mathrm{Fun}^{\times}(I,C) \subset \mathrm{Fun}(I,C)$ does NOT preserve limits.

Question. Can we find reasonable conditions on C or I, weaker than C being complete, that guarantee the inclusion does preserve limits? Is it enough if C has all finite powers, including a terminal object? Also, we're interested in the case where I is the opposite of the category of finitely generated free abelian groups, because then $\mathrm{Fun}^{\times}(I,C)$ is $Ab(C)$.

Corollary C.0.7. *Assume C is complete.*

1. *If $F : I \to J$ preserves products, then the restriction functor*

$$F^* : \mathrm{Fun}^{\times}(J,C) \to \mathrm{Fun}^{\times}(I,C)$$

preserves limits.

2. *More generally, if $F : I \to J$ is any functor, then the restriction*

$$F^* : \mathrm{Fun}^{\times}(J,C) \to \mathrm{Fun}(I,C)$$

preserves limits.

Proof. Consider the commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}^{\times}(J,C) \arrow[r, "F^*"] \arrow[d] & \mathrm{Fun}^{\times}(I,C) \arrow[d] \\
\mathrm{Fun}(J,C) \arrow[r, "F^*"] & \mathrm{Fun}(I,C)
\end{tikzcd}$$

where the vertical arrows are the inclusions. The left (and right) arrow preserves limits since C is complete. The bottom arrow preserves limits by C.0.3. This proves the second claim. For the first claim, note that the right arrow creates limits, which forces the top arrow to send limits to limits. $\square$

Remark C.0.8. Even when C is complete and $F : I \to J$ preserves products, the restriction

$$F^* : \mathrm{Fun}^\times(J, C) \to \mathrm{Fun}^\times(I, C)$$

does NOT preserve colimits in general, since the naive (objectwise) colimit of a diagram is NOT a product-preserving functor anymore. In other words, such colimits cannot be computed in $\mathrm{Fun}(I, C)$.

Appendix D

Regular categories

In this appendix, we describe some categorical tools needed to work with Quillen's standard model structure on simplicial objects. The main references are [Bar02, chap 1.8, 6.1] [Bor94b, chap 2]. Recall the following.

Definition D.0.9. [Bar02, 1.8.8] [Bor94a, def 4.3.1] An epimorphism is called **regular** if it is the coequalizer of a pair of maps. If the category has kernel pairs, this is equivalent to being the coequalizer of its kernel pair.

Definition D.0.10. [Bar02, 1.8.9] [Bor94b, def 2.1.1] A category is called **regular** if it has kernel pairs, coequalizers of kernel pairs, and the pullback of any regular epi is still a regular epi.

There are different definitions of regular category in the literature. One may require more, as Barr does: all finite limits and coequalizers of any two parallel maps. Unless otherwise stated, we will use the weaker definition given above.

Proposition D.0.11. *[Bar02, exer 1.8.13.3]. Assume we have a diagram:*

$$\begin{tikzcd}
X \arrow[r, two heads, "e"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \arrow[ld, dashed] \\
Y \arrow[r, hook, "m"'] & Y'
\end{tikzcd}$$

where e is a regular epi and m is a mono. Then there exists a unique "diagonal fill-in" $h : X' \to Y$ making both triangles commute. (No need to assume the category is regular.)

Proof. [Bor94a, prop 4.3.6 (4)] The top row in the diagram

$$\begin{tikzcd}
K \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[r, two heads, "e"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \arrow[dl, dashed] \\
& Y \arrow[r, hook, "m"'] & Y'
\end{tikzcd}$$

is a coequalizer, since e is a regular epi. Now $mf = f'e$ coequalizes the two p_i, and so does f since m is a mono. Hence there is a unique diagonal fill-in h making the top left triangle commute. We have $mhe = mf = f'e$ which implies $mh = f'$ since e is an epi, so h makes the bottom right triangle commute as well. □

Proposition D.0.12. *A map is an iso iff it is both a mono and a regular epi. (No need to assume the category is regular.)*

Proof. [Bor94a, prop 4.3.6 (3)] Clearly an iso is a mono, as well as regular epi, since the kernel pair of an iso is

$$\begin{tikzcd}
X \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "\mathrm{id}"'] & X.
\end{tikzcd}$$

Conversely, let $f : X \to Y$ be such a map. Then there is a unique diagonal fill-in in the diagram:

$$\begin{tikzcd}
X \arrow[r, two heads, "f"] \arrow[d, "\mathrm{id}"'] & Y \arrow[d, "\mathrm{id}"] \arrow[dl, dashed] \\
X \arrow[r, hook, "f"'] & Y
\end{tikzcd}$$

and is it an inverse to f. □

Proposition D.0.13. *Assume a composite $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a regular epi. Then the last map g is also a regular epi, assuming either:*

1. *f is an epi (without assumption on the category);*
2. *or the category is regular (without assumption on f).*

Proof. (1) [Qui67, II.4, prop 2 (2)] The top row in the diagram:

$$\begin{tikzcd}
K \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[r, two heads, "gf"] \arrow[d, "f"'] & Z \arrow[d, dashed, "\alpha"] \\
& Y \arrow[ur, "g"] \arrow[r, "h"'] & W
\end{tikzcd}$$

is a coequalizer and we want to show g is a coequalizer of fp_1 and fp_2. Assume h coequalizes fp_1 and fp_2, in other words hf coequalizes p_1 and p_2, so there is a unique map α such that $\alpha gf = hf$. Since f is an epi, we have $\alpha g = h$, hence g is indeed a coequalizer of fp_1 and fp_2.

(2) [Bor94a, cor 2.1.5 (2)] Start with the same diagram but take the kernel pair of g and its coequalizer:

$$\begin{tikzcd}
X \times_Z X \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] \arrow[d, "f\times f"'] & X \arrow[r, two heads, "gf"] \arrow[d, "f"'] & Z \arrow[d, dashed, shift left, "\alpha"] \\
Y \times_Z Y \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] & Y \arrow[r, "h"'] \arrow[ur, "g"] & Coeq. \arrow[u, dashed, shift left, "\beta"]
\end{tikzcd}$$

Since g coequalizes the bottom projections, there is a unique β satisfying $g = \beta h$. By universal properties, α and β are inverse to each other, hence g is a regular epi. □

Remark D.0.14. For (2), we only used that the category has kernel pairs and their coequalizers.

Proposition D.0.15. *A pushout along a regular epi is obtained as a coequalizer. More precisely, let* $e : X \twoheadrightarrow Y$ *is a regular epi and* $f : X \to Z$ *any map.*

$$\begin{tikzcd}
K \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[r, two heads, "e"] \arrow[d, "f"'] & Y \arrow[d, dashed] \\
& Z \arrow[r, two heads] & C
\end{tikzcd}$$

Let $Z \twoheadrightarrow C$ *be the coequalizer of* fp_1 *and* fp_2. *Then* C *(with the induced map* $Y \to C$*) is a pushout of* f *and* e.

Proof. [Bor94a, prop 4.3.8 (2)] □

Appendix E

Facts about simplicial sets

In this appendix, we recall some basic facts about simplicial sets.

Proposition E.0.16. *(a) A fibration of simplicial sets* $f : X_\bullet \to Y_\bullet$ *is surjective in every level iff it is surjective in level* 0

(b) iff it is surjective on path components $\pi_0(f) : \pi_0(X_\bullet) \twoheadrightarrow \pi_0(Y_\bullet)$.

Proof. (a) By induction, assume f is surjective up to level $n-1$. Let $y_n \in Y_n$ be any n-simplex, which can be thought of as a map $y_n : \Delta[n] \to Y_\bullet$ where $\Delta[n]$ is the standard n-simplex. Its 0th face $d_0(y_n) \in Y_{n-1}$ has an f-preimage $x_{n-1} \in X_{n-1}$, by induction hypothesis. Now f is a fibration so there is a lift in the diagram

$$\begin{tikzcd}
\Delta[n-1] \arrow[r, "x_{n-1}"] \arrow[d, hook, "d^0"', "\sim"] & X_\bullet \arrow[d, two heads, "f"] \\
\Delta[n] \arrow[r, "y_n"'] \arrow[ur, dashed, "\exists x_n"] & Y\bullet
\end{tikzcd}$$

and $x_n \in X_n$ satisfies $f(x_n) = y_n$. Hence f is surjective in level n.

(b) Assume f is surjective on path components. We want to show it is surjective on vertices $f : X_0 \twoheadrightarrow Y_0$. Let $y_0 \in Y_0$ be any vertex. By assumption, there is another vertex y'_0 in the path component of y_0 and in the image of f, say $f(x'_0) = y'_0$. There is a path γ (i.e. a chain of 1-simplices) in Y from y'_0 to y_0. WLOG, γ consists of a single 1-simplex. Since f is a fibration, there is a lift in the diagram

$$\begin{tikzcd}
\Delta[0] \arrow[r, "x'_0"] \arrow[d, hook, "d^0"', "\sim"] & X_\bullet \arrow[d, two heads, "f"] \\
\Delta[1] \arrow[r, "\gamma"'] \arrow[ur, dashed, "\exists\widetilde{\gamma}"] & Y\bullet
\end{tikzcd}$$

i.e. we can lift the path with a given starting point. We have

$$f(d_0\widetilde{\gamma}) = d_0 f(\widetilde{\gamma}) = d_0\gamma = y_0'$$
$$f(d_1\widetilde{\gamma}) = d_1 f(\widetilde{\gamma}) = d_1\gamma = y_0$$

hence y_0 is in the image of f. □

Corollary E.0.17. *An acyclic fibration of simplicial sets is surjective in every level.*

We could also prove the corollary directly, by picking any n-simplex $y_n \in Y_n$ and lifting in the diagram

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d, hook] & X_\bullet \arrow[d, "f", "\sim"'] \\ \Delta[n] \arrow[r, "y_n"'] \arrow[ur, dashed, "\exists x_n"] & Y\bullet. \end{tikzcd}$$

Proposition E.0.18. *Assume C has finite limits and enough projectives. Then an acyclic fibration $f : X_\bullet \to Y_\bullet$ of simplicial objects in sC is a levelwise effective epi.*

Proof. Let P be a projective of C. By definition of acyclic fibration in sC, the map

$$\mathrm{Hom}_C(P, X_\bullet) \overset{f_*}{\to} \mathrm{Hom}_C(P, Y_\bullet)$$

is an acyclic fibration of simplicial sets, in particular a levelwise surjection. Therefore $f_n : X_n \to Y_n$ is an effective epi for each level n, by [Qui67, II.4 prop 2]. □

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