Unstable operations in the Bousfield-Kan spectral sequence for simplicial commutative $\mathbb{F}_2$-algebras

by

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Abstract

In this thesis we study the Bousfield-Kan spectral sequence (BKSS) in the Quillen model category $s\mathscr{C}om$ of simplicial commutative $\mathbb{F}_2$-algebras. We develop a theory of unstable operations for this BKSS and relate these operations with the known unstable operations on the homotopy of the target. We also prove a completeness theorem and a vanishing line theorem, and together these eliminate the possibility of convergence problems for a connected object of $s\mathscr{C}om$.

We approach the computation of the BKSS by deriving a composite functor spectral sequence (CFSS) which converges to the BKSS E_2-page. We then extend this construction to an infinite sequence of CFSSs, with each abutting to the E_2-page of the last. Equipping each of these CFSSs with a theory of unstable spectral sequence operations, we are able to calculate the Bousfield-Kan E_2-page in the most important case, that of a connected sphere in $s\mathscr{C}om$. We use this calculation to describe the E_1-page of a May-Koszul spectral sequence which computes the BKSS E_2-page for any connected object of $s\mathscr{C}om$. We conclude by making two conjectures which would, together, allow for a full computation of the BKSS for a connected sphere in $s\mathscr{C}om$.

Thesis Supervisor: Haynes Miller
Title: Professor of Mathematics

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Contents

Chapter 1

Introduction

The primary object of study in this thesis is the Bousfield-Kan spectral sequence (BKSS) in the Quillen model category $s\mathscr{C}om$ of simplicial non-unital commutative $\mathbb{F}_2$-algebras. This spectral sequence calculates the homotopy groups of the homology completion $X^\wedge$ of $X \in s\mathscr{C}om$, with E_2-page given by certain derived functors applied to the André-Quillen cohomology groups $H^*_{\mathscr{C}om}X$. The approach we take in this thesis is two-fold. On the one hand, we develop an extensive theory of spectral sequence operations on the BKSS. On the other hand, we use composite functor spectral sequences (CFSSs) to calculate the derived functors that form the E_2-page.

In §1.1 we recall certain aspects of the theory of the BKSS of a pointed connected topological space, including a CFSS due to Miller for the computation of its E_2-page. There are a number of useful analogies to be drawn between this classical theory and the content of this thesis.

After giving a little of the necessary algebraic and topological background in §1.2, we discuss in §1.3 the BKSS in $s\mathscr{C}om$, and introduce the unstable spectral sequence operations referred to in the title of this thesis. There are three types of operations appearing at E_2. Higher divided power operations and a commutative product arise as the Koszul dual operations to Goerss' operations on André-Quillen cohomology, and an action of the Steenrod algebra emerges as an artifact of the positive characteristic. The δ-operations and the Steenrod operations are unstable as described in §1.3, and we also describe an elegant relation between the two types of operations. We also explain how the δ-operations and product relate to the natural homotopy operations on the target, a relationship clarified by a completeness theorem and a vanishing line theorem which, together, eliminate the potential for convergence problems.

We describe the other part of our approach in §§1.4-1.6. The derived functors that form the BKSS E_2-page may be analyzed using a sequence of composite functor spectral sequences. As we explain in §1.4, the algebraically rich BKSS E_2-page is the target of the first CFSS, and a key aspect of our approach is to extend this rich structure into this CFSS, producing therein a theory of unstable operations. This structure alone does not suffice for us to make our computations, and in §1.5 we generalize the construction, forming an infinite regress of CFSSs, each calculating the E_2-page of the last, and each possessing a theory of unstable spectral sequence operations.

In §1.6, we explain how it is possible to use this immense amount of structure to make calculations, including a calculation of the BKSS E_2-page for a sphere in the category $s\mathscr{C}om$. This special case is important for the more general calculation of the BKSS E_2 page, as it is involved in the description the E_1-page of a May-Koszul spectral sequence which calculates the BKSS E_2 page for a general connected simplicial algebra.

Finally, in §1.7, we discuss some conjectures which would unify the two parts of our approach. Were these conjectures verified, we would be able to give a complete description of the differentials available in the BKSS of a commutative algebra sphere. These conjectures are supported by the calculation we have given of the E_2-page in this case.

1.1. The classical Bousfield-Kan spectral sequence

The homotopy theory $s\mathscr{C}om$ has much in common with that of pointed connected topological spaces, and before we introduce our main results, we briefly recall the analogous classical theory in this section. The intention of this thesis is to produce an enriched version in the model category $s\mathscr{C}om$ of this classical theory.

Suppose that X is a pointed connected topological space with π_*X finitely generated in each degree. The (absolute) *Bousfield-Kan spectral sequence of* X over $\mathbb{F}_2$ is a second quadrant spectral sequence

$$[E_2X]^s_t \cong \mathrm{Ext}^s_{\mathcal{K}}(\overline{H}^*(X;\mathbb{F}_2), \overline{H}^*(S^t;\mathbb{F}_2)) \implies \pi_{t-s}X\hat{_2},$$

where $X\hat{_2}$ is the completion of X at the prime 2. Throughout this thesis, we use the notation $[E_2X]^s_t$ rather than the more standard $E_2^{s,t}$ for the pages of the spectral sequence.

At least when X is simply connected and π_*X is of finite type, one may view this spectral sequence as a tool for calculating π_*X, as $\pi_*(X\hat{_2}) \cong (\pi_*X)\hat{_2}$ determines the 2-torsion in π_*X. Under certain hypotheses (satisfied for example when $X = S^n$ for $n \geq 1$) the BKSS admits a vanishing line at E_2 [5], and is thus strongly convergent.

The non-abelian derived functors Ext^s are calculated in the category $\mathcal{K}$ of non-unital unstable algebras over the Steenrod algebra (c.f. [51, §1.4]). If we write $\mathcal{V}^+$ for the category of cohomologically graded vector spaces

$$W = \bigoplus_{n\geq 1} W^n,$$

the objects of $\mathcal{K}$ are graded non-unital $\mathbb{F}_2$-algebras $W \in \mathcal{V}^+$ equipped with an unstable left action of the Steenrod algebra, i.e. maps:

$$\mathrm{Sq}^i : W^t \longrightarrow W^{t+i},$$
$$\mu : W^t \otimes W^{t'} \longrightarrow W^{t+t'},$$

satisfying the usual properties — Adem relations, unstableness relations, and the Cartan formula. We take a moment to introduce notation, defining the functor of *indecomposables* $Q^{\mathcal{K}} : \mathcal{K} \longrightarrow \mathcal{V}^+$ by the formula

$$W \overset{Q^{\mathcal{K}}}{\longmapsto} W/\left(\mathrm{im}\Big(W \otimes W \overset{\mu}{\to} W\Big) \oplus \bigoplus_{i\geq 1} \mathrm{im}\Big(W \overset{\mathrm{Sq}^i}{\to} W\Big)\right) \in \mathcal{V}^+.$$

The BKSS E_2-page can be rewritten as the dual left derived functors

$$[E_2X]^s_t \cong H^s_{\mathcal{K}}(\overline{H}^*(X;\mathbb{F}_2))_t := \mathbf{D}((\mathbb{L}_s Q^{\mathcal{K}})(\overline{H}^*(X,\mathbb{F}_2))^t),$$

where we write $\mathbf{D}V$ for the linear dual of a vector space V, and insist that $\mathbf{D}$ interchanges homological and cohomological dimensions. We will use notation following this pattern for the rest of the thesis.

One useful idea is to search for operations which act on the BKSS. Spectral sequence operations are typically used to produce new elements on the E_2-page and to compute differentials on those elements. Bousfield and Kan [11, §14] construct a Lie bracket:

$$[E_rX]^s_t \otimes [E_rX]^{s'}_{t'} \longrightarrow [E_rX]^{s+s'+1}_{t+t'} \quad \text{for } 1 \leq r \leq \infty,$$

with the bracket on E_r satisfying a Leibniz formula and inducing the bracket on E_{r+1}.

There are two reasons why one might expect such a Lie algebra structure. First, the commutative operad $\mathscr{C}$ and the Lie operad $\mathscr{L}$ are Koszul dual, and even though the theory of Koszul homology is complicated by the non-zero characteristic, there is an action of $\mathscr{L}$ on the derived functors calculating E_2. Next, there is a graded Lie algebra structure on homotopy groups given by the Whitehead bracket [56]:

$$[\,,] : \pi_n X^{\wedge}_2 \otimes \pi_{n'} X^{\wedge}_2 \longrightarrow \pi_{n+n'-1} X^{\wedge}_2,$$

and one may ask whether or not this action preserves the filtration, in which case it would define a Lie algebra structure on E_∞. Bousfield and Kan answer this question in the affirmative by proving that the bracket at E_∞ is compatible with the Whitehead bracket, and they also show that the pairing given at E_2 has the correct homological description.

This appears to be as far as it is possible to pursue this strategy, as at both E_2 and E_∞ we lose hope of finding structure that can be readily described. We do not expect to extract further structure on E_2 using the Steenrod algebra action in $\mathcal{K}$, or at least not any that can be described so explicitly. The Steenrod algebra $\mathcal{A}_2$ suffers from the inhomogeneity $\mathrm{Sq}^0 = 1$. Were it a homogeneous Koszul algebra (in the sense of [46]), then its Koszul dual would at very least act on $\mathrm{Ext}_{\mathcal{A}_2}(\mathbb{F}_2, M)$ for an $\mathcal{A}_2$-module M, but even this is not the case. There is no particular reason to think that the situation should be any better for the non-abelian derived functors defining E_2. Moreover, we simply do not understand the natural operations that exist on π_* in enough detail to expect to see uniform structure appearing on E_∞. After all, by the Hilton-Milnor Theorem [44, §4], all natural operations on the homotopy groups of pointed spaces are composites of the Whitehead bracket and unary operations, and a natural homotopy operation $\pi_n X \longrightarrow \pi_m X$ is equivalent to an element of $\pi_m S^n$.

Before we break from our extended analogy, we will discuss the considerable task of calculating the E_2-page of this classical BKSS. Performing this calculation is at least as difficult as the calculation of the E_2-page for the classical (stable) Adams spectral sequence, which appears to be rather difficult. There is, however, the following method due to Miller [42] for extracting information about the derived functors $H^*_{\mathcal{K}}$. There is a factorization of $Q^{\mathcal{K}}$ into

$$\mathcal{K} \xrightarrow{Q^{\mathscr{C}om}} \Sigma\mathcal{U} \xrightarrow{Q^{\Sigma\mathcal{U}}} \mathcal{V}^+,$$

where $\Sigma\mathcal{U}$ is the algebraic category whose objects are vector spaces $V \in \mathcal{V}^+$ equipped with a left action of $\mathcal{A}_2$ such that $\mathrm{Sq}^i : V^n \longrightarrow V^{n+i}$ is zero unless $0 \leq i < n$. This modified unstableness condition is necessary in order that $Q^{\mathscr{C}om}$ satisfies an acyclicity condition, so that for $W \in \mathcal{K}$ there is a composite functor spectral sequence

$$[E_2^{\mathrm{cf}} W]_t^{s_2,s_1} = H^{s_2}_{\Sigma\mathcal{U}}(H_*^{\mathscr{C}om}(W))_t^{s_1} \implies H^{s_1+s_2}_{\mathcal{K}}(W)_t.$$

This spectral sequence was an integral part of Miller's proof of the Sullivan conjecture. The functor $H_*^{\mathscr{C}om}$ appearing in the above description is the *André-Quillen homology* functor on $s\mathscr{C}om$.

1.2. The various categories $s\mathcal{C}$

In this thesis we will use quite a number of categories of universal algebras, such as the category $\mathscr{C}om$ of non-unital commutative $\mathbb{F}_2$-algebras, or the category $\mathscr{L}ie$ of Lie algebras over $\mathbb{F}_2$. While we introduce certain general notions we will write $\mathcal{C}$ for any one of these categories.

For any category $\mathcal{C}$ of universal algebras, the category $s\mathcal{C}$ of simplicial objects in $\mathcal{C}$ is a Quillen model category [48]. These model categories have much in common with the category of topological spaces. For example, an object $X \in s\mathcal{C}$ possesses homotopy groups $\pi_* X$ and homology groups $H_*^{\mathcal{C}} X$. We use the pragmatic notion of homology that appears in the spectral sequences that appear in this context, and it does not always coincide with Quillen's notion of homology *derived abelianization*. The cohomology groups $H^*_{\mathcal{C}} X$ are defined to be the linear duals of the homology groups.

In §3, we recall the definition of spheres and Eilenberg-Mac Lane objects in $s\mathcal{C}$. These play the same role in $s\mathcal{C}$ as their namesakes in the category of pointed topological spaces, which is to represent the homotopy and cohomology functors on the homotopy category of $s\mathcal{C}$, respectively. We also present a unified treatment of homotopy and cohomology operations (and of homology co-operations) for such categories.

In §5 and §6 we present a number of existing examples of homotopy and cohomology operations in a common framework, with the construction of cohomology operation following Goerss' method from [33]. In particular, it will be useful for us to understand the well-known cohomology operations for simplicial Lie algebras in this same framework, and in Appendix A . yeah say it

In §4, we recall Radulescu-Banu's [49] cosimplicial resolution of $X \in s\mathscr{C}om$, which we denote by $\mathfrak{X} \in cs\mathscr{C}om$. The resolution $\mathfrak{X}$ is suitable for the construction of a BKSS for X. This construction is rather more difficult than that of Bousfield and Kan's $\mathbb{F}_2$-resolution, as the naïve monadic cobar construction in $s\mathcal{C}$ is not homotopically correct. The totalization of $\mathfrak{X}$ is the *homology completion* $X^\wedge$ *of* X, and the (absolute) BKSS is the spectral sequence associated with the totalization tower. In §4.1 we perform the homotopical algebra needed to identify the E_1 and E_2-pages arising from Radulescu-Banu's resolution.

1.3. The Bousfield-Kan spectral sequence in $s\mathscr{C}om$

At this point we depart from generalities, turning to the homotopy theory of simplicial non-unital commutative algebras in earnest. We will restrict to the *connected* objects $X \in s\mathscr{C}om$, which simplifies various aspects of our analysis. As in the classical case, one must know how the homotopy groups of the homology completion $X^\wedge$ determine those of X. We will demonstrate (Theorem 4.4) that $X^\wedge$ is equivalent to X as long as X is connected. Moreover, we will prove in Theorem 15.3 that the BKSS admits a vanishing line from E_2 in this case, and thus strongly converges to the homotopy of X.

As forecast by the discussion in §1.1, it will help to know a little about the natural operations on the homotopy of simplicial $\mathbb{F}_2$-algebras in advance. Fortunately, we have the explicit description of homotopy operations which is lacking in the category of pointed spaces, since they have been completely calculated by Dwyer [26] (and were studied earlier by Bousfield [8, 6] and Cartan [14]). In summary, $\pi_* X$ supports operations

$$\delta_i : \pi_n X \longrightarrow \pi_{n+i} X, \text{ defined when } 2 \leq i \leq n,$$
$$\mu : \pi_n X \otimes \pi_{n'} X \longrightarrow \pi_{n+n'} X,$$

with μ a graded non-unital commutative algebra product, and the δ_i satisfying various compatibilities which we discuss in detail in §5.4. In fact, these δ-operations satisfy a δ-Adem relation *which is homogeneous*, and there is a corresponding unital associative algebra Δ. Note that $\pi_* X$ is *not* a left module over the algebra Δ, because the operations are not defined in every dimension. This situation can not be remedied simply be defining the missing operations to be zero, as doing so is incompatible with the Adem relations on homotopy. Instead, we must adopt language for such situations, saying that Δ has a partially defined unstable left action on $\pi_* X$. In general, unstable homotopy operations will be partially defined, whereas unstable cohomology operations will be everywhere defined but vanish in certain ranges.

Goerss [33] described the analogue for $s\mathscr{C}om$ of the category $\mathcal{K}$, and all of the natural operations on the André-Quillen cohomology $H^*_{\mathscr{C}om} X$ of $X \in s\mathscr{C}om$ are generated by:

$$P^i : H^n_{\mathscr{C}om} X \longrightarrow H^{n+i+1}_{\mathscr{C}om} X;$$
$$[\,,] : H^n_{\mathscr{C}om} X \otimes H^m_{\mathscr{C}om} X \longrightarrow H^{n+m+1}_{\mathscr{C}om} X;$$
$$\beta : H^0_{\mathscr{C}om} X \longrightarrow H^1_{\mathscr{C}om} X.$$

As we restrict to connect objects of $s\mathscr{C}om$, the operation β can be ignored. These operations satisfy various compatibilities which we recount in detail in §6.6, and we will denote by $\mathcal{W}(0)$ the category whose objects are vector spaces $W \in \mathcal{V}^+$ equipped with the P^i-operations and the bracket. The bracket satisfies the Jacobi identity but falls just short of being a Lie algebra pairing as $[x, x]$ is not always zero. The P^i satisfy a P-Adem relation *that is homogeneous*. The evident unital associative algebra P acting on $H^*_{\mathscr{C}om}$ is a homogeneous Koszul algebra, the *P-algebra* with Koszul dual the algebra Δ, and indeed, this is how it was originally described by Goerss. In §4.1, we identify the E_2-page (for connected $X \in s\mathscr{C}om$ with $\pi_* X$ of finite type) as the non-abelian derived functors

$$[E_2 \mathcal{X}]^s_t \cong H^s_{\mathcal{W}(0)}(H^*_{\mathscr{C}om} X)_t.$$

This description begets a laundry list of operations that we expect to see on E_2. The cohomology of a Lie algebra enjoys an action of the commutative operad (the Koszul dual of the Lie operad). As we are working in positive characteristic, they also support an action of the homogeneous Steenrod algebra $\mathcal{A} := E_0\mathcal{A}_2$, due to Priddy [47]. We will discuss these operations in detail in §6.8 and Appendix A.1. We construct in Proposition 8.9 the corresponding natural 'horizontal' operations on E_2:

$$\mathrm{Sq}^j_{\mathrm{h}} : (H^s_{\mathcal{W}(0)}X)_t \longrightarrow (H^{s+j}_{\mathcal{W}(0)}X)_{2t+1};$$
$$\mu : (H^s_{\mathcal{W}(0)}X)_t \otimes (H^{s'}_{\mathcal{W}(0)}X)_{t'} \longrightarrow (H^{s+s'+1}_{\mathcal{W}(0)}X)_{t+t'+1}.$$

Moreover, we construct in Proposition 8.2 natural vertical operations constituting a (partially defined) action of the Koszul dual Δ of the P-algebra:

$$\delta^{\mathrm{v}}_i : (H^s_{\mathcal{W}(0)}X)_t \longrightarrow (H^{s+1}_{\mathcal{W}(0)}X)_{t+i+1} \;\text{ defined for } 2 \leq i < t.$$

These operations satisfy various compatibilities (c.f. §8.4 and Propositions 8.2 and 8.9).

Although the product and δ-operations on E_2 look encouraging, there is the following issue: if $x \in [E_2\mathcal{X}]^s_t$ is a permanent cycle detecting a class $\overline{x} \in \pi_*X$, then at least when $s \geq 2$ there are more operations $\delta^{\mathrm{v}}_2 x, \ldots, \delta^{\mathrm{v}}_{t-1}x$ defined on E_2 than there are operations $\delta_2\overline{x}, \ldots, \delta_{t-s}\overline{x}$ defined on homotopy. Moreover, the Steenrod operations at E_2 have no counterpart in homotopy.

This situation is quite reminiscent of that described by Dwyer [25], who works in the spectral sequence of a cosimplicial simplicial coalgebra (such as the Eilenberg-Moore spectral sequence). In such a spectral sequence, one expects to find Steenrod operations at E_2 but finds *too many.* Dwyer constructs δ-operations and differentials mapping the excess Steenrod operations to the δ operations. In this way, the excess Steenrod operations fail to be defined at E_∞, and the δ-operations become zero by E_∞, an excellent resolution to this problem.

Unfortunately, we cannot use Dwyer's operations. Indeed, although the linear dual of a cosimplicial simplicial coalgebra is a cosimplicial simplicial algebra (of which the resolution $\mathcal{X}$ is an example), the choice of filtration direction is transposed. Instead, we perform analogous constructions in the dual setting, and describe in §10.6 a theory of operations on the spectral sequence of a cosimplicial simplicial $\mathbb{F}_2$-algebra, which may be of independent interest. While defining these operations is a good first step, they are not yet what we require, as it happens that they can be lifted one filtration higher when we are working in the BKSS. In §10 and §11, we explain how to construct operations

$$\delta_i^{\mathrm{v}} : [E_r\mathcal{X}]_t^s \longrightarrow [E_r\mathcal{X}]_{t+i+1}^{s+1},$$
$$\mathrm{Sq}_{\mathrm{h}}^j : [E_r\mathcal{X}]_t^s \longrightarrow [E_r\mathcal{X}]_{2t+1}^{s+j},$$
$$\mu : [E_r\mathcal{X}]_t^s \otimes [E_r\mathcal{X}]_{t'}^{s'} \longrightarrow [E_r\mathcal{X}]_{t+t'+1}^{s+s'+1},$$

with the δ_i^{v} potentially multi-valued functions, defined when $2 \leq i \leq \max\{n, t-(r-1)\}$, and single-valued whenever $i \leq \min\{n+1, t+1-2(r-1)\}$, and the $\mathrm{Sq}_{\mathrm{h}}^j$ potentially multi-valued functions with indeterminacy vanishing by E_{2r-2}, and which equal zero unless $\min\{t,r\} < j \leq s+1$. All of the functions that are defined on E_2 are single-valued, and indeed, they coincide with the operations defined on $H^*_{\mathcal{W}(0)}$, as we show in Proposition 11.2.

For $x \in [E_r\mathcal{X}]_t^s$ such that $\delta_i^{\mathrm{v}} x$ is defined:

$$d_r\delta_i^{\mathrm{v}}(x) + \delta_i^{\mathrm{v}}(d_r x) = \begin{cases} \mathrm{Sq}_{\mathrm{h}}^{t-i+2}(x), & \text{if } i > t-s \text{ and } r = t-i+1; \\ \mu(x \otimes d_r x), & \text{if } i = t-s,\ s = 0 \text{ and } r \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

This is Corollary 11.6. In particular, if $i > t-s$ and $x \in [E_{t-i+1}\mathcal{X}]_t^s$ survives to $[E_{t-i+2}\mathcal{X}]_t^s$, then d_{t-i+1} maps $\delta_i^{\mathrm{v}} x$ to $\mathrm{Sq}_{\mathrm{v}}^{t-i+2}x$. These formulae explain how the Sq_{v} serve to absorb differentials supported by the excess δ^{v}.

1.4. The first composite functor spectral sequence

Now that we have a theory of the operations available on the BKSS in $s\mathscr{C}om$, we turn to the question of calculating it. If we hope to imitate Miller's use of a composite functor spectral sequence (CFSS), using the factorization

$$Q^{\mathcal{W}(0)} = Q^{\mathcal{L}(0)} \circ Q^{\mathcal{U}(0)} : \left(\mathcal{W}(0) \overset{Q^{\mathcal{U}(0)}}{\longrightarrow} \mathcal{L}(0) \overset{Q^{\mathcal{L}(0)}}{\longrightarrow} \mathcal{V}^+\right),$$

where $\mathcal{L}(0)$ is the category whose objects are graded vector spaces $W \in \mathcal{V}^+$ which are Lie algebras under a bracket which shifts gradings,

$$W^t \otimes W^{t'} \longrightarrow W^{t+t'+1}.$$

We write $\mathcal{U}(0)$ for the category whose objects are vector spaces $V \in \mathcal{V}^+$ equipped with an unstable action of the P-algebra given by operations

$$P : V^t \longrightarrow V^{t+i+1}$$

which are zero unless $2 \le i \le t$. The functor $Q^{\mathcal{U}(0)}$ is defined for $W \in \mathcal{W}(0)$ by

$$W \longmapsto W/\bigoplus_{i\ge 2} \operatorname{im}(W \xrightarrow{P^i} W).$$

As the category $\mathcal{L}(0)$ is not an abelian category, it is a more technical task to form a CFSS, and we use the method of Blanc and Stover [3]. The key idea in their presentation is that the derived functors $H_*^{\mathcal{U}(0)} := \mathbb{L}_* Q^{\mathcal{U}(0)}$ take values in the category $\mathcal{W}(1)$ of $\mathcal{L}(0)$-Π-algebras, as they are calculated as the homotopy of an object of $s\mathcal{L}(0)$. The first CFSS takes the following form for $W \in \mathcal{W}(0)$:

$$[E_2^{\mathrm{cf}} W]_t^{s_2,s_1} = H^*_{\mathcal{W}(1)}(H_*^{\mathcal{U}(0)} W)_t^{s_2,s_1} \implies (H^*_{\mathcal{W}(0)} W)_t^{s_1+s_2}.$$

We will now unpack this somewhat dense expression, and explain how various unstable operations defined at the E_2-page and the target interact with the spectral sequence.

Objects of the category $\mathcal{W}(1)$ are certain bigraded Lie algebras with a certain partially defined right action of the Λ-algebra (c.f. §1.5). In §9.2 we calculate the structure of $H_*^{\mathcal{U}(0)} W$ as an object of $\mathcal{W}(1)$ by explicit chain-level computation, after defining in §9.1 an unstable version of Priddy's Koszul resolution [46] for the functors $H_*^{\mathcal{U}(0)}$.

The linear duals $H^*_{\mathcal{U}(0)} W$ admit an unstable partially defined left Δ-algebra action, since the algebras Δ and P are Koszul dual, and by Proposition 12.9 there is a commuting diagram (for $2 \le i < t$):

$$\begin{tikzcd}
(H^*_{\mathcal{W}(0)} W)_t^s \arrow[r, "\text{edge hom}"] \arrow[d, "\delta_i^{\mathrm{v}}"'] & {[E_2^{\mathrm{cf}} W]_t^{0,s}} \arrow[r, tail] & (H^*_{\mathcal{U}(0)} W)_t^s \arrow[d, "\delta_i^{\mathrm{v}}"] \\
(H^*_{\mathcal{W}(0)} W)_{t+i+1}^{s+1} \arrow[r, "\text{edge hom}"] & {[E_2^{\mathrm{cf}} W]_{t+i+1}^{0,s+1}} \arrow[r, tail] & (H^*_{\mathcal{U}(0)} W)_{t+i+1}^{s+1}
\end{tikzcd}$$

In this sense the δ_i^{v}-operations on the BKSS E_2-page are compatible with the CFSS.

The BKSS E_2-page also supports products and horizontal Steenrod operations, and we should attempt to identify them in the CFSS. The functor $H^*_{\mathcal{W}(1)}$ may also be viewed as a Lie algebra cohomology functor, so that we expect horizontal Steenrod operations and products to appear in $[E_2^{\mathrm{cf}} W]_t^{s_2,s_1}$. We use a new definition of these operations that fits into the framework set out in §6 (deferring to Appendix A the work of showing that these operations coincide with those constructed by Priddy [47].)

Moreover, just as we expected δ_i^{v} operations on $H^*_{\mathcal{W}(0)}$, we expect a 'vertical' left action of the homogeneous Steenrod algebra on $H^*_{\mathcal{W}(1)}$, as it is Koszul dual to the Λ-algebra. Indeed, we construct in Proposition 8.9 such operations on the derived functors $H^*_{\mathcal{W}(1)}$. Moreover, Proposition 8.6 applies to $H^*_{\mathcal{W}(1)}$ just as it applies to $H^*_{\mathcal{W}(0)}$, yielding horizontal Steenrod operations and products, so that ultimately we obtain operations

$$\mathrm{Sq}^i_{\mathrm{v}} : [E_2^{\mathrm{cf}}W]^{s_2,s_1}_t \longrightarrow [E_2^{\mathrm{cf}}W]^{s_2+1,s_1+i-1}_{2t+1},$$
$$\mathrm{Sq}^j_{\mathrm{h}} : [E_2^{\mathrm{cf}}W]^{s_2,s_1}_t \longrightarrow [E_2^{\mathrm{cf}}W]^{s_2+j,2s_1}_{2t+1},$$
$$\mu : [E_2^{\mathrm{cf}}W]^{s_2,s_1}_t \otimes [E_2^{\mathrm{cf}}W]^{p_2,p_1}_q \longrightarrow [E_2^{\mathrm{cf}}W]^{s_2+p_2+1,s_1+p_1}_{t+q+1},$$

with both the horizontal and vertical Steenrod operations satisfying their own unstableness conditions.

Now suppose that $x \in [E_2^{\mathrm{cf}}W]^{s_2,s_1}_t$ is a permanent cycle detecting an element $\overline{x} \in (H^*_{\mathcal{W}(0)}W)^{s_2+s_1}_t$. The $s_2{+}s_1{-}1$ operations $\mathrm{Sq}^3_{\mathrm{h}}\overline{x}, \ldots, \mathrm{Sq}^{s_2+s_1+1}_{\mathrm{h}}\overline{x}$ are the potentially non-zero Steenrod operations on $\overline{x}$. The $s_1{-}2$ vertical operations $\mathrm{Sq}^3_{\mathrm{v}}x, \ldots, \mathrm{Sq}^{s_1}_{\mathrm{v}}x$ and the $s_2{+}1$ horizontal operations $\mathrm{Sq}^1_{\mathrm{h}}x, \ldots, \mathrm{Sq}^{s_2+1}_{\mathrm{h}}x$ are the potentially non-zero Steenrod operations on x. This is quite reminiscent of Singer's framework [52] (c.f. §13.1), and in §13.2 we use Singer's methods to extend the operations on $[E_2^{\mathrm{cf}}W]$ to the entire CFSS. The upshot is that if $x \in [E_2^{\mathrm{cf}}W]^{s_2,s_1}_t$ is a permanent cycle, then so are all of the above mentioned $\mathrm{Sq}^i_{\mathrm{v}}x$ and $\mathrm{Sq}^j_{\mathrm{h}}x$, and moreover,

$$\mathrm{Sq}^i_{\mathrm{v}}x \text{ detects } \mathrm{Sq}^i_{\mathrm{h}}\overline{x}\ (3 \le i \le s_1) \text{ and } \mathrm{Sq}^j_{\mathrm{h}}x \text{ detects } \mathrm{Sq}^{s_1+i}_{\mathrm{h}}\overline{x}\ (1 \le j \le s_2+1).$$

That is, the horizontal and vertical Steenrod operations *combined* account for the horizontal Steenrod operations on the target. We examine how this plays out for admissible sequences of Steenrod operations in Theorem 8.15.

1.5. Higher composite functor spectral sequences

We have constructed a comprehensive theory of the operations in the first CFSS, but it may still be the case that the $H^*_{\mathcal{W}(1)}$ is as difficult to calculate as $H^*_{\mathcal{W}(0)}$, which would mean that the CFSS is of little use for the calculation of $H^*_{\mathcal{W}(0)}$. Rather than being discouraged, we will turn this similarity to our advantage by iterating our approach. In §7 we extend the constructions summarized in §1.4, defining algebraic categories $\mathcal{W}(n)$ and $\mathcal{L}(n)$ for $n \geq 1$ such that $\mathcal{W}(n)$ is the category of $\mathcal{L}(n-1)$-Π-algebras and there are factorizations

$$Q^{\mathcal{W}(n)} = Q^{\mathcal{L}(n)} \circ Q^{\mathcal{U}(n)} : \Big(\mathcal{W}(n) \overset{Q^{\mathcal{U}(n)}}{\longrightarrow} \mathcal{L}(n) \overset{Q^{\mathcal{L}(n)}}{\longrightarrow} \mathcal{V}^+_n\Big).$$

There are CFSSs for $W \in \mathcal{W}(n)$:

$$[E_2^{\mathrm{cf}}W]^{s_{n+2},\ldots,s_1}_t = H^*_{\mathcal{W}(n+1)}(H^{\mathcal{U}(n)}_* W)^{s_{n+2},s_{n+1},\ldots,s_1}_t \implies (H^*_{\mathcal{W}(n)}W)^{s_{n+2}+s_{n+1},s_n,\ldots,s_1}_t,$$

and we equip each of these spectral sequences with a theory of operations which generalize those discussed in §1.4.

say koszul change

At this point it is useful to summarize the definitions. For $n \geq 1$, let $\mathcal{U}(n)$ denote the category whose objects are vector spaces $V \in \mathcal{V}_n^+$ equipped with *linear* right λ-operations

$$(-)\lambda_i : V^t_{s_n,\ldots,s_1} \longrightarrow V^{2t+1}_{s_n+i,2s_{n-1},\ldots,2s_1} \tag{1.1}$$

defined whenever $0 \leq i < s_{n+1}$ and not all of $i, s_n, \ldots, s_1$ are zero. Let $\mathcal{L}(n)$ denote the category whose objects are $V \in \mathcal{V}_n^+$ equipped with a (typically non-linear) λ-operation as in (1.1) defined whenever $i = s_{n+1}$ and not all of $i, s_n, \ldots, s_1$ are zero, which acts as a partial restriction for a Lie algebra bracket

$$[\,,] : V^t_{s_n,\ldots,s_1} \otimes V^{t'}_{s'_n,\ldots,s'_1} \longrightarrow V^{t+t'+1}_{s_n+s'_n,\ldots,s_1+s'_1}.$$

Finally, let $\mathcal{W}(n)$ be the category whose objects are simultaneously objects of $\mathcal{U}(n)$ and $\mathcal{L}(n)$ subject to certain compatibilities.

The functor $H_*^{\mathcal{U}(n)}W$ may be calculated by an unstable Koszul resolution, and both its linear dual $H^*_{\mathcal{U}(n)}W$ and the functor $H^*_{\mathcal{W}(n)}W$ are naturally objects of $\mathcal{M}_{\mathrm{v}}(n+1)$, the category whose objects are graded vector spaces $M \in \mathcal{V}_+^{n+1}$ with an unstable left action of the Steenrod algebra, operations

$$\mathrm{Sq}^i_{\mathrm{v}} : M_t^{s_{n+1},\ldots,s_1} \longrightarrow M_{2t+1}^{s_{n+1}+1,s_n+i-1,2s_{n-1},\ldots,2s_1},$$

which are zero except when $1 \leq i \leq s_n$ and $i-1, s_{n-1}, \ldots, s_1$ are not all zero. This structure is derived in §8.2, using the Koszul duality between the Λ-algebra and the homogeneous Steenrod algebra. This differs from the analogous constructions for $\mathcal{W}(0)$- and $\mathcal{U}(0)$-cohomology, in that $H^*_{\mathcal{W}(0)}$ supports one fewer vertical δ-operation than $H^*_{\mathcal{U}(0)}$.

On the other hand, as in the $n = 0$ case, $H^*_{\mathcal{W}(n)}$ is an example of (partially restricted) Lie algebra cohomology, so that 'horizontal' Steenrod operations and products appear. In §8.3 we define these operations:

$$\mathrm{Sq}^j_{\mathrm{h}} : (H^*_{\mathcal{W}(n)}X)_t^{s_{n+1},\ldots,s_1} \longrightarrow (H^*_{\mathcal{W}(n)}X)_{2t+1}^{s_{n+1}+j,2s_n,\ldots,2s_1},$$
$$\mu : (H^*_{\mathcal{W}(n)}X)_t^{s_{n+1},\ldots,s_1} \otimes (H^*_{\mathcal{W}(n)}X)_q^{p_{n+1},\ldots,p_1} \longrightarrow (H^*_{\mathcal{W}(n)}X)_{t+q+1}^{s_{n+1}+p_{n+1}+1,s_n+p_n,\ldots,s_1+p_1},$$

so that $\mathcal{W}(n)$-cohomology is also a certain type of unstable algebra over the homogeneous Steenrod algebra, with the horizontal Steenrod action. We write $\mathcal{M}_{\mathrm{h}}(n+1)$ for the resulting category of $\mathcal{V}_+^{n+1}$-graded unstable algebras over the homogeneous Steenrod algebra.

We identify in §8.4 the relations between the $\mathcal{M}_{\mathrm{v}}(n+1)$- and $\mathcal{M}_{\mathrm{h}}(n+1)$-operations, which leads to the definition of an algebraic category $\mathcal{M}_{\mathrm{hv}}(n+1)$ in which $\mathcal{W}(n)$-cohomology takes values for $n \geq 1$.

Consider again the CFSS for $W \in \mathcal{W}(n)$:

$$[E_2^{\mathrm{cf}} W]_t^{s_{n+2},\ldots,s_1} = H^*_{\mathcal{W}(n+1)}(H_*^{\mathcal{U}(n)} W)_t^{s_{n+2},s_{n+1},\ldots,s_1} \implies (H^*_{\mathcal{W}(n)} W)_t^{s_{n+2}+s_{n+1},s_n,\ldots,s_1}.$$

The target is an object of $\mathcal{M}_{\mathrm{hv}}(n+1)$, while the E_2-page is an object of $\mathcal{M}_{\mathrm{hv}}(n+2)$. We prove in Proposition 12.9 that there is a commuting diagram relating the $\mathcal{M}_{\mathrm{v}}(n+1)$-structures on $H^*_{\mathcal{W}(n)} W$ and $H^*_{\mathcal{U}(n)} W$ under the edge homomorphism. As in the $n = 0$ case, after extending the $\mathcal{M}_{\mathrm{hv}}(n+2)$-structure on E_2 to the whole spectral sequence, this structure converges to the $\mathcal{M}_{\mathrm{h}}(n+1)$-structure on the target.

1.6. Computing with the composite functor spectral sequences

So far, we have not explained how the CFSSs may be used for calculation. First, we make the following simple observation. Suppose we wish to calculate the group $(H^*_{\mathcal{W}(n)} W)_t^{s_{n+1},s_n,\ldots,s_1}$ for a given choice of indices. The part of the E_2-page that contributes to this particular group is the following direct sum indexed by pairs of indices s'_{n+2}, s'_{n+1} such that $s'_{n+2} + s'_{n+1} = s_{n+1}$:

$$\bigoplus \Big(H^*_{\mathcal{W}(n+1)}(H_*^{\mathcal{U}(n)} W)_t^{s'_{n+2},s'_{n+1},s_n,\ldots,s_1} \Big)$$

Now in each summand, either $s'_{n+1} = 0$ or $s'_{n+2} < s_{n+1}$. Except for the challenges of understanding the differentials and hidden extensions of algebraic structure, it suffices then to calculate the groups

$$(H^*_{\mathcal{W}(n+k)} H_*^{\mathcal{U}(n+k-1)} \cdots H_*^{\mathcal{U}(n)} W)_t^{s'_{n+k+1},\ldots,s'_{n+1},s_n,\ldots,s_1}$$

for all $k \geq 1$ and for all indices $s'_{n+k+1} + \cdots + s'_{n+1} = s_{n+1}$ satisfying either $s'_{n+k+1} = 0$ or $s'_{n+k} = 0$. It is easy to calculate these groups in either case, as long as we understand the derived functors

$$H_*^{\mathcal{U}(n+k-1)} \cdots H_*^{\mathcal{U}(n)} W$$

as objects of $\mathcal{W}(n+k)$. We undertake these calculations in §9.2. When $s'_{n+k+1} = 0$ there are no derived functors being taken, and when $s'_{n+k} = 0$ the derived functors may be calculated simply as the cohomology of a (constant, not simplicial) partially restricted Lie algebra.

With this computation in mind we define the Chevalley-Eilenberg-May complex of a partially restricted Lie algebra in §A.3. This complex interpolates between the Chevalley-Eilenberg complex for the homology of Lie algebras and May's $\overline{X}$ complex [39] for the homology of restricted Lie algebras.

This method is employed to prove Theorems 14.4 and 14.6, which together imply Corollary 14.7, that $\mathcal{M}_{\mathrm{hv}}(n+1)$ is the category of $\mathcal{W}(n)$-cohomology algebras for $n \geq 1$. That is, the $\mathcal{M}_{\mathrm{hv}}(n+1)$-structure is *all* of the natural structure on $\mathcal{W}(n)$-cohomology for $n \geq 1$.

Finally, we are able to use all of this structure together to calculate, at least as a vector space, the BKSS E_2-page for the commutative algebra T-sphere $\mathbb{S}_T^{\mathscr{C}om}$ whenever $T \geq 1$. That is, we calculate the derived functors $H^*_{\mathcal{W}(0)}W$, where $W = H^*_{\mathscr{C}om}\mathbb{S}_T^{\mathscr{C}om}$ is a one-dimensional trivial object concentrated in dimension $T \geq 1$.

Finally, we derive in §15.1 a convergent spectral sequence which calculates the E_2-page for any connected $X \in s\mathscr{C}om$ of finite type, which we name the May-Koszul spectral sequence. Its E_1-page may be described in terms of the BKSS E_2-pages of the spheres (using Theorem 14.6), and information about the E_2-operations in the BKSS for a sphere passes over to information about the general BKSS E_2-page via the May-Koszul spectral sequence.

1.7. The Bousfield-Kan spectral sequence for $\mathbb{S}_T^{\mathscr{C}om}$

In §14.6, we present a small model for the BKSS E_1-page for a commutative algebra sphere. Given our knowledge of the operations on the BKSS, of the E_2-page for $\mathbb{S}_T^{\mathscr{C}om}$ and of the homotopy groups $\pi_*\mathbb{S}_T^{\mathscr{C}om}$ (c.f. §5.4), a natural goal is the complete computation of the BKSS for $\mathbb{S}_T^{\mathscr{C}om}$. In §16.1, we make two conjectures which would together allow us to make this complete computation. It turns out that E_2 is not the right place to start this computation, and we need to consider classes on E_1 and d_1 differentials in order to see the full picture. The problem is that certain relations involving the δ^{v}- and Sq_{h}-operations only hold from E_2. The conjectures we make would overcome these problems, and would lead to the description given in §16.2 of the full structure of the BKSS for $\mathbb{S}_T^{\mathscr{C}om}$. meep moop

1.8. Overview

The primary object of study in this thesis is the Bousfield-Kan spectral sequence (BKSS) in the Quillen model category $s\mathscr{C}om$ of simplicial non-unital commutative $\mathbb{F}_2$-algebras. This spectral sequence, which we discuss in §4, calculates the homotopy groups of the homology completion $X^\wedge$ of $X \in s\mathscr{C}om$, with E_2-page given by certain derived functors applied to the André-Quillen cohomology groups $H^*_{\mathscr{C}om}X$. change up whole s tion

In §4 we work directly with the Adams tower to show that whenever X is connected, $X^\wedge$ is equivalent to X. Together with the vanishing line we prove in §15, this shows that whenever X is connected, the BKSS is strongly convergent to the homotopy of X.

In §3 we give an introduction to the theory of homotopy and cohomology algebras and homology coalgebras. In §§5-6 we construct a framework in which a number of classically known homotopy and cohomology operations may be considered together. In §§7-8 we construct a number of homotopy and cohomology operations in preparation for the following chapters, and in §9 we study the unstable Koszul resolutions related to certain of these operations.

We define and study three families of unstable spectral sequence operations on the BKSS. Our approach is to perform a generic construction of spectral sequence operations in a cosimplicial simplicial vector space in §10, and then perform a shift in filtration using properties of Radulescu-Banu's resolution in §11.

In §12 we define a sequence of composite functor spectral sequences (CFSSs) which we use in §14 to make calculations of the BKSS E_2-page in the most important case, when X is a *sphere* in $s\mathscr{C}om$. In order for these spectral sequences to be of any use, we must equip them in §13 with various unstable spectral sequence operations, using a technique due to Singer.

We define a May-Koszul spectral sequence in §15 which converges to the BKSS E_2-page for any connected simplicial algebra X, and describe the May-Koszul E^1-page using the data of the BKSS E_2-pages for spheres. Using this spectral sequence one can transfer information about the spectral sequence for spheres to the general setting.

Using the operations on the BKSS we conjecture the full structure of the spectral sequence for a sphere in $s\mathscr{C}om$ in §16.

Chapter 2

Background and conventions

2.1. Universal algebras

In this thesis we will be dealing with various categories $\mathcal{C}$ of universal graded algebras over $\mathbb{F}_2$, which we will refer to as algebraic categories. The relevant examples include a number of categories of graded associative algebras, commutative algebras and Lie algebras, categories of graded unstable modules and unstable algebras. We'll give the background on such categories of universal algebras in this section.

For us, an *algebraic category* is a category whose objects are G-graded $\mathbb{F}_2$-vector spaces $X = \{X_g\}_{g \in G}$, for some set G of gradings, equipped with a set of operators of the form $X_{g_1} \times \cdots \times X_{g_n} \longrightarrow X_k$ (with $n \geq 1$) satisfying a set identities, and whose morphisms are graded vector space maps preserving this structure. These defining maps will be referred to as the $\mathcal{C}$-*structure maps*. This is similar to the definition given in [3, §2.1] of a *category of universal graded algebras.*

It need not be true that all of the $\mathcal{C}$-structure maps must be (multi-)linear in a given presentation of an algebraic category $\mathcal{C}$, but we will always assume that $\mathcal{C}$ is monadic over the category of G-graded $\mathbb{F}_2$-vector spaces. That is, the forgetful functor $U^{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{V}$ will admit a left adjoint $F^{\mathcal{C}} : \mathcal{V} \longrightarrow \mathcal{C}$, and the natural comparison functor from $\mathcal{C}$ to the category of algebras over the monad $U^{\mathcal{C}}F^{\mathcal{C}}$ on $\mathcal{V}$ will be an equivalence.

In our examples, the monad $U^{\mathcal{C}}F^{\mathcal{C}}$ will admit an augmentation (of monads) $\epsilon : U^{\mathcal{C}}F^{\mathcal{C}} \longrightarrow \mathrm{id}$, reflecting homogeneity in the relations defining $\mathcal{C}$. This augmentation has the monad unit $\eta : \mathrm{id} \longrightarrow U^{\mathcal{C}}F^{\mathcal{C}}$ as a section, and may be thought of as *projection onto generators*.

We will generally omit the functor $U^{\mathcal{C}}$ from our notation, writing $F^{\mathcal{C}}$ as shorthand for either the monad $U^{\mathcal{C}}F^{\mathcal{C}}$ on $\mathcal{V}$ or the comonad $F^{\mathcal{C}}U^{\mathcal{C}}$ on $\mathcal{C}$. We will refer to elements of a free construction $F^{\mathcal{C}}V$ using notation such as $f(v_i)$, thought of as a composite f of $\mathcal{C}$-structure maps applied to generators $v_i \in V \subseteq F^{\mathcal{C}}(V)$. We will say that $f(v_i)$ is a $\mathcal{C}$-*expression.* In this language, the linear maps

$$F^{\mathcal{C}}F^{\mathcal{C}}V \xrightarrow{\mu} F^{\mathcal{C}}V, \quad V \xrightarrow{\eta} F^{\mathcal{C}}V \quad \text{and} \quad F^{\mathcal{C}}V \xrightarrow{\epsilon} V,$$

constituting the augmented monad $F^{\mathcal{C}}$ on $\mathcal{V}$ may be described as follows: μ collapses a $\mathcal{C}$-expression in $\mathcal{C}$-expressions into a single $\mathcal{C}$-expression; η sends a vector v to the $\mathcal{C}$-expression v; and ϵ projects a $\mathcal{C}$-expression onto those summands to which no (non-trivial) operations have been applied. For $X \in \mathcal{C}$, the comonad structure maps in $\mathcal{C}$,

$$F^{\mathcal{C}}F^{\mathcal{C}}X \xleftarrow{\Delta} F^{\mathcal{C}}X \quad \text{and} \quad X \xleftarrow{\rho} F^{\mathcal{C}}X,$$

are as follows: on an expression $f(x_i)$, $\Delta = F^{\mathcal{C}}\eta$ returns the same expression $f(x_i)$ in which the $x_i \in X$ are viewed as elements of $F^{\mathcal{C}}X$, and ρ is the evaluation map equivalent to the $\mathcal{C}$-structure on X.

2.2. The functor $Q^{\mathcal{C}}$ of indecomposables

Using the augmentation $\epsilon : F^{\mathcal{C}} \longrightarrow \mathrm{id}$ of monads on $\mathcal{V}$, any $V \in \mathcal{V}$ becomes an $F^{\mathcal{C}}$-algebra, i.e. an object of $\mathcal{C}$. We denote this functor $K^{\mathcal{C}} : \mathcal{V} \longrightarrow \mathcal{C}$; it sends $V \in \mathcal{V}$ to *the trivial object on* V, which is V equipped with coaction map the projection $\epsilon : F^{\mathcal{C}}V \longrightarrow V$. Whenever we say *trivial* in this thesis, we will mean *having no non-zero operations*, and not *equal to zero*.

In each of our examples, $K^{\mathcal{C}}$ has a left adjoint, $Q^{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{V}$, which sends $X \in \mathcal{C}$ to *the quotient of* X *by the image of its non-trivial operations*. The functor $Q^{\mathcal{C}}$ sends $X \in \mathcal{C}$ to the coequalizer in $\mathcal{V}$ of $\rho, \epsilon : F^{\mathcal{C}}X \longrightarrow X$.

Note that $F^{\mathcal{C}}$ is a section of $Q^{\mathcal{C}}$, since $Q^{\mathcal{C}}F^{\mathcal{C}}$ is adjoint to $U^{\mathcal{C}}K^{\mathcal{C}} = \mathrm{id}$.

2.3. Quillen's model structure on $s\mathcal{C}$ and the bar construction

For any of the algebraic categories $\mathcal{C}$ appearing in this thesis we use Quillen's simplicial model category structure on the category $s\mathcal{C}$ of simplicial objects of $\mathcal{C}$ [48], [42], [3]. In this structure, the weak equivalences (fibrations) are the maps which are weak equivalences (fibrations) of simplicial abelian groups, so that every object is fibrant.

A simplicial object $X \in s\mathcal{C}$ is *almost free* if there are subspaces $V_n \subseteq X_n$ for each $n \geq 0$ such that the composite $F^{\mathcal{C}}V_n \longrightarrow F^{\mathcal{C}}X_n \xrightarrow{\rho} X_n$ is an isomorphism for all n, and such that the subspaces V_n are preserved by all of the degeneracies and face maps of X except for d_0. An almost free object is cofibrant, and every cofibrant object is a retract of an almost free object [42, §3].

There is a richer notion, that of an *almost free map*, which is a map $X \longrightarrow Y$ in $s\mathcal{C}$

such that Y_n contains a subspace V_n for each n such that the V_n are preserved by all faces and degeneracies except for d_0, and such that the natural map $X_n \sqcup F^{\mathcal{C}}(V_n) \longrightarrow Y_n$ is an isomorphism for each n. An almost free map is a cofibration, and every cofibration is a retract of an almost free map.

A *cofibrant replacement functor* for $s\mathcal{C}$ is an endofunctor f of $s\mathcal{C}$ equipped with a natural acyclic fibration $\epsilon : f \Rightarrow \mathrm{id}$ such that the image of f consists only of cofibrant objects. One classical such functor is the standard *comonadic simplicial bar construction* arising from the $F^{\mathcal{C}} \dashv U^{\mathcal{C}}$ adjunction. As a functor $B^{\mathcal{C}} : \mathcal{C} \longrightarrow s\mathcal{C}$ it is defined by iterated application of the comonad $F^{\mathcal{C}}$ to $X \in \mathcal{C}$:

$$B^{\mathcal{C}}_s X = (F^{\mathcal{C}})^{s+1} X,$$

with face maps given by $d_i = (F^{\mathcal{C}})^i \rho$, and degeneracies by $s_i = (F^{\mathcal{C}})^i \Delta$. This object is almost free, with $B^{\mathcal{C}}_s X$ generated by its subspace $V_s = (F^{\mathcal{C}})^s X$, moreover, it is standard [4, §4] that the augmentation $B^{\mathcal{C}} X \longrightarrow X$ is an acyclic fibration. This functor may be prolonged to a functor $B^{\mathcal{C}} : s\mathcal{C} \longrightarrow ss\mathcal{C}$, and by taking the diagonal we obtain an endofunctor $B^{\mathcal{C}}$ of $s\mathcal{C}$. A standard spectral sequence argument shows that this endofunctor is a cofibrant replacement functor.

2.4. Categories of graded $\mathbb{F}_2$-vector spaces and linear dualization

In this section we introduce notation for the key categories of graded vector spaces. We will write $\mathcal{V}$ for a generic category of graded vector spaces or for the category of ungraded vector spaces as convenient.

Write $\mathcal{V}^q_r$ for the category of vector spaces with r non-negative homological gradings and q non-negative cohomological gradings, so that an object V of $\mathcal{V}^q_r$ decomposes as

$$V = \bigoplus_{s_r,\dots,s_1,t_q,\dots,t_1 \geq 0} V^{t_q,\dots,t_1}_{s_r,\dots,s_1}.$$

The category $\mathcal{V}^q_r$ is equipped with a tensor product:

$$(U \otimes V)^{t_q,\dots,t_1}_{s_r,\dots,s_1} = \bigoplus_{s'_i+s''_i=s_i,\, t'_j+t''_j=t_j} U^{t'_q,\dots,t'_1}_{s'_r,\dots,s'_1} \otimes V^{t''_q,\dots,t''_1}_{s''_r,\dots,s''_1}.$$

We will often discuss maps between graded vector spaces which do not preserve degrees. Although we could encode such maps as grading-preserving maps between appropriate suspensions, it will not be helpful to be so systematic. For example, we will often write $V \otimes V \longrightarrow V$ for a map which in fact adds one to certain gradings of V, and will avoid confusion by explicitly stating the effect of such a map on degrees.

We will often need to consider the linear dual of a vector space V, and the standard symbol, V^*, will cause ambiguity, due to our already intensive use of superscripts. Instead we opt for a modifier written prefix, defining the dualization functor $\mathbf{D} : (\mathcal{V}_r^q)^{\mathrm{op}} \longrightarrow \mathcal{V}_q^r$ by:

$$(\mathbf{D}V)^{s_r,\ldots,s_1}_{t_q,\ldots,t_1} := \hom(V^{t_q,\ldots,t_1}_{s_r,\ldots,s_1}, \mathbb{F}_2).$$

We will shortly define cohomology functors $H^*_{\mathcal{C}}X := \mathbf{D}H^{\mathcal{C}}_*X$, and we will use the position of the asterisk to indicate which of homology and cohomology we mean. This is not precisely an exception to our convention, but was worth mentioning.

Often, the vector spaces we are interested in will support an *extra* grading, the *quadratic grading*, so called because certain operations derived from an underlying quadratic operation tend to double this extra grading. We do not think of the quadratic grading as either homological or cohomological, so we write it prefix:

$$V = \bigoplus\nolimits_{k\geq 1} q_k V.$$

We write $\mathrm{q}\mathcal{V}_r^q$ for the category of objects of $\mathcal{V}_r^q$ equipped with this extra grading.

A common pattern for us will be to consider vector spaces with r non-negative homological gradings and a single *strictly positive* cohomological grading:

$$V = \bigoplus_{s_r,\ldots,s_1\geq 0,\, t\geq 1} V^t_{s_r,\ldots,s_1},$$

and we write $\mathcal{V}_r^+$ for the category of such objects. For the rest of this chapter we will often use gradings of this type, simply because they will be used so extensively later in the thesis.

Similarly, there is a category $\mathcal{V}^r_+$, and dualization is a functor $\mathbf{D} : (\mathcal{V}_r^+)^{\mathrm{op}} \longrightarrow \mathcal{V}^r_+$.

2.5. The Dold-Kan correspondence

In this thesis we will use each of the following five chain complexes in $\mathrm{ch}_+\mathcal{V}_r^+$ associated with a simplicial graded vector-space $V \in s\mathcal{V}_r^+$:

$$\begin{aligned}
C_nV &:= V_n && \text{with differential } d = \textstyle\sum_{i=0}^n d_i;\\
N_nV &:= \textstyle\bigcap_{0<i\leq n} \ker\,(d_i : V_n \longrightarrow V_{n-1}) && \text{with differential } d = d_0;\\
N_n^-V &:= \textstyle\bigcap_{0\leq i<n} \ker\,(d_i : V_n \longrightarrow V_{n-1}) && \text{with differential } d = d_n;\\
\mathrm{Deg}_nV &:= \textstyle\sum_{0\leq i<n} \mathrm{im}(s_i : V_{n-1} \longrightarrow V_n) && \text{with differential } d = \textstyle\sum_{i=0}^n d_i.\\
N_n^{\div}V &:= V_n/\mathrm{Deg}_nV && \text{with differential } d = \textstyle\sum_{i=0}^n d_i.
\end{aligned}$$

There are evident inclusions of N_*V and N_*^-V into C_*V, and a projection of C_*V onto $N_*^{\div}V$, and all of these maps are weak equivalences. Moreover, the composite $N_*V \longrightarrow N_*^{\div}V$ is an isomorphism (as is the composite from N_*^-V). It will be helpful to have an explicit formula for the composite

$$C_*V \longrightarrow N_n^{\div}V \cong N_nV.$$

Lemma 2.1. *The* normalization *map*

$$\mathrm{nml} = (\mathrm{id} + s_0d_1)(\mathrm{id} + s_1d_2)\cdots(\mathrm{id} + s_{n-1}d_n) : V_n \longrightarrow V_n$$

is an idempotent chain complex endomorphism with image N_*V and kernel the degenerate n-simplices of V, so that there is a commuting diagram

$$\begin{tikzcd}
N_nV \arrow[r, tail] \arrow[d, equal] & C_nV \arrow[dl, "\mathrm{nml}"'] \arrow[dr, two heads] & \\
N_nV \arrow[r, tail] \arrow[rr, bend right, "\cong"'] & C_nV \arrow[r, two heads] & N_n^{\div}V
\end{tikzcd}$$

Proof. It is obvious that nml restricts to the identity on N_nV, and that $(\mathrm{nml} - \mathrm{id})$ has image consisting of degenerate simplices. By the simplicial identities, for $1 \le i \le n$:

$$d_i(\mathrm{id} + s_{i-1}d_i) = d_i + d_is_{i-1}d_i = d_i + \mathrm{id}\, d_i = 0.$$

As for $1 \le j < i$, we also have

$$d_i(\mathrm{id} + s_{j-1}d_j) = d_i + s_{j-1}d_{i-1}d_j = (\mathrm{id} + s_{j-1}d_j)d_i,$$

this proves that $d_i \circ \mathrm{nml} = 0$ for $1 \le i \le n$, or that nml has image inside N_nV. Thus nml is an idempotent with image N_nV. As $N_nV \longrightarrow N_n^{\div}V$ is as isomorphism, the rest is easy. $\square$

Each N_nV retains the internal gradings of V, and the functor N_* appears in the celebrated Dold-Kan correspondence [34, §III.2]:

Proposition 2.2 (The Dold-Kan correspondence)**.** *There is an adjoint equivalence of categories:*

$$N_* : s\mathcal{V}_r^+ \rightleftarrows \mathrm{ch}_+\mathcal{V}_r^+ : \Gamma,$$

under which the homotopy groups of $V \in s\mathcal{V}_r^+$ (as a simplicial set) are naturally isomorphic to the homology groups of N_*V:

$$(\pi_nV)^t_{s_r,\ldots,s_1} \cong (H_nN_*V)^t_{s_r,\ldots,s_1}.$$

A cycle in N_nV is an element $x \in V_n$ such that $d_i x = 0$ for $0 \le i \le n$. We write ZN_nV for this group of cycles, referring to elements of ZN_*V as *normalized cycles*. Note that $ZN_*^-V = ZN_*V$ is the same group of normalized cycles.

For x a cycle in any of the four homotopy equivalent chain complexes calculating π_*V, we will write $\overline{x}$ for the equivalence class of x in π_*V.

It will often be helpful to remove the notational distinction between the chain complex dimension n and the other homological dimensions $s_r, \ldots, s_1$. That is, we may view π_*V as a single object of $\mathcal{V}^+_{r+1}$, defined by

$$(\pi_*V)^t_{s_{r+1},\ldots,s_1} := (\pi_{s_{r+1}}V)^t_{s_r,\ldots,s_1}.$$

Now for any collection of indices $s_{r+1}, \ldots, s_1 \ge 0$ and $t \ge 1$, define:

$$\begin{array}{rl}
\mathbb{K}^t_{s_{r+1},s_r,\ldots,s_1} & = \Gamma\Big(\cdots \longleftarrow 0 \longleftarrow \mathbb{F}_2\{z\} \longleftarrow 0 \longleftarrow 0 \longleftarrow \cdots \Big), \\
& \quad \scriptstyle \text{degrees:} \quad s_{r+1}-1 \qquad s_{r+1} \qquad s_{r+1}+1 \qquad s_{r+1}+2 \\
C\mathbb{K}^t_{s_{r+1},s_r,\ldots,s_1} & = \Gamma\Big(\cdots \longleftarrow 0 \longleftarrow \mathbb{F}_2\{dh\} \longleftarrow \mathbb{F}_2\{h\} \longleftarrow 0 \longleftarrow \cdots \Big).
\end{array}$$

Here z and h denote are both to lie in internal cohomological grading t and homological gradings $s_r, \ldots, s_1$. There is an evident inclusion $\imath n : \mathbb{K}^t_{s_{r+1},s_r,\ldots,s_1} \longrightarrow C\mathbb{K}^t_{s_{r+1},s_r,\ldots,s_1}$. For any $V \in s\mathcal{V}^+_r$, we can identify the subspaces of cycles and boundaries with hom-sets:

$$\begin{aligned}
\hom_{s\mathcal{V}^+_r}(\mathbb{K}^t_{s_{r+1},\ldots,s_1}, V) &\cong (ZN_{s_{r+1}}V)^t_{s_r,\ldots,s_1} \text{ and} \\
\hom_{s\mathcal{V}^+_r}(C\mathbb{K}^t_{s_{r+1},\ldots,s_1}, V) &\cong (N_{s_{r+1}+1}V)^t_{s_r,\ldots,s_1}.
\end{aligned}$$

Under these isomorphisms the chain complex differential $N_{s_{r+1}+1}V \longrightarrow ZN_{s_{r+1}}V$ corresponds to $\imath n^*$. In fact, $\mathbb{K}^t_{s_{r+1},\ldots,s_1}$ represents $\pi_*(-)t_{s_{r+1},\ldots,s_1}$ in the homotopy category of $s\mathcal{V}^+_r$: in a category of simplicial vector spaces, the distinction between spheres and Eilenberg-Mac Lane spaces disappears.

A dual theory exists for cosimplicial vector spaces U. We mention the cochain complexes

$$\begin{array}{ll}
C^nU := U^n & \text{with differential } d = \sum_{i=0}^{n+1} d^i; \\
N^nU := U^n / \sum_{0<i\le n} \mathrm{im}(d^i : U_{n-1} \longrightarrow U_n) & \text{with differential } d = d^0; \\
N^n_{\subseteq}U := \bigcap_{0\le i\le n-1} \ker\left(s^i : U_n \longrightarrow U_{n-1}\right) & \text{with differential } d = \sum_{i=0}^{n+1} d^i.
\end{array}$$

There are chain complex maps whose composite is an isomorphism:

$$N^nU \longleftarrow C^nU \longleftarrow N^n_{\subseteq}U,$$

and an explicit normalization map

$$\mathrm{nml} = (\mathrm{id} + d^n s^{n-1}) \cdots (\mathrm{id} + d^2 s^1)(\mathrm{id} + d^1 s^0) : C^n U \longrightarrow N^n_{\subseteq} U$$

with properties dual to the simplicial version. The cohomology of any of these three equivalent cochain complexes defines the *cohomotopy* $\pi^* U$ of U.

Homotopy and cohomotopy correspond under dualization as follows. If $V \in s\mathcal{V}$, then $C^* \mathbf{D} V = \mathbf{D} C_* V$, and there is a natural isomorphism $\pi^* \mathbf{D} V \longrightarrow \mathbf{D} \pi_* V$ given by:

$$H^* C^* \mathbf{D} V = H^* \mathbf{D} C_* V \longrightarrow \mathbf{D} H_* C_* V, \quad \overline{\alpha} \longmapsto \text{“}\overline{v} \longmapsto \alpha(v)\text{”}.$$

2.6. Skeletal filtrations of almost free objects

Suppose that $X \in s\mathcal{C}$ is almost free on generating subspaces $V_s \subseteq X_s$. Miller [42, p. 55] defines a filtration of X by almost free subobjects

$$0 \rightarrowtail F_0 X \rightarrowtail F_1 X \rightarrowtail F_2 X \rightarrowtail \cdots \rightarrowtail \operatorname{colim} F_m X = X$$

as follows. For each $m, i \geq 0$, write $F_m V_i$ for the subspace of V_i spanned by the degeneracies of elements of V_j such that $j \leq \min\{m, i\}$. Then write $F_m X$ for the subobject of X which is almost free on the subobjects $F_m V_i$. The inclusions of these subobjects are almost free maps, and the colimit is evidently X.

Lemma 2.3. *For each $m \geq 0$, $\mathrm{nml}(V_m) \subset V_m$, and V_m has direct sum decomposition*

$$V_m = (V_m \cap N_m X) \oplus (V_m \cap \mathrm{Deg}_m X),$$

natural in maps of almost free objects preserving the chosen almost free subspaces, and such that $V_m \cap N_m X = \mathrm{im}\Big(V_m \xrightarrow{\mathrm{nml}} V_m\Big)$. Moreover, the map

$$\big(\textstyle\sum_{i=0}^{m-1} s_i\big) : V_{m-1}^{\oplus m} \longrightarrow F_{m-1} V_m$$

is injective.

Proof. The final statement is implied by [42, Fact 3.9]. That nml preserves V_m is clear from its defining formula. The direct sum decomposition and the fact about $V_m \cap N_m X$ both follow from previous observations about the idempotent nml on X_m, in particular that it has image $N_m X$ and kernel $\mathrm{Deg}_m X$. $\square$

2.7. Dold's Theorem

According to Dold [22] (c.f. [17, Lemma 3.1]):

Theorem 2.4 (Dold's Theorem)**.** *Suppose that $F : s\mathcal{V}_r^+ \longrightarrow s\mathcal{V}_r^+$ is a functor preserving weak equivalences, for example, the prolongation of an endofunctor of $\mathcal{V}_r^+$. Then there is a functor $\mathcal{F} : \mathcal{V}_{r+1}^+ \longrightarrow \mathcal{V}_{r+1}^+$ such that the following diagram commutes:*

$$\begin{tikzcd} s\mathcal{V}_r^+ \arrow[r, "F"] \arrow[d, "\pi_*"] & s\mathcal{V}_r^+ \arrow[d, "\pi_*"] \\ \mathcal{V}_{r+1}^+ \arrow[r, "\mathcal{F}"] & \mathcal{V}_{r+1}^+ \end{tikzcd}$$

Moreover, if F is naturally equivalent to a composite $F_2 \circ F_1$, then $\mathcal{F}$ is naturally isomorphic to $\mathcal{F}_2 \circ \mathcal{F}_1$.

The idea here is that the functor π_* induces an equivalence between the homotopy category of $s\mathcal{V}_r^+$ and $\mathcal{V}_{r+1}^+$. In fact, the inverse equivalence can be lifted to a functor into $s\mathcal{V}_r^+$, namely

$$V \longmapsto \Gamma V, \quad \mathcal{V}_{r+1}^+ \longrightarrow s\mathcal{V}_r^+,$$

where we view V as a trivial chain complex. Then $\mathcal{F}$ can be constructed as $\mathcal{F}V := \pi_*(F\Gamma V)$.

2.8. Homology and cohomology functors $H_*^{\mathcal{C}}$ and $H_{\mathcal{C}}^*$

In this thesis we will always define the $\mathcal{C}$-*homology* of $X \in s\mathcal{C}$ by the formula:

$$H_*^{\mathcal{C}}X := \pi_*(Q^{\mathcal{C}}B^{\mathcal{C}}X) = H_*N_*(Q^{\mathcal{C}}B^{\mathcal{C}}X).$$

These homology functors are well defined, as the $Q^{\mathcal{C}} \dashv K^{\mathcal{C}}$ adjunction is a Quillen adjunction (that $K^{\mathcal{C}}$ preserves fibrations and acyclic fibrations is immediate), and indeed we are free to use any cofibrant replacement in place of $B^{\mathcal{C}}X$.

It is not always entirely appropriate to call these functors homology. Indeed, Quillen [48, §II.5] defines *homology* to be the left derived functors of the abelianization functor, and it is not true in all of our examples that $Q^{\mathcal{C}}$ models the abelianization functor. Goerss [33, §4] explains that this does occur when $\mathcal{C}$ is the category of non-unital commutative algebras, but it does not occur when $\mathcal{C}$ is the category of restricted Lie algebras [21].

When $\mathcal{C}$ is monadic over $\mathcal{V}_r^+$, we may view the groups $H_*^{\mathcal{C}}X$ together as an object of $\mathcal{V}_{r+1}^+$. That is, each homology group $H_s^{\mathcal{C}}X$ retains the gradings of X, and a new homological grading is added (to the left of the existing homological gradings). We will sometimes avoid substituting into the asterisk, writing $(H_*^{\mathcal{C}}X)^t_{s_{r+1},\ldots,s_1}$ in place of $(H_{s_{r+1}}^{\mathcal{C}}X)^t_{s_r,\ldots,s_1}$.

We define the $\mathcal{C}$-*cohomology* $H^*_{\mathcal{C}}X$ of X to be $\mathbf{D}(H^{\mathcal{C}}_*X)$, or equivalently the cohomotopy groups $\pi^*\mathbf{D}(Q^{\mathcal{C}}X)$ of the dual cosimplicial object. As we dualize to obtain cohomology, the cohomological gradings and homological gradings are swapped, and $H^*_{\mathcal{C}}X$ may be viewed as an object of $\mathcal{V}^{r+1}_+$.

Lemma 2.5. *Suppose that $X \in s\mathcal{C}$ is almost free with generating subspaces $V_n \subseteq X_n$. Then any homology class in $H^{\mathcal{C}}_nX \cong \pi_nQ^{\mathcal{C}}X$ can be represented by the image in $Q^{\mathcal{C}}X_n$ of an element of $V_n \cap N_nX$.*

Proof. This follows from Lemma 2.3 — simply represent the class in question by an element of V_n, and then apply the natural map nml. □

This lemma states that we may find representatives for any homology class in the *subobject* $V_n \cap N_nX$ of X_n, while for other applications it will be preferable simply to pass to the *quotient* V_n of X_n. Trivially:

Lemma 2.6. *Suppose that X is almost free with generating subspaces $V_n \subseteq X_n$. Then the simplicial object $\{(Q^{\mathcal{C}}X)_n\}$ may be identified with the collection of vector spaces $\{V_n\}$, using the following composite as the zeroth face map of $\{V_n\}$:*

$$V_n \xrightarrow{d_0} X_n \cong F^{\mathcal{C}}V_{n-1} \xrightarrow{\epsilon} V_{n-1},$$

and using the other structure maps of X, which by assumption preserve the generating subspaces, as the other structure maps of $\{V_n\}$.

2.9. The action of Σ_2 on $V^{\otimes 2}$

For any vector space $V \in \mathcal{V}$, the tensor power $V^{\otimes 2} := V \otimes V$ has an action of Σ_2 given by the map T interchanging the two factors. We will write S_2V for the coinvariants and S^2V for the invariants of this action:

$$S_2V := (V \otimes V)_{\Sigma_2} := (V \otimes V)/\Sigma_2;$$
$$S^2V := (V \otimes V)^{\Sigma_2} := \{x \in V \otimes V \mid x = Tx\}.$$

The *trace map* is the natural linear map

$$\mathrm{tr} := (1+T) : S_2V \longrightarrow S^2V,$$

and we write

$$\Lambda^2V := \mathrm{im}(\mathrm{tr}) \cong S_2V/\ker(\mathrm{tr})$$

Thus, we may view $\Lambda^2 V$ either as a *subobject* of $S^2 V$ of as the *quotient* of $S_2 V$ by the subspace generated by elements of the form $v \otimes v$.

For any $V \in \mathcal{V}$ there is a natural map

$$S_2 \mathbf{D} V \longrightarrow \mathbf{D} S^2 V, \quad \alpha \otimes \beta \longmapsto \text{“} v \otimes w \longmapsto \alpha(v)\beta(w) \text{”}.$$

It is an isomorphism when V is finite-dimensional.

Suppose that V and W are $\mathbb{F}_2$-vector spaces, and $p : S_2 V \longrightarrow W$ is a linear map. A *quadratic refinement of* p is a function $\sigma : V \longrightarrow W$ satisfying, for $v_1, v_2 \in V$ and $\alpha \in \mathbb{F}_2$:

$$\sigma(v_1 + v_2) = \sigma(v_1) + \sigma(v_2) + p(v_1 \otimes v_2) \text{ and } \sigma(\alpha v_1) = \alpha^2 \sigma(v_1).$$

In fact, the second condition is redundant (over $\mathbb{F}_2$), and these conditions are equivalent to the following condition. For any set B, define $\Lambda^2 B$ to be the set of subsets of B of cardinality exactly two. The equivalent condition is that, for every collection of vectors $v_b \in V$ and of coefficients $\alpha_b \in \mathbb{F}_2$ indexed by a set B, in which all but finitely many of the α_b are zero, the following equation holds:

$$\sigma\Big(\sum_{b \in B} \alpha_b v_b\Big) = \sum_{b \in B} \alpha_b^2 \sigma(v_b) + \sum_{\{b,c\} \in \Lambda^2 B} \alpha_b \alpha_c p(v_b \otimes v_c).$$

If $f : S^2 V \longrightarrow W$ is a linear map, the function $v \longmapsto f(v \otimes v)$ is a quadratic refinement of $\mathrm{tr} \circ f$, and indeed:

Proposition 2.7. *For any linear map $p : S_2 V \longrightarrow W$, extensions of p to a linear map $f : S^2 V \longrightarrow W$ are in natural bijection with quadratic refinements of p.*

Proof. Suppose that V has basis $\{v_b \mid b \in B\}$. Then $S^2 V$ has basis the set

$$\{\mathrm{tr}(v_b \otimes v_c) \mid \{b,c\} \in \Lambda^2 B\} \cup \{v_b \otimes v_b \mid b \in B\}.$$

This is easy to check for V finite dimensional, and extends to the infinite dimensional case as S^2 preserves filtered colimits, and we may calculate V as the colimit

$$\operatorname*{colim}_{B' \subseteq B} \mathbb{F}_2 \langle B' \rangle = V.$$

In particular, an extension f of p is determined by the quadratic refinement $v \longmapsto f(v \otimes v)$. Thus, as long as we can produce an extension f with $\sigma(v) = f(v \otimes v)$ for any quadratic refinement σ of p, we will have the natural construction we need.

What remains to prove is that the linear map f defined on this basis by

$$\mathrm{tr}(v_b \otimes v_c) \longmapsto p(v_b \otimes v_c), \quad v_b \otimes v_b \longmapsto \sigma(v_b)$$

does in fact have the property that $f(v \otimes v) = \sigma(v)$ for all $v \in V$. Indeed, if we write v in terms of the chosen basis as $v = \sum_{b \in B} \alpha_b v_b$, then

$$v \otimes v = \sum_{b \in B} \alpha_b^2 v_b + \sum_{\{b,c\} \in \Lambda^2 B} \alpha_b \alpha_c \,\mathrm{tr}(v_b \otimes v_c),$$

and we can apply our definition of the linear map f to this expansion directly, obtaining

$$f(v \otimes v) := \sum_{b \in B} \alpha_b^2 \sigma(v_b) + \sum_{\{b,c\} \in \Lambda^2 B} \alpha_b \alpha_c p(v_b \otimes v_c) = \sigma(v). \qquad \square$$

Corollary 2.8. *There is a natural linear map* $\sqrt{-} : S^2 V \longrightarrow V$*, the* square root map*, uniquely determined by the requirements:*

$$\sqrt{v_1 \otimes v_2 + v_2 \otimes v_1} = 0, \quad \sqrt{v \otimes v} = v \text{ for all } v_1, v_2, v \in V.$$

Proof. This map is the unique extension of $0 : S_2 V \longrightarrow V$ corresponding to the quadratic refinement $\mathrm{id} : V \longrightarrow V$ of 0. $\square$

The evocative square root symbol is doubly appropriate, as if V is dual to a finite-dimensional vector space $U \in \mathcal{V}$, the linear dual of the square root map,

$$\mathbf{D}V \longrightarrow \mathbf{D}S^2 V \xleftarrow{\cong} S_2 \mathbf{D}V$$

equals the *squaring map* $U \longrightarrow S_2 U$, defined by $u \longmapsto u \otimes u$.

2.10. Lie algebras in characteristic 2

As we work in characteristic 2, there is more than one available notion of a Lie algebra. An *$S(\mathscr{L})$-algebra* is a vector space L equipped with a bracket $L \otimes L \longrightarrow L$ satisfying the Jacobi identity and the (anti)-symmetry condition $[x, y] = [y, x]$. A *Lie algebra* (or $\Lambda(\mathscr{L})$-algebra) is a vector space L equipped with a bracket $L \otimes L \longrightarrow L$ satisfying the Jacobi identity and the alternating condition $[x, x] = 0$. Finally, a *restricted Lie algebra* [20, 13] (or $\Gamma(\mathscr{L})$-algebra) is a Lie algebra equipped with a *squaring* or *restriction* function $(-)^{[2]} : L \longrightarrow L$, satisfying the axioms

$$(x_1 + x_2)^{[2]} = x_1^{[2]} + x_2^{[2]} + [x_1, x_2] \text{ and } [x_1^{[2]}, x_2] = [x_1, [x_1, x_2]].$$

The alternating condition implies the (anti)-symmetry condition, and these three types of Lie algebras form a hierarchy: a restricted Lie algebra is in particular a Lie algebra, and a Lie algebra is in particular an $S(\mathscr{L})$-algebra.

We will write $\mathscr{L}ie$ for the category of ungraded Lie algebras, and $\mathscr{L}ie^r$ for the category of ungraded restricted Lie algebras.

Fresse [31] explains how to construct the monads $S(\mathscr{L})$, $\Lambda(\mathscr{L})$ and $\Gamma(\mathscr{L})$ on $\mathcal{V}$ which give rise to these structures, starting with the Lie operad $\mathscr{L}$. For $V \in \mathcal{V}$, it is standard that the functor

$$S(\mathscr{L}) : V \longmapsto \bigoplus_{n\geq 1} (\mathscr{L}(n) \otimes V^{\otimes n})_{\Sigma_n}$$

inherits the structure of a monad from the composition maps of $\mathscr{L}$. Fresse observes that the functor

$$\Gamma(\mathscr{L}) : V \longmapsto \bigoplus_{n\geq 1} (\mathscr{L}(n) \otimes V^{\otimes n})^{\Sigma_n}$$

may also be equipped with a monad structure, such that the trace map $S(\mathscr{L}) \longrightarrow \Gamma(\mathscr{L})$ is a map of monads, and that an intermediate monad may be defined by

$$\Lambda(\mathscr{L}) : V \longmapsto \mathrm{im}\big(\mathrm{tr} : S(\mathscr{L})(V) \longrightarrow \Gamma(\mathscr{L})(V)\big).$$

These monads give rise to the three indicated forms of Lie algebras in characteristic 2. Each of these functors supports a *quadratic grading*:

$$\mathrm{q}_k(\Gamma(\mathscr{L})V) := (\mathscr{L}(k) \otimes V^{\otimes k})^{\Sigma_k}, \text{ etc.},$$

and since $\mathscr{L}(2)$ is one-dimensional, there are natural identifications:

$$\mathrm{q}_2(S(\mathscr{L})V) \cong S_2V, \quad \mathrm{q}_2(\Lambda(\mathscr{L})V) \cong \Lambda^2V, \quad \text{and} \quad \mathrm{q}_2(\Gamma(\mathscr{L})V) \cong S^2V.$$

One can identify an $S(\mathscr{L})$-algebra with the corresponding map $S_2L \longrightarrow L$, a $\Lambda(\mathscr{L})$-algebra with the map $\Lambda^2L \longrightarrow L$, and a $\Gamma(\mathscr{L})$-algebra with the map $S^2L \longrightarrow L$, which is to say, for instance, that a map $S_2V \longrightarrow V$ admits at most one extension to a $S(\mathscr{L})$-algebra structure map $S(\mathscr{L})V \longrightarrow V$. By pulling back along the natural maps

$$S_2V \longrightarrow \Lambda^2V \longrightarrow S^2V$$

one can demote a restricted Lie algebra to a Lie algebra, or a Lie algebra to an $S(\mathscr{L})$-algebra.

A *restrictable ideal* in a Lie algebra L is a Lie ideal I of L, equipped with a restriction function $(-)^{[2]} : I \longrightarrow I$, satisfying the following axioms, for $x_1, x_2 \in I$ and $x_3 \in L$:

$$(x_1 + x_2)^{[2]} = x_1^{[2]} + x_2^{[2]} + [x_1, x_2] \quad \text{and} \quad [x_1^{[2]}, x_3] = [x_1, [x_1, x_3]].$$

In fact, let PRL denote the category of *partially restricted Lie algebras*, whose objects are pairs of vector spaces (L_+, L_0), equipped with a Lie algebra structure on $L_+ \oplus L_0$ in which L_+ is a restrictable ideal, and whose maps are Lie algebra maps preserving the decomposition and commuting with the partial restrictions. This category is monadic over $\mathcal{V} \times \mathcal{V}$, the category of pairs of vector spaces, and the value of monad F^{PRL} on (V_+, V_0) is just an appropriately chosen subalgebra of $\Gamma(\mathscr{L})(V_+ \oplus V_0)$. We will refer to homogeneous elements of L_+ as *restrictable*, and homogeneous elements of L_0 as *non-restrictable*.

In §7.1 we will define various categories of graded partially restricted Lie algebras, where membership of the restrictable ideal is determined by the non-vanishing of certain gradings.

2.11. Non-unital commutative algebras

In this thesis we will work with *non-unital* commutative algebras except when we specify otherwise. As for Lie algebras, there are three different notions of non-unital commutative algebra available in characteristic 2. A *commutative algebra* (or $S(\mathscr{C})$-algebra) is a vector space A equipped with an associative commutative pairing $A \otimes A \longrightarrow A$. We will work with these often, and will write $\mathscr{C}om$ for the category of such algebras. In fact, we will so often discuss simplicial non-unital commutative algebras that we will refer to them simply as *simplicial algebras*.

An *exterior algebra* (or $\Lambda(\mathscr{C})$-algebra) is a commutative algebra A with the property that $x^2 = 0$ for all $x \in A$. A *divided power algebra* (or $\Gamma(\mathscr{C})$-algebra) is a commutative algebra A equipped with *divided power* operations, as described in [31, 1.2.2] or [33, §2]. In characteristic 2, these operations are all determined by a single operation, the *divided square* $\gamma_2 : A \longrightarrow A$, which satisfies

$$\gamma_2(xy) = x^2\gamma_2(y), \ \gamma_2(\lambda x) = \lambda^2\gamma_2(x) \text{ and } \gamma_2(x + y) = \gamma_2(x) + \gamma_2(y) + xy.$$

Note that the second condition is in fact extraneous over $\mathbb{F}_2$, and that the last condition implies that a divided power algebra is exterior. Thus, $\gamma_2(xy) = x^2\gamma_2(y) = 0$.

There is a notion of a *divided power ideal* of a commutative algebra: an ideal I of a commutative algebra A equipped with a compatible divided power structure on I. In this case, I is necessarily exterior, although for $x \in A$ and $y \in I$, $\gamma_2(xy) = x^2\gamma_2(y)$ need not be zero.

Again, Fresse [31] explains how to construct the monads $F^{\mathscr{C}om} := S(\mathscr{C})$, $\Lambda(\mathscr{C})$ and

$\Gamma(\mathscr{C})$ on $\mathcal{V}$ which give rise to these structures, using the commutative operad $\mathscr{C}$ instead of $\mathscr{L}$. Again, there is a quadratic grading definable on these three monads, and each monad is generated in degree 2, so that a commutative algebra may be thought of as a map $S_2L \longrightarrow L$, an exterior algebra as a map $\Lambda^2 L \longrightarrow L$, and a divided power algebra as a map $S^2L \longrightarrow L$.

The coproduct $A \sqcup B$ in the category of non-unital commutative algebras is the direct sum $A \oplus (A \otimes B) \oplus B$, with the obvious product. Moreover, the *smash coproduct* (to be defined in general in §3.5) is simply $A \veebar B := A \otimes B$. Indeed, coproducts and smash coproducts in *all three* of the above categories are given by these formulae.

2.12. First quadrant cohomotopy spectral sequences

Suppose that $V_{p,q}$ is a bisimplicial vector space, ungraded for now. We will follow the standard conventions, those of [52], in defining the cohomotopy spectral sequence of V, which calculates the cohomotopy of the diagonal $|V|$ of V. For more detail, see [52, §1.15].

There is a double chain complex $C_{p,q}V := C^{\mathrm{h}}_p C^{\mathrm{v}}_q V = V_{p,q}$, where we have decorated the functors C^{v} and C^{h} in order to distinguish them from the functor C_{**} being introduced, and to distinguish the coordinates — we will always refer to p as the *horizontal* coordinate and q as the *vertical* coordinate. The total complex TV, along with one of its two canonical increasing filtrations, is defined by

$$(TV)_n := \bigoplus_{i=0}^{n} C_{i,n-i}V, \qquad F_p(TV)_n := \bigoplus_{i=0}^{p} C_{i,n-i}V.$$

The dual total complex $\mathbf{D}TV$ admits a decreasing filtration defined by

$$F^p\mathbf{D}TV := \ker\big(\mathbf{D}TV \twoheadrightarrow \mathbf{D}F_{p-1}TV\big).$$

Correspondingly, $H^*(\mathbf{D}TV) \cong \pi^*(\mathbf{D}|V|)$ is equipped with a decreasing filtration. This filtration is evidently *finite* (eventually stabilizing in any given dimension), *exhaustive* (having union $H^*(\mathbf{D}TV)$) and *Hausdorff* (having intersection zero), and one defines

$$[E_0\pi^*(\mathbf{D}|V|)]^{p,q} := F^p\pi^{p+q}(\mathbf{D}|V|)/F^{p+1}\pi^{p+q}(\mathbf{D}|V|).$$

Then, there is a spectral sequence with

$$[E_2V]^{p,q} = \pi^p_{\mathrm{h}}\pi^q_{\mathrm{v}}(\mathbf{D}V),$$

and differential $d_r : [E_rV]^{p,q} \longrightarrow [E_rV]^{p+r,q-r+1}$ so that $[E_{r+1}V]$ is the cohomology of the cochain complex $([E_rV]; d_r)$, and for each fixed p and q,

$$[E_rV]^{p,q} \text{ stabilizes to } [E_\infty V]^{p,q} \cong [E_0\pi^*(\mathbf{D}|V|)]^{p,q} \text{ as } r \to \infty.$$

Typically, V will admit an *augmentation* to a simplicial object $V_{-1} \in s\mathcal{V}$, *inducing a weak equivalence* $|V| \xrightarrow{\sim} V_{-1}$. All of our augmentations are horizontal maps to a vertical object, i.e. an augmentation is a simplicial (in q) map:

$$d_0^{\mathrm{h}} : V_{0,q} \longrightarrow V_{-1,q} \text{ coequalizing } d_0^{\mathrm{h}}, d_1^{\mathrm{h}} : V_{1,q} \longrightarrow V_{0,q}.$$

In this case, we view the spectral sequence as a tool for the calculation of the cohomotopy $\pi^*(\mathbf{D}V_{-1})$, via isomorphisms

$$[E_\infty V]^{p,q} \cong [E_0\pi^*(\mathbf{D}V_{-1})]^{p,q}.$$

If V is instead a bisimplicial *graded* vector space $V \in ss\mathcal{V}_h^c$, then we may regard $[E_rV]$ as an element of $\mathcal{V}_c^{h+2}$. That is:

$$[E_rV]^{p,q,s_h,\ldots,s_1}_{t_c,\ldots,t_1} := [E_r(V^{t_c,\ldots,t_1}_{s_h,\ldots,s_1})]^{p,q}.$$

In our application of these conventions we will actually have $V \in \mathcal{V}_h^+$, and will write $p = s_{h+2}$ and $q = s_{h+1}$. We will even sometimes have a quadratic grading on V, which will transfer to a further grading on the spectral sequence, so our spectral sequences will appear in the format

$$\mathrm{q}_k[E_rV]^{s_{h+2},\ldots,s_1}_t := [E_r(\mathrm{q}_kV^t_{s_h,\ldots,s_1})]^{s_{h+2},s_{h+1}}.$$

2.13. Second quadrant homotopy spectral sequences

Suppose that

$$0 = \mathcal{T}_{-1} \twoheadleftarrow \mathcal{T}_0 \twoheadleftarrow \mathcal{T}_1 \twoheadleftarrow \mathcal{T}_2 \twoheadleftarrow \cdots \twoheadleftarrow \mathcal{T}_\infty$$

is a tower of surjections of chain complexes, with $\mathcal{T}_\infty$ the inverse limit. Then $\mathcal{T}_\infty$ has a canonical decreasing filtration:

$$F^m = F^m\mathcal{T}_\infty := \ker(\mathcal{T}_\infty \longrightarrow \mathcal{T}_{m-1})$$

and we define, with the conventional suspensions:

$$[E_0]^s := \Sigma^s \mathrm{ker}\left(\mathcal{T}_s \longrightarrow \mathcal{T}_{s-1}\right);$$
$$[E_1]^s := H_*[E_0]^s = H_{*-s}\mathrm{ker}\left(\mathcal{T}_s \longrightarrow \mathcal{T}_{s-1}\right).$$

From this data we may derive the following diagram, in which any pair of composable maps that consists of a monomorphism then an epimorphism is a short exact sequence of chain complexes:

$$\begin{tikzcd}
 & 0 & \Sigma^{-0}[E_0]^0 & \Sigma^{-1}[E_0]^1 & \Sigma^{-2}[E_0]^2 & \Sigma^{-3}[E_0]^3 & \\
 & \mathcal{T}_\infty \arrow[u, two heads] \arrow[d, tail] \arrow[r, equal] & F^0 \arrow[u, two heads] \arrow[d, tail] & F^1 \arrow[l, tail] \arrow[u, two heads] \arrow[d, tail] & F^2 \arrow[l, tail] \arrow[u, two heads] \arrow[d, tail] & F^3 \arrow[l, tail] \arrow[u, two heads] \arrow[d, tail] & \\
\cdots & \mathcal{T}_\infty \arrow[d, two heads] \arrow[r, equal] & \mathcal{T}_\infty \arrow[d, two heads] \arrow[r, equal] & \mathcal{T}_\infty \arrow[d, two heads] \arrow[r, equal] & \mathcal{T}_\infty \arrow[d, two heads] \arrow[r, equal] & \mathcal{T}_\infty \arrow[d, two heads] & \cdots \\
 & 0 \arrow[r, equal] & \mathcal{T}_{-1} & \mathcal{T}_0 \arrow[l, two heads] & \mathcal{T}_1 \arrow[l, two heads] & \mathcal{T}_2 \arrow[l, two heads] & \\
 & 0 \arrow[u, tail] & 0 \arrow[u, tail] & \Sigma^{-0}[E_0]^0 \arrow[u, tail] & \Sigma^{-1}[E_0]^1 \arrow[u, tail] & \Sigma^{-2}[E_0]^2 \arrow[u, tail] &
\end{tikzcd}$$

Taking homology, each short exact sequence of chain complexes creates a long exact sequence, and we obtain two exact couples (c.f. [29] or [41, §2.2]), which we juxtapose, using dotted maps to indicate boundary homomorphisms:

$$\begin{tikzcd}
 & 0 & \Sigma^{-0}H[E_1]^0 \arrow[dr, dotted] & \Sigma^{-1}H[E_1]^1 \arrow[dr, dotted] & \Sigma^{-2}H[E_1]^2 \arrow[dr, dotted] & \Sigma^{-3}H[E_1]^3 & \\
 & H\mathcal{T}_\infty \arrow[u] \arrow[d] \arrow[r, equal] & HF^0 \arrow[u] \arrow[d] & HF^1 \arrow[l] \arrow[u] \arrow[d] & HF^2 \arrow[l] \arrow[u] \arrow[d] & HF^3 \arrow[l] \arrow[u] \arrow[d] & \\
\cdots & H\mathcal{T}_\infty \arrow[d] \arrow[r, equal] & H\mathcal{T}_\infty \arrow[d] \arrow[r, equal] & H\mathcal{T}_\infty \arrow[d] \arrow[r, equal] & H\mathcal{T}_\infty \arrow[d] \arrow[r, equal] & H\mathcal{T}_\infty \arrow[d] & \cdots \\
 & 0 \arrow[r, equal] & H\mathcal{T}_{-1} \arrow[uu, dotted, bend left] \arrow[dr, dotted] & H\mathcal{T}_0 \arrow[l] \arrow[uu, dotted, bend left] \arrow[dr, dotted] & H\mathcal{T}_1 \arrow[l] \arrow[uu, dotted, bend left] \arrow[dr, dotted] & H\mathcal{T}_2 \arrow[l] \arrow[uu, dotted, bend left] & \\
 & 0 \arrow[u] & 0 \arrow[u] & \Sigma^{-0}H[E_1]^0 \arrow[u] & \Sigma^{-1}H[E_1]^1 \arrow[u] & \Sigma^{-2}H[E_1]^2 \arrow[u] &
\end{tikzcd}$$

The vertical boundary homomorphisms $H\mathcal{T}_m \longrightarrow \Sigma HF^{m+1}$ in fact form a morphism of exact couples (c.f. [29]), as follows from Verdier's octahedral axiom (in the homotopy category of chain complexes, c.f. [38, Appendix A.1]) or a diagram chase. Moreover, the two resulting spectral sequences have the same E_1-page, so that they are identical (c.f. [29, §6]). This common spectral sequence is simply the spectral sequence of the decreasing filtration F^m on the complex $\mathcal{T}_\infty$ (c.f. [41, §2.2], [7]). The intended target $H\mathcal{T}_\infty$ has an exhaustive decreasing filtration, defined in either of two equivalent ways:

$$F^m(H\mathcal{T}_\infty) := \mathrm{im}(HF^m \longrightarrow H\mathcal{T}_\infty) = \mathrm{ker}\left(H\mathcal{T}_\infty \longrightarrow H\mathcal{T}_{m-1}\right),$$

and one writes $[E_0 H\mathcal{T}_\infty]^s_t = F^s H_{t-s}\mathcal{T}_\infty / F^{s+1} H_{t-s}\mathcal{T}_\infty$.

One context in which we may make these constructions is when given any sequence of maps

$$0 = \mathbb{T}_{-1} \longleftarrow \mathbb{T}_0 \longleftarrow \mathbb{T}_1 \longleftarrow \cdots$$

in $s\mathcal{V}$. Such a tower may be converted into a homotopy equivalent tower of surjections

$$0 = \mathbb{T}'_{-1} \twoheadleftarrow \mathbb{T}'_0 \twoheadleftarrow \mathbb{T}'_1 \twoheadleftarrow \cdots$$

and we may perform the above constructions with $\mathcal{T}_m := C_*\mathbb{T}'_m$. Homotopy equivalent towers will produce isomorphic spectral sequences from E_1. From this perspective, a straightforward way to give a map of spectral sequences that *shifts filtration* is simply to give a map of such towers with the corresponding shift.

Suppose now that V is an object of $(s\mathcal{V})^{\Delta_+}$, the category of coaugmented cosimplicial objects in the category of simplicial vector spaces. We think of the cosimplicial direction as *horizontal* and the simplicial direction as *vertical*, so that the *coaugmentation* of V is a (horizontal) map from a (vertical) simplicial object $V^{-1} \in s\mathcal{V}$, i.e. a simplicial (in t) map:

$$d^0_{\mathrm{h}} : V^{-1}_t \longrightarrow V^0_t \text{ equalizing } d^0_{\mathrm{h}}, d^1_{\mathrm{h}} : V^0_t \longrightarrow V^1_t.$$

There is a cochain-chain complex

$$(CV)^s_t := C^s_{\mathrm{h}} C^{\mathrm{v}}_t V = V^s_t,$$

with s the horizontal and t the vertical coordinate, whose differential is the sum of the horizontal and vertical differentials. The *total complex* TV is a chain complex with a canonical decreasing filtration, defined by

$$(TV)_n = \textstyle\prod_{t-s=n} CV^s_t, \qquad d = d_{\mathrm{h}} + d^{\mathrm{v}}, \qquad (F^m TV)_n = \prod_{\substack{t-s=n \\ s\geq m}} CV^s_t.$$

This filtration of TV corresponds to the tower of surjections of chain complexes defined by

$$(\mathcal{T}_m V)_n := (TV/F^{m+1}TV)_n \cong \textstyle\prod_{\substack{t-s=n \\ s\leq m}} CV^s_t,$$

which has inverse limit $\mathcal{T}_\infty V = TV$. Again, the two evident filtrations of $H_*(TV) =$

$H_*(\mathcal{T}_\infty V)$ coincide, and the resulting spectral sequences coincide, satisfying

$$\begin{aligned}
[E_0 V]^s_t &:= C^s_{\mathrm{h}} C^{\mathrm{v}}_t V, \qquad && d_0 = d^{\mathrm{v}};\\
[E_1 V]^s_t &:= C^s_{\mathrm{h}} \pi^{\mathrm{v}}_t V, \qquad && d_1 = d_{\mathrm{h}};\\
[E_2 V]^s_t &:= \pi^s_{\mathrm{h}} \pi^{\mathrm{v}}_t V.
\end{aligned}$$

The differential is of the form $d_r : [E_r V]^s_t \longrightarrow [E_r V]^{s+r}_{t+r-1}$, and as ever, $[E_{r+1}V]$ is the homology of the chain complex $([E_r V]; d_r)$. We will work with this spectral sequence in detail, and will need the following explicit description of the higher pages:

$$\begin{aligned}
[Z_r V]^s_t &:= \left\{ x \in (F^s TV)_{t-s} \;\middle|\; dx \in (F^{s+r}TV)_{t-s-1} \right\};\\
[E_r V]^s_t &:= [Z_r V]^s_t / \left(d\left([Z_{r-1}V]^{s-r+1}_{t-r+2}\right) + [Z_{r-1}V]^{s+1}_{t+1} \right).
\end{aligned}$$

The spectral sequence will sometimes admit a *vanishing line of slope* α on E_2, i.e. there will exist a constant c such that:

$$[E_2 V]^s_t = 0 \text{ for } s > c + \alpha(t-s).$$

In this case, the filtration on $H_*(TV)$ is Hausdorff and finite, and for each fixed s and t:

$$[E_r V]^s_t \text{ stabilizes to } [E_\infty V]^s_t \cong [E_0 H_*(TV)]^s_t \text{ as } r \to \infty.$$

The coaugmentation induces a map $V^{-1} \xrightarrow{\sim} \operatorname{Tot} V$ where $\operatorname{Tot} V$ is the totalization of V in the simplicial model category $s\mathcal{V}$ [34, VII.5]. Bousfield explains how this relates to the totalization tower [34, VII.5] of V:

Lemma 2.9 [7, Lemma 2.2]**.** *There are natural chain maps $N_* \operatorname{Tot}_m V \longrightarrow \mathcal{T}_m V$ for $m \leq \infty$ which induce an isomorphism of towers $\pi_* \operatorname{Tot}_m V \longrightarrow H_* \mathcal{T}_m V$. In particular $H_*(TV) \cong \pi_* \operatorname{Tot} V$.*

Not only then do we have a tower under $\mathcal{T}_\infty V \simeq C_* \operatorname{Tot} V$, but $\operatorname{Tot} V$ accepts the coaugmentation map from V^{-1}. Of course, the coaugmentation map need not be surjective, but if we factor it as a composite

$$V^{-1} \overset{\sim}{\rightarrowtail} r(V^{-1}) \twoheadrightarrow \operatorname{Tot} V$$

we may form the following diagram by demanding that the vertical composites be strict fiber sequences

$$\begin{tikzcd}
 & r(V^{-1}) \arrow[r, equal] \arrow[d, tail] & \mathrm{Fib}^0 \arrow[d, tail] & \mathrm{Fib}^1 \arrow[l, tail] \arrow[d, tail] & \mathrm{Fib}^2 \arrow[l, tail] \arrow[d, tail] & \mathrm{Fib}^3 \arrow[l, tail] \arrow[d, tail] & \\
\cdots & r(V^{-1}) \arrow[r, equal] \arrow[d, two heads] & r(V^{-1}) \arrow[r, equal] \arrow[d, two heads] & r(V^{-1}) \arrow[r, equal] \arrow[d, two heads] & r(V^{-1}) \arrow[r, equal] \arrow[d, two heads] & r(V^{-1}) \arrow[d, two heads] & \cdots \\
 & 0 \arrow[r, equal] & \mathrm{Tot}_{-1} & \mathrm{Tot}_0 \arrow[l, two heads] & \mathrm{Tot}_1 \arrow[l, two heads] & \mathrm{Tot}_2 \arrow[l, two heads] &
\end{tikzcd}$$

and applying the functor C_*.

We will in general hope that $V^{-1} \longrightarrow \mathrm{Tot}\, V$ will be a weak equivalence, and to investigate whether or not this is so, it will be helpful to be able to identify the fibers Fib^m up to homotopy. For this we recall a useful relationship between cosimplicial objects and cubical diagrams, explained by Sinha in [53, Theorem 6.5], and expanded on by Munson-Volić [43]. We will only present that part of the theory that we need, and refer the reader to [35], [45] or [43] for the theory of cubical diagrams and their homotopy total fibers. For $n \geq 0$ let $[n] = \{0, \ldots, n\}$, and define $\mathcal{P}[n] = \{S \subseteq [n]\}$ to be the poset category whose morphisms are the inclusions $S \subseteq S'$, so that an $(n+1)$-*cubical diagram* in $s\mathcal{V}$ is a functor $\mathcal{P}[n] \longrightarrow s\mathcal{V}$.

Sinha describes a diagram of inclusions of categories

$$\begin{tikzcd}
\mathcal{P}[-1] \arrow[r, "\tau"] \arrow[drr, "h_{-1}"'] & \mathcal{P}[0] \arrow[r, "\tau"] \arrow[dr, "h_0"] & & \mathcal{P}[1] \arrow[r, "\tau"] \arrow[dl, "h_1"'] & \mathcal{P}[2] \arrow[r, "\tau"] \arrow[dll, "h_2"] & \cdots \\
 & & \Delta_+ & & &
\end{tikzcd}$$

The augmented cosimplicial simplicial vector space $V : \Delta_+ \longrightarrow s\mathcal{V}$ may be pulled back along h_m to form an $(m+1)$-cubical diagram $h_m^* V$. After noting that V is Reedy fibrant (c.f. [10, X.4.9]), Sinha explains that there are natural weak equivalences

$$\mathrm{Fib}^{m+1} \sim \mathrm{hofib}(V^{-1} \longrightarrow \mathrm{Tot}_m V) \xrightarrow{\sim} \mathrm{hototfib}(h_m^* V)$$

under which the inclusion $\mathrm{Fib}^m \longleftarrow \mathrm{Fib}^{m+1}$ is identified with the map

$$\mathrm{hototfib}(h_m^* V) \longrightarrow \mathrm{hototfib}(\tau^* h_m^* V) = \mathrm{hototfib}(h_{m-1}^* V).$$

As $h_{-1}^* V$ is the 0-cube with value V^{-1}, the tower of homotopy total fibers is identified up to homotopy with the tower of the Fib^m.

Chapter 3

Homotopy operations and cohomology operations

Let $\mathcal{C}$ be a category of universal graded algebras, monadic over $\mathcal{V}_r^+$. Our goal is to understand construct operations on the homotopy and cohomology of an object of $s\mathcal{C}$. In §3.2 and §3.3, we set out dual frameworks in which these operations can be organized, and in 3.10 and 3.5, we describe some useful chain level operations that we will use to construct cohomology operations in §6.

3.1. The spheres in $s\mathcal{C}$ and their mapping cones

Using the forgetful functor $U^{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{V}_r^+$, for any $X \in s\mathcal{C}$ we may define the homotopy groups $\pi_* X$ of X, which we view together as an object of $\mathcal{V}_{r+1}^+$. By the definition of the model structure on $s\mathcal{C}$, the functor $\pi_* : s\mathcal{C} \longrightarrow \mathcal{V}_{r+1}^+$ is homotopical, which is to say that it inverts weak equivalences. For any set of indices $t \geq 0$ and $s_{r+1}, \ldots, s_1 \geq 0$, write:

$$\begin{aligned} \mathbb{S}^{\mathcal{C},t}_{s_{r+1},\ldots,s_1} &:= F^{\mathcal{C}}\mathbb{K}^t_{s_{r+1},\ldots,s_1}; \text{ and} \\ C\mathbb{S}^{\mathcal{C},t}_{s_{r+1},\ldots,s_1} &:= F^{\mathcal{C}}C\mathbb{K}^t_{s_{r+1},\ldots,s_1} \end{aligned}$$

These are the *spheres in* $s\mathcal{C}$ and *cones on spheres in* $s\mathcal{C}$ respectively, and we write

$$\mathrm{sph}(\mathcal{C}) := \left\{\mathbb{S}^{\mathcal{C},t}_{s_{r+1},\ldots,s_1} \,|\, t \geq 0,\ s_{r+1}, \ldots, s_1 \geq 0\right\}$$

for the set of spheres in $s\mathcal{C}$. Note that we were very literal here — the spheres in $s\mathcal{C}$ are precisely this set of objects, and not, say, the cofibrant objects in $s\mathcal{C}$ which are weakly equivalent to some $\mathbb{S}^{\mathcal{C},t}_{s_{r+1},\ldots,s_1}$. For $S \in \mathrm{sph}(\mathcal{C})$ we write CS for the corresponding cone.

For any $S \in \mathrm{sph}(\mathcal{C})$, there is an evident cofibration $\imath n : S \longrightarrow CS$. Indeed, for any sphere $S = \mathbb{S}^{\mathcal{C},t}_{s_{r+1},\dots,s_1}$, S contains a distinguished normalized cycle, the *fundamental cycle*:

$$z \in (ZN_{s_{r+1}}S)^t_{s_r,\dots,s_1},$$

and the cone CS contains a distinguished normalized chain, the *cone on* z:

$$h \in (N_{s_{r+1}+1}CS)^t_{s_r,\dots,s_1},$$

and $\imath n$ is defined by the requirement that $\imath n(z) = d_0 h$. For any $X \in s\mathcal{C}$, by adjunction:

$$\hom_{s\mathcal{C}}(\mathbb{S}^{\mathcal{C},t}_{s_{r+1},\dots,s_1}, X) \cong (ZN_{s_{r+1}}X)^t_{s_r,\dots,s_1} \text{ and}$$
$$\hom_{s\mathcal{C}}(C\mathbb{S}^{\mathcal{C},t}_{s_{r+1},\dots,s_1}, X) \cong (N_{s_{r+1}+1}X)^t_{s_r,\dots,s_1},$$

and indeed $\imath n^*$ plays the same role as above, representing the differential of N_*X. Moreover, in the homotopy category corresponding to the above model category structure, $\mathbb{S}^{\mathcal{C},t}_{n,s_r,\dots,s_1}$ represents $\pi_n(-)^t_{s_r,\dots,s_1}$ (c.f. [33, §1] or [3, §3.1.1]) which is why we refer to the objects $\mathbb{S}^{\mathcal{C},t}_{n,s_r,\dots,s_1}$ as spheres.

3.2. Homotopy groups and $\mathcal{C}$-Π-algebras

By virtue of the algebraic structure possessed by $X \in s\mathcal{C}$, the homotopy groups π_*X possess certain natural algebraic structure, that of a $\mathcal{C}$-Π-algebra. Indeed, as any given homotopy group is a representable functor on the homotopy category, natural N-ary operations on homotopy groups

$$(\pi_*X)^{t^1}_{s^1_{r+1},\dots,s^1_1} \times \cdots \times (\pi_*X)^{t^N}_{s^N_{r+1},\dots,s^N_1} \longrightarrow (\pi_*X)^t_{s_{r+1},\dots,s_1} \tag{3.1}$$

are in bijective correspondence with elements of the group

$$\pi_*\left(\mathbb{S}^{\mathcal{C},t^1}_{s^1_{r+1},\dots,s^1_1} \sqcup \cdots \sqcup \mathbb{S}^{\mathcal{C},t^N}_{s^N_{r+1},\dots,s^N_1}\right)^t_{s_{r+1},\dots,s_1}. \tag{3.2}$$

Blanc and Stover [3] define a new category of graded universal algebras, the category $\pi\mathcal{C}$ of $\mathcal{C}$-Π-algebras, monadic over $\mathcal{V}^+_{r+1}$, whose objects are graded vector spaces $V \in \mathcal{V}^+_{r+1}$ with a structure map

$$V^{t^1}_{s^1_{r+1},\dots,s^1_1} \times \cdots \times V^{t^N}_{s^N_{r+1},\dots,s^N_1} \longrightarrow V^t_{s_{r+1},\dots,s_1} \tag{3.3}$$

for every such homotopy class, satisfying certain natural compatibilities.

It is a standard formalism to encode these compatibilities as follows. A *model* [3] in $s\mathcal{C}$ is an almost free object of $s\mathcal{C}$ which is weakly equivalent to a coproduct of spheres (for example, $F^{\mathcal{C}}\Gamma V$ for any $V \in \mathcal{V}^+_{r+1}$ viewed as a chain complex with zero differential). A *finite model* is a model in which this coproduct is finite. Let Π be the $\mathcal{V}$-enriched category with objects the finite models in $s\mathcal{C}$, and morphisms

$$\hom_\Pi(M, M') := \hom_{\mathrm{ho}(s\mathcal{C})}(M, M').$$

Then the category of $\mathcal{C}$-Π-algebras may be defined as the category of $\mathcal{V}$-enriched functors $\Pi^{\mathrm{op}} \longrightarrow \mathcal{V}$ that send finite coproducts into products (where by $\mathcal{V}$ we mean the category of ungraded $\mathbb{F}_2$-vector spaces). The category of $\mathcal{C}$-Π-algebras is monadic over $\mathcal{V}^+_{r+1}$, with forgetful functor $U^{\pi\mathcal{C}}$ defined on a functor $A \in \pi\mathcal{C}$ by:

$$(U^{\pi\mathcal{C}}A)^t_{s_{r+1},\ldots,s_1} := A(\mathbb{S}^{\mathcal{C},t}_{s_{r+1},\ldots,s_1}),$$

and each of the structure maps (3.3) on $U^{\pi\mathcal{C}}A$ is induced by the corresponding homotopy class (3.2), viewed as a map in Π.

One obtains the free $\mathcal{C}$-Π-algebra on a graded vector space $V \in \mathcal{V}^+_{r+1}$ using Dold's Theorem (2.4). That is, one views V as a chain complex in $\mathrm{ch}_+\mathcal{V}^+_r$ with zero differential, and applies the Dold-Kan correspondence and $\mathcal{C}$-free functor, obtaining an object $F^{\mathcal{C}}\Gamma V \in s\mathcal{C}$, and then

$$F^{\pi\mathcal{C}}V = \pi_*(F^{\mathcal{C}}\Gamma V).$$

Moreover, as $F^{\mathcal{C}}$ is an augmented monad, so is $F^{\pi\mathcal{C}}$, via the map

$$F^{\pi\mathcal{C}}V = \pi_*(F^{\mathcal{C}}\Gamma V) \xrightarrow{\pi_*\epsilon} \pi_*(\Gamma V) = H_*V = V,$$

and in particular, there is an adjunction $Q^{\pi\mathcal{C}} \dashv K^{\pi\mathcal{C}}$.

The theory above has the upshot that understanding the category $\pi\mathcal{C}$ is equivalent to calculating the homotopy groups of the finite models. In many cases, this can be performed by calculating the homotopy of individual spheres, and then using a *Hilton-Milnor Theorem* (c.f. §5.5) or *Künneth Theorem* (c.f. Proposition 5.5) to bootstrap up to a calculation on all finite models.

Lemma 3.1. *For any model A in $s\mathcal{C}$, the Hurewicz map $\pi_*A \longrightarrow \pi_*Q^{\mathcal{C}}A$ descends to an isomorphism*

$$\gamma : Q^{\pi\mathcal{C}}\pi_*A \longrightarrow \pi_*Q^{\mathcal{C}}A \cong H^{\mathcal{C}}_*A.$$

Proof. A is (cofibrant and) homotopic to a coproduct of spheres, and as such may be taken to be equal to a coproduct of spheres. As π_*A is free on generators in correspondence with the sphere summands, $Q^{\pi\mathcal{C}}\pi_*A$ is simply the vector space with basis their fundamental classes, which is isomorphic to $H^{\mathcal{C}}_*A$. □

3.3. Cohomology groups and $\mathcal{C}$-H^*-algebras

It will in general be preferable for us to consider algebraic structure on cohomology, rather than coalgebraic structure on homology: algebra is in general a more familiar subject than coalgebra, and cohomology has the advantage that it consists of representable functors. Another advantage is that the theory of cohomology and $\mathcal{C}$-H^*-algebras is dual to the theory of homotopy groups and $\mathcal{C}$-Π-algebras, and §3.3 can be (and has been) obtained from §3.2 by appropriate dualization. On the other hand, using cohomology groups has the disadvantages associated with double-dualization.

For any set of indices $t \geq 0$ and $s_{r+1}, \ldots, s_1 \geq 0$, write:

$$\begin{aligned} \mathbb{K}^{\mathcal{C},t}_{s_{r+1},\ldots,s_1} &:= K^{\mathcal{C}}\mathbb{K}^{t}_{s_{r+1},\ldots,s_1}; \text{ and} \\ C\mathbb{K}^{\mathcal{C},t}_{s_{r+1},\ldots,s_1} &:= K^{\mathcal{C}}C\mathbb{K}^{t}_{s_{r+1},\ldots,s_1} \end{aligned}$$

The $\mathbb{K}^{\mathcal{C},t}_{s_{r+1},\ldots,s_1}$ are the *Eilenberg-Mac Lane objects in* $s\mathcal{C}$. In the homotopy category of $s\mathcal{C}$, the object $\mathbb{K}^{\mathcal{C},t}_{n,s_r,\ldots,s_1}$ represents the contravariant functor $H^n_{\mathcal{C}}(-)^{s_r,\ldots,s_1}_t : s\mathcal{C} \longrightarrow \mathcal{V}$, c.f. [33, Proposition 4.3].

By virtue of the algebraic structure possessed by X, the cohomology groups $H^*_{\mathcal{C}}X$ possess certain natural algebraic structure, that of a $\mathcal{C}$-H^*-algebra. As for $\mathcal{C}$-Π-algebras, natural N-ary operations on cohomology groups

$$(H^*_{\mathcal{C}}X)^{s^1_{r+1},\ldots,s^1_1}_{t^1} \times \cdots \times (H^*_{\mathcal{C}}X)^{s^N_{r+1},\ldots,s^N_1}_{t^N} \longrightarrow (H^*_{\mathcal{C}}X)^{s_{r+1},\ldots,s_1}_{t} \tag{3.4}$$

are in bijective correspondence with elements of the group

$$H^*_{\mathcal{C}}\Big(\mathbb{K}^{\mathcal{C},t^1}_{s^1_{r+1},\ldots,s^1_1} \times \cdots \times \mathbb{K}^{\mathcal{C},t^N}_{s^N_{r+1},\ldots,s^N_1}\Big)^{s_{r+1},\ldots,s_1}_{t}. \tag{3.5}$$

The category of $\mathcal{C}$-H^*-algebras, monadic over $\mathcal{V}^{r+1}_+$, has objects graded vector spaces $V \in \mathcal{V}^{r+1}_+$ with a structure map

$$V^{s^1_{r+1},\ldots,s^1_1}_{t^1} \times \cdots \times V^{s^N_{r+1},\ldots,s^N_1}_{t^N} \longrightarrow V^{s_{r+1},\ldots,s_1}_{t} \tag{3.6}$$

for every such cohomology class, satisfying certain natural compatibilities.

The formalism required to express these compatibilities is as follows. A *generalized Eilenberg-Mac Lane object*, or *GEM*, in $s\mathcal{C}$ is an almost free object of $s\mathcal{C}$ which is weakly equivalent to a product of Eilenberg-Mac Lane objects $\mathbb{K}^{\mathcal{C},t}_{s_{r+1},\ldots,s_1}$. A *finite GEM* is a GEM in which this product is finite. Let $\mathbb{K}$ be the $\mathcal{V}$-enriched category with objects the finite

GEMs in $s\mathcal{C}$, and morphisms

$$\hom_{\mathbb{K}}(M, M') := \hom_{\mathrm{ho}(s\mathcal{C})}(M, M').$$

Then the category of $\mathcal{C}$-H^*-algebras may be defined as the category of $\mathcal{V}$-enriched functors $\mathbb{K} \longrightarrow \mathcal{V}$ that preserve finite products. The category of $\mathcal{C}$-H^*-algebras is monadic over $\mathcal{V}_+^{r+1}$, with forgetful functor defined on a functor $h : \mathbb{K} \longrightarrow \mathcal{V}$ by:

$$(U^{H\mathcal{C}} h)_t^{s_{r+1},\ldots,s_1} := h(\mathbb{K}^{\mathcal{C},t}_{s_{r+1},\ldots,s_1}),$$

and each of the structure maps (3.6) on $U^{H\mathcal{C}} h$ is induced by the corresponding cohomology class (3.5), viewed as a map in $\mathbb{K}$.

One obtains the free $\mathcal{C}$-H^*-algebra on a graded vector space $V \in \mathcal{V}_+^{r+1}$ *of finite type* as follows. One views $\mathbf{D}V$ as a chain complex in $\mathrm{ch}_+\mathcal{V}_r^+$ with zero differential, and applies the Dold-Kan correspondence and $K^{\mathcal{C}}$, obtaining an object $K^{\mathcal{C}}\Gamma\mathbf{D}V \in s\mathcal{C}$. Then:

$$F^{H\mathcal{C}} V = H^*_{\mathcal{C}} K^{\mathcal{C}} \Gamma \mathbf{D} V.$$

Moreover, $F^{H\mathcal{C}}$ is an augmented monad: one applies $H^*_{\mathcal{C}}$ to the natural collapse map $F^{\mathcal{C}}\Gamma V \longrightarrow K^{\mathcal{C}}\Gamma V$, to obtain

$$K^{H\mathcal{C}} V \cong H^*_{\mathcal{C}} F^{\mathcal{C}} \Gamma \mathbf{D} V \longleftarrow H^*_{\mathcal{C}} K^{\mathcal{C}} \Gamma \mathbf{D} V =: F^{H\mathcal{C}} V.$$

and in particular, there is an adjunction $Q^{H\mathcal{C}} \dashv K^{H\mathcal{C}}$.

These definitions simplify when we apply them to the dual of a vector space $U \in \mathcal{V}^+_{r+1}$ of finite type:

$$F^{H\mathcal{C}}(\mathbf{D}U) := \pi^* \mathbf{D} Q^{\mathcal{C}} c K^{\mathcal{C}} \Gamma \mathbf{D}^2 U \xleftarrow{\cong} \pi^* \mathbf{D} Q^{\mathcal{C}} c K^{\mathcal{C}} \Gamma U \xrightarrow{\cong} \mathbf{D} \pi_* Q^{\mathcal{C}} c K^{\mathcal{C}} \Gamma U$$

suggesting that the functor $F^{H\mathcal{C}}$ is altogether of the wrong variance. It is preferable to work with the functor

$$C^{H\mathcal{C}-\mathrm{coalg}} U := \pi_* Q^{\mathcal{C}} c K^{\mathcal{C}} \Gamma U$$

discussed in §3.7.

To dualize a paragraph from §3.2: the theory above has the upshot that understanding the category $H\mathcal{C}$ is equivalent to calculating the cohomology groups of finite GEMs. In many cases, this can be performed by calculating the cohomology of individual Eilenberg-Mac Lane objects, and then using a *Hilton-Milnor Theorem* (c.f. [33, §11] and §6.6) or *Künneth Theorem* (c.f. Theorems 6.15 and 14.6) to bootstrap up to a calculation on all finite GEMs.

3.4. The reverse Adams spectral sequence

We will now give a description of Miller's reverse Adams spectral sequence [42, §4], which was used by Goerss [33, Chapter V] to calculate the cohomology of Eilenberg-Mac Lane objects in $s\mathscr{C}om$.

Suppose that $X \in s\mathcal{C}$, and consider the *bisimplicial* object $Q^{\mathcal{C}}(B^{\mathcal{C}}_p X)_q \in ss\mathcal{V}^+_n$. There is a first quadrant cohomotopy spectral sequence

$$[E_2\, Q^{\mathcal{C}} B^{\mathcal{C}} X]^{p,q,s_n,\dots,s_1}_t = \pi^p_{\mathrm{h}} \pi^q_{\mathrm{v}} (\mathbf{D} Q^{\mathcal{C}} B^{\mathcal{C}} X)^{s_n,\dots,s_1}_t$$

converging to $H^*_{\mathcal{C}} X := \pi^{p+q} \mathbf{D} Q^{\mathcal{C}} |B^{\mathcal{C}} X|$. For each fixed p,

$$\begin{aligned} \pi^*(\mathbf{D} Q^{\mathcal{C}} B^{\mathcal{C}}_p X) &\cong \mathbf{D} \pi_*(Q^{\mathcal{C}} B^{\mathcal{C}}_p X) \\ &\cong \mathbf{D} Q^{\pi\mathcal{C}} \pi_*(B^{\mathcal{C}}_p X) \\ &\cong \mathbf{D} Q^{\pi\mathcal{C}} B^{\pi\mathcal{C}}_p \pi_*(X), \end{aligned}$$

where the second isomorphism is that of Lemma 3.1, so that

$$[E_2\, Q^{\mathcal{C}} B^{\mathcal{C}} X]^{p,q,s_n,\dots,s_1}_t \cong (H^p_{\pi\mathcal{C}} \pi_* X)^{q,s_n,\dots,s_1}_t.$$

When $\mathcal{C} = \mathscr{C}om$, this is precisely the spectral sequence used by Goerss in [33, Chapter V] to calculate the cohomology of Eilenberg-Mac Lane objects in $s\mathscr{C}om$. In this thesis, we will use this spectral sequence only for certain low-dimensional calculations. Goerss equipped the reverse Adams spectral sequence with certain spectral sequence operations [33, §14], work which can be framed using the external operations, due to Singer, reprised in §13.

In this thesis we study a Bousfield-Kan spectral sequence (BKSS), which is also known as an unstable Adams spectral sequence, for the category $\mathscr{C}om$. The operations defined in §11 for this spectral sequence make for another point of comparison. Loosely, we find that the operations on the BKSS are in a sense Koszul dual to the operations on the reverse Adams spectral sequence.

3.5. The smash coproduct

For X_1 and X_2 objects of any algebraic category, for example $\mathcal{C}$, $\pi\mathcal{C}$ or $H\mathcal{C}$ (to be defined shortly), we define the *smash coproduct* $X_1 \veebar X_2$ to be the kernel of the natural map $X_1 \sqcup X_2 \longrightarrow X_1 \times X_2$. When $X_1 = X_2 = X$, $X \veebar X$ has a natural action of Σ_2, and we write

$X \veebar^{\Sigma_2} X$ for the subobject of invariant elements under this action.

When X_1 and X_2 are objects of $s\mathcal{C}$, taking this strict fiber is in fact homotopically correct, since the map $X_1 \sqcup X_2 \longrightarrow X_1 \times X_2$ is always a fibration, and indeed:

Proposition 3.2. *For X_1 and X_2 in $s\mathcal{C}$, the natural $\mathcal{C}$-Π-algebra map*

$$\pi_*(X_1 \times X_2) \longrightarrow \pi_* X_1 \times \pi_* X_2$$

is an isomorphism. If X_1 and X_2 are models in $s\mathcal{C}$, the natural $\mathcal{C}$-Π-algebra map

$$\pi_* X_1 \sqcup \pi_* X_2 \longrightarrow \pi_*(X_1 \sqcup X_2)$$

is an isomorphism, and there is an isomorphism of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \pi_* X_1 \veebar \pi_* X_2 \arrow[r] \arrow[d, "\cong"'] & \pi_* X_1 \sqcup \pi_* X_2 \arrow[r] \arrow[d, "\cong"'] & \pi_* X_1 \times \pi_* X_2 \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & \pi_*(X_1 \veebar X_2) \arrow[r] & \pi_*(X_1 \sqcup X_2) \arrow[r] & \pi_*(X_1 \times X_2) \arrow[r] & 0
\end{tikzcd}$$

Proof. The first claim is easy: the forgetful functor is a right adjoint, and π_* preserves products (of vector spaces). Consider the commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_* X_1 \veebar \pi_* X_2 \arrow[r] & \pi_* X_1 \sqcup \pi_* X_2 \arrow[r] \arrow[d, "i"] & \pi_* X_1 \times \pi_* X_2 \arrow[r] \arrow[d, "\cong"'] & 0 \\
& \pi_*(X_1 \veebar X_2) \arrow[r] & \pi_*(X_1 \sqcup X_2) \arrow[r] & \pi_*(X_1 \times X_2) &
\end{tikzcd}$$

in which the top row is a short exact sequence, and the bottom row is just a three term excerpt of the homotopy long exact sequence of the fiber sequence defining $X_1 \veebar X_2$. If i were an isomorphism, the bottom row would also be short exact, and a simple diagram chase would show that i restricts to the isomorphism we desire.

If X_1 and X_2 are models, the displayed map i is an isomorphism, since both source and target represent the free $\mathcal{C}$-Π-algebra on generators corresponding to the sphere summands of X_1 and X_2 taken together. $\square$

3.6. Cofibrant replacement via the small object argument

The homotopy of an object X of $s\mathcal{C}$ was defined simply by application of the forgetful functor $U^{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{V}$, a definition which is tautologically homotopically correct. On the other hand, in order to define the homology $H_*^{\mathcal{C}} X$, as the left Quillen functor $Q^{\mathcal{C}}$ does not preserve all weak equivalences, we must perform a cofibrant replacement before applying $Q^{\mathcal{C}}$. While the comonadic bar construction $B^{\mathcal{C}}$ described in §2.3 suffices to define the groups $H_*^{\mathcal{C}} X$, it lacks the structure that we will need at various points in this thesis.

Radulescu-Banu's innovation [49] was to explain that the cofibrant replacement functor $c : s\mathcal{C} \longrightarrow s\mathcal{C}$ constructed by Quillen's *small object argument* [48], which by design already possesses a natural acyclic fibration $\epsilon : c \longrightarrow \mathrm{id}$, in fact admits the full structure of a comonad, with diagonal $\beta : c \longrightarrow cc$. As explained by Blumberg and Riehl [4, Remark 4.12]:

Proposition 3.3. *The endofunctor $Q^{\mathcal{C}}cK^{\mathcal{C}}$ of $s\mathcal{V}$ admits the structure of a comonad, via the maps*

$$Q^{\mathcal{C}}cK^{\mathcal{C}} \overset{Q^{\mathcal{C}}(\beta)}{\longrightarrow} Q^{\mathcal{C}}ccK^{\mathcal{C}} \overset{Q^{\mathcal{C}}c(\eta)}{\longrightarrow} Q^{\mathcal{C}}cK^{\mathcal{C}}Q^{\mathcal{C}}cK^{\mathcal{C}} \quad \textit{and} \quad Q^{\mathcal{C}}cK^{\mathcal{C}} \overset{Q^{\mathcal{C}}(\epsilon)}{\longrightarrow} Q^{\mathcal{C}}K^{\mathcal{C}} \cong \mathrm{id},$$

where η denotes the unit of the $Q^{\mathcal{C}} \dashv K^{\mathcal{C}}$ adjunction.

The functor c of the small object argument depends on the choice of sets of generating cofibrations and acyclic cofibrations. It will be helpful in our applications to include in the set of generating cofibrations the following important cofibrations:

(1) the inclusion of 0 into any sphere $\mathbb{S}^{\mathcal{C},t}_{s_{r+1},\ldots,s_1}$;

(2) the cofibration $\mathbb{S}^{\mathcal{C}}_t \sqcup \mathbb{S}^{\mathcal{C}}_{t'} \longrightarrow J_{t,t'}$ defined in §11.5;

(3) the cofibration $\mathbb{S}^{\mathcal{C}}_t \longrightarrow \Theta_{t,i}$ defined in §11.7; and

(4) for each cofibration $A \longrightarrow B$ just mentioned, the map $\Delta^1 \otimes A \longrightarrow \Delta^1 \otimes B$ formed using the standard closed simplicial model category structure [48, II.4] on $s\mathcal{C}$.

It will be helpful to have included these maps, because of the following facts about the small object argument functor cX. It is constructed as the colimit of a (transfinite) sequence of cofibrations:

$$\begin{tikzcd}
0 = c_0X \arrow[r, tail] \arrow[drr, two heads] & c_1X \arrow[r, tail] \arrow[dr] & c_2X \arrow[r, tail] \arrow[d] & c_3X \arrow[r, tail] \arrow[dl] & \cdots \\
 & & X & &
\end{tikzcd}$$

and given an element $f : A \longrightarrow B$ of the chosen set of generating cofibrations and a commuting square

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "f"] & c_nX \arrow[d] \\
B \arrow[r] & X
\end{tikzcd}$$

there is a canonical choice of map $B \longrightarrow c_{n+1}X$ making

$$\begin{tikzcd}
A \arrow[r] \arrow[dd, tail, "f"] & c_nX \arrow[d] \\
 & c_{n+1}X \arrow[d] \\
B \arrow[r] \arrow[ur] & X
\end{tikzcd}$$

commute. Indeed, the map $c_n X \longrightarrow c_{n+1} X$ is constructed by attaching a copy of B along the image of A in $c_n X$, for each such commuting square.

We will use this canonical lift later, and so establish a little notation. There is a function

$$\hom_{s\mathscr{C}om}(\mathbb{S}_t^{\mathcal{C}}, X) \longrightarrow \hom_{s\mathscr{C}om}(\mathbb{S}_t^{\mathcal{C}}, c_1 X)$$

denoted $\alpha \longmapsto \widetilde{\alpha}$, natural in $X \in s\mathcal{C}$, and which provides a section of

$$\hom_{s\mathscr{C}om}(\mathbb{S}_t^{\mathcal{C}}, cX) \xrightarrow{\epsilon_*} \hom_{s\mathscr{C}om}(\mathbb{S}_t^{\mathcal{C}}, X).$$

We define $\widetilde{\alpha}$ to be the canonical lift corresponding to the square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, tail] & c_0 X := 0 \arrow[d] \\ \mathbb{S}_t^{\mathcal{C}} \arrow[r, "\alpha"] & X \end{tikzcd}$$

Finally, we note that Radulescu-Banu's construction has a convenient (albeit not crucial) consequence for the construction of homotopy cofibers in $s\mathcal{C}$. Quillen's small object argument actually provides a functorial factorization

$$X \rightarrowtail c^{\text{fac}}(g) \overset{\sim}{\twoheadrightarrow} Y$$

of any map $g : X \longrightarrow Y$ in $s\mathcal{C}$, and in this notation, one might say that we have been writing cX as shorthand for $c^{\text{fac}}(0 \longrightarrow X)$. There is a commuting square

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d] & cY \arrow[d, equal] \\ cX \arrow[r, "cg"] & cY \end{tikzcd}$$

which by functoriality induces a commuting diagram (ignoring the dotted map):

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d] & ccY \arrow[r, two heads, "\epsilon_{cY}"] \arrow[d, "m"'] & cY \arrow[d, equal] \arrow[l, dotted, bend right, start anchor=south west, "\beta"'] \\ cX \arrow[r, tail] & c^{\text{fac}}(cg) \arrow[r, two heads, "\sim"'] & cY \end{tikzcd}$$

Radulescu-Banu's diagonal β is the dotted map in this diagram, and as it is a comonad diagonal $\epsilon_{cY} \circ \beta = \mathrm{id}_{cY}$, so that $m \circ \beta$ is a section of the acyclic fibration $c^{\text{fac}}(cg) \overset{\sim}{\twoheadrightarrow} cY$.

We may define the homotopy cofiber of g as the pushout

$$\begin{tikzcd} cX \arrow[r, tail] \arrow[d] & c^{\text{fac}}(cg) \arrow[d] \\ 0 \arrow[r] & \mathrm{hocof}(g) \end{tikzcd}$$

and by virtue of the construction just given:

Proposition 3.4. *There is a construction in $s\mathcal{C}$ of the homotopy cofiber* $\mathrm{hocof}(g)$ *of a map* $g : X \longrightarrow Y$*, implemented by natural maps*

$$cX \xrightarrow{cg} cY \longrightarrow \mathrm{hocof}(g).$$

This is in contrast to the standard situation, where there is at best a natural zig-zag, even from cY to $\mathrm{hocof}(g)$.

3.7. Homology groups and $\mathcal{C}$-H_*-coalgebras

There is a commuting diagram

$$\begin{tikzcd} s\mathcal{V}_r^+ \arrow[r, "Q^{\mathcal{C}}cK^{\mathcal{C}}"] \arrow[d, "\pi_*"] & s\mathcal{V}_r^+ \arrow[d, "\pi_*"] \\ \mathcal{V}_{r+1}^+ \arrow[r, "C^{H\mathcal{C}-\mathrm{coalg}}"] & \mathcal{V}_{r+1}^+ \end{tikzcd}$$

in which we are using Dold's Theorem (2.4) to *define* $C^{H\mathcal{C}-\mathrm{coalg}}$, the *cofree* $\mathcal{C}$-H_**-coalgebra comonad*. By Proposition 3.3 and the naturality of Dold's Theorem, this is a comonad on $\mathcal{V}_{r+1}^+$. A $\mathcal{C}$-H_*-coalgebra is simply a coalgebra over this monad, i.e. any $h \in \mathcal{V}_{r+1}^+$ equipped with a coaction map $h \longrightarrow C^{H\mathcal{C}-\mathrm{coalg}}h$ satisfying the standard compatibilities. The homology $H_*^{\mathcal{C}}X$ of $X \in s\mathcal{C}$ is a $\mathcal{C}$-H_*-coalgebra with coaction map

$$\pi_*(Q^{\mathcal{C}}cX) \xrightarrow{\pi_*(Q^{\mathcal{C}}(\beta))} \pi_*(Q^{\mathcal{C}}ccX) \xrightarrow{\pi_*(Q^{\mathcal{C}}c(\eta))} \pi_*(Q^{\mathcal{C}}cK^{\mathcal{C}}Q^{\mathcal{C}}cX) = C^{H\mathcal{C}-\mathrm{coalg}}(\pi_*(Q^{\mathcal{C}}cX)).$$

If $X \sim K^{\mathcal{C}}V$ for some $V \in s\mathcal{V}_r^+$, then $H_*^{\mathcal{C}}X \cong C^{H\mathcal{C}-\mathrm{coalg}}(\pi_*(V))$, and the coaction map of $H_*^{\mathcal{C}}X$ is none other than the diagonal map of the comonad.

The comparison maps of §3.3 give the dual of a $\mathcal{C}$-H_*-algebra of finite type a $\mathcal{C}$-H^*-algebra structure.

Proposition 3.5. *If $V \in \mathcal{V}$ and $X, X' \in H\mathcal{C}{-}\mathrm{coalg}$ are of finite type, then there are natural isomorphisms:*

$$\mathbf{D}C^{H\mathcal{C}-\mathrm{coalg}}V \cong F^{H\mathcal{C}}\mathbf{D}V;$$
$$Q^{H\mathcal{C}}\mathbf{D}X \cong \mathbf{D}\,\mathrm{Pr}^{H\mathcal{C}-\mathrm{coalg}}X;$$
$$\mathbf{D}(X \times X') \cong \mathbf{D}X \sqcup \mathbf{D}X';$$
$$\mathbf{D}(X \barwedge X') \cong \mathbf{D}X \veebar \mathbf{D}X';$$

where the primitives $\mathrm{Pr}^{H\mathcal{C}-\mathrm{coalg}}X$ are defined §3.8, and the smash product $X \barwedge X'$ in §3.9.

3.8. The Hurewicz map, primitives and homology completion

For any $X \in s\mathcal{C}$, there is a map $\pi_* X \longrightarrow H_*^{\mathcal{C}} X$, the *Hurewicz map*, defined as the composite

$$\pi_* X \cong \pi_*(cX) \longrightarrow \pi_*(Q^{\mathcal{C}} cX).$$

Indeed, the Hurewicz map provides a coaugmentation of the comonad $C^{H\mathcal{C}-\mathrm{coalg}}$, the natural transformation $a : \mathrm{id} \longrightarrow C^{H\mathcal{C}-\mathrm{coalg}}$ of endofunctors of $\mathcal{V}_{r+1}^{+}$ defined by

$$V \cong \pi_*(cK^{\mathcal{C}}\Gamma V) \longrightarrow \pi_*(Q^{\mathcal{C}} cK^{\mathcal{C}}\Gamma V) = F^{\pi\mathcal{C}} V.$$

One reading of this observation is:

Lemma 3.6. *If $X \in s\mathcal{C}$ is in the image of $K^{\mathcal{C}}$, then $Q^{\mathcal{C}} X = U^{\mathcal{C}} X$, and the Hurewicz map of X is a section of the composite*

$$H_*^{\mathcal{C}} X := \pi_* Q^{\mathcal{C}} cX \xrightarrow{(Q\epsilon)_*} \pi_* Q^{\mathcal{C}} X = \pi_* X.$$

Given that the comonad $C^{H\mathcal{C}-\mathrm{coalg}}$ has a coaugmentation, we may define the *primitives* of a $\mathcal{C}$-H_*-coalgebra H as the equalizer (in $s\mathcal{V}$):

$$\mathrm{Pr}^{H\mathcal{C}-\mathrm{coalg}}(H) \longrightarrow H \underset{\mathrm{coact}}{\overset{a}{\rightrightarrows}} CH.$$

We will briefly defer the proof of:

Proposition 3.7. *The Hurewicz map $\pi_* X \longrightarrow H_*^{\mathcal{C}} X$ factors through $\mathrm{Pr}^{H\mathcal{C}-\mathrm{coalg}}(H_*^{\mathcal{C}} X)$, and if X is GEM, the map $\pi_* X \longrightarrow \mathrm{Pr}^{H\mathcal{C}-\mathrm{coalg}}(H_*^{\mathcal{C}} X)$ is an isomorphism. In particular, for any $V \in \mathcal{V}_{r+1}^{+}$, $\mathrm{Pr}^{H\mathcal{C}-\mathrm{coalg}}(C^{H\mathcal{C}-\mathrm{coalg}} V) \cong V$.*

Radulescu-Banu [49] has constructed a cosimplicial resolution $\mathcal{X}^\bullet$ of an object $X \in s\mathcal{C}$ by GEMs, and defined the *homology completion of* X to be the totalization $X^{\hat{}} := \mathrm{Tot}(\mathcal{X}^\bullet)$.

This construction is the analogue of Bousfield and Kan's R-completion functor on simplicial sets [12], a construction that has proven extremely useful in classical homotopy theory. There is an additional difficulty, however, in constructing the cosimplicial resolution $\mathcal{X}^\bullet$, which is not present in the classical context: since not all simplicial algebras are cofibrant, the naïve cosimplicial resolution (with the coaugmentation drawn dashed)

$$X \dashrightarrow K^{\mathcal{C}} Q^{\mathcal{C}} X \rightrightarrows (K^{\mathcal{C}} Q^{\mathcal{C}})^2 X \;\substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow}\; (K^{\mathcal{C}} Q^{\mathcal{C}})^3 X \;\cdots.$$

fails to be homotopically correct, and as $Q^{\mathcal{C}} K^{\mathcal{C}} = \mathrm{id}$, fails to hold any interest whatsoever.

Radulescu-Banu's innovation was to explain that the cofibrant replacement functor $c : s\mathcal{C} \longrightarrow s\mathcal{C}$ constructed by Quillen's small object argument [48] admits a comonad diagonal $\beta : c \longrightarrow cc$ (already used in §3.7) and can thus be mixed into the cosimplicial resolution, making it homotopically correct.

In detail, the diagonal is needed in order to define the coface maps in Radulescu-Banu's resolution, the coaugmented cosimplicial object

$$\mathcal{X}^\bullet : \qquad cX \dashrightarrow cK^{\mathcal{C}}Q^{\mathcal{C}}cX \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; c(K^{\mathcal{C}}Q^{\mathcal{C}}c)^2X \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; c(K^{\mathcal{C}}Q^{\mathcal{C}}c)^3X \;\cdots .$$

Instead of simply using the unit and counit of the adjunction respectively, one uses the composites discussed in §3.7:

$$c \xrightarrow{\beta} cc \xrightarrow{c\eta c} cK^{\mathcal{C}}Q^{\mathcal{C}}c \quad \text{and} \quad Q^{\mathcal{C}}cK^{\mathcal{C}} \xrightarrow{Q\epsilon K} Q^{\mathcal{C}}K^{\mathcal{C}} \longrightarrow \mathrm{id}.$$

By an application of Dold's Theorem (2.4), if $X \longrightarrow Y$ is a weak equivalence, so is $\mathcal{X}^s \longrightarrow \mathcal{Y}^s$ for each s. Both $\mathcal{X}$ and $\mathcal{Y}$, being group-like, are automatically Reedy fibrant (cf. [10, X.4.9]), so that the map of completions $X\hat{} \longrightarrow Y\hat{}$ is a weak equivalence. This construction is explained and generalized by Blumberg and Riehl [4, §4].

Comments in [4, §4] show that the coaugmented cosimplicial $\mathcal{C}$-H_*-coalgebra $H_*^{\mathcal{C}}\mathcal{X}^\bullet$ is weakly equivalent to its coaugmentation $H_*^{\mathcal{C}}X$ *as a vector space* (c.f. §4.1), which starts to explain the title *homology completion*. One says that X is *homology complete* when the map $cX \longrightarrow X\hat{} := \mathrm{Tot}(\mathcal{X}^\bullet)$ is an equivalence.

In Theorem 4.4 we specialize to the case when $\mathcal{C}$ is either the category $\mathscr{C}om$ of ungraded non-unital commutative algebras the category $\mathscr{L}ie^r$ of ungraded restricted Lie algebras, and prove that the completion $X\hat{}$ is weakly equivalent to X when X is connected. Analogous results for topological Quillen homology may be found in [18].

A question analogous to questions studied in [18] and [**2010arXiv1001.1556H**] is whether the homotopy category of connected objects of $s\mathcal{C}$ is equivalent to the homotopy category of cosimplicial $Q^{\mathcal{C}}cK^{\mathcal{C}}$-coalgebras. We have not investigated this question.

Proof of Proposition 3.7. The maps $d^0, d^1 : \mathcal{X}^0 \longrightarrow \mathcal{X}^1$ induce respectively the coaugmentation and coaction maps for $\pi_*\mathcal{X}^0 = H_*^{\mathcal{C}}X$ on homotopy, while $d^0 : \mathcal{X}^{-1} \longrightarrow \mathcal{X}^0$ induces the Hurewicz map. The very existence of this diagram then shows that the Hurewicz map factors through the primitives. The observation that this cosimplicial object has extra codegeneracies when $X = K^{\mathcal{C}}V$ (c.f. [4, §4]) completes the proof. □

3.9. The smash product of homology coalgebras

For $X_1, X_2 \in H\mathcal{C}-$coalg connected homology coalgebras, we define the *smash product* $X_1 \bar{\wedge} X_2$ to be the cokernel of the natural map $X_1 \sqcup X_2 \longrightarrow X_1 \times X_2$.

The theory changes a little in form after passing from homotopy to homology, and in order to obtain a result analogous to Proposition 3.2, we must introduce the *left derived smash product* in $\mathcal{C}$. For A_1 and A_2 in $s\mathcal{C}$, the natural map $A_1 \sqcup A_2 \longrightarrow A_1 \times A_2$ is a surjection, and so in general very far from a cofibration. We define the *left derived smash product* $A_1 \bar{\wedge}^{\mathrm{L}} A_2$ to be the homotopy cofiber of this map. In light of Proposition 3.4, there are natural maps

$$c(A_1 \sqcup A_2) \longrightarrow c(A_1 \times A_2) \longrightarrow A_1 \bar{\wedge}^{\mathrm{L}} A_2,$$

and this *cofiber sequence* induces a homology long exact sequence (c.f. [33, Proposition 4.6]).

The following result and its proof are dual to Proposition 3.2 and its proof.

Proposition 3.8. *For X_1 and X_2 in $s\mathcal{C}$, the natural $\mathcal{C}$-H_*-coalgebra map*

$$H_*^{\mathcal{C}}(X_1 \sqcup X_2) \longleftarrow H_*^{\mathcal{C}} X_1 \sqcup H_*^{\mathcal{C}} X_2$$

is an isomorphism. If X_1 and X_2 are GEMs in $s\mathcal{C}$, the natural $\mathcal{C}$-H_*-coalgebra map

$$H_*^{\mathcal{C}} X_1 \times H_*^{\mathcal{C}} X_2 \longleftarrow H_*^{\mathcal{C}}(X_1 \times X_2)$$

takes part in an isomorphism of short exact sequences:

$$\begin{tikzcd}
0 & H_*^{\mathcal{C}} X_1 \bar{\wedge} H_*^{\mathcal{C}} X_2 \arrow[l] & H_*^{\mathcal{C}} X_1 \times H_*^{\mathcal{C}} X_2 \arrow[l] & H_*^{\mathcal{C}} X_1 \sqcup H_*^{\mathcal{C}} X_2 \arrow[l] & 0 \arrow[l] \\
0 & H_*^{\mathcal{C}}(X_1 \bar{\wedge}^{\mathrm{L}} X_2) \arrow[l] \arrow[u, "\cong"] & H_*^{\mathcal{C}}(X_1 \times X_2) \arrow[l] \arrow[u, "\cong"] & H_*^{\mathcal{C}}(X_1 \sqcup X_2) \arrow[l] \arrow[u, "\cong"] & 0 \arrow[l]
\end{tikzcd}$$

3.10. The quadratic part of a $\mathcal{C}$-expression

In this thesis, we will often use a method of constructing cohomology operations used by Goerss in [33, §5], and here we will set up a framework that can be applied to each case. We continue to suppose that $\mathcal{C}$ is an algebraic category, monadic over $\mathcal{V}$, a category of graded vector spaces, satisfying the assumptions of §2.1.

For $V \in \mathcal{V}$, the diagonal map $\Delta : V \longrightarrow V \oplus V$ of V induces a *diagonal map* $F^{\mathcal{C}} V \longrightarrow F^{\mathcal{C}}(V \oplus V) \cong (F^{\mathcal{C}}(V))^{\sqcup 2}$, and writing i_1 and i_2 for the two summand inclusions $F^{\mathcal{C}}(V) \longrightarrow (F^{\mathcal{C}}(V))^{\sqcup 2}$, consider the map

$$(F^{\mathcal{C}}(\Delta) + i_1 + i_2) : F^{\mathcal{C}} V \longrightarrow (F^{\mathcal{C}} V)^{\sqcup 2}.$$

This map factors through $(F^{\mathcal{C}}V)^{\veebar 2}$, and is symmetric. We name this factoring the *cross terms*:

$$\mathrm{cr} : F^{\mathcal{C}}V \longrightarrow (F^{\mathcal{C}}V) \veebar^{\Sigma_2} (F^{\mathcal{C}}V),$$

as it measures the non-linearity in an expression in $F^{\mathcal{C}}V$. We will give an example in each of the categories $\mathscr{C}om$, $\mathscr{L}ie$ and $\mathscr{L}ie^r$, in each case using subscripts to denote membership of the first or second copy of V:

$$\begin{aligned}
\mathscr{C}om: &\qquad \mathrm{cr}(vw) = (v_1+v_2)(w_1+w_2) + v_1w_1 + v_2w_2 = v_1w_2 + w_1v_2;\\
\mathscr{L}ie: &\qquad \mathrm{cr}([v,w]) = [v_1+v_2, w_1+w_2] + [v_1,w_1] + [v_2,w_2] = [v_1,w_2] + [w_1,v_2];\\
\mathscr{L}ie^r: &\qquad \mathrm{cr}(v^{[2]}) = v_1^{[2]} + v_2^{[2]} + (v_1+v_2)^{[2]} = [v_1,v_2].
\end{aligned}$$

For certain categories of interest to us we will define a *decomposition map*, natural and symmetric in $X_1, X_2 \in \mathcal{C}$:

$$j_{\mathcal{C}} : Q^{\mathcal{C}}(X_1 \veebar X_2) \longrightarrow Q^{\mathcal{C}}(X_1) \otimes Q^{\mathcal{C}}(X_2).$$

When $\mathcal{C} = \mathscr{C}om$, $X_1 \veebar X_2 \cong X_1 \otimes X_2$ and $Q(X_1 \veebar X_2) \cong QX_1 \otimes QX_2$, and we choose the identity map of this object as decomposition map $j_{\mathscr{C}om}$. In other words, the map $j_{\mathscr{C}om}$ is defined by $x_1x_2 \longmapsto x_1 \otimes x_2$ whenever $x_1 \in X_1$ and $x_2 \in X_2$.

When $\mathcal{C} = \mathscr{L}ie$ or $\mathcal{C} = \mathscr{L}ie^r$, we define the decomposition map by

$$j_{\mathcal{L}(n)} : [x_1, \cdots, x_a]^{[2^r]} \longmapsto \begin{cases} x_1 \otimes x_2, & \text{if } r = 0,\ a = 2,\ z_1 \in X_1,\ z_2 \in X_2,\\ 0, & \text{otherwise,}\end{cases}$$

where by $[x_1, \cdots, x_a]^{[2^r]}$ we mean the r-fold restriction ($r = 0$ when $\mathcal{C} = \mathscr{L}ie$) of some bracketing of various $x_1, \ldots, x_a$ from X_1 and X_2, with at least one z_k must lie in each of X_1 and X_2. Any element of the smash coproduct may be written as a sum of such expressions, so there is at most one map $j_{\mathscr{C}om}$ satisfying this equation. That this map is well defined is less obvious, but nonetheless routine.

Finally, we define the *quadratic part* map $\mathrm{qu}_{\mathcal{C}}$ to be the composite

$$\mathrm{qu}_{\mathcal{C}} : \left(F^{\mathcal{C}}V \xrightarrow{\mathrm{cr}} (F^{\mathcal{C}}V)^{\veebar 2} \longrightarrow Q^{\mathcal{C}}((F^{\mathcal{C}}V) \veebar^{\Sigma_2} (F^{\mathcal{C}}V)) \xrightarrow{j_{\mathcal{C}}} S^2(Q^{\mathcal{C}}F^{\mathcal{C}}V) = S^2V\right).$$

Lemma 3.9. *Suppose $V \in \mathcal{V}$. Then:*

(1) $\mathrm{qu}_{\mathscr{C}om}$ *is the composite* $F^{\mathscr{C}om}V \twoheadrightarrow S_2V \xrightarrow{\mathrm{tr}} S^2V$*;*

(2) $\mathrm{qu}_{\mathscr{L}ie}$ *is the composite* $F^{\mathscr{L}ie}V \twoheadrightarrow \Lambda^2V \xrightarrow{\mathrm{tr}} S^2V$*;*

(3) $\mathrm{qu}_{\mathscr{L}ie^r}$ *is the projection* $F^{\mathscr{L}ie^r}V \twoheadrightarrow S^2V$.

Proof. These are simple observations, and an example is more useful than a proof: consider the expression $e := u + vw + xy^2 \in F^{\mathscr{C}om}V$ where u, v, w, x, y are in V. Then

$$\begin{aligned} \mathrm{cr}(e) \ &= v_1 w_2 + w_1 v_2 + x_1 y_2^2 + y_1^2 x_2, \text{ and} \\ \mathrm{qu}_{\mathcal{C}}(e) &:= j_{\mathscr{C}om}(\mathrm{cr}(e)) \\ &= v_1 \otimes w_2 + w_1 \otimes v_2 + x_1 \otimes y_2^2 + y_1^2 \otimes x_2 \\ &= v_1 \otimes w_2 + w_1 \otimes v_2 \in S^2(Q^{\mathscr{C}om} F^{\mathscr{C}om} V). \end{aligned}$$

Parts *(2)* and *(3)* are a light modification of a part of Proposition 7.5. $\square$

In each category of interest to us, the following equation of maps $F^{\mathcal{C}} F^{\mathcal{C}} V \longrightarrow S^2 V$ will always be satisfied:

$$\mathrm{qu}_{\mathcal{C}} \circ \mu_V = \mathrm{qu}_{\mathcal{C}} \circ \epsilon_{F^{\mathcal{C}} V} + \mathrm{qu}_{\mathcal{C}} \circ F^{\mathcal{C}} \epsilon_V,$$

where μ and ϵ stand for the multiplication and augmentation of the augmented monad $U^{\mathcal{C}} F^{\mathcal{C}}$. This is another expression of homogeneity in the relations defining $\mathcal{C}$, which states that if $f(g_i)$ is a $\mathcal{C}$-expression in various $\mathcal{C}$-expressions $g_i(v_{ij})$, then

$$\mathrm{qu}(fg_i)(v_{ij}) = \mathrm{qu}(f\epsilon(g_i))(v_{ij}) + \epsilon(f)(\mathrm{qu}(g_i)(v_{ij})).$$

For an example when $\mathcal{C} = \mathscr{C}om$, we specify an expression $f(g_1, g_2, g_3) := g_1 g_2 + g_3 \in F^{\mathcal{C}} F^{\mathcal{C}} V$ in expressions $g_i := v_{i1} v_{i2} + v_{i3} \in F^{\mathcal{C}} V$ for each $i = 1, 2, 3$. Then

$$\begin{aligned} \mathrm{qu}(fg_i)(v_{ij}) &= \mathrm{qu}((v_{11}v_{12} + v_{13})(v_{21}v_{22} + v_{23}) + (v_{31}v_{32} + v_{33})) = \mathrm{tr}(v_{13} \otimes v_{23} + v_{31} \otimes v_{32}), \\ \mathrm{qu}(f\epsilon(g_i))(v_{ij}) &= \mathrm{qu}((v_{13})(v_{23}) + (v_{33})) = \mathrm{tr}(v_{13} \otimes v_{23}), \text{ and} \\ \epsilon(f)(\mathrm{qu}(g_i)(v_{ij})) &= \mathrm{qu}(v_{31}v_{32} + v_{33}) = \mathrm{tr}(v_{31} \otimes v_{32}). \end{aligned}$$

Chapter 4

The Bousfield-Kan spectral sequence

In this chapter, we will write $\mathcal{C}$ for any category of universal graded $\mathbb{F}_2$-algebras satisfying the standing assumptions of §2.1. The *Bousfield-Kan spectral sequence of* $X \in s\mathcal{C}$ is the second quadrant homotopy spectral sequence (c.f. §2.13) of Radulescu-Banu's resolution $\mathfrak{X} \in cs\mathcal{C}$ of X recalled in §3.8. The key objective of this thesis is to understand this spectral sequence when $\mathcal{C} = \mathscr{C}om$.

Our first step, in §4.1, is to identify the E_2-page as appropriate derived functors. Before we turn to the calculation of these derived functors in later chapters, we consider the convergence target, $\operatorname{Tot}\mathfrak{X} =: X^{\wedge}$. From §4.2 to the end of this section, we will give a proof of Theorem 4.4 — that the completion $X^{\wedge}$ is weakly equivalent to X when $\mathcal{C}$ is either $\mathscr{C}om$ or $\mathscr{L}ie^r$ and X is connected.

Although Theorem 4.4 alone does not fully resolve the question of the convergence of the BKSS, we will prove in §15.2 that if $X \in s\mathscr{C}om$ is a connected object with $H^*_{\mathscr{C}om}$ of finite type, the spectral sequence supports a vanishing line at E_2, so that there are no convergence problems whatsoever when $X \in s\mathscr{C}om$ is connected.

4.1. Identification of E_1 and E_2

In light of §3.6 and §3.7, applying the functor $H_*^{\mathcal{C}}$ to $\mathfrak{X}$ yields the monadic cobar resolution of $H_*^{\mathcal{C}}X$ in the category $H\mathcal{C}-$coalg, obtained by repeated application of the monad on $H\mathcal{C}-$coalg of the adjunction

$$U^{H\mathcal{C}-\text{coalg}} : H\mathcal{C}-\text{coalg} \rightleftarrows \mathcal{V}_1 : C^{H\mathcal{C}-\text{coalg}}.$$

In more detail, we have a map of coaugmented cosimplicial objects

$$\begin{tikzcd}
cX \arrow[r, dashed] \arrow[d] & cK^{\mathcal{C}}Q^{\mathcal{C}}cX \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & c(K^{\mathcal{C}}Qc^{\mathcal{C}})^2X \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & \cdots \\
Q^{\mathcal{C}}cX \arrow[r, dashed] & Q^{\mathcal{C}}cK^{\mathcal{C}}Q^{\mathcal{C}}cX \arrow[r, shift left] \arrow[r, shift right] & Q^{\mathcal{C}}c(K^{\mathcal{C}}Q^{\mathcal{C}}c)^2X \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots
\end{tikzcd}$$

and if we abbreviate the monad $C^{H\mathcal{C}-\mathrm{coalg}}U^{H\mathcal{C}-\mathrm{coalg}}$ on $H\mathcal{C}-$coalg to $\overline{C}$, applying π_* to this diagram we obtain a cosimplicial *Hurewicz map*:

$$\begin{tikzcd}
\pi_*\mathcal{X}^\bullet : & \pi_*X \arrow[r, dashed] \arrow[d] & \pi_*\mathcal{X}^0 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & \pi_*\mathcal{X}^1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\cong"] & \cdots \\
\mathrm{Pr}(H_*^{\mathcal{C}}X^\bullet) : & \mathrm{Pr}(H_*^{\mathcal{C}}X) \arrow[r, dashed] \arrow[d, tail] & \mathrm{Pr}(\overline{C}H_*^{\mathcal{C}}X) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, tail] & \mathrm{Pr}(\overline{C}^2H_*^{\mathcal{C}}X) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, tail] & \cdots \\
H_*^{\mathcal{C}}X^\bullet : & H_*^{\mathcal{C}}X \arrow[r, dashed] & \overline{C}H_*^{\mathcal{C}}X \arrow[r, shift left] \arrow[r, shift right] & \overline{C}^2H_*^{\mathcal{C}}X \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots
\end{tikzcd}$$

The indicated maps are isomorphisms since each $\mathcal{X}^s$ for $s \geq 0$ is a GEM, thanks to Proposition 3.7. In particular, we see that:

$$[E_1\mathcal{X}]^s_t \cong (\mathrm{Pr}^{H\mathcal{C}-\mathrm{coalg}}(\overline{C}^{s+1}H_*^{\mathcal{C}}X))_t;$$
$$[E_2\mathcal{X}]^s_t \cong ((\mathbb{R}^s\,\mathrm{Pr}^{H\mathcal{C}-\mathrm{coalg}})H_*^{\mathcal{C}}X)_t.$$

Corollaries 6.9 and 6.17 and Proposition 3.5 show that

Theorem 4.1. *If $\mathcal{C}$ is either $\mathscr{C}om$ or $\mathscr{L}ie^r$, and X is connected with $H^*_{\mathcal{C}}X$ of finite type, then $H^*_{\mathcal{C}}\mathcal{X}^s$ is of finite type for each s, and:*

$$[E_1\mathcal{X}]^s_t \cong (C^*\mathbf{D}Q^{H\mathcal{C}}B^{H\mathcal{C}}H^*_{\mathcal{C}}X)^s_t;$$
$$[E_2\mathcal{X}]^s_t \cong (H^*_{H\mathcal{C}}H^*_{\mathcal{C}}X)^s_t.$$

4.2. The Adams tower

Bousfield and Kan defined the *Bousfield-Kan spectral sequence*, or *unstable Adams spectral sequence*, of a simplicial set in two different ways. Their earlier approach [9] was to define the *derivation of a functor with respect to a ring.* This approach constructs the *Adams tower* over the simplicial set in question, and lends itself well to connectivity analyzes. Their latter approach, [12], to give a cosimplicial resolution of a simplicial set by simplicial R-modules, lends itself more to the analysis of the E_2-page, and is directly analogous to Radulescu-Banu's construction described in §3.8.

Since the release of [9] and [12], the relationship between the two approaches has been clarified by the introduction of cubical homotopy theory [35]. In this section we will define the Adams tower of a simplicial algebra using a construction analogous to Bousfield and Kan in [9], and then to relate it to Radulescu-Banu's construction using the theory of cubical diagrams.

For brevity, write $K := K^{\mathcal{C}}$ and $Q := Q^{\mathcal{C}}$. For any functor $F : s\mathcal{C} \longrightarrow s\mathcal{C}$, we define the r^{th} *derivation R_rF of F with respect to homology* as follows. The definition is recursive, and again involves repeated application of the cofibrant replacement functor c:

$$\begin{aligned}
(R_0F)(X) &:= F(cX),\\
(R_sF)(X) &:= \operatorname{hofib}\big((R_{s-1}F)(cX) \xrightarrow{(R_{s-1}F)(\eta_{cX})} (R_{s-1}F)(KQcX)\big),
\end{aligned}$$

where η is the unit of the adjunction $Q \dashv K$, i.e. the natural surjection onto indecomposables, and hofib is any fixed functorial construction of the homotopy fiber. These functors fit into a tower via the following composite natural transformations:

$$\delta : \left((R_sF)(X) \longrightarrow (R_{s-1}F)(cX) \overset{(R_{s-1}F)(\epsilon)}{\longrightarrow} (R_{s-1}F)(X)\right).$$

We have thus constructed a tower

$$\cdots \longrightarrow (R_2F)X \longrightarrow (R_1F)X \longrightarrow (R_0F)X = FcX,$$

which is natural in the object X and the functor F. The functors R_rF are homotopical as long as F preserves weak equivalences between cofibrant objects. Employing the shorthand

$$R_sX := (R_s\mathrm{id})X,$$

we define *the Adams tower of X* to be the tower

$$\cdots \longrightarrow R_2X \longrightarrow R_1X \longrightarrow R_0X = cX.$$

For example, $(R_2F)(X)$ is constructed by the following diagram in which every composable pair of parallel arrows is *defined* to be a homotopy fiber sequence.

$$\begin{tikzcd}
(R_2F)(X) \arrow[d] & & \\
(R_1F)(cX) \arrow[r] \arrow[d] & FcccX \arrow[r] \arrow[d] & FcKQccX \arrow[d] \\
(R_1F)(KQcX) \arrow[r] & FccKQcX \arrow[r] & FcKQcKQcX
\end{tikzcd}$$

In general, $(R_{n+1}F)(X)$ is the homotopy total fiber of an $(n+1)$-cubical diagram:

$$(R_{n+1}F)(X) := \mathrm{hototfib}\big((R^{\square}_{n+1}F)X\big).$$

See [35], [45] or [43] for the general theory of cubical diagrams. Before defining the cubical diagram $(R^{\square}_{n+1}F)X$, we set notation: for $n \geq 0$ let $[n] = \{0, \ldots, n\}$, and define $\mathcal{P}[n] = \{S \subseteq [n]\}$ to be the poset category whose morphisms are the inclusions $S \subseteq S'$. Then an $(n+1)$-cube in $s\mathcal{C}$ is a functor $\mathcal{P}[n] \longrightarrow s\mathcal{C}$, and the $(n+1)$-cubical diagram $(R^{\square}_{n+1}F)X : \mathcal{P}[n] \longrightarrow s\mathcal{C}$ is the functor defined on objects by:

$$S \longmapsto Fc(KQ)^{\chi_n}c(KQ)^{\chi_{n-1}}c\cdots c(KQ)^{\chi_0}cX \quad \text{where} \quad \chi_i := \begin{cases} 1, & \text{if } i \in S; \\ 0, & \text{if } i \notin S, \end{cases}$$

such that for $S \subseteq S'$, the map $((R^{\square}_{n+1}F)X)(S) \longrightarrow ((R^{\square}_{n+1}F)X)(S')$ is given by applying the counit $\eta : 1 \longrightarrow KQ$ in those locations indexed by $S' \setminus S$.

Radulescu-Banu defines the homology completion of X to be the totalization

$$X\hat{} := \mathrm{Tot}(\mathfrak{X}^\bullet) = \mathrm{holim}(\mathrm{Tot}_n(\mathfrak{X}^\bullet)),$$

and the BKSS to be the spectral sequence of the Tot tower

$$\cdots \longrightarrow \mathrm{Tot}_n(\mathfrak{X}^\bullet) \longrightarrow \mathrm{Tot}_{n-1}(\mathfrak{X}^\bullet) \longrightarrow \cdots$$

under cX. Our goal in this section is to prove

Proposition 4.2. *There is a natural zig-zag of weak equivalences of towers*

$$\{R_{n+1}X\}_n \simeq \{\mathrm{hofib}(cX \longrightarrow \mathrm{Tot}_n(\mathfrak{X}^\bullet))\}_n\,.$$

That is, the Tot *tower induces the Adams tower by taking homotopy fibers, and thus the spectral sequence of the* Tot *tower coincides with the spectral sequence of the Adams tower.*

As $\mathfrak{X}^{-1}$ equals cX, the tower $\mathrm{hofib}(\mathfrak{X}^{-1} \longrightarrow \mathrm{Tot}_n\mathfrak{X}^\bullet)$ appearing in §2.13 is one of the towers in Proposition 4.2. This proposition explains the relevance of the Adams tower to the cosimplicial resolution, and thus its relevance to the BKSS which was defined as the spectral sequence of this cosimplicial object.

Proof of Proposition 4.2. By the discussion of the Tot tower in §2.13, it will suffice to construct a weak equivalence $h_n^*\mathfrak{X}^\bullet \longrightarrow (R^{\square}_n\mathrm{id})(X)$ of $(n+1)$-cubes. The $(n+1)$-cubical diagram

$h_n^*\mathfrak{X}^\bullet$ is defined on objects by

$$(h_n^*\mathfrak{X}^\bullet)(S) := c(KQc)^{\chi_n}(KQc)^{\chi_{n-1}}\cdots(KQc)^{\chi_0}X \quad \text{where} \quad \chi_i := \begin{cases} 1, & \text{if } i \in S, \\ 0, & \text{if } i \notin S, \end{cases}$$

and the map $(h_n^*\mathfrak{X}^\bullet)(S) \longrightarrow (h_n^*\mathfrak{X}^\bullet)(S \sqcup \{i\})$, for $i \notin S$, may be described as follows. Let j be the smallest element of $S \sqcup \{n+1\}$ exceeding i, so that

$$(h_n^*\mathfrak{X}^\bullet)(S) := \begin{cases} c(KQc)^{\chi_n}\cdots(KQc)^{\chi_{j+1}}(KQ\underline{c})(KQc)^{\chi_{i-1}}\cdots(KQc)^{\chi_0}X, & \text{if } j \le n; \\ \underline{c}(KQc)^{\chi_{i-1}}\cdots(KQc)^{\chi_0}X, & \text{if } j = n+1. \end{cases}$$

In the expression for either case, we have distinguished one of the applications of c with an underline, and the map to $(h_n^*\mathfrak{X}^\bullet)(S \sqcup \{i\})$ is induced by the composite $\underline{c} \longrightarrow cc \longrightarrow cKQc$ of the diagonal of the comonad c with the unit of the monad KQ.

We now define maps $(h_n^*\mathfrak{X}^\bullet)(S) \longrightarrow ((R_{n+1}^{\square}\mathrm{id})X)(S)$ for $S = \{j_0 < j_1 < \cdots < j_r\} \subseteq \{0, \ldots, n\}$. The only difference between the domain and codomain is that in $((R_{n+1}^{\square}\mathrm{id})X)(S)$, all $n+2$ applications of c are present, whereas in $(h_n^*\mathfrak{X}^\bullet)(S)$, only $r+2$ appear. The required map is then

$$\beta^{n-j_r}KQ\beta^{j_r-j_{r-1}-1}KQ\beta^{j_{r-1}-j_{r-2}-1}KQ\cdots KQ\beta^{j_1-j_0-1}KQ\beta^{j_0}X,$$

which is to say that we apply the iterated diagonal the appropriate number of times in each c appearing in the domain. As β is coassociative, this definition is unambiguous, and the resulting maps assemble to a weak equivalence of $(n+1)$-cubes. $\square$

4.3. Connectivity estimates and homology completion

In this section we will make the following connectivity estimates in the Adams tower:

Proposition 4.3. *Suppose that $\mathcal{C}$ is one of the categories $\mathscr{C}om$ or $\mathscr{L}ie^r$, that $X \in s\mathcal{C}$ is connected, and that $t \ge 1$ and $q \ge 2$. Then there is some $f(q,t) \ge t$ such that the map $\pi_q(R_{f(q,t)}X) \longrightarrow \pi_q(R_tX)$ is zero.*

Propositions 4.2 and 4.3 together imply the following conjecture of Radulescu-Banu:

Theorem 4.4. *If either $\mathcal{C} = \mathscr{C}om$ or $\mathcal{C} = \mathscr{L}ie^r$ and $X \in s\mathcal{C}$ is connected, then X is naturally equivalent to its homology completion $X^\wedge$.*

Proof of Theorem 4.4. The fiber sequences $R_{n+1}X \longrightarrow cX \longrightarrow \mathrm{Tot}_n\mathfrak{X}^\bullet$ fit together into a tower of fiber sequences. Taking homotopy limits, one obtains a natural fiber sequence

$$\operatorname{holim}(R_n X) \longrightarrow cX \longrightarrow X\hat{}.$$

We need to show that $\operatorname{holim}(R_n X)$ has zero homotopy groups. Applying [34, Proposition 6.14], there is a short exact sequence

$$0 \longrightarrow \lim{}^1 \pi_{q+1}(R_n X) \longrightarrow \pi_q(\operatorname{holim}(R_n X)) \longrightarrow \lim \pi_q(R_n X) \longrightarrow 0.$$

Proposition 4.3 implies that for each q, the tower $\{\pi_q(R_n X)\}_n$ has zero inverse limit and satisfies the Mittag-Leffler condition (c.f. [10, p. 264]), so that the $\lim^1$ groups appearing also vanish. □

The application of the small object argument functor c adds to the difficulty of proving the connectivity estimates of Proposition 4.3. We circumvent the difficulty of working with c by shifting to the standard bar construction $B^{\mathcal{C}}$ on $s\mathcal{C}$, which we abbreviation to b.

We define recursively a somewhat less homotopical version $\mathcal{R}_s F$ of the derivations $R_s F$:

$$\begin{aligned}
(\mathcal{R}_0 F)(X) &:= F(X), \\
(\mathcal{R}_s F)(X) &:= \ker\left((\mathcal{R}_{s-1}F)(bX) \xrightarrow{(\mathcal{R}_{s-1}F)(\eta_{bX})} (\mathcal{R}_{s-1}F)(KQbX)\right).
\end{aligned}$$

There are three differences between this definition and that of $R_s F$: here, there is one fewer cofibrant replacement applied, we use b instead of c, and we take *strict* fibers, not homotopy fibers. While these functors are not generally homotopical, we define *the modified Adams tower of X* to be the tower

$$\cdots \xrightarrow{\delta} \mathcal{R}_2 X \xrightarrow{\delta} \mathcal{R}_1 X \xrightarrow{\delta} \mathcal{R}_0 X = X,$$

where $\mathcal{R}_s X$ is again shorthand for $(\mathcal{R}_s \mathrm{id})X$, and the tower maps δ are defined as before.

Proposition 4.5. *There is a natural zig-zag of weak equivalences of towers between the Adams tower of X and the modified Adams tower of X. In particular, the modified Adams tower is homotopical.*

Proof. Let $\mathsf{CR}(\mathsf{s}\mathcal{C})$ be the category of cofibrant replacement functors in $s\mathcal{C}$. That is, an object of $\mathsf{CR}(\mathsf{s}\mathcal{C})$ is a pair, (f, ϵ), such that $f : s\mathcal{C} \longrightarrow s\mathcal{C}$ is a functor whose image consists only of cofibrant objects, and $\epsilon : f \Rightarrow \mathrm{id}$ is a natural acyclic fibration. Morphisms in $\mathsf{CR}(\mathsf{s}\mathcal{C})$ are natural transformations which commute with the augmentations. For any $(f, \epsilon) \in \mathsf{CR}(\mathsf{s}\mathcal{C})$

we obtain an alternative definition of the derivations of a functor $F : s\mathcal{C} \longrightarrow s\mathcal{C}$:

$$(R_0^f F)(X) := F(fX), \quad (R_s^f F)(X) := \mathrm{hofib}((R_{s-1}^f F)(fX) \longrightarrow (R_{s-1}^f F)(KQfX)).$$

These functors are natural in f, so that a morphism in $\mathsf{CR}(\mathsf{s}\mathcal{C})$ induces a weak equivalence of towers. Our proposed zig-zag of towers is:

$$R_s = R_s^c \mathrm{id} \longleftarrow R_s^{b\circ c}\mathrm{id} \longrightarrow R_s^b \mathrm{id} \xleftarrow{\gamma_s} \mathcal{R}_s b \xrightarrow{\mathcal{R}_s\epsilon} \mathcal{R}_s \mathrm{id} = \mathcal{R}_s$$

The maps with domain $R_s^{b\circ c}\mathrm{id}$ are induced by the maps $\epsilon c : b \circ c \longrightarrow c$ and $b\epsilon : b \circ c \longrightarrow b$ and are evidently natural weak equivalences of towers. The map $\gamma_0 : (\mathcal{R}_0 b)X \longrightarrow (R_0^b \mathrm{id})X$ is the identity of bX, and the map $\mathcal{R}_0\epsilon : (\mathcal{R}_0 b)X \longrightarrow (R_0^b\mathrm{id})X$ is $\epsilon : bX \longrightarrow X$. Thereafter, γ_s and $\mathcal{R}_s\epsilon$ are defined recursively:

$$\begin{array}{rl}
(\mathcal{R}_{s+1}\mathrm{id})X := & \ker\left((\mathcal{R}_s\mathrm{id})(bX) \longrightarrow (\mathcal{R}_s\mathrm{id})(KQbX)\right) \\
& \uparrow \text{induced by } (\mathcal{R}_s\epsilon, \mathcal{R}_s\epsilon) \\
(\mathcal{R}_{s+1}b)X := & \ker\left((\mathcal{R}_s b)(bX) \longrightarrow (\mathcal{R}_s b)(KQbX)\right) \\
& \downarrow \text{incl.} \\
& \mathrm{hofib}((\mathcal{R}_s b)(bX) \longrightarrow (\mathcal{R}_s b)(KQbX)) \\
& \downarrow \text{induced by } (\gamma_s, \gamma_s) \\
(R_{s+1}^b)X := & \mathrm{hofib}((R_s^b\mathrm{id})(bX) \longrightarrow (R_s^b\mathrm{id})(KQbX))
\end{array}$$

Lemma 4.6 shows that the kernels taken are actually kernels of surjective maps, and by induction on s, the maps γ_s and $\mathcal{R}_s\epsilon$ are weak equivalences. □

The connectivity result will rely on the observation that any element in the s^{th} level of the modified tower maps down to an $(s+1)$-fold expression in X. In order to formalize this, when $\mathcal{C} = \mathscr{C}om$, we let $P^s : s\mathcal{C} \longrightarrow s\mathcal{C}$ be the "s^{th} power" functor, the prolongation of the endofunctor $Y \longmapsto Y^s$ of $\mathcal{C}$, where $Y^s = \mathrm{im}(\mathrm{mult} : Y^{\otimes s} \longrightarrow Y)$. When $\mathcal{C} = \mathscr{C}om$, we define $P^s := \Gamma^s$, the s^{th} term in the lower central series filtration (c.f. [13]). Then we have:

Lemma 4.6. *Suppose that either $\mathcal{C} = \mathscr{C}om$ or $\mathcal{C} = \mathscr{L}ie^r$. The functors $\mathcal{R}_r$, $\mathcal{R}_r b$ and $\mathcal{R}_r P^s$ preserve surjective maps and there is a commuting diagram of functors:*

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{R}_r \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{R}_2 \arrow[r] \arrow[d] & \mathcal{R}_1 \arrow[r] \arrow[d] & \mathcal{R}_0 \arrow[d, equal] \\
\cdots \arrow[r, tail] & P^{r+1} \arrow[r, tail] & \cdots \arrow[r, tail] & P^3 \arrow[r, tail] & P^2 \arrow[r, tail] & \mathrm{id}
\end{tikzcd}$$

Proof. As b and P^s preserve surjections, we need only check the claims about $\mathcal{R}_r X$ for

$X \in s\mathcal{C}$, which is constructed as the subobject

$$\mathcal{R}_r X := \bigcap_{i=1}^{r} \ker\left(b^{r-i}\eta b^i : b^r X \longrightarrow b^{r-i}KQb^iX\right)$$

of b^rX. For the rest of this proof only we write $\bar{F}$ as shorthand for $F^{\mathcal{C}}$. In dimension n, this is the following subset of $(b^rX)_n := (\bar{F}^{n+1})^r X_n$:

$$(\mathcal{R}_r X)_n := \bigcap_{i=1}^{r} \ker\left(\bar{F}^{(r-i)(n+1)}\eta\bar{F}^{i(n+1)} : (\bar{F}^{n+1})^r X_n \longrightarrow (\bar{F}^{n+1})^{r-i}KQ(\bar{F}^{n+1})^i X_n\right).$$

Whichever of $\mathscr{C}om$ or $\mathscr{L}ie^r$ we are working with, it is possible to construct monomial bases for $\bar{F}V$ once a basis of V has been chosen. For given n and r, first choose a basis of X_n; build from it a monomial basis of $\bar{F}X_n$; build from this a monomial basis of $\bar{F}^2X_n$; etc. Continue until we have a monomial basis of $\bar{F}^{r(n+1)}X_n = (b^rX)_n$. The effect of the map $\bar{F}^{(r-i)(n+1)}\eta\bar{F}^{i(n+1)}$ on monomials is either to annihilate them or leave them unchanged, depending on whether any non-trivial constructions were employed at the $((n+1)i)^{\text{th}}$ stage. Thus, the subset $(\mathcal{R}_rX)_n$ has basis those iterated monomials in which some non-trivial construction was used in the $((n+1)i)^{\text{th}}$ for $1 \le i \le r$. The image of such a monomial in X_n lies in P^r.

To see that $\mathcal{R}_r$ preserves surjections: if $f : X \longrightarrow Y$ is a surjection, choose a basis $B \sqcup B'$ of X_n for which f maps the B bijectively onto a basis of Y_n and B' maps to zero. We may continue this pattern at each stage of the construction of iterated monomial bases of $\bar{F}^{r(n+1)}X_n$ and $\bar{F}^{r(n+1)}Y_n$. That is, we may choose a basis $C \sqcup C'$ of $\bar{F}^{r(n+1)}X_n$ such that the monomials in C only involve the elements of B and map under f bijectively onto a basis of $\bar{F}^{r(n+1)}Y_n$, and such that each monomial in C' involves some element of B', and so vanishes under f. This pattern is further preserved in passing to the monomial bases just derived for $(\mathcal{R}_rX)_n$ and $(\mathcal{R}_rY)_n$, proving the claim that $\mathcal{R}_r$ preserves surjections. □

We are now able to state and prove the key connectivity result in detail:

Lemma 4.7. *Suppose that $X \in s\mathcal{C}$ is connected, $t \ge 1$ and $s \ge 2$. If $\mathcal{C} = \mathscr{C}om$, then $(\mathcal{R}_tP^s)(X)$ is $(s-t)$-connected. If $\mathcal{C} = \mathscr{L}ie^r$, then $(\mathcal{R}_tP^s)(X)$ is $(\log_2(s)+1-t)$-connected.*

Proof. We will prove this by induction on t. The induction step is simple: by Lemma 4.6, there is a short exact sequence:

$$0 \longrightarrow (\mathcal{R}_tP^s)(X) \longrightarrow (\mathcal{R}_{t-1}P^s)(bX) \longrightarrow (\mathcal{R}_{t-1}P^s)(KQbX) \longrightarrow 0.$$

Now both bX and $KQbX$ are connected, as they have $\pi_0(bX) = \pi_*X$ is zero by assumption, and $\pi_0(KQbX) = Q\pi_*X$. By induction we can bound the connectivity of $(\mathcal{R}_{t-1}P^s)(bX)$

and $(\mathcal{R}_{t-1}P^s)(KQbX)$, and the associated long exact sequence shows that $(\mathcal{R}_t P^s)(X)$ has a connectivity bound at most one degree lower.

For the base case, $t = 1$, as $P^s(KQ-) = 0$ for $s \geq 2$:

$$(\mathcal{R}_1 P^s)X := \ker\left(P^s(bX) \longrightarrow P^s(KQbX)\right) = P^s(bX).$$

When $\mathcal{C} = \mathscr{L}ie^r$, a modification [13, 4.3] of a theorem of Curtis [19, §5] states that $P^s(bX)$ is $\log_2(s)$-connected. When $\mathcal{C} = \mathscr{C}om$, we must demonstrate then that $P^s(bX)$ is $(s-1)$-connected. For this we use a truncation of Quillen's fundamental spectral sequence, as presented in [33, Theorem 6.2]: the filtration

$$P^s(bX) \supset P^{s+1}(bX) \supset P^{s+2}(bX) \supset \cdots$$

of $P^s(bX)$ yields a convergent spectral sequence $[E_0 P^s(bX)]^p_q \implies \pi_q(P^s(ba))$, with:

$$[E_0 P^s(bX)]^p_q = \begin{cases} N_q\big((Q^{\mathscr{C}om} bX)^{\otimes p}_{\Sigma_p}\big), & \text{if } p \geq s; \\ 0, & \text{if } p < s. \end{cases}$$

As $\pi_0(Q^{\mathscr{C}om} bX) = Q^{\mathscr{C}om}(\pi_0 bX) = Q^{\mathscr{C}om}(0) = 0$, the $t = 1$ result follows from [24, Satz 12.1]: if V is a connected simplicial vector space then $V^{\otimes p}_{\Sigma_p}$ is $(p-1)$-connected. □

Before we can give the proof of Proposition 4.3, we need the following *twisting lemma*, analogous to that of [9]. Before stating it, we note that $(\mathcal{R}_s\mathcal{R}_t)X$ and $\mathcal{R}_{s+t}X$ are equal by construction.

Lemma 4.8. *The maps $\mathcal{R}_i\delta : \mathcal{R}_n X \longrightarrow \mathcal{R}_{n-1}X$ are homotopic for $0 \leq i < n$.*

Proof. We may reindex the twisting lemma as follows: the maps

$$\mathcal{R}_s\delta, \mathcal{R}_{s-1}\delta : \mathcal{R}_{s+t}X \longrightarrow \mathcal{R}_{s+t-1}X$$

are homotopic whenever $s, t \geq 1$. Now $\mathcal{R}_{s+t}X$ is constructed as the subalgebra

$$\mathcal{R}_{s+t}X := \bigcap_{i=1}^{s+t} \ker\left(b^{s+t-i}\eta b^i : b^{s+t}X \longrightarrow b^{s+t-i}KQb^iX\right)$$

of the iterated bar construction $b^{s+t}X$, and for $0 \leq i < s+t$, $\mathcal{R}_i\delta$ is the restriction of the map $b^i\epsilon b^{s+t-i-1} : b^{s+t}X \longrightarrow b^{s+t-1}X$. Proposition 4.9 gives an explicit simplicial homotopy between the maps $b^s\epsilon b^{t-1}$ and $b^{s-1}\epsilon b^t$. Moreover, the naturality of the construction of Proposition 4.9 implies that this homotopy does indeed restrict to a homotopy of maps $\mathcal{R}_{s+t}X \longrightarrow \mathcal{R}_{s+t-1}X$. □

Now that we have the twisting lemma, Proposition 4.3 follows:

Proof of Proposition 4.3. By Proposition 4.5, it is enough to prove that for any $q \geq 0$ and $t \geq 1$, $\pi_q(\mathcal{R}_{f(q,t)}X) \longrightarrow \pi_q(\mathcal{R}_t X)$ is zero for some $f(q,t) \geq t$. Apply $\mathcal{R}_t-$ to the diagram of functors constructed in 4.6 and apply the result to X to obtain a commuting diagram of functors

$$\begin{tikzcd} \mathcal{R}_{f(q,t)}X \arrow[r, "\mathcal{R}_t\delta"] \arrow[d] & \cdots \arrow[r, "\mathcal{R}_t\delta"] & \mathcal{R}_{t+1}X \arrow[r, "\mathcal{R}_t\delta"] \arrow[d] & \mathcal{R}_t X \arrow[d, equal] \\ \mathcal{R}_t P^{f(q,t)-t+1}X \arrow[r] & \cdots \arrow[r] & \mathcal{R}_t P^2 X \arrow[r] & \mathcal{R}_t P^1 X \end{tikzcd}$$

By the twisting lemma, 4.8, the composite along the top row is homotopic to the map of interest, and factors through $\mathcal{R}_t P^{f(q,t)-t+1}X$. If we choose $f(q,t) = 2t+q-1$ when $\mathcal{C} = \mathscr{C}om$ and $f(q,t) = 2^{t+q-1} + t - 1$ when $\mathcal{C} = \mathscr{L}ie^r$, then Lemma 4.7 shows that $\mathcal{R}_t P^{f(q,t)-t+1}X$ is q-connected. $\square$

4.4. Iterated simplicial bar constructions

We will now state and prove a useful result on iterated simplicial bar constructions, used in the proof of the twisting lemma. The result here applies in general in the category $\mathcal{C}$ of algebras over a monad. Establishing notation, for any simplicial object X in $\mathcal{C}$, we will write

$$d^X_{i,q} : X_q \longrightarrow X_{q-1} \text{ and } s^X_{i,q} : X_q \longrightarrow X_{q+1}$$

for the i^{th} face and degeneracy maps out of X_q. Suppose that G and G' are endofunctors of $\mathcal{C}$, that $\Phi : G \longrightarrow G'$ is a natural transformation, and that $C, C' \in \mathcal{C}$ are objects. Write $[\Phi] : \hom_{\mathcal{C}}(C, C') \longrightarrow \hom_{\mathcal{C}}(GC, G'C')$ for the operator sending $m : C \longrightarrow C'$ to the diagonal composite in the commuting square

$$\begin{tikzcd} GC \arrow[r, "\Phi_C"] \arrow[d, "Gm"'] \arrow[dr, "{[\Phi]m}"] & G'C \arrow[d, "G'm"] \\ GC' \arrow[r, "\Phi_{C'}"'] & G'C' \end{tikzcd}$$

There is an (augmented) simplicial endofunctor, $\mathfrak{b} \in s(\mathcal{C}^{\mathcal{C}})$, derived from the unit and counit of the adjunction:

$$\begin{tikzcd} \mathrm{id} & (F^{\mathcal{C}})^1 \arrow[l, dashed, "\mathfrak{d}_{0,0}"'] \arrow[r, "\mathfrak{s}_{0,0}"] & (F^{\mathcal{C}})^2 \arrow[l, shift right=2, "\mathfrak{d}_{0,1}"'] \arrow[l, shift left=2, "\mathfrak{d}_{1,1}"] \arrow[r, shift left=1, "\mathfrak{s}_{0,1}"] \arrow[r, shift right=1, "\mathfrak{s}_{1,1}"'] & (F^{\mathcal{C}})^3 \arrow[l, shift right=3, "\mathfrak{d}_{0,2}"'] \arrow[l, "\mathfrak{d}_{1,2}"] \arrow[l, shift left=3, "\mathfrak{d}_{2,2}"] & \cdots \end{tikzcd}$$

The simplicial bar construction $b = B^{\mathcal{C}}$ on $s\mathcal{C}$ is the diagonal of the bisimplicial object obtained by levelwise application of $\mathfrak{b}$. That is, for $X \in s\mathcal{C}$, bX is the simplicial object with

$(bX)_q := (F^{\mathcal{C}})^{q+1}X_q$, and with
$$d^{bX}_{i,q} := [\mathfrak{d}_{i,q}]d^X_{i,q}.$$
The augmentation $\epsilon : b \longrightarrow \mathrm{id}$ is defined on level q by
$$\epsilon_q = \mathfrak{d}_{0,0}\mathfrak{d}_{0,1}\cdots\mathfrak{d}_{0,q} : (F^{\mathcal{C}})^{q+1} \longrightarrow \mathrm{id}.$$
We can now construct the simplicial homotopy needed for the twisting lemma, 4.8.

Proposition 4.9. *The natural transformations ϵ_b and $b\epsilon$ from $b^2 : s\mathcal{C} \longrightarrow s\mathcal{C}$ to $b : s\mathcal{C} \longrightarrow s\mathcal{C}$ are naturally simplicially homotopic.*

Proof. Write $K = b^2X$ and $L = bX$ for the source and target of these maps respectively. Noting the formulae
$$[\mathfrak{d}_{iq}]^2 = [\mathfrak{d}_{q+i,2q} \circ \mathfrak{d}_{i,2q+1}] \text{ and } [\mathfrak{s}_{iq}]^2 = [\mathfrak{s}_{q+i+2,2q+2} \circ \mathfrak{s}_{i,2q+1}],$$
we can describe the simplicial structure maps in K and L as follows:
$$\begin{aligned} d^L_{iq} &= [\mathfrak{d}_{iq}]d^X_{iq} \\ s^L_{iq} &= [\mathfrak{s}_{iq}]s^X_{iq} \\ d^K_{iq} &= [\mathfrak{d}_{q+i,2q} \circ \mathfrak{d}_{i,2q+1}]d^X_{iq} \\ s^K_{iq} &= [\mathfrak{s}_{q+i+2,2q+2} \circ \mathfrak{s}_{i,2q+1}]s^X_{iq} \end{aligned}$$
We can now state an explicit simplicial homotopy between the two maps of interest. Using precisely the notation of [40, §5], we define $h_{jq} : K_q \longrightarrow L_{q+1}$, for $0 \leq j \leq q$, by the formula
$$h_{jq} := [\mathfrak{d}_{j+1,q+2} \circ \cdots \circ \mathfrak{d}_{j+1,2q+1}]s^X_{jq}.$$
We first check that these maps satisfy the defining identities for the notion of simplicial homotopy, numbered (1)-(5) as in [40, §5]. Each identity can be checked in two parts (a)-(b):

(1) We must check that $d^L_{i,q+1}h_{j,q} = h_{j-1,q-1}d^K_{i,q}$ whenever $0 \leq i < j \leq q$, i.e.:
   - (a) $d^X_{i,q+1}s^X_{j,q} = s^X_{j-1,q-1}d^X_{i,q}$, and
   - (b) $\mathfrak{d}_{i,q+1}\mathfrak{d}_{j+1,q+2}\cdots\mathfrak{d}_{j+1,2q+1} = \mathfrak{d}_{j,q+1}\cdots\mathfrak{d}_{j,2q-1}\mathfrak{d}_{q+i,2q}\mathfrak{d}_{i,2q+1}$.

(2) We must check that $d^L_{j+1,q+1}h_{j,q} = d^L_{j+1,q+1}h_{j+1,q}$ whenever $0 \leq j \leq q-1$, i.e.:
   - (a) $d^X_{j+1,q+1}s^X_{j,q} = d^X_{j+1,q+1}s^X_{j+1,q}$, and
   - (b) $\mathfrak{d}_{j+1,q+1}\mathfrak{d}_{j+1,q+2}\cdots\mathfrak{d}_{j+1,2q+1} = \mathfrak{d}_{j+1,q+1}\mathfrak{d}_{j+2,q+2}\cdots\mathfrak{d}_{j+2,2q+1}$.

(3) We must check that $d^L_{i,q+1} h_{j,q} = h_{j,q-1} d^K_{i-1,q}$ whenever $0 \le j < i-1 \le q$, i.e.:

   (a) $d^X_{i,q+1} s^X_{j,q} = s^X_{j,q-1} d^X_{i-1,q}$, and

   (b) $\mathfrak{d}_{i,q+1}\mathfrak{d}_{j+1,q+2}\cdots\mathfrak{d}_{j+1,2q+1} = \mathfrak{d}_{j+1,q+1}\cdots\mathfrak{d}_{j+1,2q-1}\mathfrak{d}_{q+i-1,2q}\mathfrak{d}_{i-1,2q+1}$.

(4) We must check that $s^L_{i,q+1} h_{j,q} = h_{j+1,q+1} s^K_{i,q}$ whenever $0 \le i \le j \le q$, i.e.:

   (a) $s^X_{i,q+1} s^X_{j,q} = s^X_{j+1,q+1} s^X_{i,q}$, and

   (b) $\mathfrak{s}_{i,q+1}\mathfrak{d}_{j+1,q+2}\cdots\mathfrak{d}_{j+1,2q+1} = \mathfrak{d}_{j+2,q+3}\cdots\mathfrak{d}_{j+2,2q+3}\mathfrak{s}_{q+i+2,2q+2}\mathfrak{s}_{i,2q+1}$.

(5) We must check that $s^L_{i,q+1} h_{j,q} = h_{j,q+1} s^K_{i-1,q}$ whenever $0 \le j < i \le q+1$, i.e.:

   (a) $s^X_{i,q+1} s^X_{j,q} = s^X_{j,q+1} s^X_{i-1,q}$, and

   (b) $\mathfrak{s}_{i,q+1}\mathfrak{d}_{j+1,q+2}\cdots\mathfrak{d}_{j+1,2q+1} = \mathfrak{d}_{j+1,q+3}\cdots\mathfrak{d}_{j+1,2q+3}\mathfrak{s}_{q+i+1,2q+2}\mathfrak{s}_{i-1,2q+1}$.

Each of these equations follows from the simplicial identities, proving that the h_{jq} form a homotopy. Finally, we check that this homotopy is indeed a homotopy between the two maps of interest:

$$\begin{aligned} d^L_{0,q+1} h_{0,q} &= [\mathfrak{d}_{0,q+1}\mathfrak{d}_{1,q+2}\cdots\mathfrak{d}_{1,2q+1}](d^X_{0,q+1} s^X_{0q}) \\ &= [\mathfrak{d}_{0,q+1}\mathfrak{d}_{0,q+2}\cdots\mathfrak{d}_{0,2q+1}]\mathrm{id}_{X_q} \end{aligned}$$

is the action of $\epsilon_{(bX)}$ in level q, and similarly,

$$\begin{aligned} d^L_{q+1,q+1} h_{q,q} &= [\mathfrak{d}_{q+1,q+1}\mathfrak{d}_{q+1,q+2}\cdots\mathfrak{d}_{q+1,2q+1}](d^X_{q+1,q+1} s^X_{qq}) \\ &= [\mathfrak{d}_{q+1,q+1}\mathfrak{d}_{q+1,q+2}\cdots\mathfrak{d}_{q+1,2q+1}]\mathrm{id}_{X_q} \end{aligned}$$

is the action of $b\epsilon_X$ in level q. □

Chapter 5

Constructing homotopy operations

5.1. Higher simplicial Eilenberg-Mac Lane maps

In what follows, we will often have a natural map G whose domain and codomain both support a switch map T, obtained by interchanging tensor factors. Furthermore, we will so often use the expression TGT that we introduce the shorthand $\omega G := TGT$. Although this notation is potentially ambiguous, whenever we write σGH, for functions G and H, we mean $(\sigma G)H$, not $\sigma(GH)$.

Let $\{\nabla_k\}$ be a higher simplicial Eilenberg-Mac Lane map [26, §3], i.e. a collection of maps

$$\nabla_k : (CU \otimes CV)_{i+k} \longrightarrow N(U \otimes V)_i \quad \text{defined for } 0 \leq k \leq i$$

natural in simplicial vector spaces U and V, such that for $k \geq 0$, the identity

$$(1+\omega)\nabla_k = \phi^k + \begin{cases} \nabla_{k-1}\partial + \partial\nabla_{k-1}, & \text{if } k \geq 1, \\ \nabla, & \text{if } k = 0, \end{cases}$$

holds on classes of simplicial dimension at least $2k$, where:

(1) $\nabla : CU \otimes CV \longrightarrow N(U \times V)$ is the *Eilenberg-Mac Lane shuffle map*, also known as the *Eilenberg-Zilber map*, a chain homotopy equivalence inducing the identity in simplicial dimension zero; and
(2) ϕ^k is the map $(CU \otimes CV)_{i+k} \longrightarrow N(U \otimes V)_i$ which vanishes except on $U_k \otimes V_k$, where its value is just the projection $U_k \otimes V_k \longrightarrow N(U \otimes V)_k$.

Note that as ϕ^0 commutes with symmetry isomorphisms, so does ∇.

5.2. External unary homotopy operations

In this section we recall the definition of certain homotopy operations with domain π_*V for any $V \in s\mathcal{V}$, implicit in [26, §4] (c.f. [8, 6], [14]) and explicit in [33, §3], using the functions

$$a \longmapsto \nabla_{n-i}(a \otimes a), \quad N_nV \longrightarrow N_{n+i}(S_2V).$$

By postcomposing with the maps $S_2V \longrightarrow \Lambda^2V \longrightarrow S^2V$, we obtain functions from N_nV to $N_{n+i}(\Lambda^2V)$ and $N_{n+i}(S^2V)$.

Proposition 5.1 [26, Lemma 4.1], [33, §3]**.** *These functions descend to well defined homotopy operations:*

$$\begin{aligned}
\delta_i^{\mathrm{ext}} &: \pi_nV \longrightarrow \pi_{n+i}(S_2V), && \text{defined when } 2 \le i \le n,\\
\lambda_i^{\mathrm{ext}} &: \pi_nV \longrightarrow \pi_{n+i}(\Lambda^2V), && \text{defined when } 1 \le i \le n,\\
\sigma_i^{\mathrm{ext}} &: \pi_nV \longrightarrow \pi_{n+i}(S^2V), && \text{defined when } 1 \le i \le n.
\end{aligned}$$

The function $N_nV \longrightarrow N_n(S^2V)$ given by $\overline{a} \longmapsto \overline{a \otimes a}$ yields a well defined homotopy operation $\sigma_0^{\mathrm{ext}} : \pi_nV \longrightarrow \pi_n(S^2V)$. These operations are linear whenever $i < n$. For all $n \ge 0$, the map $\sigma_n^{\mathrm{ext}} : \pi_nV \longrightarrow \pi_{2n}(S^2V)$ satisfies

$$\sigma_n^{\mathrm{ext}}(\overline{x} + \overline{y}) = \sigma_n^{\mathrm{ext}}(\overline{x}) + \sigma_n^{\mathrm{ext}}(\overline{y}) + \overline{(1+T)\nabla(x \otimes y)} \quad \textit{for } x, y \in ZN_nV.$$

Proof. Although all of the operations are defined in the cited references, we will be a little more explicit about the definition of σ_0^{ext}, and the final equation of the proposition.

As described in [33, §3], we might choose to define σ_0^{ext} using a universal example, for which the cycle

$$z \otimes z \in ZN_n(S^2\mathbb{K}_n) \cong \mathbb{F}_2$$

is the only possible representative, demonstrating that the formula $\overline{a} \longmapsto \overline{a \otimes a}$ yields the correct (well defined) operation. To check that $\sigma_0^{\mathrm{ext}} : \pi_0V \longrightarrow \pi_0S^2V$ satisfies the stated equation, we need only check that it holds on $z_1 + z_2 \in ZN_0(\mathbb{K}_0 \oplus \mathbb{K}_0) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$. But

$$\sigma_0^{\mathrm{ext}}(z_1 + z_2) - \sigma_0^{\mathrm{ext}}(z_1) - \sigma_0^{\mathrm{ext}}(z_2) = z_1 \otimes z_2 + z_2 \otimes z_1 = (1+T)\nabla(z_1 \otimes z_2),$$

as ∇ is the identity in dimension zero.

To explain the equation when $n \ge 1$, as $\sigma_n^{\mathrm{ext}}(\overline{x}) := \overline{(1+T)\nabla_0(x \otimes x)}$, we obtain

$$\sigma_n^{\mathrm{ext}}(\overline{x} + \overline{y}) - \sigma_n^{\mathrm{ext}}(\overline{x}) - \sigma_n^{\mathrm{ext}}(\overline{y}) = \overline{(1+T)\nabla_0(1+T)(x \otimes y)}$$

and using the symmetry $T\big((1+T)(x\otimes y)\big) = (1+T)(x\otimes y)$ and the fact that ϕ^0 vanishes on $(1+T)(x\otimes y)$,

$$(1+T)\nabla_0(1+T)(x\otimes y) = (1+\omega\nabla_0)(1+T)(x\otimes y) = \nabla(1+T)(x\otimes y). \qquad \square$$

5.3. External binary homotopy operations

We will now give an account of various natural external homotopy operations, most of which are binary operations, induced by the Eilenberg-Mac Lane shuffle map $\nabla : N_*(V)\otimes N_*(V) \longrightarrow N_*(V\otimes V)$, which is also known as the Eilenberg-Zilber map. These operations are well known, but we make a point of giving them the following unified treatment:

Proposition 5.2. *There is a natural commuting diagram:*

$$\begin{array}{ccc}
S_2(\pi_*V) & \xrightarrow{\widetilde{\nabla}} & \pi_*(S_2V) \\
\big\downarrow{\scriptstyle \mathrm{proj}} & & \big\downarrow{\scriptstyle \pi_*(\mathrm{proj})} \\
\Lambda^2(\pi_*V) & \xrightarrow{\widetilde{\nabla}} & \pi_*(\Lambda^2V) \\
\big\downarrow{\scriptstyle \mathrm{incl}} & & \big\downarrow{\scriptstyle \pi_*(\mathrm{incl})} \\
S^2(\pi_*V) & \xrightarrow{\widetilde{\nabla}} & \pi_*(S^2V)
\end{array}$$

For cycles $x, y \in ZN_*(V)$ and $z \in ZN_n(V)$, the upper horizontal is determined by

$$\overline{x}\otimes\overline{y} \longmapsto \overline{x\otimes y},$$

and the lower horizontal is determined by

$$\overline{x}\otimes\overline{y} + \overline{y}\otimes\overline{x} \longmapsto \overline{\nabla(x\otimes y + y\otimes x)} \text{ and } \overline{z}\otimes\overline{z} \longmapsto \sigma_n^{\mathrm{ext}}(\overline{z}).$$

Proof. During this proof, write $\widetilde{\nabla}_U$, $\widetilde{\nabla}_M$ and $\widetilde{\nabla}_L$ for the upper, middle and lower horizontal maps. We must demonstrate: that $\widetilde{\nabla}_U$ is well defined; that

$$\ker\left(\pi_*(\mathrm{proj})\circ\widetilde{\nabla}_U\right) \supseteq \ker\left(\mathrm{proj}\right),$$

so that there is a unique map $\widetilde{\nabla}_M$ for which the upper square commutes; and that one may extend the composite $\pi_*(\mathrm{tr})\circ\widetilde{\nabla}_U$ along the trace map $S_2(\pi_*A) \longrightarrow S^2(\pi_*A)$ using the operations σ_n^{ext}. A simple diagram chase would then reveal that the bottom square must also commute.

As ∇ is a chain map, it produces a well defined map $(\pi_*V)^{\otimes 2} \longrightarrow \pi_*(V^{\otimes 2})$, and the fact that $\nabla = \omega\nabla$ implies that this map descends to a well defined map ∇_U.

The kernel of the projection $S_2(\pi_*V) \longrightarrow \Lambda^2(\pi_*V)$ is spanned by classes of the form

$\overline{x} \otimes \overline{x}$, and the image under $\pi_*(\mathrm{proj}) \circ \widetilde{\nabla}_U$ of such a class equals $\overline{x \otimes x}$ which is zero as $x \otimes x \in \Lambda^2 V$ is zero. This proves the inclusion of kernels.

Finally, to extend the composite $\pi_*(\mathrm{tr}) \circ \widetilde{\nabla}_U$ to $S^2(\pi_* V)$, we simply need the operations σ_n^{ext} to satisfy the equations of Proposition 2.7, which are part of Proposition 5.1. □

5.4. Homotopy operations for simplicial commutative algebras

Suppose that $A \in s\mathscr{C}om$ is a simplicial non-unital commutative algebra, with multiplication map $\mu : S_2 A \longrightarrow A$. Then by composition with the map $\pi_*(\mu) : \pi_*(S_2 A) \longrightarrow \pi_* A$, one obtains unary operations:

$$\delta_i := \pi_*(\mu) \circ \delta_i^{\mathrm{ext}} : \pi_n A \longrightarrow \pi_{n+i} A, \text{ defined when } 2 \leq i \leq n,$$

and a pairing

$$\mu := \pi_*(\mu) \circ \widetilde{\nabla} : S_2(\pi_* A) \longrightarrow \pi_* A.$$

Proposition 5.3 [26]**.** *These operations have the following properties:*

(1) the pairing μ equips $\pi_* A$ with the structure of a non-unital commutative algebra;

(2) the ideal $\bigoplus_{n\geq 1} \pi_n A$ is an exterior algebra;

(3) the ideal $\bigoplus_{n \geq 2} \pi_n A$ is a divided power algebra, with divided square given by the top δ-operation, *i.e. $x \longmapsto \delta_n x$ for $x \in \pi_n A$;*

(4) the non-top operations, *$\delta_i : \pi_n A \longrightarrow \pi_{n+i} A$ for $2 \leq i < n$, are linear;*

(5) for $x \in \pi_n A$, $y \in \pi_m A$ and $2 \leq i \leq n$

$$\delta_i(xy) = \begin{cases} y^2 \delta_i(x), & \textit{if } m = 0; \\ 0, & \textit{otherwise}; \end{cases}$$

(6) the δ-Adem relations *hold: if $\delta_i \delta_j x$ is defined, and $i < 2j$, then*

$$\delta_i \delta_j x := \sum_{s=\lceil (i+1)/2 \rceil}^{\lfloor (i+j)/3 \rfloor} \binom{j+s-i-1}{j-s} \delta_{i+j-s} \delta_s x.$$

A few comments are in order. Firstly, the proposition distinguishes between the *top* and *non-top* δ-operations, as they have rather different behaviour — this will be a recurring pattern. Secondly, it is not immediately obvious that the δ-Adem relations make sense, in that it is not obvious that every term in the right hand side is defined. This does indeed happen, by Lemma 5.4 (to follow).

We may define an associative unital algebra Δ to be the algebra generated by δ_i for $i \geq 2$, subject to relations

$$\delta_i \delta_j := \sum_{s=\lceil (i+1)/2 \rceil}^{\lfloor (i+j)/3 \rfloor} \binom{j+s-i-1}{j-s} \delta_{i+j-s} \delta_s \text{ when } i < 2j.$$

We will say that a sequence $I = (i_\ell, \ldots, i_1)$ of integers $i_j \geq 2$ is *δ-admissible* if $i_{j+1} \geq 2i_j$ for $1 \leq j < \ell$. For any sequence $I = (i_\ell, \ldots, i_1)$, write δ_I for the composite $\delta_{i_\ell} \cdots \delta_{i_1}$. This δ-Adem relation allows us to write any δ_I in Δ as a sum of composites δ_J in which J is δ-admissible. In fact, it follows from [33, Proposition 2.7] that the algebra Δ has an *admissible basis*, consisting of those $\delta_I = \delta_{i_\ell} \cdots \delta_{i_1}$ with I a δ-admissible sequence.

It then makes sense to make the following definition. Suppose that I is any non-empty sequence of integers at least 2, and J is a sequence of integers no less than two. Then we will say that I *produces* J *in* Δ, denoted $I \overset{\Delta}{\to} J$, if δ_J appears with non-zero coefficient when δ_I is written in the δ-admissible basis of Δ. In this case, J must be δ-admissible and I must be δ-inadmissible unless $J = I$.

Proposition 5.3 does *not* state that $\pi_* A$ is a left module over Δ, since the δ-operations are not always defined (or even linear). We define

$$m(I) := \max\{(i_1),\, (i_2 - i_1),\, (i_3 - i_2 - i_1),\, \ldots,\, (i_\ell - \cdots - i_1)\},$$

following the convention that $\max(\emptyset) = -\infty$, for any sequence I of integers $i_j \geq 2$ (or more generally, for any sequence of non-negative integers). The intent of this definition is that the composite δ_I, by which we mean

$$\pi_n A \xrightarrow{\delta_{i_1}} \pi_{n+i_1} A \xrightarrow{\delta_{i_2}} \cdots \xrightarrow{\delta_{i_\ell}} \pi_{n+i_1+\cdots+i_\ell} A,$$

is defined if and only if $n \geq m(I)$. Note that when I is a non-empty *δ-admissible* sequence,

$$m(I) = i_\ell - i_{\ell-1} - \cdots - i_1 =: e(I),$$

the *Serre excess of* I. Moreover, if I is δ-admissible, then for any expression $\delta_{i_\ell} \cdots \delta_{i_1} x$ there is some k with $0 \leq k \leq \ell$ such that each of the k operations $\delta_{i_\ell} \cdots \delta_{i_{\ell-k+1}}$ are acting as top operations, and each of the remaining $\ell - k$ are acting as non-top operations.

The following lemma assures us that the δ-Adem relations make sense as they appear in *(6)*.

Lemma 5.4. *If $I \overset{\Delta}{\to} J$, then $m(I) \geq m(J)$.*

Proof. It is enough to show this result when I and J are distinct and have length two, in light of the evident algorithm for expressing δ_I in terms of admissible composites. In the length two case it can be checked directly from the format of the δ-Adem relation, and the inequality is in fact strict (unless I is itself δ-admissible). $\square$

Finally, one should note that these operations generate all of the operations in the category $\pi\mathscr{C}om$, and that all of the relations between the operations in $\pi\mathscr{C}om$ are implied by those presented here. Goerss [33, §2] presents this information as follows. First, he observes that there is a *Künneth Theorem* available:

Proposition 5.5. *Suppose that A_1 and A_2 are models in $s\mathscr{C}om$. Then $\pi_*(A_1 \sqcup A_2)$, which is the coproduct of $\pi_* A_1$ and $\pi_* A_2$ in $\pi\mathscr{C}om$, may be calculated as the non-unital commutative algebra coproduct of $\pi_* A_1$ and $\pi_* A_2$.*

After giving the calculation on a single sphere, the homotopy of finite models (which is the structure defining the category $\pi\mathscr{C}om$) will be determined by this proposition, and the calculation for a single sphere is the following:

Proposition 5.6 [33, Proposition 2.7]**.** *For $n \geq 0$, let $\imath_n$ be the fundamental class in $\pi_n(\mathbb{S}_n^{\mathscr{C}om})$. There are isomorphisms of non-unital commutative algebras:*

$$\begin{aligned}
&\pi_*(\mathbb{S}_0^{\mathscr{C}om}) \cong S(\mathscr{C})[\imath_0] = F^{\mathscr{C}om}[\imath_0];\\
&\pi_*(\mathbb{S}_n^{\mathscr{C}om}) \cong \Lambda(\mathscr{C})[\delta_I(\imath_n) \mid I \text{ is } \delta\text{-admissible, } e(I) \leq n] \ \ \textit{for } n \geq 1;\\
&\pi_*(\mathbb{S}_n^{\mathscr{C}om}) \cong \Gamma(\mathscr{C})[\delta_I(\imath_n) \mid I \text{ is } \delta\text{-admissible, } e(I) < n] \ \ \textit{for } n \geq 2.
\end{aligned}$$

5.5. Homotopy operations for simplicial Lie algebras

Suppose that $L \in s\mathscr{L}ie$ is a simplicial Lie algebra with bracket $[\,,] : \Lambda^2 L \longrightarrow L$. There are unary operations

$$\lambda_i := \pi_*([\,,]) \circ \lambda_i^{\text{ext}} : \pi_n L \longrightarrow \pi_{n+i} L, \text{ defined when } 1 \leq i \leq n,$$

which we write *on the right* as $x \longmapsto x\lambda_i$, and a bracket

$$[\,,] := \pi_*([\,,]) \circ \widetilde{\nabla} : \Lambda^2(\pi_* L) \longrightarrow \pi_* L.$$

Alternatively, one can suppose that $L \in s\mathscr{L}ie^r$ is a simplicial restricted Lie algebra with bracket $[\,,] : S^2 L \longrightarrow L$, and construct operations:

$$\begin{gathered}
\lambda_i := \pi_*([\,,]) \circ \sigma_i^{\text{ext}} : \pi_n L \longrightarrow \pi_{n+i} L, \text{ defined when } 0 \leq i \leq n, \text{ and}\\
[\,,] := \pi_*([\,,]) \circ \widetilde{\nabla} : S^2(\pi_* L) \longrightarrow \pi_* L.
\end{gathered}$$

Proposition 5.7 [13], [20, §8]. *For $L \in s\mathscr{L}ie$, these operations satisfy:*

(1) the bracket gives $\pi_* L$ the structure of a Lie algebra;

(2) the ideal $\bigoplus_{n\geq 1} \pi_n L$ is a restricted Lie algebra, with restriction given by the top λ-operation, *i.e. $x^{[2]} = \lambda_n x$ for $x \in \pi_n L$;*

(3) the non-top *operations, $\lambda_i : \pi_n L \longrightarrow \pi_{n+i} L$ for $1 \leq i < n$, are linear;*

(4) for $x \in \pi_* L$, $y \in \pi_n L$ and $1 \leq i \leq n$:

$$[x, y\lambda_i] = \begin{cases} [y, [x, y]], & \textit{if } i = n; \\ 0, & \textit{otherwise}; \end{cases}$$

(5) the Λ-Adem relations *hold: if $x\lambda_j\lambda_i$ is defined, and $i > 2j$, then*

$$x\lambda_j\lambda_i = \sum_{k=0}^{(i-2j)/2-1} \binom{i-2j-2-k}{k} x\lambda_{i-j-1-k}\lambda_{2j+1+k}.$$

For $L \in s\mathscr{L}ie^r$, we may omit ((1)), modify ((2)) to state that the whole of $\pi_* L$ is restricted, and modify ((3))-((5)) to include λ_0.

Similar comments apply as for commutative algebras, for example, one needs Lemma 5.8 (to follow) to understand why this unstable relation makes sense.

The well known Λ-*algebra* is the unital associative algebra generated by λ_i for $i \geq 0$, subject to relations

$$\lambda_j\lambda_i = \sum_{k=0}^{(i-2j)/2-1} \binom{i-2j-2-k}{k} \lambda_{i-j-1-k}\lambda_{2j+1+k} \text{ for } i > 2j.$$

We say that a sequence $I = (i_\ell, \ldots, i_1)$ of non-negative integers is Λ-*admissible* if $i_{j+1} \leq 2i_j$ for $1 \leq j < \ell$. For any sequence $I = (i_\ell, \ldots, i_1)$, if we write λ_I for the element $\lambda_{i_1} \cdots \lambda_{i_\ell}$ in Λ, then the Λ-algebra has the evident admissible basis, and we may make sense of the symbol $I \xrightarrow{\Lambda} J$. Note that the ordering of the generators in λ_I is opposite the ordering for the δ_I, to be consistent with the fact that we write the λ-operations on $\pi_* L$ *on the right*. Thus, we may think of λ_I as the composite operator

$$\pi_n L \xrightarrow{\lambda_{i_1}} \pi_{n+i_1} L \xrightarrow{\lambda_{i_2}} \cdots \xrightarrow{\lambda_{i_\ell}} \pi_{n+i_1+\cdots+i_\ell} L,$$

again defined only when $m(I) \leq n$, so that $\pi_* L$ is *not* a right module over Λ. We will say, however, that it is an *unstable partial right* Λ-*module*. Note that when I is a non-empty Λ-*admissible* sequence, $m(I) = i_1$, not the Serre excess, reflecting the observation that when $x\lambda_{i_1} \cdots \lambda_{i_\ell}$ is a Λ-admissible composite, the top (i.e. restriction) operations which appear are applied first. The following lemma assures us that the Λ-Adem relations make sense in *(5)*.

Lemma 5.8. *If $I \stackrel{\Lambda}{\to} J$, then $m(I) \geq m(J)$, and J does not contain zero unless I does.*

These operations generate all of the operations in each of the categories $\pi\mathscr{L}ie$ and $\pi\mathscr{L}ie^r$, and the relations presented here are sufficient, as:

Proposition 5.9 ([20, Theorem 8.8 and proof], [13]). *For $V \in \mathcal{V}_1$, choose a homogeneous basis of V, and construct from it a monomial basis B of $\Lambda(\mathscr{L})V$ (such as any choice of Hall basis). Then:*

$$\begin{aligned} F^{\pi\mathscr{L}ie}V &= \mathbb{F}_2\left\{\lambda_I b \;\middle|\; \begin{matrix} b \in B_t,\ I\ \Lambda\text{-admissible with } m(I) \leq t, \\ I \text{ does not contain } 0 \end{matrix}\right\} \text{ and;} \\ F^{\pi\mathscr{L}ie^r}V &= \mathbb{F}_2\left\{\lambda_I b \;\middle|\; \begin{matrix} b \in B_t,\ I\ \Lambda\text{-admissible with } m(I) \leq t, \\ I \text{ does not contain } 0 \text{ when } t = 0 \end{matrix}\right\} \\ &= \mathbb{F}_2\left\{\lambda_I\big(b^{[2^r]}\big) \;\middle|\; \begin{matrix} b \in B_t,\ I\ \Lambda\text{-admissible with } m(I) < t, \\ r \geq 0 \end{matrix}\right\}. \end{aligned}$$

For the sake of interest, we can emulate Goerss' method of calculating the cohomology of GEMs in $s\mathscr{C}om$ (c.f. §6.6, [32] and [33, §11]) by giving a Hilton-Milnor decomposition for the calculation of the free $\mathscr{L}ie$-Π-algebra on a finite-dimensional object of $\mathcal{V}_1$, using [50, Proposition 3.1]. For any $i \geq 0$, write $\Sigma^i\mathbb{F}_2 \in \mathcal{V}_1$ for a one-dimensional vector space concentrated in homological dimension i. For any finite collection of indices $i_1, \ldots, i_n \geq 0$, we would like to calculate:

$$F^{\pi\mathscr{L}ie}(\Sigma^{i_1}\mathbb{F}_2 \oplus \cdots \oplus \Sigma^{i_n}\mathbb{F}_2) = \pi_* F^{\mathscr{L}ie}(\mathbb{K}_{i_1} \oplus \cdots \oplus \mathbb{K}_{i_n}).$$

and we obtain a decomposition of $\pi_* F^{\mathscr{L}ie}(\mathbb{K}_{i_1} \oplus \cdots \oplus \mathbb{K}_{i_n})$ as follows. For any monomial b is the free Lie algebra on $\{x_1, \ldots, x_n\}$ and any collection of n vector spaces $A_1, \ldots, A_n$, there is a corresponding tensor product $w_b(A_1, \ldots, A_n)$. For example, one defines

$$w_{[[x_2,x_1],x_3]} := A_2 \otimes A_1 \otimes A_3.$$

Moreover, for each monomial b there is an evident function

$$w_b(A_1, \ldots, A_n) \longrightarrow F^{\mathscr{L}ie}(A_1 \oplus \cdots \oplus A_n),$$

given in our example by $a_2 \otimes a_1 \otimes a_3 \longmapsto [[a_2, a_1], a_3]$.

Iteration of the procedure described in [44, §4.3], using the formula of [50, Proposition 3.1], we obtain a Hall basis B of the free Lie algebra on $\{x_1, \ldots, x_n\}$, with the property that the resulting map

$$\bigoplus_{b\in B} F^{\mathscr{L}ie} w_b(A_1,\ldots,A_n) \longrightarrow F^{\mathscr{L}ie}(A_1\oplus\cdots\oplus A_n)$$

is an isomorphism, natural in $A_1,\ldots,A_n$. Thus, there is an isomorphism in $s\mathcal{V}_1$:

$$\bigoplus_{b\in B} F^{\mathscr{L}ie} w_b(\mathbb{K}_{i_1},\ldots,\mathbb{K}_{i_n}) \longrightarrow F^{\mathscr{L}ie}(\mathbb{K}_{i_1}\oplus\cdots\oplus\mathbb{K}_{i_n}).$$

Moreover, if we follow [33, §11] by writing $j_k(b)$ for the number of appearances of x_k in the monomial b, there is a homotopy equivalence

$$w_b(\mathbb{K}_{i_1},\ldots,\mathbb{K}_{i_n}) \simeq \mathbb{K}_{\sum_{k=1}^n J_k(b) i_k}.$$

Thus, on homotopy there is a decomposition:

$$\bigoplus_{b\in B} F^{\pi\mathscr{L}ie}\Sigma^{(\sum_{k=1}^n j_k(b) i_k)}\mathbb{F}_2 \xrightarrow{\cong} F^{\pi\mathscr{L}ie}(\Sigma^{i_1}\mathbb{F}_2\oplus\cdots\oplus\Sigma^{i_n}\mathbb{F}_2),$$

under which the fundamental class of a summand on the left maps to the corresponding Lie bracket of fundamental classes on the right. This proves the first part of Proposition 5.9.

Chapter 6

Constructing cohomology operations

6.1. Higher cosimplicial Alexander-Whitney maps

Let $\{D^k\}$ be a special cosimplicial Alexander-Whitney map [55, Proposition 5.2], i.e. maps

$$D^k : (CR \otimes CS)^{i+k} \longrightarrow C(R \otimes S)^i \quad \text{for } i, k \geq 0,$$

natural in cosimplicial vector spaces R, S, with the properties:

(1) $dD^k + D^k d = (1 + \omega)D^{k-1}$ for $k \geq 1$;

(2) D^0 is a chain homotopy equivalence inducing the identity in dimension zero;

(3) the restriction of D^k to $C^i R \otimes C^j S$ is zero unless $i \geq k$ and $j \geq k$; and

(4) D^k maps $C^k R \otimes C^k S$ identically onto $C^k(R \otimes S)$.

It is a natural convention to define $D^k = 0$ for all $k, i \in \mathbb{Z}$, in which case the relation $dD^k + D^k d = (1 + \omega)D^{k-1}$ holds for any k.

Maps dual to these are described in detail, under the name *special cup-k product*, by Singer in [52, Definitions 1.91 and 1.94], and were developed originally in [23]. Indeed, we will use these maps later, and denote them

$$(D^k)^\star : C(U \otimes V)_i \longrightarrow (CU \otimes CV)_{i+k} \quad \text{for } i, k \geq 0,$$

natural in $U, V \in s\mathcal{V}$. The sense in which these maps are dual to the D^k is captured in the following commuting diagram (for $i, k \geq 0$):

$$\begin{tikzcd} (C\mathbf{D}U \otimes C\mathbf{D}V)^{i+k} \arrow[r, "D^k"] \arrow[d] & C(\mathbf{D}U \otimes \mathbf{D}V)^i \arrow[d] \\ (\mathbf{D}(CU \otimes CV))^{i+k} \arrow[r, "((D^k)^\star)^*"] & (\mathbf{D}C(U \otimes V))^i \end{tikzcd}$$

This is the first instance of a notational convention we will use occasionally in what follows. We consider the operations D^k to be 'of primary interest' in this thesis, and so we prefer not to adorn the symbol D^k. However, we would like to have access to operations $(D^k)^\star$ *of which the* D^k *are the duals* in the sense of the above commuting square, and not the other way around. So, we use a star as opposed to an asterisk when writing $(D^k)^\star$.

6.2. External unary cohomotopy operations

In this section we recall the definition of certain cohomotopy operations with domain π^*U for any $U \in c\mathcal{V}$, using the functions

$$\alpha \longmapsto D^{n-i}(\alpha \otimes \alpha) + D^{n-i+1}(\alpha \otimes d\alpha), \quad C^n U \longrightarrow C^{n+i}(S_2 U).$$

The same arguments as in [52, §1.12] show that

Proposition 6.1. *These functions descend to well defined linear operations:*

$$\mathrm{Sq}^k_{\mathrm{ext}} : \pi^n U \longrightarrow \pi^{n+k}(S_2 U), \quad \textit{zero unless } 0 \le k \le n.$$

If $U = \mathbf{D}V$ for some $V \in s\mathcal{V}$, then we may use the natural transformation $S_2\mathbf{D} \longrightarrow \mathbf{D}S^2$ to form the following composite, also denoted $\mathrm{Sq}^k_{\mathrm{ext}}$:

$$\pi^n \mathbf{D}V \xrightarrow{\mathrm{Sq}^k_{\mathrm{ext}}} \pi^{n+k} S_2 \mathbf{D}V \longrightarrow \pi^{n+k}\mathbf{D}S^2 V.$$

This will be part of the process we use shortly to define cohomology operations.

6.3. Linearly dual homotopy operations

Whenever $V \in s\mathcal{V}$ has $\pi_* V$ of finite type, the linear maps $\mathrm{Sq}^k_{\mathrm{ext}} : \pi^n \mathbf{D}V \longrightarrow \pi^{n+k}\mathbf{D}S^2 V$ induce dual operators

$$\pi_*(S^2 V) \longrightarrow \pi_{*-k} V.$$

Following [33, §3], one can do much better than just this observation, giving a direct definition of such operations, valid for any $V \in s\mathcal{V}$, whose duals are the $\mathrm{Sq}^k_{\mathrm{ext}}$.

Again, the cohomotopy operation $\mathrm{Sq}^k_{\mathrm{ext}}$ is of primary interest, and we prefer to allow it its standard symbol (albeit with the attached subscript). On the other hand, we are about to produce a homotopy operation of which it is the dual, so we will use a star and not an asterisk. That is, for *any* $V \in s\mathcal{V}$, there is an operation

$$(\mathrm{Sq}^k_{\mathrm{ext}})^\star : \pi_*(S^2 V) \longrightarrow \pi_{*-k} V$$

such that the following diagram commutes:

$$\begin{tikzcd}
\pi^{n+k}\mathbf{D}S^2V & & \pi^n\mathbf{D}V \arrow[ll, "((\mathrm{Sq}^k_{\mathrm{ext}})^\star)^*"'] \arrow[dll, "\mathrm{Sq}^k_{\mathrm{ext}}"] \\
\pi^{n+k}S_2\mathbf{D}V \arrow[u] & &
\end{tikzcd}$$

In order to define these precursor homotopy operations, Goerss [33, Proposition 3.7] observes that any element of $x \in \pi_m(S^2V)$ can be written as a sum

$$x = \sum_j \pi_*(1+T)(y_j \otimes z_j) + \sum_{0 \le k \le \lfloor m/2 \rfloor} \sigma_k(w_k),$$

where $w_k \in \pi_{m-k}V$ for $0 \le k \le \lfloor m/2 \rfloor$, and $y_j, z_j \in \pi_*V$, and that we may *define*:

$$(\mathrm{Sq}^k_{\mathrm{ext}})^\star(x) := w_k.$$

One might only need to determine the operations $(\mathrm{Sq}^k_{\mathrm{ext}})^\star$ for $k < m/2$, so that when m is even we may ignore the dual of the top operation, $(\mathrm{Sq}^{m/2}_{\mathrm{ext}})^\star$. In this case, it is more convenient to rewrite the key equation as:

$$x = \widetilde{\nabla}(v) + \sum_{0 \le k < m/2} \sigma_k(w_k),$$

where $v \in (S^2(\pi_*V))_m$ and $w_k \in \pi_{m-k}V$ for $0 \le k < m/2$.

6.4. External binary cohomotopy operations

Again, the arguments of [52, §1.12] imply:

Proposition 6.2. *Suppose that $U \in c\mathcal{V}$. Then there is a pairing*

$$\mu_{\mathrm{ext}} : S_2(\pi^*U) \longrightarrow \pi^*(S_2U),$$

defined by $x \otimes y \longmapsto D^0(x \otimes y)$, with the property that $\mu_{\mathrm{ext}}(\alpha \otimes \alpha) = \mathrm{Sq}^k_{\mathrm{ext}}\alpha$ for $\alpha \in \pi^kU$.

Unsurprisingly, these bear relation to the homotopy operation $\widetilde{\nabla} : \pi_*S^2V \longrightarrow S^2\pi_*V$, via a commuting diagram, for any $V \in s\mathcal{V}$:

$$\begin{tikzcd}
\mathbf{D}\pi_*S^2V & \mathbf{D}S^2\pi_*V \arrow[l, "(\widetilde{\nabla})^*"'] \\
\pi^*S_2\mathbf{D}V \arrow[u] & S_2\pi^*\mathbf{D}V \arrow[l, "\mu_{\mathrm{ext}}"'] \arrow[u]
\end{tikzcd}$$

We might have denoted $\widetilde{\nabla}$ by $\mu^\star_{\mathrm{ext}}$, but decided against the idea.

6.5. Chain level structure for cohomology operations; the maps $\xi_{\mathcal{C}}$ and $\psi_{\mathcal{C}}$

We will now give generalizations of Goerss' constructions in [33, §5] which often yield useful structure on the complexes calculating $\mathcal{C}$-cohomology. Suppose that $X \in s\mathcal{C}$ is almost free, with $V_s \subseteq X_s$ the freely generating subspace. Then for each s, the functor

$$\mathrm{Hom}_{\mathcal{C}}(X_s, -) \cong \mathrm{Hom}_{\mathcal{V}}(V_s, U^{\mathcal{C}}-)$$

is naturally an $\mathbb{F}_2$-vector space. Writing $\varphi_s = F^{\mathcal{C}}(\Delta) : X_s \longrightarrow X_s \sqcup X_s$, the addition operation on $\mathrm{Hom}_{\mathcal{C}}(X_s, -)$ is given by $(f, g) \longmapsto (f \sqcup g) \circ \varphi_s$. Now let $\overline{\xi}_{\mathcal{C}}$ be the sum of $((d_0 \sqcup d_0)\varphi_s)$ and $(\varphi_{s-1} d_0)$ in the $\mathbb{F}_2$-vector space $\mathrm{Hom}_{\mathcal{C}}(X_s, X_{s-1} \sqcup X_{s-1})$. It is completely formal to check that $\overline{\xi}_{\mathcal{C}}$ maps to zero in the group

$$\mathrm{Hom}_{\mathcal{C}}(X_s, X_{s-1} \times X_{s-1}) = \mathrm{Hom}_{\mathcal{C}}(X_s, X_{s-1}) \times \mathrm{Hom}_{\mathcal{C}}(X_s, X_{s-1}),$$

and thus $\overline{\xi}_{\mathcal{C}}$ factors through a unique map $\xi_{\mathcal{C}} : X_s \longrightarrow X_{s-1} \veebar X_{s-1}$. Furthermore, $\xi_{\mathcal{C}}$ enjoys the symmetry $\tau\xi_{\mathcal{C}} = \xi_{\mathcal{C}}$, and it is again formal to verify the analogue of [33, Lemma 5.5]:

Lemma 6.3. *When the equation $\mathrm{qu}_{\mathcal{C}} \circ \mu_V = \mathrm{qu}_{\mathcal{C}} \circ \epsilon_{F^{\mathcal{C}}V} + \mathrm{qu}_{\mathcal{C}} \circ F^{\mathcal{C}}\epsilon_V$ of §3.10 is satisfied, the map $Q^{\mathcal{C}}\xi_{\mathcal{C}}$ induces a chain map of degree -1 on normalized complexes:*

$$N_s(Q^{\mathcal{C}}X) \longrightarrow N_{s-1}((Q^{\mathcal{C}}(X \veebar X))^{\Sigma_2}).$$

The composite

$$\psi_{\mathcal{C}} : \left(N_s(Q^{\mathcal{C}}X) \xrightarrow{Q^{\mathcal{C}}\xi_{\mathcal{C}}} N_{s-1}((Q^{\mathcal{C}}(X \veebar X))^{\Sigma_2}) \xrightarrow{j_{\mathcal{C}}} N_{s-1}(S^2(Q^{\mathcal{C}}X)) \right),$$

is essentially $\mathrm{qu}_{\mathcal{C}}$, in that if $v \in V_s \cap N_s X$ represents an element of $N_s Q^{\mathcal{C}}X$, writing $d_0 v = f(w_j)$ for $w_j \in V_{s-1}$, we have $\psi_{\mathcal{C}}(v) = \mathrm{qu}_{\mathcal{C}}(f)(w_j) \in S^2(V_{s-1})$.

The typical use of this structure is to define cohomology operations using the external cohomotopy operations defined above, i.e. natural operations on $H^*_{\mathcal{C}}X = \pi^*(\mathbf{D}(Q^{\mathcal{C}}B^{\mathcal{C}}X))$ defined by the composites:

$$H^{n_1}_{\mathcal{C}}X \otimes H^{n_2}_{\mathcal{C}}X \xrightarrow{\mu_{\mathrm{ext}}} \pi^{n_1+n_2}\mathbf{D}(S^2(Q^{\mathcal{C}}B^{\mathcal{C}}X)) \xrightarrow{\psi^*_{\mathcal{C}}} H^{n_1+n_2+1}_{\mathcal{C}}X,$$
$$H^n_{\mathcal{C}}X \xrightarrow{\mathrm{Sq}^k_{\mathrm{ext}}} \pi^{n+k}\mathbf{D}(S^2(Q^{\mathcal{C}}B^{\mathcal{C}}X)) \xrightarrow{\psi^*_{\mathcal{C}}} H^{n+k+1}_{\mathcal{C}}X.$$

These operations are the duals of natural homology co-operations, defined using the maps of §6.3 and §5.3:

$$(S^2 H^{\mathcal{C}}_* X)_n \xleftarrow{\widetilde{\nabla}} \pi_n(S^2(Q^{\mathcal{C}} B^{\mathcal{C}} X)) \xleftarrow{\psi^*_{\mathcal{C}}} H^{\mathcal{C}}_{n+1} X,$$
$$H^{\mathcal{C}}_n(X) \xleftarrow{(\mathrm{Sq}^k_{\mathrm{ext}})^\star} \pi_{n+k}(S^2(Q^{\mathcal{C}} B^{\mathcal{C}} X)) \xleftarrow{\psi^*_{\mathcal{C}}} H^{\mathcal{C}}_{n+k+1}(X).$$

Instead of proving Lemma 6.3, we will prove the more general:

Proposition 6.4. *Suppose that $\theta : F^{\mathcal{C}} V \longrightarrow GV$ is a natural transformation from $F^{\mathcal{C}}$ to another endofunctor G of $\mathcal{V}$ satisfying the condition:*

$$\theta \circ \mu_V = \theta \circ \epsilon_{F^{\mathcal{C}} V} + \theta \circ F^{\mathcal{C}} \epsilon_V : F^{\mathcal{C}} F^{\mathcal{C}} V \longrightarrow GV.$$

Write $\widetilde{\theta} : Q^{\mathcal{C}} X_s \longrightarrow G(Q^{\mathcal{C}} X_{s-1})$ for the following

$$Q^{\mathcal{C}} X_s \xrightarrow{\cong} V_s \xrightarrow{d_0} F^{\mathcal{C}} V_{s-1} \xrightarrow{\theta} GV_{s-1} \xrightarrow{\cong} G(Q^{\mathcal{C}} X_{s-1}).$$

Then $d_0 \circ \widetilde{\theta} = \widetilde{\theta} \circ (d_0 + d_1)$, and $d_j \circ \widetilde{\theta} = \widetilde{\theta} \circ d_{j+1}$ for $j \geq 1$, so that $\widetilde{\theta}$ restricts to a degree -1 chain map on $N_s Q^{\mathcal{C}} X \longrightarrow N_{s-1} Q^{\mathcal{C}} X$ and also on $C_s Q^{\mathcal{C}} X \longrightarrow C_{s-1} Q^{\mathcal{C}} X$.

Note that $\widetilde{\theta}$ depends on the almost free structure chosen.

Proof. In order to see that $d_j \circ \widetilde{\theta} = \widetilde{\theta} \circ d_{j+1}$ for $j \geq 1$, we examine the diagram

$$\begin{tikzcd}
V_s \arrow[r, "d_0"] \arrow[d, dotted, "d_{j+1}"] & F^{\mathcal{C}} V_{s-1} \arrow[r, "\theta"] \arrow[d, "d_j"] & GV_{s-1} \arrow[r, "\cong"] \arrow[d, dotted, "d_j"] & GQ^{\mathcal{C}} X_{s-1} \arrow[d, "GQ^{\mathcal{C}}(d_j)"] \\
V_{s-1} \arrow[r, "d_0"] & F^{\mathcal{C}} V_{s-2} \arrow[r, "\theta"] & GV_{s-2} \arrow[r, "\cong"] & GQ^{\mathcal{C}} X_{s-2}
\end{tikzcd}$$

The dotted vertical arrows are available since X is almost free. That the left square commutes is a simplicial identity, and the center square commutes by naturality of θ. In order to show that $d_0 \circ \widetilde{\theta} = \widetilde{\theta} \circ (d_0 + d_1)$, we use the following diagram, which commutes except for the leftmost square:

$$\begin{tikzcd}
V_s \arrow[r, "d_0"] \arrow[d, dotted, "d_1 + \epsilon\circ d_0"'] & F^{\mathcal{C}} V_{s-1} \arrow[r, "\theta"] \arrow[d, "F^{\mathcal{C}}(\epsilon\circ d_0)"] & GV_{s-1} \arrow[r, "\cong"] \arrow[d, dotted, "G(\epsilon\circ d_0)"] & GQ^{\mathcal{C}} X_{s-1} \arrow[d, "GQ^{\mathcal{C}}(d_0)"] \\
V_{s-1} \arrow[r, "d_0"] & F^{\mathcal{C}} V_{s-2} \arrow[r, "\theta"] & GV_{s-2} \arrow[r, "\cong"] & GQ^{\mathcal{C}} X_{s-2}
\end{tikzcd}$$

To show that the outer rectangle commutes, it is enough to see that the two composites $V_s \longrightarrow F^{\mathcal{C}} V_{s-2}$ are coequalized by θ. Using the simplicial identity $d_0 d_1 = d_0 d_0$, we are trying to show that $\theta d_0 d_0 + \theta d_0 \epsilon d_0$ and $\theta F^{\mathcal{C}}(\epsilon d_0) d_0$ are the same map from V_s to GV_{s-2}. Even more, we will show that $\theta d_0 + \theta d_0 \epsilon$ and $\theta F^{\mathcal{C}}(\epsilon d_0)$ are the same map $F^{\mathcal{C}} V_{s-1}$ to GV_{s-2}.

Starting with an expression $f(v_i)$ in various $v_i \in V_{s-1}$, we calculate $\theta d_0 f(v_i) = \theta(f d_0 v_i)$, $\theta d_0 \epsilon f(v_i) = \theta(\epsilon(f)(d_0 v_i))$ and $\theta F^{\mathcal{C}}(\epsilon d_0)(f(v_i)) = \theta(f)(\epsilon(d_0 v_i))$. That these three terms add to zero was the requirement specified for θ. $\square$

6.6. Cohomology operations for simplicial commutative algebras

Goerss [33, §5] defines cohomology operations, natural in $A \in s\mathscr{C}om$:

$$\begin{gathered} P^i = \psi^*_{\mathscr{C}om} \circ \mathrm{Sq}^i_{\mathrm{ext}} : H^n_{\mathscr{C}om} A \longrightarrow H^{n+i+1}_{\mathscr{C}om} A; \text{ and} \\ [\,,] = \psi^*_{\mathscr{C}om} \circ \mu_{\mathrm{ext}} : H^n_{\mathscr{C}om} A \otimes H^m_{\mathscr{C}om} A \longrightarrow H^{n+m+1}_{\mathscr{C}om} A. \end{gathered}$$

He also defines a natural operation $\beta : H^0_{\mathscr{C}om} A \longrightarrow H^1_{\mathscr{C}om} A$. Note that as a result of the use of $\psi^*_{\mathscr{C}om}$, these operations have a grading shift.

Proposition 6.5 [33, §5]**.** *These operations have the following properties:*

(1) the bracket gives $H^*_{\mathscr{C}om} A$ the structure of an $S(\mathscr{L})$-algebra (with grading shift);

(2) the operation β acts as a restriction defined only in dimension zero, so that for $x, y \in H^0_{\mathscr{C}om} A$ and $z \in H^*_{\mathscr{C}om} A$:

$$\beta(x + y) = \beta(x) + \beta(y) = [x, y], \quad \textit{and} \quad [\beta(x), z] = [x, [x, z]];$$

(3) the self-bracket operation on $H^*_{\mathscr{C}om} A$ equals the top P-operation*:*

$$P^n x = [x, x] \quad \textit{for } x \in H^n_{\mathscr{C}om} A;$$

(4) if $x \in H^n_{\mathscr{C}om} A$, then $P^i x = 0$ unless $2 \le i \le n$;

(5) every P-operation is linear;

(6) there holds the following Cartan formula*: for all $x, y \in H^*_{\mathscr{C}om} A$ and $i \ge 0$,*

$$[x, P^i y] = 0;$$

(7) the P-Adem relations *hold: if $i \ge 2j$, then*

$$P^i P^j x = \sum_{s=i-j+1}^{i+j-2} \binom{2s-i-1}{s-j} P^{i+j-s} P^s x.$$

In this case, *(7) does* state that $H^*_{\mathscr{C}om} A$ is a left module over $\mathcal{P}$, the *Steenrod algebra for commutative algebras over* $\mathbb{F}_2$ of $\mathcal{P}$-algebra. This is the unital associative algebra generated by symbols P^i for $i \geq 0$, modulo the two sided ideal generated by P^0, P^1, and the evident *P-Adem relations*.

A sequence $I = (i_\ell, \ldots, i_1)$ of integers $i_j \geq 2$ is *P-admissible* if $i_{j+1} < 2i_j$ for $1 \leq j < \ell$. For any sequence $I = (i_\ell, \ldots, i_1)$, write P^I for the monomial $P^{i_\ell} \cdots P^{i_1}$ in $\mathcal{P}$. It follows from [33, Theorem I] that $\mathcal{P}$ has an *admissible basis*, consisting of those $P^I = P^{i_\ell} \cdots P^{i_1}$ with I a P-admissible sequence. Again, we will say that I *produces* J *in* $\mathcal{P}$, denoted $I \xrightarrow{\mathcal{P}} J$ if, when P_I is written in the P-admissible basis of $\mathcal{P}$, P_J appears with non-zero coefficient. In this case, unless $J = I$, J is P-admissible and I is P-inadmissible.

We define

$$\overline{m}(I) := \max\{(i_1),\, (i_2 - i_1 - 1),\, (i_3 - i_2 - i_1 - 2),\, \ldots,\, (i_\ell - \cdots - i_1 - \ell + 1)\},$$

for a rather different purpose than in §5.4 and §5.5: although the composite

$$P^I : \left(H^n_{\mathscr{C}om} A \xrightarrow{P^{i_1}} H^{n+i_1+1}_{\mathscr{C}om} A \xrightarrow{P^{i_2}} \cdots \xrightarrow{P^{i_\ell}} H^{n+i_1+\cdots+i_\ell+\ell}_{\mathscr{C}om} A\right)$$

is always defined, it is *forced to be zero* (by *(4)* alone) except when $n \geq \overline{m}(I)$.

As in §5.4 and §5.5, if a non-empty sequence I is *P-admissible*, we can identify which term is largest in the maximum defining $\overline{m}(I)$, and calculate that $\overline{m}(I) = i_1$. More explicitly, in a non-vanishing admissible expression $P^{i_\ell} \cdots P^{i_1} x$, for $x \in H^n_{\mathscr{C}om} A$, the only P-operations that can be a top operation is P^{i_1}.

The following result shows that whenever an expression $P^J x$ is forced to be zero by *(4)* and we reduce $P^J x$ to a sum of P-admissible composites, then *(4)* forces all of the resulting summands to be zero.

Lemma 6.6. *If $I \xrightarrow{\mathcal{P}} J$, then $\overline{m}(J) \geq \overline{m}(I)$, with strict inequality when I and J are distinct and of length two.*

The main theorem of Goerss' memoir is that these operations generate all of the operations in the category $H\mathscr{C}om$, and that all the relations between them are implied by those presented here. In [33, Chapter V], Goerss shows that the listed operations completely capture the cohomology of an object $\mathbb{K}^{\mathscr{C}om}_n$. He proves a Hilton-Milnor Theorem [32], which he uses in [33, §11] to bootstrap up to a calculation of the cohomology of any GEM in $s\mathscr{C}om$, namely [33, Theorem I]. The result states that whenever $V \in \mathcal{V}^1$, is a vector space of finite type, not only is $F^{\mathscr{C}om} V$ generated by V under the operations of Proposition 6.5, it is as large as is conceivable given the relations presented. We will present, in Proposition 6.8, a partial version of his result.

It is interesting to observe that $\mathcal{P}$, the Steenrod algebra for commutative algebras, is in fact *Koszul dual* (c.f. [46]) to Δ, the algebra which possesses an unstable partial left action on the homotopy of a simplicial algebra. Indeed, Goerss *calculates* $\mathcal{P}$ as the Koszul dual of Δ, using a reverse Adams spectral sequence due to Miller [42] (c.f. §3.4). We will explore this duality further when we consider the Bousfield-Kan spectral sequence.

6.7. The categories $\mathcal{W}(0)$ and $\mathcal{U}(0)$

Suppose that $A \in s\mathscr{C}om$ is *connected*, i.e. that $\pi_0 A = 0$. Then $H^0_{\mathscr{C}om} A = Q^{\mathscr{C}om}\pi_0 A = 0$, so that the operation β can be ignored. This is a convenient by-product of working with the cohomology of connected objects, although the real reason that we do so is that doing so avoid completion and convergence problems. If we say that $V \in \mathcal{V}^1$ is *connected* when $V^0 = 0$, we may identify the full subcategory of $\mathcal{V}^1$ on the connected objects with $\mathcal{V}^+$.

Goerss' result proves that the monad $F^{H\mathscr{C}om}$ on $\mathcal{V}^1$ preserves $\mathcal{V}^+$ (and indeed it is a general fact that no non-trivial natural cohomology operations decrease dimension). We will write $\mathcal{W}(0)$ for the category of connected $\mathscr{C}om$-H^*-algebras, so that the monad $F^{\mathcal{W}(0)}$ is simply the restriction of $F^{H\mathscr{C}om}$ to $\mathcal{V}^+$. The way that we will report Goerss' result here is to explain how the monad $F^{\mathcal{W}(0)}$ may be constructed on objects of $\mathcal{V}^+$ of finite type.

Let the *category of unstable $\mathcal{P}$-modules*, denoted $\mathcal{U}(0)$, be the category whose objects are $\mathcal{V}^+_0$-graded $\mathcal{P}$-modules in which P^i acts with grading $i+1$ by everywhere defined maps

$$P^i : V^n \longrightarrow V^{n+i+1}$$

which equal zero unless $2 \leq i \leq n$. Recall that we have *already* imposed the P-Adem relations and set P^0 and P^1 to be zero in $\mathcal{P}$.

Proposition 6.7. *The monad $F^{\mathcal{U}(0)}$ may be defined by*

$$\begin{aligned} F^{\mathcal{U}(0)}V &:= (\mathcal{P} \otimes V) \,/\, \mathbb{F}_2\{P^I \otimes v \mid V \in V^n,\ \overline{m}(I) > n\} \\ &= (\mathcal{P} \otimes V) \,/\, \mathbb{F}_2\{P^I \otimes v \mid V \in V^n,\ \overline{m}(I) > n,\ I \text{ is } P\text{-admissible}\}. \end{aligned}$$

Proof. This follows from Goerss' [33, Theorem I] and Lemma 6.6. □

Now an object of $\mathcal{W}(0)$ is in particular an object of $\mathcal{U}(0)$. It is also a (degree shifted) $S(\mathscr{L})$-algebra. Thus, there is a natural map

$$F^{\mathcal{U}(0)}F^{S(\mathscr{L})}V \longrightarrow F^{\mathcal{W}(0)}V.$$

This map is not an isomorphism, but it follows from [33, Theorem I] that it is surjective. Moreover, our final reading of Goerss' result is:

Proposition 6.8. *For $V \in \mathcal{V}^+$ of finite type, $F^{\mathcal{W}(0)}V \in \mathcal{V}^+$ is the coequalizer:*

$$\mathrm{coeq}\left(\mathcal{P} \otimes F^{S(\mathscr{L})}V \overset{\mathrm{sb}_1}{\underset{\mathrm{sb}_2}{\rightrightarrows}} \mathcal{P} \otimes F^{S(\mathscr{L})}V \twoheadrightarrow F^{\mathcal{U}(0)}F^{S(\mathscr{L})}V \right),$$

where the maps sb_1 *and* sb_2 *are defined on* $\mathcal{P} \otimes (F^{S(\mathscr{L})}V)^m$ *by*

$$\mathrm{sb}_1(P^I \otimes x) = P^I \otimes [x, x] \quad \text{and} \quad \mathrm{sb}_2(P^I \otimes x) = P^I P^m \otimes x.$$

Choose a homogeneous basis of V, construct from it a monomial basis of $\Lambda(\mathscr{L})V$ (such as any choice of Hall basis), and then lift these monomials in the evident way to a collection B of elements of $S(\mathscr{L})V$. Then a basis of $F^{\mathcal{W}(0)}V$ is

$$\left\{ P^I b \;\middle|\; b \in B,\ \overline{m}(I) \leq |b|,\ I \text{ is } P\text{-admissible} \right\}.$$

Corollary 6.9. *Suppose that $V \in \mathcal{V}^+$ is of finite type. Then so is $F^{\mathcal{W}(0)}V$.*

The following observations will be useful for the calculation of the cohomology of objects of $\mathcal{W}(0)$.

Lemma 6.10. *The monads $F^{\mathcal{U}(0)}$ and $F^{\mathcal{W}(0)}$ on $\mathcal{V}^+$ may be promoted to monads on the category $\mathrm{q}\mathcal{V}^+$, by insisting that quadratic gradings add when taking brackets and double when applying P-operations.*

It is typical to think of $V \in \mathcal{V}^+$ as an object of $\mathrm{q}\mathcal{V}^+$ concentrated in quadratic grading one when considering $F^{\mathcal{W}(0)}V$.

An object of $\mathcal{W}(0)$ is in particular an object of $\mathcal{U}(0)$, and (as all of the P-operations are linear), we can define a functor $Q^{\mathcal{U}(0)} : \mathcal{W}(0) \longrightarrow \mathcal{V}_0^+$ which takes the quotient by the image of the P-operations. Moreover:

Lemma 6.11. *For $X \in \mathcal{W}(0)$, the vector space $Q^{\mathcal{U}(0)}X \in \mathcal{V}^+$ inherits a (grading shifted) Lie algebra structure from the bracket of X, yielding a factorization:*

$$Q^{\mathcal{W}(0)} = Q^{\Lambda(\mathscr{L})} \circ Q^{\mathcal{U}(0)} : \left(\mathcal{W}(0) \longrightarrow \Lambda(\mathscr{L}) \longrightarrow \mathcal{V}_0^+ \right).$$

Moreover the composite $Q^{\mathcal{U}(0)} \circ F^{\mathcal{W}(0)}$ equals the free construction $F^{\mathcal{L}(0)}$.

Proof. One checks that the bracket is well defined in the quotient, and that taking the quotient by the top P-operation imposes the relation $[x, x] = 0$, to create a $\Lambda(\mathscr{L})$-algebra from the pre-existing $S(\mathscr{L})$-algebra structure. The final claim follows from Proposition 6.8. $\square$

6.8. Cohomology operations for simplicial (restricted) Lie algebras

A standard definition of the cohomology of a simplicial Lie algebra $L \in s\mathscr{L}ie$ or $s\mathscr{L}ie^r$ is presented in [47] as follows. Let UL be the simplicial primitively Hopf algebra obtained by applying the universal enveloping algebra functor levelwise, or the *restricted* universal enveloping algebra functor when working in $s\mathscr{L}ie^r$. Applying the Eilenberg-Mac Lane suspension functor ([47, §2.3], [42, §5] or [40, p. 87]), one defines (using a subscript $\bar{W}$ to avoid confusion):

$$H^*_{\bar{W}}L := \begin{cases} \pi^*\mathbf{D}\bar{W}UL, & \text{if } * > 0; \\ 0, & \text{if } * = 0. \end{cases}$$

We discuss universal enveloping algebra functors in Appendix A.1. The suspension $\bar{W}$ destroys the associative algebra structure but leaves a simplicial cocommutative coalgebra structure on $\bar{W}UL$, with diagonal we denote by Δ. Homotopy operations for simplicial cocommutative coalgebras are well known, being the mode of definition of the cup product and Steenrod operations present in the category $\mathcal{K}$ discussed in §1.1 of unstable algebras over the Steenrod algebra, and can be constructed using Propositions 6.1 and 6.2:

$$\mathrm{Sq}^k := \Delta^* \circ \mathrm{Sq}^k_{\mathrm{ext}} : \left(\pi^n\mathbf{D}(\bar{W}UL) \xrightarrow{\mathrm{Sq}^k_{\mathrm{ext}}} \pi^{n+k}\mathbf{D}S^2(\bar{W}UL) \xrightarrow{\Delta^*} \pi^{n+k}\mathbf{D}(\bar{W}UL)\right);$$
$$\mu := \Delta^* \circ \mu_{\mathrm{ext}} : \left(S_2(\pi^*\mathbf{D}(\bar{W}UL))^n \xrightarrow{\mu_{\mathrm{ext}}} \pi^n S_2\mathbf{D}(\bar{W}UL) \longrightarrow \pi^n\mathbf{D}S^2(\bar{W}UL) \xrightarrow{\Delta^*} \pi^n\mathbf{D}(\bar{W}UL)\right).$$

The operations here make $H^*_{\bar{W}}L$ a module over the homogeneous Steenrod algebra discussed in §1.3, which is the usual mod 2 Steenrod algebra 'with Sq^0 set to zero'. That is, the homogeneous Steenrod algebra is the unital associative algebra $\mathcal{A}$ generated by symbols Sq^j for $j \geq 1$, subject to the *homogeneous* Sq-*Adem relation*:

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{k=1}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k \text{ for } i < 2j.$$

We only ever work with the homogeneous Steenrod algebra and the homogeneous Sq-Adem relation, and so may omit the word homogeneous if we desire.

This algebra is Koszul dual to the *opposite* of the Λ-algebra (c.f. 5.5). There is an index shift in this duality, so that Sq^i corresponds to λ_{i-1} for $i \geq 1$ [46, §7.1].

In [47], Priddy concentrates on simplicial *restricted* Lie algebras L, and works out all of the natural operations on $H^*_{\bar{W}}$ and the relations between them. Moreover, he gives a spectral sequence argument showing that the two notions of cohomology are isomorphic, *with a shift in degree* arising from the use of $\bar{W}$: $H^n_{\bar{W}}L \cong H^{n-1}_{\mathscr{L}ie^r}L$ for $n \geq 1$.

For our purposes it is better to work in the framework set out in §6.5, giving an alternative definition of Priddy's operations. This alternative definition will fit more readily into the spectral sequence arguments we intend to make.

For now, let $\mathcal{C}$ stand either for $\mathscr{L}ie^r$ or $\mathscr{L}ie$. Our definition of the cohomology operations is:

$$\mathrm{Sq}^k := \psi_{\mathcal{C}}^* \circ \mathrm{Sq}_{\mathrm{ext}}^{k-1} : H_{\mathcal{C}}^n L \xrightarrow{\mathrm{Sq}_{\mathrm{ext}}^{k-1}} \pi^{n+k-1}\mathbf{D}(S^2(Q^{\mathcal{C}}B^{\mathcal{C}}L)) \xrightarrow{\psi_{\mathcal{C}}^*} H_{\mathcal{C}}^{n+k}L,$$
$$\mu := \psi_{\mathcal{C}}^* \circ \mu_{\mathrm{ext}} : (S_2 H_{\mathcal{C}}^* L)^n \xrightarrow{\mu_{\mathrm{ext}}} \pi^n\mathbf{D}(S^2(Q^{\mathcal{C}}B^{\mathcal{C}}L)) \xrightarrow{\psi_{\mathcal{C}}^*} H_{\mathcal{C}}^{n+1}L.$$

We will check the properties of these operations using a spectral sequence argument similar to Priddy's, although we will need to give a richer construction of the spectral sequence in order to extract information about the operations. This work will be deferred until Appendix A, and will prove:

Proposition 6.12. *There are commuting diagrams:*

$$\begin{tikzcd} H_{\mathcal{C}}^n L \arrow[r, "\psi_{\mathcal{C}}^*\circ\mathrm{Sq}_{\mathrm{ext}}^{k-1}"] \arrow[d, "\cong"] & H_{\mathcal{C}}^{n+k}L \arrow[d, "\cong"] \\ H_{\bar{W}}^{n+1}L \arrow[r, "\Delta^*\circ\mathrm{Sq}_{\mathrm{ext}}^{k}"] & H_{\bar{W}}^{n+k+1}L \end{tikzcd} \qquad \begin{tikzcd} H_{\mathcal{C}}^{n_1}L\otimes H_{\mathcal{C}}^{n_2}L \arrow[r, "\psi_{\mathcal{C}}^*\circ\mu_{\mathrm{ext}}"] \arrow[d, "\cong"] & H_{\mathcal{C}}^{n_1+n_2+1}L \arrow[d, "\cong"] \\ H_{\bar{W}}^{n_1+1}L\otimes H_{\bar{W}}^{n_2+1}L \arrow[r, "\Delta^*\circ\mu_{\mathrm{ext}}"] & H_{\bar{W}}^{n_1+n_2+2}L \end{tikzcd}$$

That is, the two definitions of Sq^k *coincide, as do the two definitions of* μ.

Given the use of suspension $\bar{W}$, one expects the notion of *top Steenrod operation* to be different to that in other settings, and in this context we say that $\mathrm{Sq}^{n+1} : H_{\mathcal{C}}^n L \longrightarrow H_{\mathcal{C}}^{2n+1}L$ is the top operation.

Proposition 6.13 [47, §5.3]**.** *These operations have the following properties:*

(1) the product μ *gives* $H_{\mathcal{C}}^*L$ *the structure of a commutative algebra (with grading shift);*

(2) the squaring operation on $H_{\mathcal{C}}^*L$ *equals the* top Steenrod operation*:*

$$\mathrm{Sq}^{n+1}x = x^2 \quad \text{for } x \in H_{\mathcal{C}}^n L;$$

(3) if $x \in H_{\mathcal{C}}^n L$*, then* $\mathrm{Sq}^i x = 0$ *unless* $1 \leq i \leq n+1$*;*

(4) every Steenrod operation is linear;

(5) the Cartan formula *holds: for all* $x, y \in H_{\mathcal{C}}^* L$ *and* $i \geq 0$*,*

$$\mathrm{Sq}_{\mathrm{h}}^i(xy) = \textstyle\sum_{k=1}^{i-1}(\mathrm{Sq}^k x)(\mathrm{Sq}^{i-k}y);$$

(6) the homogeneous Sq*-Adem relations hold, making* $H_{\mathcal{C}}^*L$ *a left* $\mathcal{A}$*-module.*

This fact follows from Proposition A.3. We will also use the following calculation:

Proposition 6.14. *If* $\mathcal{C} = \mathscr{L}ie$ *(as opposed to* $\mathscr{L}ie^r$*), then* $\mathrm{Sq}^1 = 0$. *In particular, for* $x \in H^0_{\mathscr{L}ie}X$, $x^2 = 0$.

Proof. It is enough to prove this for the universal example $\imath_n \in H^*_{\mathscr{L}ie}\mathbb{K}_n^{\mathscr{L}ie}$. The reverse Adams spectral sequence (§3.4) is of the form

$$E_2^{p,q} = (H^p_{\pi\mathscr{L}ie}\mathbb{K}_{0,n}^{\pi\mathscr{L}ie})^q \implies H^{p+q}_{\mathscr{L}ie}\mathbb{K}_n^{\mathscr{L}ie}.$$

The point now is that $\mathbb{K}_{0,n}^{\pi\mathscr{L}ie}$, which is just a constant object in $s(\pi\mathscr{L}ie)$ with value a one-dimensional Lie algebra in internal dimension n, is actually free as an object of $\pi\mathscr{L}ie$ below internal dimension $n+1$. This is simply because there is no λ_0 operation defined in $\pi\mathscr{L}ie$. One may thus construct pas-à-pas a simplicial resolution of $\mathbb{K}_{0,n}^{\pi\mathscr{L}ie}$ (a process described in [1]) which in positive simplicial dimension is concentrated in internal dimension at least $n+1$, implying that $E_2^{p,q} = 0$ when $p \geq 1$ and $q \leq n$. Moreover, $E_2^{0,q} = 0$ unless $q = n$, showing that $H^{n+1}_{\mathscr{L}ie}\mathbb{K}_n^{\mathscr{L}ie} = 0$. This group contains $\mathrm{Sq}^1\imath_n$. $\square$

A sequence $I = (i_\ell, \ldots, i_1)$ of integers $i_j \geq 1$ is Sq-*admissible* if $i_{j+1} \geq 2i_j$ for $1 \leq j < \ell$. For any sequence $I = (i_\ell, \ldots, i_1)$, write Sq^I for the monomial $\mathrm{Sq}^{i_\ell} \cdots \mathrm{Sq}^{i_1}$. The homogeneous Steenrod algebra has the expected admissible basis, and we say that I *produces* J *in* $\mathcal{A}$, denoted $I \overset{\mathrm{Sq}}{\to} J$ if Sq^J appears in the Sq-admissible expansion of Sq^I.

We use the function m defined in §5.4, this time noting that the composite

$$\mathrm{Sq}^I : \left(H^n_{\mathscr{C}om}A \xrightarrow{\mathrm{Sq}^{i_1}} H^{n+i_1}_{\mathscr{C}om}A \xrightarrow{\mathrm{Sq}^{i_2}} \cdots \xrightarrow{\mathrm{Sq}^{i_\ell}} H^{n+i_1+\cdots+i_\ell}_{\mathscr{C}om}A\right)$$

is *forced to be zero* by *(3)* alone except when $n \geq m(I) - 1$.

If a non-empty sequence I is Sq-*admissible*, we have

$$m(I) = e(I) = i_\ell - i_{\ell-1} - \cdots - i_1,$$

the Serre excess of I. We now have enough notation available to describe the category $H\mathscr{L}ie^r$, using Priddy's calculations. The results are similar to those in §5.4 on the category $\pi\mathscr{C}om$. There is again a *Künneth Theorem*:

Proposition 6.15. *Suppose that* K_1 *and* K_2 *are finite GEMs in* $s\mathscr{L}ie^r$. *Then* $H^*_{\mathscr{L}ie^r}K_1$ *and* $H^*_{\mathscr{L}ie^r}K_2$ *in* $H\mathscr{L}ie^r$ *are of finite type, and their coproduct* $H^*_{\mathscr{L}ie^r}(K_1 \times K_2)$ *in* $H\mathscr{L}ie^r$ *may be calculated as the non-unital (grading shifted) commutative algebra coproduct of* $H^*_{\mathscr{L}ie^r}K_1$ *and* $H^*_{\mathscr{L}ie^r}K_2$.

Proof. We rely on Proposition 6.12 and the following calculation:

$$\begin{aligned}
H^*_{\bar{W}}(K_1 \times K_2) &:= \pi^*\mathbf{D}\bar{W}U(K_1 \times K_2) \\
&\cong \pi^*\mathbf{D}\bar{W}(UK_1 \otimes UK_2) \\
&\cong \pi^*\mathbf{D}(\bar{W}UK_1 \otimes \bar{W}UK_2) \\
&\cong \mathbf{D}(\pi_*\bar{W}UK_1 \otimes \pi_*\bar{W}UK_2) \\
&\supseteq H^*_{\bar{W}}K_1 \otimes H^*_{\bar{W}}K_2.
\end{aligned}$$

This containment is in fact an equality when $H^*_{\bar{W}}K_1$ and $H^*_{\bar{W}}K_2$ are both of finite type, in which case $H^*_{\bar{W}}(K_1 \times K_2)$ is also of finite type, and the isomorphism is proved. Thus, by induction on the total number of factors $\mathbb{K}_n^{\mathscr{L}ie^r}$ of K_1 and K_2, we only need to check that $H^*_{\bar{W}}\mathbb{K}_n^{\mathscr{L}ie^r}$ is of finite type for any $n \geq 0$. This is implied by a calculation of Priddy [47, 6.1] which we recall in Proposition 6.16. $\square$

After giving the calculation on a single Eilenberg-Mac Lane object, the cohomology of finite GEMs, and thus the category $H\mathscr{L}ie^r$ is determined by Proposition 6.15 and the Cartan formula. The structure defining $\pi\mathscr{L}ie^r$ is then well understood in light of:

Proposition 6.16 [47, 6.1]**.** *For $n \geq 0$, let $\imath$ be the fundamental class in $H^n_{\mathscr{L}ie^r}(\mathbb{K}_n^{\mathscr{L}ie^r})$. Then, as non-unital (degree shifted) commutative algebras:*

$$H^n_{\mathscr{L}ie^r}(\mathbb{K}_n^{\mathscr{L}ie^r}) \cong S(\mathscr{C})\big[\mathrm{Sq}^I\imath \;\big|\; I \text{ is Sq-admissible, } e(I) \leq n\big].$$

Corollary 6.17. *Suppose that $V \in \mathcal{V}^+$ is of finite type. Then so is $F^{H\mathscr{L}ie^r}V$. That is, the restriction of the monad $F^{H\mathscr{L}ie^r}V$ on $\mathcal{V}^1$ to $\mathcal{V}^+$ preserves objects of finite type.*

The case of simplicial Lie algebras mimics that of simplicial commutative algebras: for Lie algebras, the homogeneous Steenrod algebra acts on cohomology, and is Koszul dual to the opposite of the Λ-algebra, which possesses an unstable partial left action on homotopy. Further material on the cohomology of Lie algebras is deferred to Appendix A.

Chapter 7

Homotopy operations for partially restricted Lie algebras

7.1. The categories $\mathcal{L}(n)$ of partially restricted Lie algebras

For each $n \geq 0$, we will be interested in certain categories of Lie algebras monadic over $\mathcal{V}_n^+$, with a grading shift. Broadly, a $\mathcal{V}_n^+$-graded Lie algebra is a graded vector space $L \in \mathcal{V}_n^+$ with a structure map $\Lambda^2 L \longrightarrow L$ which shifts gradings as follows

$$L^t_{s_n,\ldots,s_1} \otimes L^{t'}_{s'_n,\ldots,s'_1} \longrightarrow L^{t+t'+1}_{s_n+s'_n,\ldots,s_1+s'_1}.$$

If we wished to be precise we could view the Lie operad as an operad in $(\mathcal{V}_n^+, \otimes)$ such that

$$\mathscr{L}(r) = (\mathscr{L}(r))^{r-1}_{0,\ldots,0},$$

and then a $\mathcal{V}_n^+$-graded Lie algebra would be an algebra over the corresponding monad $\Lambda(\mathscr{L})$ on $\mathcal{V}_n^+$. In our context, the Lie operad arises as the Koszul dual of the commutative operad, through the constructions in [33, §5], and the use of the operadic bar construction (c.f. [30, §3]) explains the shift. See [30, §5.3.4] for a discussion of Koszul duality of operads in positive characteristic. From this point forward we will simply think of such a Lie algebra as a vector space $L \in \mathcal{V}_n^+$ with a map $\Lambda^2 L \longrightarrow L$ shifting degrees as described.

A $\mathcal{V}_n^+$-graded *partially restricted* Lie algebra is to be a $\mathcal{V}_n^+$-graded Lie algebra such that certain graded parts admit a restriction operation. Specifically, there is to be defined a restriction operation

$$(-)^{[2]} : L^t_{s_n,\ldots,s_1} \longrightarrow L^{2t+1}_{2s_n,\ldots,2s_1}$$

whenever not all of $s_n, \ldots, s_1$ are zero. We will denote the category of such objects $\mathcal{L}(n)$.

It is monadic over $\mathcal{V}_n^+$, with an adjunction

$$F^{\mathcal{L}(n)} : \mathcal{V}_n^+ \rightleftarrows \mathcal{L}(n) : U^{\mathcal{L}(n)}.$$

The monad of this adjunction may be constructed as an appropriately chosen submonad of $\Gamma(\mathscr{L}) : \mathcal{V}_n^+ \longrightarrow \mathcal{V}_n^+$ (with $\mathscr{L}$ shifted as above), containing $\Lambda(\mathscr{L})$. As such, the free construction $F^{\mathcal{L}(n)}V$ admits a quadratic grading as in §2.10, which we denote $\mathrm{q}_k F^{\mathcal{L}(n)}V$.

7.2. Homotopy operations for $s\mathcal{L}(n)$

We will now state precisely how much of the structure given in §5.5 carries over to our new setting. If $L \in s\mathcal{L}(n)$, we may restrict the structure map $[\,,] : F^{\mathcal{L}(n)}L \longrightarrow L$ to a map

$$[\,,] : \mathrm{q}_2 F^{\mathcal{L}(n)}L \longrightarrow L, \quad \text{with} \quad \Lambda^2 L \subseteq \Sigma^{-1}\mathrm{q}_2 F^{\mathcal{L}(n)}L \subseteq S^2 L,$$

where the desuspension acts in the cohomological degree t. Only certain of the external homotopy operations $\pi_* V \longrightarrow \pi_* S^2 V$ defined in §5.2 factor through $\pi_*\Sigma^{-1}\mathrm{q}_2 F^{\mathcal{L}(n)}V$, and similarly for the operations of §5.3. One readily checks that the operations that factor in this way are:

$$\sigma_i^{\mathrm{ext}} : \pi_n V \longrightarrow \pi_{n+i}(\Sigma^{-1}\mathrm{q}_2 F^{\mathcal{L}(n)}V)$$

defined only when $0 \le i \le n$ and $i, s_1, \ldots, s_n$ are not all zero, and

$$\widetilde{\nabla} : \mathrm{q}_2 F^{\mathcal{L}(n+1)}(\pi_* V) \longrightarrow \pi_*(\mathrm{q}_2 F^{\mathcal{L}(n)}V).$$

The resulting operations on $\pi_* L$, for $L \in s\mathcal{L}(n)$, are right λ-operations

$$(-)\lambda_i : \Big((\pi_{s_{n+1}}L)^t_{s_n,\ldots,s_1} \xrightarrow{\sigma_i} (\pi_{s_{n+1}+i}(\Sigma^{-1}\mathrm{q}_2 F^{\mathcal{L}(n)}L))^{2t}_{2s_n,\ldots,2s_1} \xrightarrow{\pi_*([\,,])} (\pi_{s_{n+1}+i}L)^{2t+1}_{2s_n,\ldots,2s_1}\Big)$$

defined whenever $0 \le i \le s_{n+1}$ and not all of $i, s_n, \ldots, s_1$ equal zero, and a bracket:

$$\begin{aligned}[\,,] : \Big((\pi_* L)^t_{s_{n+1},\ldots,s_1} \otimes (\pi_* L)^{t'}_{s'_{n+1},\ldots,s'_1} &\longrightarrow (\mathrm{q}_2 F^{\mathcal{L}(n+1)}\pi_* L)^{t+t'+1}_{s_{n+1}+s'_{n+1},\ldots,s_1+s'_1} \\ &\xrightarrow{\widetilde{\nabla}} (\pi_*\mathrm{q}_2 F^{\mathcal{L}(n)}L)^{t+t'}_{s_{n+1}+s'_{n+1},\ldots,s_1+s'_1} \xrightarrow{\pi_*([\,,])} (\pi_* L)^{t+t'+1}_{s_{n+1}+s'_{n+1},\ldots,s_1+s'_1}\Big).\end{aligned}$$

We have written the bracket as a map from $(\pi_* L)^{\otimes 2}$ to clarify the degree shift, but nevertheless, the top λ-operation, whenever it is defined, acts as a restriction for this bracket. Indeed, this set of natural operations satisfies the evident modification of Proposition 5.7 (c.f. Proposition 7.1).

7.3. The category $\mathcal{U}(n+1)$ of unstable partial right Λ-modules

For $n \geq 0$, let $\mathcal{U}(n+1)$ denote *the category of unstable partial right* Λ*-modules*, the algebraic category whose objects are vector spaces $V \in \mathcal{V}^+_{n+1}$ equipped with *linear* right λ-operations

$$(-)\lambda_i : V^t_{s_{n+1},s_n,\ldots,s_1} \longrightarrow V^{2t+1}_{s_{n+1}+i,2s_n,\ldots,2s_1}$$

defined whenever $0 \leq i < s_{n+1}$ and not all of $i, s_n, \ldots, s_1$ are zero, satisfying the unstable Λ-Adem relations of Proposition 5.7*(5)*.

We have shown that an object of $\pi\mathcal{L}(n)$ is in particular an object of $\mathcal{U}(n+1)$, indeed, the $\mathcal{U}(n+1)$-structure on $\pi_* L$ consists solely of its *non-top* λ-operations, which are linear as required.

7.4. The category $\mathcal{W}(n+1)$ of $\mathcal{L}(n)$-Π-algebras

For $n \geq 0$, let $\mathcal{W}(n+1)$ denote the algebraic category whose objects are $\mathcal{V}^+_{n+1}$-graded vector spaces which are simultaneously an object of $\mathcal{U}(n+1)$ and of $\mathcal{L}(n+1)$, such that the compatibilities of Proposition 5.7 are satisfied. Explained another way, an object of $\mathcal{W}(n+1)$ is such a vector space with the bracket and all of the λ-operations (both top and non-top) described in §7.2, subject to the compatibilities of Proposition 5.7.

This category has a number of useful properties, following from the calculations of [13], primarily:

Proposition 7.1. *The operations defined in §7.2 generate the set of natural operations on the homotopy of simplicial objects of* $\mathcal{L}(n)$ *and satisfy the compatibilities of Proposition 5.7. The category* $\mathcal{W}(n+1)$ *is isomorphic to the category* $\pi\mathcal{L}(n)$ *of* $\mathcal{L}(n)$*-*Π*-algebras. The monad* $F^{\mathcal{W}(n+1)}$ *on* $\mathcal{V}^+_{n+1}$ *factors as a composite* $F^{\mathcal{U}(n+1)} \circ F^{\mathcal{L}(n+1)}$*, with monad structure arising from a distributive law [2] of monads on* $\mathcal{V}^+_{n+1}$*.*

Proof. All of these facts are easy to prove after observing that, for $W \in s\mathcal{V}$ a coproduct of spheres, $\pi_*(F^{\mathcal{L}(n)}W)$ embeds in $\pi_*(\Gamma(\mathscr{L})W)$, which, along with $\pi_*(\Lambda(\mathscr{L})W)$, is described in Proposition 5.9 (although by $\Lambda(\mathscr{L})$ and $\Gamma(\mathscr{L})$ we mean the shifted monads of §7.1). In order to make this observation, let write $W_{\mathbf{0}}$ for $\bigoplus_{t\geq 1} W^t_{0,\ldots,0}$, the non-restrictable part of W. This is actually a sub-coproduct of W, the coproduct of those summands of W which lie in homological dimension $(0,\ldots,0)$. There is a commuting diagram of simplicial vector spaces, containing two short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & F^{\mathcal{L}(n)}W \arrow[r, "\alpha"] & \Gamma(\mathscr{L})W \arrow[dd, "\gamma"] \arrow[dr, "\rho\gamma"] & & \\
& & & \dfrac{\Gamma(\mathscr{L})W_{\mathbf{0}}}{\Lambda(\mathscr{L})W_{\mathbf{0}}} \arrow[r] & 0 \\
0 \arrow[r] & \Lambda(\mathscr{L})W_{\mathbf{0}} \arrow[r, "\beta"] & \Gamma(\mathscr{L})W_{\mathbf{0}} \arrow[ur, "\rho"] & &
\end{tikzcd}$$

where α and β are inclusions and γ and ρ are epimorphisms. On homotopy groups: β_* is injective (its source and target are well understood), so that ρ_* is surjective. Thus γ_* is surjective (after all, γ is an isomorphism in those internal degrees in which its codomain in non-zero), implying that $(\rho\gamma)_*$ is surjective. This implies that α_* is injective, as hoped. $\square$

The following two lemmas are the direct analogues of Lemmas 6.10 and 6.11:

Lemma 7.2. *For $n \geq 0$, the monads $F^{\mathcal{U}(n+1)}$ and $F^{\mathcal{W}(n+1)}$ on $\mathcal{V}^+_{n+1}$ may be promoted to monads on the category $\mathrm{q}\mathcal{V}^+_{n+1}$ by requiring that quadratic gradings add when taking brackets and double when applying λ-operations. The same holds for $\mathcal{L}(n)$ for $n \geq 0$.*

7.5. The factorization $Q^{\mathcal{L}(n)} \circ Q^{\mathcal{U}(n)}$ of $Q^{\mathcal{W}(n)}$

For $n \geq 0$, we define

$$Q^{\mathcal{U}(n+1)} := \Big(\mathcal{W}(n+1) \overset{\text{forget}}{\longrightarrow} \mathcal{U}(n+1) \overset{Q^{\mathcal{U}(n+1)}}{\longrightarrow} \mathcal{V}^+_{n+1}\Big).$$

That is, for $X \in \mathcal{W}(n+1)$ we may take the quotient by the image of the *non-top* λ-operations (which are linear, so that this operation is well defined). In fact, it is not hard to see that, for $X \in \mathcal{W}(n+1)$, $Q^{\mathcal{U}(n+1)}X$ retains the structure of an object of $\mathcal{L}(n+1)$, so that we may view $Q^{\mathcal{U}(n+1)}$ as a functor $\mathcal{W}(n+1) \longrightarrow \mathcal{L}(n+1)$.

Lemma 7.3. *For $n \geq 0$ and $X \in \mathcal{W}(n+1)$, X is in particular an object of $\mathcal{L}(n+1)$, and the vector space $Q^{\mathcal{U}(n+1)}X$ retains this structure, yielding a factorization:*

$$Q^{\mathcal{W}(n+1)} = Q^{\mathcal{L}(n+1)} \circ Q^{\mathcal{U}(n+1)} : \left(\mathcal{W}(n+1) \longrightarrow \mathcal{L}(n+1) \longrightarrow \mathcal{V}^+_{n+1}\right).$$

Moreover the composite $Q^{\mathcal{U}(n+1)} \circ F^{\mathcal{W}(n+1)}$ equals the free construction $F^{\mathcal{L}(n+1)}$.

Proof. Similar to the proof of Lemma 6.11, using the observation from the proof of Proposition 7.1 that $\pi_*(F^{\mathcal{L}(n)}W) \subseteq \pi_*(F^{\Gamma(\mathscr{L})}W)$. $\square$

This differs from the definition of $Q^{\mathcal{U}(0)} \longrightarrow \mathcal{L}(0)$, in which one takes the quotient by *all* the P-operations. Indeed, the category $\mathcal{W}(0)$ differs from the categories $\mathcal{W}(n+1)$ (for

$n \geq 0$) in a number of ways, primarily because $\mathcal{W}(0)$ is a category of cohomology algebras while the $\mathcal{W}(n+1)$ are categories of Π-algebras. If $X \in \mathcal{W}(0)$ and $Y \in \mathcal{W}(n+1)$:

(1) Y is a Lie algebra, while X is only an $S(\mathscr{L})$-algebra;

(2) the P-operations on X are always defined and vanish when out of range, while the λ operations are simply undefined when out of range;

(3) the top P-operation is the self-bracket and thus is linear, while the top λ-operation is the restriction and thus a quadratic refinement of the bracket.

Nevertheless, the two regimes share the following common ground:

Corollary 7.4. *For all $n \geq 0$, there are algebraic categories $\mathcal{W}(n)$ and $\mathcal{U}(n)$, a forgetful functor $U' : \mathcal{W}(n) \longrightarrow \mathcal{U}(n)$, and a functor $Q^{\mathcal{U}(n)} : \mathcal{W}(n) \longrightarrow \mathcal{L}(n)$, such that*

$$U^{\mathcal{L}(n)} \circ Q^{\mathcal{U}(n)} = Q^{\mathcal{U}(n)} \circ U', \quad Q^{\mathcal{W}(n)} = Q^{\mathcal{L}(n)} \circ Q^{\mathcal{U}(n)} \text{ and } Q^{\mathcal{U}(n)} \circ F^{\mathcal{W}(n)} = F^{\mathcal{L}(n)}.$$

7.6. Decomposition maps for $\mathcal{L}(n)$ and $\mathcal{W}(n)$

Here we will introduce decomposition maps for the categories $\mathcal{L}(n)$ and $\mathcal{W}(n)$, and calculate the resulting quadratic part maps. The definitions are simple enough, and the reader can verify that each is well defined. For any $n \geq 0$, the following formulae define decomposition maps $j_{\mathcal{C}} : Q^{\mathcal{C}}(X \veebar Y) \longrightarrow Q^{\mathcal{C}}X \otimes Q^{\mathcal{C}}Y$:

$$j_{\mathcal{W}(0)} : P^{i_\ell} \cdots P^{i_1}[z_1, \cdots, z_a] \longmapsto \begin{cases} z_1 \otimes z_2, & \text{if } \ell = 0,\ a = 2,\ z_1 \in X,\ z_2 \in Y, \\ 0, & \text{otherwise.} \end{cases}$$

$$j_{\mathcal{W}(n+1)} : [z_1, \cdots, z_a]\lambda_{i_1} \cdots \lambda_{i_\ell} \longmapsto \begin{cases} z_1 \otimes z_2, & \text{if } \ell = 0,\ a = 2,\ z_1 \in X,\ z_2 \in Y, \\ 0, & \text{otherwise.} \end{cases}$$

$$j_{\mathcal{L}(n)} : [z_1, \cdots, z_a]^{[2^r]} \longmapsto \begin{cases} z_1 \otimes z_2, & \text{if } r = 0,\ a = 2,\ z_1 \in X,\ z_2 \in Y, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 7.5. *Suppose $V \in \mathcal{V}_n^+$. Then:*

(1) $\mathrm{qu}_{\mathcal{L}(n)}$ *is the composite* $F^{\mathcal{L}(n)}V \twoheadrightarrow \mathrm{q}_2 F^{\mathcal{L}(n)}V \subseteq \Sigma S^2 V$*;*

(2) $\mathrm{qu}_{\mathcal{W}(n)}$ *is the composite* $F^{\mathcal{W}(n)}V \twoheadrightarrow F^{\mathcal{L}(n)}V \twoheadrightarrow \mathrm{q}_2 F^{\mathcal{L}(n)}V \subseteq \Sigma S^2 V$*.*

Proof. Consider the case $\mathcal{W}(n+1)$ for $n \geq 0$. As $\mathrm{qu}_{\mathcal{C}}$ vanishes except on quadratic grading

2, one only checks terms $[x, y]$, $x^{[2]}$, and $x\lambda_i$ (a $\mathcal{U}(n+1)$-operation, not the restriction):

$$\begin{aligned}\mathrm{qu}_{\mathcal{W}(n+1)}([x,y]) &= j_{\mathcal{W}(n+1)}([x_1+x_2, y_1+y_2]+[x_1,y_1]+[x_2,y_2]) \\ &= j_{\mathcal{W}(n+1)}([x_1,y_2]+[x_2,y_1]) = x\otimes y + y\otimes x,\end{aligned}$$

which is precisely the representation of $[x, y]$ in $\mathrm{q}_2 F^{\mathcal{L}(n+1)}V \subseteq \Sigma(V^{\otimes 2})$. Similarly, $\mathrm{qu}_{\mathcal{W}(n+1)}(x\lambda_i)$ vanishes (as $\mathcal{U}(n+1)$-operations are linear), while $\mathrm{qu}_{\mathcal{W}(n+1)}(x^{[2]})$ equals $x \otimes x$ as desired. The other cases, including the case of $\mathcal{W}(0)$, are barely any different. $\square$

Chapter 8

Operations on $\mathcal{W}(n)$- and $\mathcal{U}(n)$-cohomology

8.1. Vertical δ-operations on $H^*_{\mathcal{W}(0)}$ and $H^*_{\mathcal{U}(0)}$

We will now define natural homomorphisms for $V \in \mathcal{V}^+$

$$\theta_i : (F^{\mathcal{W}(0)}V)^{t+i+1} \longrightarrow V^t, \text{ for } 2 \leq i < t.$$

There are natural homomorphisms into the quadratic grading 2 part of $F^{\mathcal{W}(0)}V$:

$$\begin{aligned} P^i : V^t &\longrightarrow \mathrm{q}_2(F^{\mathcal{W}(0)}V)^{t+i+1}, \quad \text{for } 2 \leq i < t \\ [\,,] : (S_2V)^t &\longrightarrow \mathrm{q}_2(F^{\mathcal{W}(0)}V)^{t+1}, \end{aligned}$$

and for given $m \geq 1$, the degree m, quadratic grading 2 part $\mathrm{q}_2(F^{\mathcal{W}(0)}V)^m$ decomposes as

$$\mathrm{q}_2(F^{\mathcal{W}(0)}V)^m = \mathrm{im}\big((S_2V)^{m-1} \xrightarrow{[,]} \mathrm{q}_2F^{\mathcal{W}(0)}V\big) \oplus \bigoplus_{2\leq i<(m-1)/2} \mathrm{im}\big(V^{m-i-1} \xrightarrow{P^i} \mathrm{q}_2F^{\mathcal{W}(0)}V\big).$$

Moreover, each map $P^i : V^t \longrightarrow \mathrm{q}_2(F^{\mathcal{W}(0)}V)^{t+i+1}$ appearing in this decomposition is an isomorphism onto its image, so that for $2 \leq i < t$ we may construct θ_i as the composite

$$\theta_i : \left((F^{\mathcal{W}(0)}V)^{t+i+1} \xrightarrow{\text{proj}} (\mathrm{q}_2F^{\mathcal{W}(0)}V)^{t+i+1} \xrightarrow{\text{proj}} \mathrm{im}(P^i) \xrightarrow{(P^i)^{-1}} V^t \right).$$

Here we have projected onto the quadratic filtration 2 part, and then further onto the relevant summand in its natural decomposition. Note that although $P^t : V^t \longrightarrow \mathrm{q}_2(F^{\mathcal{W}(0)}V)^{2t+1}$ is a non-trivial linear map when $t \geq 2$, its image is entangled with the image of the bracket, and we are not able to split it off. Thus we are not able to improve on the bounds $2 \leq i < t$.

Proposition 8.1. *There is a linear map $\theta_i^\star : V_t \longrightarrow (C^{H\mathscr{C}om-\mathrm{coalg}}V)_{t+i+1}$ for $2 \leq i < t$, natural in $V \in \mathcal{V}_+$, such that the following diagram commutes:*

$$\begin{tikzcd}
\mathbf{D}((C^{H\mathscr{C}om-\mathrm{coalg}}V)_{t+i+1}) \arrow[r, "(\theta_i^\star)^*"] & \mathbf{D}V \\
(F^{\mathcal{W}(0)}\mathbf{D}V)^{t+i+1} \arrow[u] \arrow[ur, "\theta_i"'] &
\end{tikzcd}$$

Proof. When V is of finite type, as $F^{\mathcal{W}(0)}$ preserves vector spaces of finite type, we may simply define $\theta_i^\star$ to be the dual of θ_i. This is natural on vector spaces of finite type, and any vector space is the filtered colimit of such. $\square$

In fact, whenever $t \geq 2$ we may define a non-linear function

$$\theta_t^\star : V_t \longrightarrow (C^{H\mathscr{C}om-\mathrm{coalg}}V)_{2t+1}$$

which completes the collection of functions $\theta_i^\star$, but not by this method: we use the upcoming Proposition 11.14 to *define* this top $\theta_i^\star$. That we need to do this is a disadvantage of working with cohomology algebras, as opposed to homology coalgebras.

Proposition 8.2. *Suppose that $X \in s\mathcal{W}(0)$ is almost free, so that we may identify $H^*_{\mathcal{W}(0)}X$ with $\pi^*\mathbf{D}Q^{\mathcal{W}(0)}X$. Then for $2 \leq i < t$, the chain map $\widetilde{\theta}_i$ of Proposition 6.4 induces a linear operation*

$$\delta_i^{\mathrm{v}} : (H^*_{\mathcal{W}(0)}X)^s_t \longrightarrow (H^*_{\mathcal{W}(0)}X)^{s+1}_{t+i+1}.$$

These operations are natural in maps preserving the generating subspaces, and satisfy the unstable δ-Adem relation of Proposition 5.3((6)).

If X is of finite type, this statement may be amended to include a (potentially non-linear) operation

$$\delta_t^{\mathrm{v}} : (H^*_{\mathcal{W}(0)}X)^0_t \longrightarrow (H^*_{\mathcal{W}(0)}X)^1_{2t+1}$$

induced by the function $\theta_t^\star$.

For *any* $X \in s\mathcal{W}(0)$, the bar construction $B^{\mathcal{W}(0)}X$ of X has a natural almost free structure, so that Proposition 8.2 may be used to construct natural operations on $H^*_{\mathcal{W}(0)}X$.

Proof of Proposition 8.2. The finite type assumption is needed in order to define the operation δ_t^{v}, as it is not induced by a chain map on $N_*Q^{\mathcal{W}(0)}X$. Instead, it is induced by the potentially non-linear function

$$\widetilde{\theta}_t^\star : (N^0\,\mathrm{Pr}^{H\mathscr{C}om-\mathrm{coalg}}\,\mathbf{D}X)_t \longrightarrow (N^1\,\mathrm{Pr}^{H\mathscr{C}om-\mathrm{coalg}}\,\mathbf{D}X)_{2t+1}$$

induced by the function $\theta_t^\star$ defined using Proposition 11.14. We will give the proof that the operations δ_i^{v} with $2 \leq i < t$ are well defined and satisfy the δ-Adem relations, in the cohomological variance. By working with homology coalgebras, and thus avoiding double-dualization, the proof given below extends to encompass the extra operation δ_t^{v} in dimension zero. This exercise is left to the reader.

The conditions of Proposition 6.4 are satisfied with $\theta = \theta_i$ and G is the identity functor. The condition

$$\theta \circ \mu_V = \theta \circ \epsilon_{F^{\mathcal{C}}V} + \theta \circ F^{\mathcal{C}}\epsilon_V : F^{\mathcal{C}}F^{\mathcal{C}}V \longrightarrow V$$

just states that given an iterated expression in $F^{\mathcal{C}}F^{\mathcal{C}}V$, the two obvious ways to produce a summand of the form $P^i v$ under the map $\mu : F^{\mathcal{C}}F^{\mathcal{C}}V \longrightarrow F^{\mathcal{C}}V$ are the only ways, due to the homogeneity of the P-Adem relations.

It just remains to prove the δ-Adem relations, which we will do using the technique of [46], the point being that the algebra of δ-operations is Koszul dual to $\mathcal{P}$. For this, we define a map θ_{ij}, whenever $i < 2j$, $2 \leq j < t$ and $2 \leq i < t+j+1$:

$$\theta_{ij} : \left((F^{\mathcal{W}(0)}V)^{t+i+j+2} \overset{\text{proj}}{\twoheadrightarrow} (\mathrm{q}_4 F^{\mathcal{W}(0)}V)^{t+i+j+2} \overset{\text{proj}}{\twoheadrightarrow} \mathrm{im}(P^{i,j}) \overset{(P^{i,j})^{-1}}{\longrightarrow} V^t \right),$$

where we have split off the image of $P^{i,j} = P^i P^j$ as before. This is possible since neither P^j nor P^i are entangled with the bracket in these ranges. We may identify $Q^{\mathcal{W}(0)}X_s$ with V_s, at the cost of replacing d_0 with $\epsilon \circ d_0$, as in Lemma 2.6. Define $\widetilde{\theta_{ij}}$ to be the composite $V_{s+1} \xrightarrow{d_0} FV_s \xrightarrow{\theta_{ij}} V_s$. This will be the nullhomotopy giving the δ-Adem relation. As in the proof of Proposition 6.4, we have $d_k \circ \widetilde{\theta_{ij}} = \widetilde{\theta_{ij}} \circ d_{k+1}$ for $k \geq 1$, and $\widetilde{\theta_{ij}}$ has nullhomotopy the sum

$$\epsilon d_0 \widetilde{\theta_{ij}} + \widetilde{\theta_{ij}}(\epsilon d_0 + d_1) = (\epsilon d_0 \theta_{ij} + \theta_{ij} d_0 \epsilon + \theta_{ij} d_0) d_0.$$

The δ-Adem relation will follow from

$$\theta_{ij} d_0 = \left(\epsilon d_0 \theta_{ij} + \theta_{ij} d_0 \epsilon + \textstyle\sum_{(\alpha,\beta) \xrightarrow{\mathcal{P}} (i,j)} \theta_\beta d_0 \theta_\alpha \right) : FV_{s+1} \longrightarrow V_s,$$

This identity states the following: if $V \in \mathcal{V}_0^+$, and $f(g_k)$ is a nested $\mathcal{W}(0)$-expression with $g_k \in F^{\mathcal{W}(0)}V$ and $f(g_k) \in F^{\mathcal{W}(0)}F^{\mathcal{W}(0)}V$, then if we write $d_0 : F^{\mathcal{W}(0)}F^{\mathcal{W}(0)}V \longrightarrow F^{\mathcal{W}(0)}V$ for the monad product map, there are only three ways that one may obtain expressions of the form $P^i P^j v$ in $d_0(f(g_k))$: for some k, $g_k = P^i P^j v$, and f adds no further operations to this term; $f = P^i P^j g_k$ for some k for which $g_k = v$ is a unit expression; or for some k, $g_k = P^\beta v$, and f has $P^\alpha(g_k)$ as a summand. In this last case, after applying d_0, we may need to rearrange the composite $P^\alpha P^\beta v$ using the P-Adem relations, and we sum over those (α, β) producing a summand $P^i P^j v$.

This shows that the proposed nullhomotopy equals

$$\textstyle\sum_{(\alpha,\beta)\overset{\mathcal{P}}{\to}(i,j)} \widetilde{\theta_\beta} \circ \widetilde{\theta_\alpha},$$

and as Goerss [33] *constructs* the P-algebra as the Koszul dual, in the sense of [46], of the Δ-algebra, so that $(\alpha,\beta) \overset{\mathcal{P}}{\to} (i,j)$ if and only if $(i,j) \overset{\Delta}{\to} (\alpha,\beta)$, so that the nullhomotopy equals the desired sum:

$$\textstyle\sum_{(i,j)\overset{\Delta}{\to}(\alpha,\beta)} \widetilde{\theta_\beta} \circ \widetilde{\theta_\alpha}. \qquad \square$$

The same constructions work in the category $s\mathcal{U}(0)$ of simplicial unstable $\mathcal{P}$-*modules*, the only difference being that when we define θ_i, we need not worry about Lie algebra structures, and we can define a map

$$\theta_i : (F^{\mathcal{U}(0)}V)^{t+i+1} \longrightarrow V^t$$

whenever $2 \le i \le t$, so that there is one more operation available on $H^*_{\mathcal{U}(0)}$ than on $H^*_{\mathcal{W}(0)}$, at least in dimensions $s > 0$.

It will be useful to encode this structure in a definition. Write $\mathcal{M}_{\mathrm{v}}(1)$ for the algebraic category whose objects are vector spaces $M \in \mathcal{V}^1_+$ with left δ-operations

$$\delta^{\mathrm{v}}_i : M^s_t \longrightarrow M^{s+1}_{t+i+1}, \text{ defined whenever } 2 \le i \le t,$$

satisfying unstable δ-Adem relations analogous to those of *(6)* in Proposition 5.3.

Proposition 8.3. *Suppose that $X \in s\mathcal{U}(0)$ is almost free. Then the chain maps $\widetilde{\theta_i}$ of Proposition 6.4 give $H^*_{\mathcal{U}(0)}X$ the structure of an object of $\mathcal{M}_{\mathrm{v}}(1)$, natural in maps preserving the generating subspaces. In fact, $\mathcal{M}_{\mathrm{v}}(1)$ is the category of $\mathcal{U}(0)$-H^*-algebras.*

See §9 for further discussion of this fact, and Proposition 9.1 for a restatement. It is **not true** that $H^*_{\mathcal{W}(0)}$ is an object of $\mathcal{M}_{\mathrm{v}}(1)$, a fact that we emphasize because $H^*_{\mathcal{W}(n)}$ *will* be an object of $\mathcal{M}_{\mathrm{v}}(n+1)$ for $n \ge 1$ (under definitions made in §8.2).

In order to give a basis for a free object in $\mathcal{M}_{\mathrm{v}}(1)$, for a sequence $I = (i_\ell, \ldots, i_1)$ of integers $i_j \ge 2$, we use the function

$$\overline{m}(I) := \max\{(i_1),\, (i_2 - i_1 - 1),\, (i_3 - i_2 - i_1 - 2),\, \ldots,\, (i_\ell - \cdots - i_1 - \ell + 1)\},$$

of §6.6, following the convention that $\max(\emptyset) = -\infty$, and the notion of δ-admissibility from §5.4: each $i_j \ge 2$ and $i_{j+1} \ge 2i_j$ for $1 \le j < \ell$.

Lemma 8.4. *For $V \in \mathcal{V}^1_+$ with homogeneous basis B, a basis of $F^{\mathcal{M}_{\mathrm{v}}(1)}V$ is*

$$\{\delta^{\mathrm{v}}_I b \mid b \in B^s_t,\ I\ \delta\text{-admissible with } \overline{m}(I) \le t\}.$$

We will often apply such results as these when V is concentrated in degrees V_t^0. At this point we introduce a notational abuse, identifying $\mathcal{V}_+$ with the full subcategory of $\mathcal{V}_+^1$ with objects concentrated in these degrees. The effect of this will be that we will be able to write $F^{\mathcal{M}_{\mathrm{v}}(1)}V$ for $V \in \mathcal{V}_+$. Restricting Lemma 8.4 to such cases:

Corollary 8.5. *For $V \in \mathcal{V}_+$ with homogeneous basis B, a basis of $F^{\mathcal{M}_{\mathrm{v}}(1)}V \in \mathcal{V}_+^1$ is*

$$\{\delta_I^{\mathrm{v}} b \mid b \in B_t,\ I\ \delta\text{-admissible with } \overline{m}(I) \leq t\}.$$

8.2. Vertical Steenrod operations for $H^*_{\mathcal{W}(n)}$ and $H^*_{\mathcal{U}(n)}$ when $n \geq 1$

For $V \in \mathcal{V}_n^+$, we will define natural homomorphisms

$$\theta^i : (F^{\mathcal{W}(n)}V)^{2t+1}_{s_n+i-1,2s_{n-1},\ldots,2s_1} \longrightarrow V^t_{s_n,\ldots,s_1},$$

which are defined for all $i, s_1, \ldots, s_n \geq 0$ and $t \geq 1$, but are zero except when $1 \leq i \leq s_n$ and not all of $i-1, s_{n-1}, \ldots, s_1$ are zero. These are rather easier to define than in the $n = 0$ case investigated in §8.1, as the monad $F^{\mathcal{W}(n)}$ is a simple composite $F^{\mathcal{U}(n)}F^{\mathcal{L}(n)}$ of monads when $n \geq 1$. Indeed, there are natural monomorphisms

$$(-)\lambda_{i-1} : V^t_{s_n,\ldots,s_1} \longrightarrow (\mathrm{q}_2 F^{\mathcal{W}(n)}V)^{2t+1}_{s_n+i-1,2s_{n-1},\ldots,2s_1}$$

defined only when $1 \leq i \leq n$ and $i-1, s_{n-1}, \ldots, s_1$ are not all zero, and an inclusion

$$\mathrm{incl} : \mathrm{q}_2 F^{\mathcal{L}(n)}V \longrightarrow \mathrm{q}_2 F^{\mathcal{W}(n)}V.$$

As in the $n = 0$ case, the images of the listed maps are linearly independent and span the quadratic grading 2 part of $F^{\mathcal{W}(n)}V$. We define θ^i to be zero unless $1 \leq i \leq n$ and $i-1, s_{n-1}, \ldots, s_1$ are not all zero, in which case we define it as the composite:

$$\theta^i : \left((F^{\mathcal{W}(n)}V)^{2t+1}_{s_n+i-1,2s_{n-1},\ldots,2s_1} \overset{\mathrm{proj} \circ \mathrm{proj}}{\longrightarrow\!\!\!\!\!\rightarrow} \mathrm{im}(\lambda_{i-1}) \overset{(\lambda_{i-1})^{-1}}{\longrightarrow} V^t_{s_n,\ldots,s_1} \right).$$

One can give exactly the same definitions for the free construction in $\mathcal{U}(n)$, producing functions $\theta^i : F^{\mathcal{U}(n)}V \longrightarrow V$ which are zero under the same conditions as for $\mathcal{W}(n)$.

Write $\mathcal{M}_{\mathrm{v}}(n+1)$ for the algebraic category whose objects are vector spaces $M \in \mathcal{V}_+^{n+1}$ with left Steenrod operations

$$\mathrm{Sq}^i_{\mathrm{v}} : M_t^{s_{n+1},\ldots,s_1} \longrightarrow M_{2t+1}^{s_{n+1}+1,s_n+i-1,2s_{n-1},\ldots,2s_1},$$

which are zero except when $1 \leq i \leq s_n$ and not all of $i-1, s_{n-1}, \ldots, s_1$ are zero, and which satisfy the homogeneous Sq-Adem relations. Note that in an object of $\mathcal{M}_{\mathrm{v}}(2)$, $\mathrm{Sq}^1_{\mathrm{v}} = 0$.

In the present case ($n \geq 1$) there is no disparity between the unstableness conditions on $\mathcal{W}(n)$- and $\mathcal{U}(n)$-cohomology, so that the analogue of Propositions 8.2 and 8.3 is:

Proposition 8.6. *Suppose that $X \in s\mathcal{C}$ is almost free, where $\mathcal{C}$ stands for either $\mathcal{W}(n)$ or $\mathcal{U}(n)$ with $n \geq 1$. Then the chain maps $\widetilde{\theta^i}$ of Proposition 6.4 give $H^*_{\mathcal{C}}X$ the structure of an object of $\mathcal{M}_{\mathrm{v}}(n+1)$, natural in maps of almost free objects preserving the generating subspaces. Again, $\mathcal{M}_{\mathrm{v}}(n+1)$ is the category of $\mathcal{U}(n)$-H^*-algebras.*

In order to give a basis for a free object in $\mathcal{M}_{\mathrm{v}}(n+1)$, for a sequence $I = (i_\ell, \ldots, i_1)$ of integers $i_j \geq 1$, we define

$$\underline{m}(I) := \max\{(i_1),\, (i_2 - i_1 + 1),\, (i_3 - i_2 - i_1 + 2),\, \ldots,\, (i_\ell - \cdots - i_1 + (\ell - 1))\}.$$

Recall that I is Sq-*admissible* if each $i_j \geq 1$ and $i_{j+1} \geq 2i_j$ for $1 \leq j < \ell$.

Lemma 8.7. *For $V \in \mathcal{V}^{n+1}_+$ with homogeneous basis B, a basis of $F^{\mathcal{M}_{\mathrm{v}}(n+1)}V$ is*

$$\left\{ \mathrm{Sq}^J_{\mathrm{v}} b \;\middle|\; \begin{array}{c} b \in B^{s_{n+1},\ldots,s_1}_t,\ J \text{ Sq-admissible with } \underline{m}(J) \leq s_n, \\ \text{if } s_{n-1} = \cdots = s_1 = 0 \text{ then } J \text{ does not contain } 1 \end{array} \right\}.$$

Performing the same abuse of notation as in Corollary 8.5:

Corollary 8.8. *For $V \in \mathcal{V}^n_+$ with homogeneous basis B, a basis of $F^{\mathcal{M}_{\mathrm{v}}(n+1)}V \in \mathcal{V}^{n+1}_+$ is*

$$\left\{ \mathrm{Sq}^J_{\mathrm{v}} b \;\middle|\; \begin{array}{c} b \in B^{s_n,\ldots,s_1}_t,\ J \text{ Sq-admissible with } \underline{m}(J) \leq s_n, \\ \text{if } s_{n-1} = \cdots = s_1 = 0 \text{ then } J \text{ does not contain } 1 \end{array} \right\}.$$

8.3. Horizontal Steenrod operations and a product for $H^*_{\mathcal{W}(n)}$

For any $n \geq 0$, we will construct operations on the homology $H^*_{\mathcal{W}(n)}$ arising from the $S(\mathscr{L})$-algebra structure or Lie algebra structures.

Indeed, suppose that $X \in s\mathcal{W}(n)$ is almost free. Then $Q^{\mathcal{U}(n)}X \in s\mathcal{L}(n)$ is also almost free, on essentially the same generating subspaces. Thus, the cohomotopy of $Q^{\mathcal{W}(n)}X = Q^{\mathcal{L}(n)}Q^{\mathcal{U}(n)}X$ is an instance of simplicial partially restricted Lie algebra cohomology. Cohomology operations of this type are discussed in §6.8 and Appendix A. In the present context, we have two equivalent definitions, one using $\psi_{\mathcal{L}(n)}$ and one using $\psi_{\mathcal{W}(n)}$, and until Appendix

A, we will use $\psi_{\mathcal{W}(n)}$, defining operations

$$\mathrm{Sq}^j_{\mathrm{h}} : \left(H^n_{\mathcal{W}(n)}(X) \xrightarrow{\mathrm{Sq}^{j-1}_{\mathrm{ext}}} \pi^{n+j-1}\mathbf{D}S^2 Q^{\mathcal{W}(n)} X \xrightarrow{\psi^*_{\mathcal{W}(n)}} H^{n+j}_{\mathcal{W}(n)}(X) \right) \text{ and}$$

$$\mu : \left(H^{n_1}_{\mathcal{W}(n)}(X) \otimes H^{n_2}_{\mathcal{W}(n)}(X) \xrightarrow{\mu_{\mathrm{ext}}} \pi^{n_1+n_2}\mathbf{D}S^2 Q^{\mathcal{W}(n)} X \xrightarrow{\psi^*_{\mathcal{W}(n)}} H^{n_1+n_2+1}_{\mathcal{W}(n)}(X) \right).$$

In more detail:

Proposition 8.9. *Fix $n \geq 0$. For $X \in s\mathcal{W}(n)$, there are natural operations*

$$\mathrm{Sq}^j_{\mathrm{h}} : (H^*_{\mathcal{W}(n)} X)^{s_{n+1},\ldots,s_1}_t \longrightarrow (H^*_{\mathcal{W}(n)} X)^{s_{n+1}+j, 2s_n, \ldots, 2s_1}_{2t+1},$$
$$\mu : (H^*_{\mathcal{W}(n)} X)^{s_{n+1},\ldots,s_1}_t \otimes (H^*_{\mathcal{W}(n)} X)^{p_{n+1},\ldots,p_1}_q \longrightarrow (H^*_{\mathcal{W}(n)} X)^{s_{n+1}+p_{n+1}+1, s_n+p_n, \ldots, s_1+p_1}_{t+q+1}$$

with the following properties

(1) the product μ gives $H^*_{\mathcal{C}} L$ the structure of a (grading shifted) $S(\mathscr{C})$-algebra;

(2) the squaring operation on $H^*_{\mathcal{C}} L$ equals the top Steenrod operation*:*

$$\mathrm{Sq}^{s_{n+1}+1} x = x^2 \quad \textit{for } x \in (H^*_{\mathcal{W}(n)} X)^{s_{n+1},\ldots,s_1}_t;$$

(3) if $x \in (H^*_{\mathcal{W}(n)} X)^{s_{n+1},\ldots,s_1}_t$, then $\mathrm{Sq}^i x = 0$ unless $1 \leq i \leq s_{n+1}+1$ and not all of $i-1, s_n, \ldots, s_1$ equal zero;

(4) if $n = 0$ then $\mathrm{Sq}^1 \equiv 0$, and $\mathrm{Sq}^2 x = 0$ for $x \in (H^*_{\mathcal{W}(0)} X)^{s_1}_t$ with $t \geq 2$;

(5) every Steenrod operation is linear;

(6) the Cartan formula *holds: for all $x, y \in H^*_{\mathcal{C}} L$ and $i \geq 0$,*

$$\mathrm{Sq}^i_{\mathrm{h}}(xy) = \textstyle\sum_{k=1}^{i-1} (\mathrm{Sq}^k x)(\mathrm{Sq}^{i-k} y);$$

(7) the homogeneous Sq*-Adem relations hold, making $H^*_{\mathcal{C}} L$ a left $\mathcal{A}$-module.*

Proof. Almost everything here follows from Proposition A.3, which demonstrates that the operations we are discussing here coincide with those defined on $H^*_{\bar{W}}$ (c.f. Propositions 6.12 and 6.13). The same technique used to prove Proposition 6.14 proves the new part of *(3)*. For *(4)*, when $n = 0$, *(3)* shows that $\mathrm{Sq}^1 x = 0$. On the other hand, to see why $\mathrm{Sq}^2 x = 0$ when $t \geq 2$ is more difficult, especially since we have not determined the category of $\mathcal{W}(0)$-Π-algebras. Nonetheless, as in the proof of Proposition 6.14, we will prove this for the universal example $\imath^s_t \in H^*_{\mathcal{W}(0)} \mathbb{K}^{\mathcal{W}(0),t}_s$. The reverse Adams spectral sequence (§3.4) can be equipped with a quadratic grading if we view the generator of $\mathbb{K}^{\mathcal{W}(0),t}_s$ as lying in quadratic grading one, and is of the form

$$\mathrm{q}_k E_2^{p,q} = \mathrm{q}_k(H^p_{\pi\mathcal{W}(0)}\mathbb{K}^{\pi\mathcal{W}(0),t}_{0,s})^q_T \implies \mathrm{q}_k(H^{p+q}_{\mathcal{W}(0)}\mathbb{K}^{\mathcal{W}(0),t}_s)_T.$$

As $\mathrm{Sq}^2\imath^s_t \in \mathrm{q}_2(H^{s+2}_{\mathcal{W}(0)}\mathbb{K}^{\mathcal{W}(0),t}_s)_{2t+1}$, we need to determine

$$\mathrm{q}_2(H^2_{\pi\mathcal{W}(0)}\mathbb{K}^{\pi\mathcal{W}(0),t}_{0,s})^s_{2t+1} \text{ and } \mathrm{q}_2(H^1_{\pi\mathcal{W}(0)}\mathbb{K}^{\pi\mathcal{W}(0),t}_{0,s})^{s+1}_{2t+1},$$

and as in the proof of Proposition 6.14, we need to see how far $\mathbb{F}_2\{\imath^t_s\}$ is from being free in $\pi\mathcal{W}(0)$. Fortunately, we only need to answer this question in quadratic grading two, and

$$F^{\pi\mathcal{W}(0)}(\mathbb{F}_2\{\imath^t_s\}) = \pi_*(\mathbb{S}^{\mathcal{W}(0),t}_s) = \pi_*(F^{\mathcal{W}(0)}\mathbb{K}^t_s).$$

Now if $t \geq 2$, $\mathrm{q}_2F^{\mathcal{W}(0)}V$ naturally decomposes as

$$\mathrm{q}_2F^{\mathcal{W}(0)}V = \mathrm{q}_2F^{S(\mathscr{L})}V \oplus P^2V \oplus \cdots \oplus P^{t-1}V,$$

and we calculate, by Proposition 5.6:

$$\mathrm{q}_2F^{\pi\mathcal{W}(0)}(\mathbb{F}_2\{\imath^t_s\})^{2t+1} = \mathbb{F}_2\{\lambda_2\imath^t_s, \lambda_3\imath^t_s, \ldots, \lambda_s\imath^t_s\}$$

That is, there are two missing λ-operations, λ_0 and λ_1, in the functor $F^{S(\mathscr{L})}$, and the presence of the operations $P^2, \ldots, P^{t-1}$ do not effect $\mathrm{q}_2F^{\pi\mathcal{W}(0)}(\mathbb{F}_2\{\imath^t_s\})$ in internal dimension $2t+1$. We now have enough information to proceed as in the proof of Proposition 6.14, since $\lambda_k\imath^t_s \in \mathrm{q}_2F^{\pi\mathcal{W}(0)}(\mathbb{F}_2\{\imath^t_s\})^{2t+1}_{s+k}$ for $2 \leq k \leq s$. □

For $n \geq 0$, write $\mathcal{M}_\mathrm{h}(n+1)$ for the algebraic category whose objects are vector spaces $M \in \mathcal{V}^{n+1}_+$ with left Steenrod operations and a commutative pairing satisfying the conditions of Proposition 8.9. We have simply shown that $H^*_{\mathcal{W}(n)}$ takes values in $\mathcal{M}_\mathrm{h}(n+1)$ — it is certainly not true that $\mathcal{M}_\mathrm{h}(n+1)$ is the category of $\mathcal{W}(n)$-H^*-algebras, as we have also seen that $H^*_{\mathcal{W}(n)}$ takes values in $\mathcal{M}_\mathrm{v}(n+1)$.

Note that the unstableness condition implies that $x^2 = 0$ whenever $x \in M^{0,\ldots,0}_t$. Indeed

Proposition 8.10. *Suppose that $n \geq 1$, and that $V \in \mathcal{V}^{n+1}_+$ has homogeneous basis B. Then $F^{\mathcal{M}_\mathrm{h}(n+1)}V$ is the quotient of the non-unital commutative algebra*

$$S(\mathscr{C})\left[\mathrm{Sq}^J_\mathrm{h}b \;\middle|\; \begin{array}{c} b \in B^{s_{n+1},\ldots,s_1}_t,\ J \text{ Sq-admissible with } e(J) \leq s_{n+1}, \\ \text{if } s_n = \cdots = s_1 = 0 \text{ then } J \text{ does not contain } 1 \end{array}\right]$$

by the relation $b^2 = 0$ if $b \in B^{0,\ldots,0}_t$. Here, $e(J) := j_\ell - j_{\ell-1} - \cdots j_1$ is the Serre excess of J.

Proof. By [47, 6.1], the true free object is a quotient of what we propose. It is in fact equal to what we propose, because the two-sided ideal in the homogeneous Steenrod algebra $\mathcal{A}$ generated by $\mathrm{Sq}_{\mathrm{h}}^1$ is spanned by those admissible sequences ending in $\mathrm{Sq}_{\mathrm{h}}^1$, so that forcing $\mathrm{Sq}_h^1 = 0$ in the relevant degrees has no unintended consequences. □

Corollary 8.11. *Suppose that $n \geq 1$. For $V \in \mathcal{V}_+^n$ with homogeneous basis B. Then $F^{\mathcal{M}_{\mathrm{h}}(n+1)}V \in \mathcal{V}_+^{n+1}$ is the non-unital commutative algebra coproduct*

$$S(\mathscr{C})\left[b \,\middle|\, \begin{array}{c} b \in B_t^{s_n,\ldots,s_1} \\ s_n,\ldots,s_1 \text{ not all zero} \end{array}\right] \sqcup \Lambda(\mathscr{C})\left[b \,\middle|\, b \in B_t^{0,\ldots,0}\right].$$

8.4. Relations between the horizontal and vertical operations

It will be helpful to be able to reduce expressions in the various available operations to a standard format,

$$\textstyle\prod_k \mathrm{Sq}_{\mathrm{h}}^{J_k}\delta_{I_k}^{\mathrm{v}}x_k \text{ when } n=0, \text{ or } \prod_k \mathrm{Sq}_{\mathrm{h}}^{J_k}\mathrm{Sq}_{\mathrm{v}}^{I_k}x_k \text{ when } n \geq 1,$$

which is possible, thanks to:

Proposition 8.12. *Suppose that $x, y \in H_{\mathcal{W}(0)}^*X$. If $\mathrm{Sq}_{\mathrm{h}}^j x \in (H_{\mathcal{W}(0)}^*X)_t^s$, then $\delta_i^{\mathrm{v}}\mathrm{Sq}_{\mathrm{h}}^j x = 0$ for $2 \leq i < t$, and if $xy \in (H_{\mathcal{W}(0)}^*X)_t^s$, then $\delta_i^{\mathrm{v}}(xy) = 0$ for $2 \leq i < t$.*

Suppose that $n \geq 1$ and $x, y \in H_{\mathcal{W}(n)}^*X$. Then $\mathrm{Sq}_{\mathrm{v}}^i\mathrm{Sq}_{\mathrm{h}}^j x = 0$ and $\mathrm{Sq}_{\mathrm{v}}^i(xy) = 0$.

Proof. For the case $n = 0$, suppose that $X \in s\mathcal{W}(0)$ is almost free on generating subspaces V_s. It is enough to prove that the composite

$$N_{s+1}(Q^{\mathcal{W}(0)}X_{s+2})^{t+i+1} \xrightarrow{\widetilde{\theta}_i} N_s(Q^{\mathcal{W}(0)}X_{s+1})^t \xrightarrow{\psi_{\mathcal{W}(n)}} N_{s-1}(S^2(Q^{\mathcal{W}(0)}X_s))^{t-1}$$

is nullhomotopic, using a similar method to that used in the proof of Proposition 8.2. For any $V \in \mathcal{V}_0^+$, there is a natural composite

$$(S_2V)^{t-1} \xrightarrow[\alpha]{[\,,]} (\mathrm{q}_2F^{\mathcal{W}(0)}V)^t \xrightarrow[\beta]{P^i} (F_{\mathcal{W}(0)}^{(4)}V)^{t+i+1},$$

whose maps we have labeled α and β for convenience. The map $\beta|_{\mathrm{im}(\alpha)}$ is not a monomorphism when $i = t-1$ is even, as in this case, for any $v \in V^{i/2}$,

$$P^i[v,v] = P^iP^{i/2}v = \textstyle\sum_{k=i/2+1}^{3i/2-2}\binom{2(k-i/2)-1}{k-i/2}P^{3i/2-k}P^kv = 0,$$

as each expression P^kv in the sum vanishes by the unstableness condition. However, $\ker(\beta|_{\mathrm{im}(\alpha)})$ is contained in $\ker(\mathrm{qu}_{\mathcal{W}(0)})$, and $\mathrm{im}(\beta\circ\alpha)$ *does* naturally split off as a direct summand of $(F^{(4)}_{\mathcal{W}(0)}V)^{t+i+1}$. We write h_i for the composite:

$$h_i : \Big((F^{\mathcal{W}(0)}V)^{t+i+1} \overset{\mathrm{proj}}{\twoheadrightarrow} (\mathrm{im}(\beta\circ\alpha))^{t+i+1} \xrightarrow{\beta^{-1}} \frac{\mathrm{im}(\alpha)}{\ker(\beta)\cap\mathrm{im}(\alpha)} \xrightarrow{\mathrm{qu}_{\mathcal{W}(0)}} (S^2V)^{t-1}\Big).$$

Identifying $Q^{\mathcal{W}(0)}X_s$ with V_s as in the proof of Proposition 8.2, the nullhomotopy associated with the composite $\widetilde{h_i} : (V_{s+1} \xrightarrow{d_0} FV_s \xrightarrow{h_i} V_s)$ is the sum

$$(\epsilon d_0 h_i + h_i d_0 \epsilon + h_i d_0)d_0,$$

and the relation we seek will follow from the identity

$$h_i d_0 = \Big(\epsilon d_0 h_i + h_i d_0\epsilon + \mathrm{qu}_{\mathcal{W}(0)} d_0\theta_i\Big) : FV_{s+1} \longrightarrow V_s,$$

as then $\psi_{\mathcal{W}(0)}\widetilde{\theta_i} = \mathrm{qu}_{\mathcal{W}(0)}d_0\theta_i d_0 = d_0\widetilde{h_i} + \widetilde{h_i}d_0$. This identity states the following: if $V \in \mathcal{V}_0^+$, and $f(g_k) \in F^{\mathcal{W}(0)}F^{\mathcal{W}(0)}V$ is a nested $\mathcal{W}(0)$-expression in various expressions $g_k \in F^{\mathcal{W}(0)}V$, then if we write $d_0 : F^{\mathcal{W}(0)}F^{\mathcal{W}(0)}V \longrightarrow F^{\mathcal{W}(0)}V$ for the monad product map, there are only three ways that one may obtain summands of the form $P^i[v_1, v_2]$ in $d_0(f(g_k)) \in (F^{\mathcal{W}(0)}V)^{t+i+1}$: for some k, $g_k = P^i[v_1, v_2]$, and f adds no further operations to this term; $f = P^i[g_{k_1}, g_{k_2}]$, where $g_{k_1} = v_1$ and $g_{k_2} = v_2$ are unit expressions; or for some k, $g_k = [v_1, v_2]$, and f has $P^i(g_k)$ as a summand.

For the case $n \geq 0$, the proof only becomes easier, the main difference being that in the corresponding composite:

$$(\mathrm{q}_2F^{\mathcal{L}(n)})^{t-1} \xrightarrow[\alpha]{} (\mathrm{q}_2F^{\mathcal{W}(0)}V)^t \xrightarrow[\beta]{\lambda_{i-1}} (F^{(4)}_{\mathcal{W}(0)}V)^{t+i-1},$$

both α and $\beta|_{\mathrm{im}(\alpha)}$ are monomorphisms. □

8.5. The categories $\mathcal{M}_{\mathrm{hv}}(n+1)$

For $n \geq 1$, let $\mathcal{M}_{\mathrm{hv}}(n+1)$ be the following algebraic category, monadic over $\mathcal{V}_+^{n+1}$. An object of $\mathcal{M}_{\mathrm{hv}}(n+1)$ is a vector space $V \in \mathcal{V}_+^{n+1}$ which is simultaneously an object of $\mathcal{M}_{\mathrm{v}}(n+1)$ and of $\mathcal{M}_{\mathrm{h}}(n+1)$, and in which

$$\mathrm{Sq}^i_{\mathrm{v}}(xy) = 0 \text{ and } \mathrm{Sq}^i_{\mathrm{v}}(\mathrm{Sq}^j_{\mathrm{h}}(x)) = 0 \;\text{ for all } x, y \in V.$$

By Proposition 8.12, for $n \geq 1$ and $X \in \mathcal{W}(n)$, $H^*_{\mathcal{W}(n)}X$ is naturally an object of $\mathcal{M}_{\mathrm{hv}}(n+1)$. We will prove in Corollary 14.7 that for $n \geq 1$, $\mathcal{M}_{\mathrm{hv}}(n+1)$ is the category $H\mathscr{C}om$ of $\mathcal{W}(n)$-cohomology algebras.

The corresponding facts are **not** true for $n = 0$, so we do not even *define* a category $\mathcal{M}_{\mathrm{hv}}(1)$.

For any $n \geq 1$, the monad $F^{\mathcal{M}_{\mathrm{hv}}(n+1)}$ factors as $F^{\mathcal{M}_{\mathrm{h}}(n+1)}F^{\mathcal{M}_{\mathrm{v}}(n+1)}$, with the evident distributive law of monads, and combining Corollary 8.8 and Proposition 8.10:

Corollary 8.13. *For $n \geq 1$ and $V \in \mathcal{V}_+^n$ with homogeneous basis B, $F^{\mathcal{M}_{\mathrm{hv}}(n+1)}V$ is the quotient of the non-unital commutative algebra*

$$S(\mathscr{C})\left[\mathrm{Sq}_{\mathrm{h}}^J\mathrm{Sq}_{\mathrm{v}}^I b \;\middle|\; \begin{array}{c} b \in B_t^{s_n,\dots,s_1},\ I, J \text{ Sq-admissible with } \underline{m}(I) \leq s_n,\ e(J) \leq \ell I \\ \text{if } s_{n-1} = \cdots = s_1 = 0 \text{ then } I \text{ does not contain } 1 \\ \text{if } s_n = \cdots = s_1 = 0 \text{ then } J \text{ does not contain } 1 \end{array}\right]$$

by the relation $b^2 = 0$ if $b \in B_t^{0,\dots,0}$.

Although we do not define $\mathcal{M}_{\mathrm{hv}}(1)$, it will be useful to have a description of the composite $F^{\mathcal{M}_{\mathrm{h}}(1)}F^{\mathcal{M}_{\mathrm{v}}(1)}$. Combining Corollary 8.5 and Proposition 8.10:

Corollary 8.14. *For $V \in \mathcal{V}_+$ with homogeneous basis B, $F^{\mathcal{M}_{\mathrm{h}}(1)}F^{\mathcal{M}_{\mathrm{v}}(1)}V$ is isomorphic to the non-unital commutative algebra coproduct*

$$S(\mathscr{C})\left[\mathrm{Sq}_{\mathrm{h}}^J\delta_I^{\mathrm{v}} b \;\middle|\; \begin{array}{c} b \in B_t,\ I \text{ non-empty, } \delta\text{-admissible with } \overline{m}(I) \leq t, \\ J \text{ Sq-admissible with } e(J) \leq \ell I,\ \text{and } 1 \notin J \end{array}\right] \sqcup \Lambda(\mathscr{C})\left[b \mid b \in B\right].$$

For elements $b_1, \dots, b_N$ of B with $b_k \in B_{t_k}$ and appropriate sequences I_k, J_k, we have

$$\textstyle\prod_{k=1}^N \mathrm{Sq}_{\mathrm{h}}^{J_k}\delta_{I_k}^{\mathrm{v}} b_k \in \left(F^{\mathcal{M}_{\mathrm{h}}(1)}F^{\mathcal{M}_{\mathrm{v}}(1)}V\right)_{-1+\sum_k(2^{\ell J_k + \ell I_k}(t_k+1))}^{-1+\sum_k(nJ_k+\ell I_k+1)}.$$

8.6. Compressing sequences of Steenrod operations

The following theorem creates a model for the convergence of a spectral sequence which we will discuss in §13. One should think of $F^{\mathcal{M}_{\mathrm{hv}}(n+1)}V$ as the E_∞-page of a first quadrant cohomotopy spectral sequence and $F^{\mathcal{M}_{\mathrm{h}}(n)}V$ as the cohomotopy of the total complex.

Theorem 8.15. *Suppose that $n \geq 1$ and $V \in \mathcal{V}_+^n$. Then there is a decreasing filtration on $F^{\mathcal{M}_{\mathrm{h}}(n)}V$, the* target filtration, *and an isomorphism*

$$f : (F^{\mathcal{M}_{\mathrm{hv}}(n+1)}V)_t^{s_{n+1},\dots,s_1} \xrightarrow{\cong} [E_0 F^{\mathcal{M}_{\mathrm{h}}(n)}V]_t^{s_{n+1},\dots,s_1},$$

defined by requiring that $f(\mathrm{Sq}_{\mathrm{v}}^I v) = \mathrm{Sq}_{\mathrm{h}}^I v$ for $v \in V$, that $f(w_1w_2) = f(w_1)f(w_1)$ for $w_1, w_2 \in F^{\mathcal{M}_{\mathrm{hv}}(n+1)}V$, and that

$$f(\mathrm{Sq}_{\mathrm{h}}^j w) = \mathrm{Sq}_{\mathrm{h}}^{j+s_n} f(w) \text{ for } w \in (F^{\mathcal{M}_{\mathrm{hv}}(n+1)}V)_t^{s_{n+1},\dots,s_1}.$$

Proof. The proposed map f is not a well defined map to $F^{\mathcal{M}_{\mathrm{h}}(n)}V$ since the Adem relations between the $\mathrm{Sq_h}$ are not preserved by the proposed map f. Write $W(V)$ for the quotient of $S(\mathscr{C})[\mathcal{A} \otimes F^{\mathcal{M}_{\mathrm{v}}(n+1)}V]$ by the *horizontal* unstableness relations and Cartan formula, so that $F^{\mathcal{M}_{\mathrm{hv}}(n+1)}V$ is obtained from $W(V)$ by taking the quotient by the two-sided ideal generated by the *horizontal* Adem relations. Then may define a map $\overline{f} : W(V) \longrightarrow F^{\mathcal{M}_{\mathrm{h}}(n)}V$ by requiring the same of $\overline{f}$ as of f. There is a decreasing filtration on $W(V)$, given by

$$F^pW(V) = \bigoplus_{s_{n+1}\geq p} \bigoplus_{s_n,\ldots,s_1\geq 0} \bigoplus_{t\geq 1} W(V)_t^{s_{n+1},\ldots,s_1}$$

and we define the *target filtration* on the target by $F^p(F^{\mathcal{M}_{\mathrm{h}}(n)}V) := \overline{f}(F^pW(V))$.

The map $\overline{f}$ fails to descend to a well-defined map $F^{\mathcal{M}_{\mathrm{hv}}(n+1)}V \longrightarrow F^{\mathcal{M}_{\mathrm{h}}(n)}V$, because it does not annihilate the Adem relations. However, we will show that it does send them into higher filtration, so that $\overline{f}$ induces a well defined map f as advertised: if $w \in W(V)_t^{s_{n+1},\ldots,s_1}$ and $i < 2j$, then

$$\begin{aligned}
&\overline{f}\Big(\mathrm{Sq}_{\mathrm{h}}^i\mathrm{Sq}_{\mathrm{h}}^j w - \textstyle\sum_{k=0}^{\lfloor i/2\rfloor} \binom{j-k-1}{i-2k}\mathrm{Sq}_{\mathrm{h}}^{i+j-k}\mathrm{Sq}_{\mathrm{h}}^k w\Big)\\
&\quad := \mathrm{Sq}_{\mathrm{h}}^{i+2s_n}\mathrm{Sq}_{\mathrm{h}}^{j+s_n}\overline{f}(w) - \textstyle\sum_{k=0}^{\lfloor i/2\rfloor} \binom{j-k-1}{i-2k}\mathrm{Sq}_{\mathrm{h}}^{i+j-k+2s_n}\mathrm{Sq}_{\mathrm{h}}^{k+s_n}\overline{f}(w)\\
&\quad = \mathrm{Sq}_{\mathrm{h}}^{i+2s_n}\mathrm{Sq}_{\mathrm{h}}^{j+s_n}\overline{f}(w) - \textstyle\sum_{k=s_n}^{\lfloor (i+2s_n)/2\rfloor} \binom{j+s_n-k-1}{i+2s_n-2k}\mathrm{Sq}_{\mathrm{h}}^{(i+2s_n)+(j+s_n)-k}\mathrm{Sq}_{\mathrm{h}}^k\overline{f}(w)\\
&\quad = \textstyle\sum_{k=0}^{s_n-1} \binom{j+s_n-k-1}{i+2s_n-2k}\mathrm{Sq}_{\mathrm{h}}^{(i+2s_n)+(j+s_n)-k}\mathrm{Sq}_{\mathrm{h}}^k\overline{f}(w)\\
&\quad =: \textstyle\sum_{k=0}^{s_n-1} \binom{j+s_n-k-1}{i+2s_n-2k}\overline{f}(\mathrm{Sq}_{\mathrm{h}}^{i+j+2(s_n-k)+1}\mathrm{Sq}_{\mathrm{v}}^k w),
\end{aligned}$$

which is in filtration $s_{n+1} + i + j + 2(s_n + 1 - k) > s_{n+1} + i + j$ (the second equation holds by simply shifting the dummy variable k, the third by an Adem relation in the codomain).

What remains is to show that f is an isomorphism as in the theorem statement, which we approach simply by choosing a set of multiplicative generators for both the domain and codomain. The domain is generated by those expressions $\mathrm{Sq}_{\mathrm{h}}^I\mathrm{Sq}_{\mathrm{v}}^J v$, for $v \in V_t^{s_n,\ldots,s_1}$ running through a basis of V, and appropriate Sq-admissible sequences J and I. The codomain is generated by expressions $\mathrm{Sq}_{\mathrm{h}}^K v$, for $v \in V_t^{s_n,\ldots,s_1}$ running through a basis of V, and appropriate Sq-admissible sequences K. It is a combinatorial exercise in the properties of admissible sequences to show that these sets of generators are put in bijection by f, and this bijection sends polynomial generators to polynomial generators and exterior generators to exterior generators. □

Chapter 9

Koszul complexes calculating $\mathcal{U}(n)$-homology

We will now discuss the Koszul resolutions that one may use to calculate $H_*^{\mathcal{U}(n)}X$ for X a (non-simplicial) object of $\mathcal{U}(n)$ or $\mathcal{W}(n)$ of finite type, using Priddy's technique [46], adapted to an unstable context, as in [20] and [33, Chapter V].

9.1. The Koszul complex and co-Koszul complex

Write $N_*^{\div}$ and C_* for the chain complexes $N_*^{\div}Q^{\mathcal{U}(n)}B^{\mathcal{U}(n)}X$ and $C_*Q^{\mathcal{U}(n)}B^{\mathcal{U}(n)}X$. We will use the convenient bar notation after which the bar construction is named, c.f. [28, §7]. Suppose that $n = 0$, then the vector space $N_s^{\div}$ is spanned by

$$\left[P^{i_{k_s+\cdots+k_1}}\cdots P^{i_{k_{s-1}+\cdots+k_1+1}}\middle|\cdots\middle|P^{i_{k_2+k_1}}\cdots P^{i_{k_1+1}}\middle|P^{i_{k_1}}\cdots P^{i_1}\right]x,$$

where $x \in X$, the expressions in each of the r spaces are P-admissible, and none of the spaces is empty (so that $k_j > 0$ for $1 \leq j \leq s$). Such an expression represents an element of the repeated free construction $Q^{\mathcal{U}(n)}B_s^{\mathcal{U}(n)}X \cong (F^{\mathcal{U}(0)})^s X$, with the requirement that no space be empty reflecting having taken the quotient by degenerate simplices. In particular, this expression equals zero unless $\overline{m}(i_{k_s+\cdots k_1},\ldots,i_1) \leq |x|$.

When $n \geq 1$, the vector space $N_*^{\div}$ is spanned by expressions

$$x\left[\lambda_{i_1}\cdots\lambda_{i_{k_1}}\middle|\lambda_{i_{k_1+1}}\cdots\lambda_{i_{k_2+k_1}}\middle|\cdots\middle|\lambda_{i_{k_{s-1}+\cdots+k_1+1}}\cdots\lambda_{i_{k_s+\cdots+k_1}}\right],$$

again without empty spaces, and subject to an admissibility condition. However, these expressions are only defined when every λ-operation appearing is *defined*. That is, if $x \in X^t_{s_n,\ldots,s_1}$, then we require $m(i_{k_s+\cdots k_1},\ldots,i_1) \leq s_n$, and for no λ_0 to appear if $s_{n-1} = \cdots =$

$s_1 = 0$.

Each of these complexes admits an increasing filtration, the length filtration, with $F_\ell N_s^{\div}$ generated by those terms in which $i_{k_s+\cdots+k_1} \leq \ell$ for each term, which is to say that there are at most ℓ generators appearing in the s free constructions in $C_s = Q^{\mathcal{U}(n)} B_s^{\mathcal{U}(n)} X \cong (F^{\mathcal{U}(n)})^s X$. Note that $F_{s-1} N_s^{\div} = 0$.

Write $E^r_{\ell,s}$ for the spectral sequence of the filtered complex $N_*^{\div} X$, so that $E^0_{\ell,p}$ is the associated graded complex. As $F_{s-1} N_s^{\div} = 0$, $E^r_{\ell,s} = 0$ for $\ell < s$, and $E^0_{s,s}$ is the subspace $F_s N_s^{\div}$ of $N^{\div}$. Priddy [46, Proof of Theorem 5.3] shows that $E^1_{\ell,s} = 0$ for $\ell > s$. Thus, the groups

$$K_s^{\mathcal{U}(n)} X := E^1_{s,s}, \text{ equipped with } d^1 : E^1_{s,s} \longrightarrow E^1_{s-1,s-1}$$

form a subcomplex of $N_*^{\div}$, the *Koszul complex*, whose inclusion is a homotopy equivalence, and E^1_{ss} is the preimage of $F_{s-1} N_{s-1}^{\div}$ under

$$d : F_s N_s^{\div} \longrightarrow F_s N_{s-1}^{\div}.$$

Rather than determining these groups directly, Priddy works with their duals, $K^*_{\mathcal{U}(n)} X$, which form a cochain complex with homology $H^*_{\mathcal{U}(n)} X$. In fact, Priddy's theory shows that the cochain complex $K^*_{\mathcal{U}(n)} X$, the *co-Koszul complex*, is actually a differential unstable left module over the same operations as its cohomology $H^*_{\mathcal{U}(n)} X$, and indeed that this (partial) module is *free*. More precisely, $K_0^{\mathcal{U}(n)} X = X$, and $K^*_{\mathcal{U}(n)} X$ is free on the subspace X^* of $K^0_{\mathcal{U}(n)} X$:

Proposition 9.1. *Suppose that $n \geq 0$, and X is an object of $\mathcal{U}(n)$ of finite type. The chain maps $\widetilde{\theta}^i$ ($\widetilde{\theta}_i$ when $n = 0$) on $C_* Q^{\mathcal{U}(n)} B^{\mathcal{U}(n)} X$ restrict to the subcomplex $K_*^{\mathcal{U}(n)} X$, and induce an $\mathcal{M}_{\mathrm{v}}(n+1)$-structure on $K^*_{\mathcal{U}(n)} X$ which commutes with the differentials. The inclusion $\mathbf{D}X \cong K^0_{\mathcal{U}(n)} X \subseteq K^*_{\mathcal{U}(n)} X$ induces an isomorphism $F^{\mathcal{M}_{\mathrm{v}}(n+1)}(\mathbf{D}X) \longrightarrow K^*_{\mathcal{U}(n)} X$. Moreover, this $\mathcal{M}_{\mathrm{v}}(n+1)$-structure on $K^*_{\mathcal{U}(n)}$ induces the $\mathcal{M}_{\mathrm{v}}(n+1)$-structure on $H^*_{\mathcal{U}(n)} X$ of Propositions 8.3 and 8.6.*

Although it is easier to calculate the co-Koszul complex, we will need to understand the Koszul complex itself in order to calculate the $\mathcal{W}(n+1)$-structure of $H_*^{\mathcal{U}(n)}$. For this, we will introduce a little notation:

Proposition 9.2. *Suppose that $X \in \mathcal{U}(0)$ has homogeneous basis B. Then $K_*^{\mathcal{U}(0)} X$ has basis*

$$\left\{ \delta_I^{\mathrm{v}\star} b \;\middle|\; b \in B^t,\, I \ \delta\text{-admissible with } \overline{m}(I) \leq t \right\},$$

where we define, for any $x \in X^t$ and I δ-admissible with $\overline{m}(I) \le t$:

$$\delta_I^{\mathrm{v}\star} x := \sum_{K \overset{\Delta}{\to} I} \left[P^{k_\ell}\middle|\cdots\middle|P^{k_1}\right] x \text{ for } x \in X^t.$$

If X is of finite type, there is a basis $\{\delta_I^{\mathrm{v}}(b^*)\}$ of $K^*_{\mathcal{U}(0)}X$ constructed using the isomorphism of Proposition 9.1, Corollary 8.5 and the basis $\{b^*\}$ of $\mathbf{D}X$ dual to B. The bases $\{\delta_I^{\mathrm{v}}(b^*)\}$ and $\{\delta_I^{\mathrm{v}\star} b\}$ are dual.

The differential of $K_*^{\mathcal{U}(0)}X$ is given by the formula:

$$d(\delta_I^{\mathrm{v}\star} x) = \sum_{\substack{K \overset{\Delta}{\to} I \\ (k_\ell,\ldots,k_2)\ \Delta\text{-admis.}}} \delta^{\mathrm{v}\star}_{(k_\ell,\ldots,k_2)}(P^{k_1}x),$$

summing over those $K = (k_\ell, \ldots, k_1)$ such that $(k_\ell, \ldots, k_2)$ is δ-admissible, and yet $K \overset{\Delta}{\to} I$.

Note that the sum defining $\delta_I^{\mathrm{v}\star} x$ is finite, simply because the Δ-algebra is graded by the sum of indices. We may assign $\delta_I^{\mathrm{v}\star} x = 0$ for $x \in X^t$ and $\overline{m}(I) > t$, if we wish, since:

Lemma 9.3. *If $I \overset{\Delta}{\to} J$, then $m(I) \ge m(J)$. This inequality is strict if I and J have length 2.*

In fact, this lemma ensures that the sum defining $\delta_I^{\mathrm{v}\star} x$ is finite. That is, any K with $K \overset{\Delta}{\to} I$ must have $m(K) \ge m(I)$ δ-Adem relations only decrease $\overline{m}$. Indeed, we may further restrict the two sums appearing in this proposition be requiring that $\overline{m}(K) \le t$ in each case, but this has no effect. Dually, in the co-Koszul complex, the operations δ_i^{v} are *undefined* when out of range.

Proof of Proposition 9.2. Firstly, we may assume that X is of finite type, as any object of $\mathcal{U}(0)$ is the union of its subobjects of finite type, and the functor $K_*^{\mathcal{U}(0)}$ preserves unions. It is enough to check that $\delta_I^{\mathrm{v}\star} b$ is in fact a member of N_*^-, not just of C_*, as then the collection $\delta_I^{\mathrm{v}\star} b$ will evidently be the dual basis to the $\delta_I^{\mathrm{v}}(b^*)$: in the sum defining $\delta_I^{\mathrm{v}\star} b$, the only δ-admissible sequence K appearing is $K = I$.

Using [46, Lemma 3.2], to check that $\delta_I^{\mathrm{v}\star} b \in N_*^-$, we only need to check that $d(\delta_I^{\mathrm{v}\star} b) \in F_{s-1}C_{s-1}$. To check this membership condition is to check that $\delta_I^{\mathrm{v}\star} b$ pairs to zero with $\mathrm{im}(d^* : \mathbf{D}(F_s N_{s-1}) \longrightarrow \mathbf{D}(F_s N_s))$. Priddy's proof shows that $\mathbf{D}(F_s N_s)$ is spanned by functionals $[(P^{k_s})^*|\cdots|(P^{k_1})^*]b^*$, which pair with the $\delta_I^{\mathrm{v}\star} c$ according to:

$$\left([(P^{k_s})^*|\cdots|(P^{k_1})^*]b^*\right)(\delta_I^{\mathrm{v}\star} c) = b^*(c) \cdot (\delta_I \text{ coeff. of } \delta_K \in \Delta \text{ written in admissibles}).$$

However, the image of d^*, as determined by Priddy, is spanned by the space of 'δ-Adem relations' (see [46, Theorem 2.5 and proof]), and these tautologically evaluate to zero on any

$\delta_I^{\mathrm{v}\star} b$. $\square$

The same analysis applies in the $n \geq 1$ case. Although we write the bar construction on the right, we end up with a left action of the homogeneous Steenrod algebra, as the homogeneous Steenrod algebra is Koszul dual to the *opposite* of the Λ-algebra with an index shift.

Proposition 9.4. *Suppose that $n \geq 1$ and $X \in \mathcal{U}(n)$ has homogeneous basis B. Then $K_*^{\mathcal{U}(n)} X$ has basis*

$$\left\{ \mathrm{Sq}_{\mathrm{v}}^{J\star} b \;\middle|\; \begin{array}{l} b \in B^t_{s_n,\ldots,s_1},\ J \text{ Sq-admissible with } \underline{m}(J) \leq s_n, \\ \text{if } s_{n-1} = \cdots = s_1 = 0 \text{ then } J \text{ does not contain } 1 \end{array} \right\}.$$

where we only define

$$\mathrm{Sq}_{\mathrm{v}}^{J\star} b := \sum_{K \overset{\mathrm{Sq}}{\to} J} b\,[\lambda_{k_1 - 1}|\cdots|\lambda_{k_\ell - 1}]$$

when J and b satisfy the conditions on $\underline{m}(J)$ and on the appearance of 1 in J. If X is of finite type, this basis is dual to the $\{\mathrm{Sq}_{\mathrm{v}}^J b^*\}$ basis of $K^*_{\mathcal{U}(n)} X$ constructed using Proposition 9.1, Corollary 8.8 and the basis $\{b^*\}$ of $\mathbf{D}X$ dual to B. The differential of $K_*^{\mathcal{U}(n)} X$ is given by the formula:

$$d(\mathrm{Sq}_{\mathrm{v}}^{J\star} x) = \sum_{\substack{K \overset{\mathrm{Sq}}{\to} J \\ (k_\ell,\ldots,k_2)\ \text{Sq-admis.}}} \mathrm{Sq}_{\mathrm{v}}^{(k_\ell,\ldots,k_2)\star}(x \lambda_{k_1 - 1}),$$

summing over $K = (k_\ell, \ldots, k_1)$ such that $(k_\ell, \ldots, k_2)$ is Sq*-admissible and yet $K \overset{\mathrm{Sq}}{\to} J$.*

As part of the omitted analysis, we would use 5.8, and the fact that the Λ-algebra and the homogeneous Steenrod algebra are Koszul dual, to show:

Lemma 9.5. *If I and J are sequences of non-negative integers (of any length), such that $J \overset{\mathrm{Sq}}{\to} I$, then $\underline{m}(J) \leq \underline{m}(I)$, and if* 1 *appears in J, it must also appear in I.*

This implies that all the summands in the above definition of $\mathrm{Sq}_{\mathrm{v}}^{J\star} x$ are indeed *defined*.

9.2. The $\mathcal{W}(n+1)$-structure on $H_*^{\mathcal{U}(n)} X$

Suppose that $X \in \mathcal{W}(n)$ for some $n \geq 0$. The form of the bases of $K_*^{\mathcal{U}(n)} X$ given in Propositions 9.2 and 9.4 imply:

Corollary 9.6. *The Koszul complex $K_*^{\mathcal{U}(n)} X$ is naturally a subcomplex of $N^- Q^{\mathcal{U}(n)} B^{\mathcal{U}(n)} X$.*

There are thus two monomorphic quasi-isomorphisms of chain complexes with homology is $H_*^{\mathcal{U}(n)} X$, and we denote their composite $\jmath$:

$$\jmath : \left(K_*^{\mathcal{U}(n)} X \subseteq N_*^- Q^{\mathcal{U}(n)} B^{\mathcal{U}(n)} X \subseteq N_*^- Q^{\mathcal{U}(n)} B^{\mathcal{W}(n)} X \right).$$

The key upshot of Corollary 9.6 is that cycles in the Koszul complex map to *normalized* cycles under $\jmath$.

Now $H_*^{\mathcal{U}(n)}X$ is an object of $\mathcal{W}(n+1)$, since it can be calculated as the homotopy of $Q^{\mathcal{U}(n)}B^{\mathcal{W}(n)}X \in s\mathcal{L}(n)$, and this structure will be needed for the composite functor spectral sequences discussed in §12. We will go some way to calculating this structure in this section. Our method will be to take cycles in the Koszul complex, map them into the large complex using $\jmath$, perform the operations in question, and then homotope the outcome back into the Koszul complex.

We will need a little notation for elements of the various bar constructions. We will label the $s+1$ free constructions in $B_s^{\mathcal{W}(n)}X$ with subscripts in angle brackets:

$$B_s^{\mathcal{W}(n)}X = F_{\langle -1\rangle}^{\mathcal{W}(n)}F_{\langle 0\rangle}^{\mathcal{W}(n)}\cdots F_{\langle s-1\rangle}^{\mathcal{W}(n)}X$$

so that we can then indicate in which free construction operations are being performed. For example, when $n=0$ and $x,y\in X$, $B_2^{\mathcal{W}(0)}X$ contains an element

$$[P^i_{\langle 0\rangle}x, P^j_{\langle 1\rangle}y]_{\langle -1\rangle} := [\eta P^i\eta^2 x, \eta^2 P^j\eta y]$$

where we write $\eta : \mathrm{id} \longrightarrow F^{\mathcal{W}(n)}$ for the unit of the monad on $\mathcal{V}_n^+$ (omitting the forgetful functor). That is: we apply P^j, not to $y\in X$, but rather to ηy, the corresponding generator of $F_{\langle 1\rangle}^{\mathcal{W}(n)}X$; we apply P^i to $\eta^2 x$, a generator of $F_{\langle 0\rangle}^{\mathcal{W}(n)}F_{\langle 1\rangle}^{\mathcal{W}(n)}X$; the bracket is taken in the outermost free construction in $B_2^{\mathcal{W}(0)}X := F_{\langle -1\rangle}^{\mathcal{W}(n)}F_{\langle 0\rangle}^{\mathcal{W}(n)}F_{\langle 1\rangle}^{\mathcal{W}(n)}X$.

With this notation in hand, the map $\jmath$ is induced by the assignment

$$\begin{aligned} [P^{i_s}|\cdots|P^{i_1}]x &\longmapsto P^{i_s}_{\langle 0\rangle}P^{i_s-1}_{\langle 1\rangle}\cdots P^{i_1}_{\langle s-1\rangle}x && \text{(if } n=0\text{)},\\ x[\lambda_{i_1}|\cdots|\lambda_{i_s}] &\longmapsto x\lambda_{i_1\langle s-1\rangle}\cdots\lambda_{i_{s-1}\langle 1\rangle}\lambda_{i_s\langle 0\rangle} && \text{(if } n\geq 1\text{)}.\end{aligned}$$

Before making calculations, we recall the formulae of [20, §8] for the Lie algebra homotopy operations discussed in §5.5. Let Sh_{pq} be the set of (p,q)-shuffles, that is, pairs (α,β) where $\alpha=(\alpha_{p-1},\ldots,\alpha_0)$ and $\beta=(\beta_{q-1},\ldots,\beta_0)$ are disjoint monotonically decreasing sequences that together partition the set $\{0,\ldots,p+q-1\}$. Let s_α denote the iterated degeneracy operator $s_{\alpha_{p-1}}\cdots s_{\alpha_0}$. Finally, let $\mathrm{Sh}_{ii}^{\div 2}$ denote the subset of Sh_{ii} consisting of those shuffles $(\alpha,\beta)\in\mathrm{Sh}_{ii}$ such that $\beta_{i-1}=2i-1$. The formulae of [20, §8], for $z\in ZK_p(X)$ and

$w \in ZK_q(X)$ cycles representing classes $\overline{z}, \overline{w} \in H_*^{\mathcal{U}(n)}X$, are as follows:

$$[\overline{z}, \overline{w}] \text{ is represented by } \sum_{(\alpha,\beta)\in \mathrm{Sh}_{pq}} [s_\beta(\jmath z), s_\alpha(\jmath w)]_{\langle -1\rangle} \in Q^{\mathcal{U}(n)}B^{\mathcal{W}(n)}_{p+q}X;$$

$$\overline{z}\lambda_i \text{ is represented by } \sum_{(\alpha,\beta)\in \mathrm{Sh}^{\div 2}_{ii}} [s_\beta(\jmath z), s_\alpha(\jmath z)]_{\langle -1\rangle} \in Q^{\mathcal{U}(n)}B^{\mathcal{W}(n)}_{p+i}X, \quad (0 < i \leq p);$$

$$\overline{z}\lambda_0 \text{ is represented by } (z)^{[2]}_{\langle -1\rangle} \in Q^{\mathcal{U}(n)}B^{\mathcal{W}(n)}_{p}X, \quad \text{(when defined)}.$$

It will be important to understand these sums. Suppose that $z \in ZK_p^{\mathcal{U}(n)}X$ (for $n \geq 1$). Then z may be written as a sum of terms of the form $x\lambda_{i_1\langle p-1\rangle}\cdots\lambda_{i_p\langle 0\rangle}$, and

Lemma 9.7. *If $(\alpha,\beta) \in \mathrm{Sh}_{pq}$, then $s_\beta(x\lambda_{i_1\langle p-1\rangle}\cdots\lambda_{i_p\langle 0\rangle}) = x\lambda_{i_1\langle \alpha_{p-1}\rangle}\cdots\lambda_{i_p\langle \alpha_0\rangle}$.*

We will also need the following consequence of the simplicial identities:

Lemma 9.8. *Choose $i \geq 1$ and $\alpha = (\alpha_{p-1}, \ldots, \alpha_0)$ with $\alpha_{p-1} > \cdots > \alpha_0 \geq 0$.*

(1) If neither $i-1$ nor $i-2$ appear in α, then $d_{i-1}s_\alpha = s_{\alpha'}d_{i'}$ for some α' and i'.

(2) If exactly one of $i-1$ and $i-2$ appears in α, then $d_{i-1}s_\alpha$ does not depend on which of $i-1$ and $i-2$ appeared in α.

Proposition 9.9. *The $\pi\mathcal{L}(n)$ bracket $H_p^{\mathcal{U}(n)}X \otimes H_q^{\mathcal{U}(n)}X \longrightarrow H_{p+q}^{\mathcal{U}(n)}X$ vanishes except when $p = q = 0$. The Lie algebra structure on $H_0^{\mathcal{U}(n)}X$ is induced by that on X: if $z, w \in X$ represent $\overline{z}, \overline{w} \in H_0^{\mathcal{U}(n)}X$, then $[\overline{x}, \overline{y}]$ is represented by the cycle $[x, y] \in ZC_0(Q^{\mathcal{U}(n)}B^{\mathcal{W}(n)}X)$.*

This theorem shows that $H_*^{\mathcal{U}(X)}$ is trivial in positive dimensions *as a Lie algebra*, but nonetheless, the *restriction* need not be trivial (c.f. Propositions 9.11 and 9.12).

Proof. We will give the proof for $n \geq 1$, but it works the same way for $n = 0$. In fact, when $n = 0$ we can ignore all discussion of top and non-top operations.

Use the abbreviation $\mathbb{B} := Q^{\mathcal{U}(n)}B^{\mathcal{W}(n)}X \in s\mathcal{L}(n)$. Then $\mathbb{B}$ is almost free on the subspaces $V_s = F^{\mathcal{W}(n)}_{\langle 0\rangle}\cdots F^{\mathcal{W}(n)}_{\langle s-1\rangle}X$. Choose representatives $z \in ZK_p^{\mathcal{U}(n)}X$ and $w \in ZK_q^{\mathcal{U}(n)}X$. For any $(\alpha,\beta) \in \mathrm{Sh}_{pq}$, the elements $s_0 s_\beta(\jmath z)$ and $s_0 s_\alpha(\jmath w)$ of $\mathbb{B}_{p+q+1}$ both lie in V_{p+q+1}, and it is only a minor abuse of notation to define:

$$a := \sum_{(\alpha,\beta)\in \mathrm{Sh}_{pq}} [s_0 s_\beta(\jmath z), s_0 s_\alpha(\jmath w)]_{\langle 0\rangle} \in C_{p+q+1}\mathbb{B}.$$

What we mean here is that the bracket of the elements $s_0 s_\beta(\jmath z)$ and $s_0 s_\alpha(\jmath w)$ of

$$V_{p+q+1} = F^{\mathcal{W}(n)}_{\langle 0\rangle}\cdots F^{\mathcal{W}(n)}_{\langle s-1\rangle}X \subseteq F^{\mathcal{W}(n)}_{\langle -1\rangle}F^{\mathcal{W}(n)}_{\langle 0\rangle}\cdots F^{\mathcal{W}(n)}_{\langle s-1\rangle}X = B^{\mathcal{W}(n)}_s X$$

is taken in the free construction $F^{\mathcal{W}(n)}_{\langle 0\rangle}$.

Using the simplicial identity $d_0 s_0 = \mathrm{id}$, we have $d_0 a = \sum [s_\beta(\jmath z), s_\alpha(\jmath w)]_{\langle -1\rangle}$, the representative given for $[\overline{z}, \overline{w}]$. Moreover, we will find that $d_i a = 0$ for $i > 0$, except when $p = q = 0$, in which case $d_1 a = [x, y]$. Thus, in either case, a is the required homotopy in $C_*\mathbb{B}$.

Using the simplicial identity $d_1 s_0 = \mathrm{id}$, we have $d_1 a = \sum [s_\beta(\jmath z), s_\alpha(\jmath w)]_{\langle 0\rangle}$. Now for every pair (α, β) indexing this sum, unless $p = q = 0$, one of α or β, say β, will contain 0. Then by Lemma 9.7, every summand in $s_\alpha(\jmath z)$ is in the image of some *non-top* $\lambda_{i\langle 0\rangle}$, and as $[x\lambda_i, y] = 0$ whenever λ_i is not a top operation, the entire expression vanishes in the construction $F^{\mathcal{W}(n)}_{\langle 0\rangle}$.

What remains is to show that $d_i a = 0$ for $2 \le i \le p + q + 1$. As $d_i s_0 = s_0 d_{i-1}$ for $i \ge 2$:

$$d_i a = \sum [s_0 d_{i-1} s_\beta(\jmath z), s_0 d_{i-1} s_\alpha(\jmath w)]_{\langle 0\rangle}.$$

For this, we will define an involution ρ_i of the set Sh_{pq} indexing the sum, for $2 \le i \le p+q+1$. If α and β do not each contain exactly one of $i-1$ and $i-2$, then ρ_i fixes (α, β). Otherwise, ρ_i interchanges the positions of $i-1$ and $i-2$ in (α, β). To avoid confusion, we note that ρ_{p+q+1} is the identity, as neither α nor β ever contain $p+q$.

If (α, β) is a fixed point of ρ_i, then one of α and β, say α, contains neither of i and $i-1$. Then by Lemma 9.8*(2)*, $d_{i-1} s_\alpha(\jmath w) = s_{\alpha'} d_{i'}(\jmath w) = 0$, as $\jmath w \in ZN^-_*\mathbb{B}$. Thus, the summands corresponding to fixed points vanish. On the other hand, given a shuffle (α, β) which is not fixed by ρ_i, Lemma 9.8**??** shows that the summand corresponding to (α, β) equals the summand corresponding to $\rho_i(\alpha, \beta)$, so these two summands cancel with each other. □

In order to state our calculation of λ_0 of $H^{\mathcal{U}(0)}_* X$ for $X \in \mathcal{W}(0)$, define

$$\mathrm{adm}_+(\Delta, t) := \{I \mid I \text{ a non-empty } \delta\text{-admissible sequence with } \overline{m}(I) \le t\}.$$

Lemma 9.10. *There is an injective function* $\mathfrak{T}_t : \mathrm{adm}_+(\Delta, t) \longrightarrow \mathrm{adm}_+(\Delta, t)$ *given by*

$$I = (i_\ell, \ldots, i_1) \overset{\mathfrak{T}_t}{\longmapsto} (t + nI + \ell, i_\ell, \ldots, i_1).$$

Proof. This is indeed a well defined injective endomorphism of the set $\mathrm{adm}_+(\Delta, t)$, in that it preserves admissibility and the condition $\overline{m}(I) \le t$. The claim about $\overline{m}(I)$ holds by definition. For δ-admissibility, as $\overline{m}(I) \le t$,

$$i_\ell \le \ell - 1 + i_{\ell-1} + \cdots + i_1 + t$$

which (even) implies the (strict) inequality

$$2i_\ell < \ell + i_\ell + \cdots + i_1 + t. \qquad \square$$

Proposition 9.11. *Suppose that $n \geq 0$, $z \in (ZK^{\mathfrak{U}(n)}_{s_{n+1}}X)^t_{s_n,\dots,s_1}$ and $1 \leq k \leq s_{n+1}$, so that $z\lambda_k$ is defined. Then $z\lambda_k = 0$ unless $n = 0$ and $k = 1$.*

When $n = 0$, $k = 1$ and $s_1 \geq 1$, λ_1 may be defined at the level of the Koszul complex as follows. The generic cycle $z \in (ZK^{\mathfrak{U}(0)}_{s_1}X)^t$ may be written as a sum

$$z = \textstyle\sum_j \delta^{\vee\star}_{I_j} x_j, \text{ with } x_j \in X^{t_j} \text{ and } I_j \in \mathrm{adm}_+(\Delta, t_j) \text{ of length } s_1.$$

Then $\overline{z}\lambda_1$ is represented by the cycle

$$\textstyle\sum_j \delta^{\vee\star}_{(\mathfrak{T}_{t_j} I_j)} x_j \in (ZK^{\mathfrak{U}(0)}_{s_1+1}X)^{2t+1}.$$

Proof. We will first prepare for the calculation of λ_1 in case $n = 0$, abbreviating s_1 to s. Note that each $\mathfrak{T}_{t_j}$ appends the same integer, t, to I_j. Write e for the proposed representative $\sum_j \delta^{\vee\star}_{(\mathfrak{T}_{t_j} I_j)} x_j$ of $\overline{z}\lambda_1$. Our first claim is that $e = P^t_{\langle 0\rangle} s_0(\jmath z)$, since

$$\sum_{j,\, K \overset{\Delta}{\to} (\mathfrak{T}_{t_j} I_j)} \left[P^{k_{s+1}}\middle|\cdots\middle|P^{k_1}\right] x_j = \sum_{j,\, H \overset{\Delta}{\to} I_j} \left[P^t\middle|P^{h_s}\middle|\cdots\middle|P^{h_1}\right] x_j.$$

The first of these two sums a priori contains more terms. However, the extra terms all vanish, by the unstableness condition. More precisely: if $(k_{s+1},\dots,k_1) \overset{\Delta}{\to} \mathfrak{T}_{t_j} I_j$ and $k_{s+1} \neq t$, then $\overline{m}(k_{s+1},\dots,k_1) > t_j$, so that $\left[P^{k_{s+1}}\middle|\cdots\middle|P^{k_1}\right] x_j = 0$. To understand this observation, as δ-Adem relations cannot increase $\overline{m}$ (Lemma 9.3), we may reduce to the case where $(k_s,\dots,k_1)$ is already δ-admissible, $t \neq k_{s+1}$, and $(k_{s+1}, k_s) \overset{\Delta}{\to} (t, k_{s+1} + k_s - t)$, where::

$$\begin{aligned}
\overline{m}(k_{s+1},\dots,k_1) &\geq \overline{m}(k_{s+1},k_s) - (k_{s-1}+1) - \cdots - (k_1+1) \\
&> \overline{m}(t, k_{s+1}+k_s-t) - (k_{s-1}+1) - \cdots - (k_1+1) \\
&\geq 2t - (k_{s+1} + \cdots + k_1 + s) \\
&= 2t - (t + i_s + \cdots + i_1 + s) = t_j.
\end{aligned}$$

where: the two non-strict inequalities are by definition of $\overline{m}$; the strict inequality follows from Lemma 9.3; the first equation holds as Δ is graded by the sum of the indices; and the second equation holds as t is the dimension of $\delta^{\vee\star}_{I_j} x_j$.

With this in hand, we return the general case, $1 \leq k \leq p$ and $n \geq 0$, our goal being to produce a nullhomotopy, except when $n = 0$ and $k = 1$, when we need a homotopy to

$P^t_{\langle 0\rangle} s_0(\jmath z)$. We proceed as in the previous proof, defining

$$a := \sum_{(\alpha,\beta)\in \mathrm{Sh}^{\div 2}_{kk}} [s_0 s_\beta(\jmath z), s_0 s_\alpha(\jmath z)]_{\langle 0\rangle} \in C_{p+k+1}\mathbb{B}^{2t+1}_{2s_n,\dots,2s_1}.$$

Then $d_0 a$ is the representative for $\overline{z}\lambda_k$, and $d_1 a = 0$ as in the previous proof (and there is no analogue here of the special case $p = q = 0$). Now consider the same involutions ρ_i as in the previous proof, now acting on Sh_{kk}. When $2 \le i < 2k$, ρ_i preserves $\mathrm{Sh}^{\div 2}_{kk}$. When $2k < i \le p + k + 1$, ρ_i is the identity, so preserves $\mathrm{Sh}^{\div 2}_{kk}$ trivially. Thus, $d_i a = 0$ for all $2 \le i \le p + k + 1$ with $i \ne 2k$, as the summands corresponding to fixed points vanish, and the cancellations still all occur within the smaller sum

$$d_i a = \sum_{(\alpha,\beta)\in \mathrm{Sh}^{\div 2}_{kk}} [s_0 d_{i-1} s_\beta(\jmath z), s_0 d_{i-1} s_\alpha(\jmath z)]_{\langle 0\rangle}.$$

To address the question of d_{2k}, we define an alternative involution $\widetilde{\rho}_{2k}$ of Sh_{kk} as follows. If α and β do not each contain exactly one of $2k-2$ and $2k-1$, then $\widetilde{\rho}_{2k}$ fixes (α,β). Otherwise, we define $\widetilde{\rho}_{2k}(\alpha,\beta) := \rho_{2k}(\beta,\alpha)$, which is to say that $\widetilde{\rho}_{2k}$ swaps *everything but* $2k-2$ and $2k-1$.

Now the summands in this formula exhibit a symmetry not present in the previous proof: z is repeated. This symmetry, along with Lemma 9.8**??**, shows that all the summands corresponding to shuffles not fixed by $\widetilde{\rho}_{2k}$ cancel out. When $k > 1$, the fixed points of $\widetilde{\rho}_{2k}$ are only those shuffles in which one of α and β contains neither $2k-2$ nor $2k-1$, and the corresponding summands vanish, by 9.8*(2)*, as in previous arguments. When $k = 1$, however, $\widetilde{\rho}_{2k}$ has an *extra* fixed point, the shuffle $((0),(1))$, which fails to differ from its image under $\widetilde{\rho}_{2k}$. In this case, then:

$$\begin{aligned} d_2 a &= [s_0 d_1 s_1(\jmath z), s_0 d_1 s_0(\jmath z)]_{\langle 0\rangle} \\ &= [s_0(\jmath z), s_0(\jmath z)]_{\langle 0\rangle} \\ &= \begin{cases} 0, & \text{if } n \ge 1, \\ P^t_{\langle 0\rangle} s_0(\jmath z), & \text{if } n = 0. \end{cases} \end{aligned}$$

That is, if $n \ge 1$, this self-bracket vanishes (an object of $\mathcal{W}(n)$ for $n \ge 1$ is a Lie algebra), while if $n = 0$, the self-bracket is equal to the top P-operation, in this case P^t.

In sum, we have shown that $d_0 a = 0$ represents $\overline{z}\lambda_i$, and that $d_i a = 0$ whenever $1 \le i \le p + k + 1$, except when $k = 1$, $i = 2$ and $n = 0$, in which case $d_2 a = e$, as hoped. $\square$

Proposition 9.12. *Suppose that $n \ge 1$, and $z \in (ZK^{\mathcal{U}(n)}_{s_{n+1}} X)^t_{s_n,\dots,s_1}$ where not all of $s_n,\dots,s_1$ equal zero. If $s_{n+1} = 0$ then $\overline{z}\lambda_0$ is represented by $z\lambda_{s_n} \in X^{2t+1}_{2s_n,\dots,2s_1}$.*

Suppose instead that $s_{n+1} > 0$, and consider a cycle

$$z = \sum_j \mathrm{Sq}_{\mathrm{v}}^{I_j\star} x_j \in (ZK^{\mathfrak{U}(n)}_{s_{n+1}}X)^t_{s_n,\ldots,s_1},$$

for various $x_j \in X$ and Sq*-admissible sequences $I_j = (i_{j,s_{n+1}}, \ldots, i_{j,1})$. Suppose further that for each summation index j, $x_j\lambda_{i-1} = 0$ whenever $i \geq i_{j,1}$. Then $\overline{z}\lambda_0 = 0$.*

Proof. Write $p := s_{n+1}$. The same homotopy a as in the previous cases shows that $\widetilde{z}\lambda_0$ is represented by $(z)^{[2]}_{\langle 0\rangle} = z\lambda_{s_n\langle 0\rangle}$ when $p > 0$, and by $z\lambda_{s_n} \in X$ when $p = 0$, so that we may restrict to the case $p > 0$. Then, $z\lambda_{s_n\langle 0\rangle}$ is the image of the following element of $ZF_{p+1}N^{\div}_p Q^{\mathfrak{U}(n)} B^{\mathfrak{U}(n)} X$:

$$\begin{aligned} E &= \sum_{j,\, K \overset{\mathrm{Sq}}{\to} I_j} x_j \left[\lambda_{k_1-1}\middle|\cdots\middle|\lambda_{k_{p-1}-1}\middle|\lambda_{k_p-1}\lambda_{s_n}\right] \\ &= \sum_{j,\, K \overset{\mathrm{Sq}}{\to} I_j} \sum_{(\alpha,\beta) \overset{\mathrm{Sq}}{\to} (s_n+1, k_p)} x_j \left[\lambda_{k_1-1}\middle|\cdots\middle|\lambda_{k_{p-1}-1}\middle|\lambda_{\beta-1}\lambda_{\alpha-1}\right], \end{aligned}$$

where the second equation holds by the Koszul duality of the Λ-algebra and the homogeneous Steenrod algebra. As homogeneous Sq-Adem relations move Sq-inadmissible sequences towards Sq-admissibility, when $p \geq 2$ we have $k_1 \geq i_{j,1}$ in each summand, and when $p = 1$ we have $\beta \geq i_{j,1}$ in each summand.

Dualizing Priddy's work, namely [46, Proof of Theorem 5.3], gives a sequence of homotopies which move this cycle into $F_pN^{\div}_p$. Indeed, given an expression

$$e = y\left[\lambda_{g_1-1}\middle|\cdots\middle|\lambda_{g_{r-2}-1}\middle|\lambda_{g_{r-1}-1}\lambda_{g_r-1}\middle|\lambda_{g_{r+1}-1}\middle|\cdots\middle|\lambda_{g_{p+1}-1}\right] \in F_{p+1}N^{\div}_p,$$

(with the composite $\lambda_{g_{r-1}-1}\lambda_{g_r-1}$ Λ-admissible), define:

$$\Gamma(e) := \begin{cases} y\left[\lambda_{g_1-1}\middle|\cdots\middle|\lambda_{g_{r-1}-1}\middle|\lambda_{g_r-1}\middle|\cdots\middle|\lambda_{g_{p+1}-1}\right], & \text{if } (g_{p+1},\ldots,g_r) \text{ is Sq-admissible;} \\ 0, & \text{otherwise.} \end{cases}$$

If we further define Γ to be zero on $F_pN^{\div}_p$, then $\Gamma : F_{p+1}N^{\div}_p \longrightarrow F_{p+1}N^{\div}_{p+1}$ may be used as a chain homotopy to compress $E \in ZF_{p+1}N^{\div}_p$ into $ZF_pN^{\div}_p$:

$$(\mathrm{id} + d\Gamma)^u E \text{ stabilizes to an element of } ZF_pN^{\div}_p \text{ as } u \longrightarrow \infty.$$

As we repeatedly apply $(\mathrm{id} + d\Gamma)$ to this e, because $a_1 \geq b_1$ whenever $(b_2, b_1) \overset{\Lambda}{\to} (a_2, a_1)$, the very leftmost λ-operation in any of the expressions that appear is λ_{m-1} for some $m \geq g_1$, and every term in $(\mathrm{id} + d\Gamma)^u e \in ZF_pN^{\div}_p$ will be of the form $y\lambda_{m-1}[\cdots]$ for some $m \geq g_1$.

Applying these observations in the very specific circumstances of this proposition, along with the earlier observation that in the sum defining the cycle E we always have $k_1 \geq i_{j,1}$ (or $\beta \geq i_{j,1}$ if 1), one derives that $(\mathrm{id} + d\Gamma)^u E = 0$, so that E is nullhomotopic. $\square$

Chapter 10

Operations on second quadrant homotopy spectral sequences

In this chapter we will produce various external operations on second quadrant homotopy spectral sequences. That is, for $X \in cs\mathcal{V}$, we will produce operations from $[E_r X]$ to each of $[E_r S_2 X]$ and $[E_r \Lambda^2 X]$

This approach leaves open a number of possibilities. If $X \in cs\mathscr{C}om$, then the structure map $\mu : S_2 X \longrightarrow X$ induces a spectral sequence map $[E_r S^2 X] \longrightarrow [E_r X]$, and so the external operations induce internal operations on $[E_r X]$. If X is a Lie algebra we may apply the analogous technique $[\,,\,] : \Lambda^2 X \longrightarrow X$. In §11, we will use these external operations in another way to produce operations on the BKSS of a commutative algebra or Lie algebra — the construction will involve a shift in filtration, which is conceivable given that Radulescu-Banu's resolution is a resolution by GEMs.

A number of authors have written about spectral sequence operations in a variety of settings. Singer's work [52] on first quadrant cohomology spectral sequences will be used extensively in §13.1, and has been extended by Turner [55]. Perhaps the closest recent examples are due to Hackney [37] and [36], who works out the operations available on the homotopy spectral sequence of a cosimplicial E_∞- or E_n-space respectively, using Bousfield and Kan's universal examples [9]. We will be working with cosimplicial simplicial *vector spaces*, and so we are able to develop a direct approach, mirroring Dwyer's work in second quadrant cohomotopy spectral sequences [25].

Dwyer's work makes an interesting point of comparison with ours. In both cases: products, Steenrod operations and higher divided powers (as in [25] and §5.4) are produced on the spectral sequence; one set of operations is not present in the target; the other set of operations is present in the target, but the unstableness conditions on the target and on E_2 do not agree; and differentials are constructed between the two varieties to simultaneously rectify these disparities. Between Dwyer's theory and the theory presented here, the roles of the two types of operations are interchanged.

10.1. Operations with indeterminacy

On pages higher than the E_2-page, some of the 'operations' $[E_rX] \longrightarrow [E_rS_2X]$ that we would like to use will in fact fail to be well defined, and in this section we will introduce the language which we will use in such situations.

We will make use of the notion of a *(potentially) multi-valued function* $f : D \longrightarrow C$, which is just a relation $f \subset D \times C$ such that for all $x \in D$ there exists some $y \in C$ for which $(x, y) \in f$. We may drop the modifier *potentially*, with the understanding that we do not insist that a multi-valued function fail to be a function. If a multi-valued function turns out to be an actual function, we will call it *single-valued*. For $x \in D$, the *set of values* of $f(x)$ is $\{y \in C \mid (x, y) \in f\}$.

In all of our examples, D and C will be vector spaces. A multi-valued function $f : D \longrightarrow C$ has *linear indeterminacy* if it is essentially a map $D \longrightarrow C/I$ for some subspace I of C. That is if there exists a subspace I of C such that for all $x \in D$, the set of values of $f(x)$ is a coset of I in C. Such a function is *linear* if $f(x + y)$ is the sum of the cosets $f(x)$ and $f(y)$ for all $x, y \in D$. Almost all of the multi-valued functions we encounter are linear, and all of the exceptions are operations at E_0 or E_1 or *top* δ-operations (c.f. §10.5 and §11.3).

In this chapter, multi-valued functions will arise in two ways. An *operation* $[E_rV] \longrightarrow [E_rS_2V]$ *with indeterminacy disappearing by* $E_{r'}$ will be an actual function

$$[E_rV] \longrightarrow [E_rS^2V]/[B_{r,r'}S^2V],$$

where $[B_{r,r'}S^2V] \subseteq [E_rS^2V]$ is the subgroup consisting of those elements which survive to $[E_rS^2V]$ and represent zero there. We view such operations as linear multi-valued functions

$$[E_rV] \longrightarrow [E_rS^2V],$$

and the external Steenrod operations that we will define in §10.4 will be examples. On the other hand, we will define in §10.5 external δ-operations which will sometimes be multi-valued, and will almost always be linear with linear indeterminacy.

10.2. Maps of mixed simplicial vector spaces

For mixed simplicial vector spaces $X, Y \in cs\mathcal{V}$, we will write $C(X \otimes Y)$ for the double complex associated with the levelwise tensor product of $C(X \otimes Y)$, so that $C(X \otimes Y)^s_t = X^s_t \otimes Y^s_t$.

We will write $C(X \otimes_{\mathrm{v}} Y)$ for the double complex with $C(X \otimes_{\mathrm{v}} Y)_t^s = \bigoplus_{t'+t''=t} X_{t'}^s \otimes X_{t''}^s$. The following vector space maps are given by prolonging D^k, ∇, ∇_k and ϕ_k *wherever these maps are defined*, and by zero elsewhere:

$$\begin{aligned}
D^k : (CX \otimes CY)_t^{s+k} &\longrightarrow C(X \otimes_{\mathrm{v}} Y)_t^s && \text{(zero unless } 0 \leq k \leq s) \\
\nabla : C(X \otimes_{\mathrm{v}} Y)_t^s &\longrightarrow C(X \otimes Y)_t^s && \text{(no condition)} \\
\nabla_k : C(X \otimes_{\mathrm{v}} Y)_{t+k}^s &\longrightarrow C(X \otimes Y)_t^s && \text{(zero unless } 0 \leq k \leq t) \\
\phi_k : C(X \otimes_{\mathrm{v}} Y)_{t+k}^s &\longrightarrow C(X \otimes Y)_t^s && \text{(zero unless } k = t \geq 0)
\end{aligned}$$

We have just committed to regarding ∇_k as zero where it is not defined. This is certainly not a natural convention, and it has somewhat untidy results, for instance:

Lemma 10.1. *Suppose that $z \in C(X \otimes_{\mathrm{v}} Y)_t^s$. Then*

$$(d\nabla_k + \nabla_k d)z = ((1+\omega)\nabla_{k+1} + \phi_{k+1})z$$

whenever $k \geq 0$ and t does not equal either of $2k$ and $2k+1$.

As discussed earlier, we will write T for any symmetry isomorphism, write "ωG" as shorthand for the function TGT, and whenever we write ωGH, we will mean $(\omega G)H$. We will also use the notation

$$X^{\otimes 2} \xrightarrow{\rho} S_2 X \quad \text{and} \quad X^{\otimes 2} \xrightarrow{\rho'} \Lambda^2$$

for the projection onto coinvariants and further onto the exterior quotient. Until §10.5, the operations that we will produce into each of $[E_r S_2 X]$ and $[E_r \Lambda^2 X]$ will be essentially the same.

10.3. An external spectral sequence pairing μ_{ext}

The easiest and most standard of our constructions is that of an external product, using the chain-level formula

$$x \otimes y \longmapsto \rho\nabla D^0(x \otimes y).$$

Both ∇ and D^0 are chain maps, and filtrations add under ∇D^0, and thus:

Proposition 10.2. *The map $\rho\nabla D^0(x \otimes y) : CX \otimes CX \longrightarrow CX$ induces a pairing*

$$\mu_{\mathrm{ext}} : [E_r X]_t^s \otimes [E_r X]_{t'}^{s'} \longrightarrow [E_r S_2 X]_{t+t'}^{s+s'}$$

for each r, satisfying the Leibniz formula. For $r \geq 2$, this map descends to the symmetric quotient $S_2[E_rX]$. Under the identifications $[E_2X]^s_t \cong \pi^s_{\mathrm{h}}\pi^{\mathrm{v}}_t X$ and $[E_2S_2X]^s_t \cong \pi^s_{\mathrm{h}}\pi^{\mathrm{v}}_t S_2X$, μ_{ext} corresponds to the composite

$$S_2\pi^*_{\mathrm{h}}\pi^{\mathrm{v}}_* X \xrightarrow{\mu_{\mathrm{ext}}} \pi^*_{\mathrm{h}}(S_2\pi^{\mathrm{v}}_* X) \xrightarrow{\pi^*_{\mathrm{h}}(\widetilde{\nabla})} \pi^*_{\mathrm{h}}\pi^{\mathrm{v}}_* S_2X.$$

10.4. External spectral sequence operations $\mathrm{Sq}^i_{\mathrm{ext}}$

Consider the chain-level map:

$$\mathrm{SQ}^{i,s} : x \longmapsto \rho\nabla(D^{s-i}(x \otimes x) + D^{s-i+1}(x \otimes dx)).$$

We will use these maps to define *external Steenrod operations* $\mathrm{Sq}^i_{\mathrm{ext}}$, the behaviour of which is rather different on E_1 than on later pages. Thus, we will state two separate Propositions that we will prove together.

Proposition 10.3. *Suppose that $r \geq 2$. The chain level operation $\mathrm{SQ}^{i,s}$ defines a linear operation with indeterminacy vanishing by E_{2r-2}:*

$$\mathrm{Sq}^i_{\mathrm{ext}} : [E_rX]^s_t \longrightarrow [E_rS_2X]^{s+i}_{2t}.$$

Now suppose that $x \in [E_rX]^s_t$. $\mathrm{Sq}^i_{\mathrm{ext}}x = 0$ unless $\min\{t, r\} \leq i \leq s$, and this vanishing occurs without indeterminacy. In any case, $\mathrm{Sq}^i_{\mathrm{ext}}x$ survives to $[E_{2r-1}S_2X]^{s+i}_{2t}$, and the following equation holds in $[E_{2r-1}S_2X]^{s+i+2r-1}_{2t+2r-2}$ (without indeterminacy):

$$d_{2r-1}(\mathrm{Sq}^i_{\mathrm{ext}}x) = \mathrm{Sq}^{i+r-1}_{\mathrm{ext}}(d_rx).$$

The top *operation $\mathrm{Sq}^s_{\mathrm{ext}}x$ is equal to the product-square $\mu^{\mathrm{ext}}(x \otimes x)$, and in particular has no indeterminacy. As for the only potentially non-zero $\mathrm{Sq}^0_{\mathrm{ext}}$ operation:*

$$\mathrm{Sq}^0_{\mathrm{ext}} : [E_rX]^s_0 \longrightarrow [E_rS_2X]^s_0 \text{ is induced by } X \xrightarrow{\text{squaring}} S_2X.$$

At E_2, there is no indeterminacy, and the operation $\mathrm{Sq}^k_{\mathrm{ext}}$ corresponds to the composite:

$$\pi^s_{\mathrm{h}}\pi^{\mathrm{v}}_t X \xrightarrow{\mathrm{Sq}^i_{\mathrm{ext}}} \pi^{s+i}_{\mathrm{h}} S_2(\pi^{\mathrm{v}}_t X) \xrightarrow{\pi^{s+i}_{\mathrm{h}}(\widetilde{\nabla})} \pi^{s+i}_{\mathrm{h}}\pi^{\mathrm{v}}_{2t} S_2X.$$

The condition $\min\{t, r\} \leq i \leq s$ may be replaced with $\min\{t+1, r\} \leq i \leq s$ after composing with $[E_rS_2X]^{s+i}_{2t} \longrightarrow [E_r\Lambda^2X]^{s+i}_{2t}$.

Proposition 10.4. *At E_1, the chain level operation $\mathrm{SQ}^{i,s}$ defines an operation*

$$\mathrm{Sq}^i_{\mathrm{ext}} : [E_r X]^s_t \longrightarrow [E_r S_2 X]^{s+i}_{2t}$$

which commutes with the differential d_1. Suppose that $x \in [E_1X]^s_t$. The top operation $\mathrm{Sq}^s_{\mathrm{ext}} x$ need not equal the product-square $\mu^{\mathrm{ext}}(x \otimes x)$ on E_1, and $\mathrm{Sq}^{s+1}_{\mathrm{ext}} x$ need not vanish, instead equalling $\mu^{\mathrm{ext}}(x \otimes d_1 x)$ on E_1. The operations need not be linear and have no indeterminacy. At least for $i > s+1$, $\mathrm{Sq}^i_{\mathrm{ext}} x = 0$. $\mathrm{Sq}^1_{\mathrm{h}} x$ is zero whenever $t \geq 1$. $\mathrm{Sq}^0_{\mathrm{h}} x = 0$ for all t.

Proof of Propositions 10.3 and 10.4. Choose a representative $x \in [Z_r X]^s_t$ of the class of interest. We readily check that $\mathrm{SQ}^{i,s}(x)$ has filtration at least $s+i$:

$$\mathrm{filt}(\rho \nabla D^{s-i}(x \otimes x) \geq s + s - (s-i) = s+i,$$
$$\mathrm{filt}(\rho \nabla D^{s-i+1}(x \otimes dx) \geq s + (s+r) - (s-i+1) = s+i+(r-1).$$

Thus, we may view $\mathrm{SQ}^{i,s}(x)$ as an element of $[Z_0 S_2 X]^{s+i}_{2t}$. A straightforward calculation shows that

$$d(\mathrm{SQ}^{i,s}(x)) = \rho \nabla D^{s-i+1}(dx \otimes dx) = \mathrm{SQ}^{i+r-1,s+r}(dx),$$

and as $x \in [Z_r X]^s_t$:

$$\mathrm{filt}(d(\mathrm{SQ}^{i,s}(x))) \geq (s+r) + (s+r) - (s-i+1) = (s+i) + (2r-1),$$

so that $\mathrm{SQ}^{i,s}(x) \in [Z_{2r-1} S_2 X]^{s+i}_{2t}$. This demonstrates the survival property, along with the formula commuting the $\mathrm{Sq}^i_{\mathrm{ext}}$ with spectral sequence differentials.

The next step is to examine the non-linearity of the operation $\mathrm{SQ}^{i,s}$, which we do using formulae analogous to [52, (1.111) and (1.112)]. That is, for $x, x' \in [Z_r X]^s_t$, one calculates

$$\begin{aligned} \mathrm{NL}(x,x') &:= \mathrm{SQ}^{i,s}(x) + \mathrm{SQ}^{i,s}(x') + \mathrm{SQ}^{i,s}(x+x') \\ &= d\rho\nabla[D^{s-i+2}(x \otimes dx') + D^{s-i+1}(x' \otimes x)] + \rho\nabla D^{s-i+2}(dx \otimes dx'). \end{aligned}$$

The first two terms of $\mathrm{NL}(x,x')$ are the boundaries of chains in filtrations satisfying

$$\mathrm{filt}(\rho\nabla D^{s-i+2}(x \otimes dx')) \geq s+s+r-(s-i+2) = (s+i-r+1) + 2(r-2) + 1,$$
$$\mathrm{filt}(\rho\nabla D^{s-i+1}(x' \otimes x)) \geq s+s-(s-i+1) = (s+i-r+1) + (r-2),$$

so that they vanish in $[E_r X]^{s+i}_{2t}$ whenever $r \geq 2$. Moreover

$$\mathrm{filt}(\rho\nabla D^{s-i+2}(dx \otimes dx')) \geq 2(s+r) - (s-i+2) = s+i+2(r-1),$$

so that the final term also vanishes in $[E_rX]^{s+i}_{2t}$ when $r \geq 2$. When $r \geq 2$, this proves that whatever indeterminacy these operations are subject to is linear, and that the operations themselves are linear.

To examine the indeterminacy, as a representative of a class in $[E_rX]^s_t$, x is only determined up to boundaries of $y \in [Z_{r-1}X]^{s-r+1}_{t-r+2}$ and elements of $[E_{r-1}X]^{s+1}_{t+1}$. The latter are irrelevant, as their effect on the value of $\mathrm{SQ}^{i,s}$ is restricted to high filtration. The boundaries dy are more problematic, but if we define

$$\mathrm{BC}(x,y) := \rho\nabla[D^{s-i-1}(y \otimes y) + D^{s-i}(y \otimes dy) + D^{s-i+1}(dy \otimes x)]$$

then this chain has boundary

$$\begin{aligned} d(\mathrm{BC}(x,y)) &= \rho\nabla[D^{s-i-1}(dy \otimes y + y \otimes dy) + D^{s-i}(dy \otimes dy) + D^{s-i+1}(dy \otimes dx)] \\ &\quad + \rho\nabla[0 + D^{s-i-1}(y \otimes dy + dy \otimes y) + D^{s-i}(dy \otimes x + x \otimes dy)] \\ &= \rho\nabla[D^{s-i}(dy \otimes x + x \otimes dy + dy \otimes dy) + D^{s-i+1}(dy \otimes dx)] \\ &= \mathrm{SQ}^{i,s}(x) - \mathrm{SQ}^{i,s}(x + dy). \end{aligned}$$

That is, $\mathrm{BC}(x,y)$ is a bounding chain for this difference, and

$$\mathrm{filt}(\mathrm{BC}(x,y)) \geq 2(s-r+1) - (s-i-1) = (s+i) - (2r-3),$$

so that $\mathrm{Sq}^i_{\mathrm{ext}}x$ has indeterminacy vanishing by $[E_{2r-2}S_2X]^{s+i}_{2t}$ as claimed. When $i = s$, this result may be improved to $\mathrm{filt}(\mathrm{BC}(x,y)) \geq 2s - (r-1)$, as in this case the lowest filtration summand in fact vanishes — this is one explanation of why the top square has no indeterminacy.

When $i \geq s+2$, we have $\mathrm{SQ}^i(x) = 0$, and even with $i = s+1$:

$$\mathrm{SQ}^{s+1,s}(x) = \rho\nabla D^0(x \otimes dx) \in F^{2s+r},$$

so that $\mathrm{Sq}^{s+1}_{\mathrm{ext}}x$ vanishes when $r \geq 2$, and $\mathrm{Sq}^{s+1}_{\mathrm{ext}}x = \mu^{\mathrm{ext}}(x \otimes d_1x)$ when $r = 1$, without indeterminacy in both cases.

We must also check that $\mathrm{Sq}^i_{\mathrm{ext}}x$ vanishes (without indeterminacy) when $i < \min\{t,r\}$. For this we use the filtration preserving operations DEL_i to be defined in §10.5. Suppose Proposition 10.7 states that

$$d(\mathrm{DEL}_{t-i+1}(x)) + \mathrm{DEL}_{t-i+1}(dx) = \mathrm{SQ}^{i,s}(x)$$

as long as $2 \leq t-i+1 \leq t+1$ (which is satisfied whenever $i < t$). Moreover, if $i < r$, then

$$\mathrm{DEL}_{t-i+1}(dx) \in F^{s+r} \subset F^{s+i+1} \text{ and } \mathrm{DEL}_{t-i+1}(x) \in F^{s},$$

so that this equation states that $\mathrm{SQ}^{i,s}(x) = 0$ in $[E_r X]^{s+i}_{2t}$, without indeterminacy.

For the statement about the top operation, one calculates that

$$\mathrm{SQ}^{s,s}(x) - \rho\nabla D^0(x \otimes x) = \rho\nabla D^1(x \otimes dx) \in F^{2s+r-1},$$

which exceeds filtration $2s$ when $r \geq 2$.

For the statement about $\mathrm{Sq}^0_{\mathrm{ext}} x$ when $t = 0$, using the specialness assumption:

$$\mathrm{SQ}^{0,s}(x) - \rho\nabla D^s(x \otimes x) = \rho\nabla D^{s+1}(x \otimes dx) \in F^{s+1},$$

so that (using the assumption that $\{D^k\}$ is special):

$$\begin{aligned}
\mathrm{SQ}^{0,s}(x) &\equiv \rho\nabla D^s(x^s_t \otimes x^s_t) && \pmod{F^{s+1}} \\
&= \rho(\phi_0 + (1{+}\omega)\nabla_0)(x^s_t \otimes_{\mathrm{v}} x^s_t) && (x^s_t \otimes_{\mathrm{v}} x^s_t \in C(X \otimes_{\mathrm{v}} X)^s_{2t}) \\
&= \rho\phi_0(x^s_t \otimes_{\mathrm{v}} x^s_t) \in C(X \otimes_{\Sigma_2} X)^s_t.
\end{aligned}$$

The statements about $[E_r\Lambda^2 X]^{s+i}_{2t}$ follow similarly, replacing DEL_i with LAM_i. □

10.5. External spectral sequence operations δ^{ext}_i

For any k (positive or otherwise) write $\mathbb{D}_k : (C(X) \otimes C(Y))_i \longrightarrow C(X \otimes Y)_{i-k}$ for the map:

$$\mathbb{D}_r(z) = \sum_{\alpha-\beta=r} \nabla_\alpha \omega^\alpha D^\beta(z).$$

Lemma 10.5. *If $x \in F_s C_n(X)$ and $y \in F_{s'} C_{n'}(X)$, then*

$$\mathbb{D}_k(x \otimes y) \in F_{\max\{s,s'\}} C_{n+n'-k}(X \otimes X).$$

Proof. We may assume that x and y are each homogeneous, with $x \in X^s_t$ and $y \in Y^{s'}_{t'}$. As $\{D^k\}$ is special, $D^\beta(x \otimes y) = 0$ unless $\beta \leq \min\{s, s'\}$, in which case

$$\mathrm{filt}(D^\beta(x \otimes y)) \geq s + s' - \beta \geq s + s' - \min\{s, s'\} = \max\{s, s'\}.$$ □

Lemma 10.6. *For all k (positive or otherwise) the equation:*

$$(d\mathbb{D}_k + \mathbb{D}_k d)(z) = ((1+\omega)\mathbb{D}_{k+1} + \nabla\omega D^{-k-1})(z)$$

holds when $z \in (CX \otimes CX)_N$ with $N > 2(k+1)$. When $N = 2(k+1)$,

$$(d\mathbb{D}_k + \mathbb{D}_k d)(z) = ((1+\omega)\mathbb{D}_{k+1} + \nabla\omega D^{-k-1} + \textstyle\sum_\alpha \phi_\alpha \omega^{\alpha+1} D^{\alpha-k-1})(z).$$

Proof. We may assume that z is homogeneous, $z \in (CX \otimes CX)_t^s$ with $N = t - s \geq 2(k+1)$. Choose α and β such that $\alpha - \beta = k$. Then $(\omega^\alpha D^\beta(z)) \in C(X \otimes_{\mathrm{v}} Y)_t^{s-\beta}$.

We will need to apply Lemma 10.1 to calculate, for $\alpha - \beta = k$:

$$(d\nabla_\alpha + \nabla_\alpha d)(\omega^\alpha D^\beta(z)) = ((1+\omega)\nabla_{\alpha+1} + \phi_{\alpha+1})(\omega^\alpha D^\beta(z)),$$

but Lemma 10.1 does not apply when $t = 2\alpha + e$ for $e \in \{0,1\}$. Fortunately, in that case $D^\beta(z)$ is zero, so the equation holds by default: after all, if $t = 2\alpha + e$, our assumed inequality on N implies:

$$\beta = \frac{t-e}{2} - k \geq \frac{t-e}{2} - \frac{t-s-2}{2} = \frac{s+2-e}{2} > \frac{s}{2}.$$

After these observations and under our conventions on the ∇_α and D^β, all but one of the following manipulations is totally formal:

$$\begin{aligned}
(d\mathbb{D}_k + \mathbb{D}_k d)(z) &:= \sum_{\alpha-\beta=k} \Big(d\nabla_\alpha \omega^\alpha D^\beta + \nabla_\alpha \omega^\alpha D^\beta d \Big)(z) \\
&= \sum_{\alpha-\beta=k} \Big((d\nabla_\alpha + \nabla_\alpha d)\omega^\alpha D^\beta + \nabla_\alpha \omega^\alpha (dD^\beta + D^\beta d) \Big)(z) \\
&= \sum_{\alpha-\beta=k,\ \alpha\geq 0} ((1{+}\omega)\nabla_{\alpha+1} + \phi_{\alpha+1})\omega^\alpha D^\beta(z) + \sum_{\alpha-\beta=k} \nabla_\alpha \omega^\alpha (1{+}\omega) D^{\beta-1}(z) \\
&= \sum_{\alpha-\beta=k+1,\ \alpha\geq 1} ((1{+}\omega)\nabla_\alpha + \phi_\alpha)\omega^{\alpha-1} D^\beta(z) + \sum_{\alpha-\beta=k+1} \nabla_\alpha \omega^\alpha (1{+}\omega) D^\beta(z)
\end{aligned}$$

Using the identity $(1+\omega)\nabla_0 + \phi_0 = \nabla$ for the first equation, and the observation that $(1+\omega)F\omega G + F(1+\omega)G = (1+\omega)(FG)$ for the second (with $F = \nabla_\alpha$ and $G = \omega^\alpha D^\beta$):

$$\begin{aligned}
(d\mathbb{D}_k + \mathbb{D}_k d)(z) - \nabla\omega D^{-k-1}(z) &= \sum_{\alpha-\beta=k+1} \Big(((1+\omega)\nabla_\alpha + \phi_\alpha)\omega\omega^\alpha D^\beta + \nabla_\alpha (1+\omega)\omega^\alpha D^\beta \Big)(z) \\
&= \sum_{\alpha-\beta=k+1} \Big((1+\omega)(\nabla_\alpha \omega^\alpha D^\beta) + \phi_\alpha \omega^{\alpha+1} D^\beta \Big)(z) \\
&= (1+\omega)\mathbb{D}_{k+1}(z) + \sum_\alpha \phi_\alpha \omega^{\alpha+1} D^{\alpha-k-1}(z)
\end{aligned}$$

When the strict inequality $t-s > 2(k+1)$ holds, due to the application of ϕ_α, each summand $\phi_\alpha \omega^{\alpha+1} D^{\alpha-k-1}(z)$ is zero unless $t = 2\alpha$, but in that case, $s = 2\alpha - N < 2(\alpha - k - 1)$, and then $D^{\alpha-k-1}(z)$ vanishes as $\{D^k\}$ is special. $\square$

We will be able to define (sometimes multi-valued) operations (for $r \geq 0$):

$$\delta_i^{\text{ext}} : [E_r X]_t^s \longrightarrow [E_r S_2 X]_{t+i}^s \quad \text{for } 2 \leq i \leq \max\{n, t-(r-1)\};$$
$$\lambda_i^{\text{ext}} : [E_r X]_t^s \longrightarrow [E_r \Lambda^2 X]_{t+i}^s \quad \text{for } 1 \leq i \leq \max\{n, t-(r-1)\};$$

using the chain-level maps $\mathrm{DEL}_i : C_* X \longrightarrow C_* S_2 X$ and $\mathrm{LAM}_i : C_* X \longrightarrow C_* \Lambda^2 X$:

$$\mathrm{DEL}_i(x) := \rho\,(\mathbb{D}_{n-i}(x \otimes x) + \mathbb{D}_{n-i-1}(dx \otimes x));$$
$$\mathrm{LAM}_i(x) := \rho'(\mathbb{D}_{n-i}(x \otimes x) + \mathbb{D}_{n-i-1}(dx \otimes x));$$

where we write $n := t - s$ in each formula. Except when $i < 2$, we can work just with the DEL_i, as in §5.2. Lemma 10.5 shows immediately that these maps preserve filtration, in the sense that $\mathrm{DEL}_i(x) \in F^s C_{n+i}(X \otimes X)$ whenever $x \in F^s C_n X$. Moving forward we will need a formula for the boundary of $\mathrm{DEL}_i(x)$:

Proposition 10.7. *For $2 \leq i \leq t+1$ and $x \in [Z_0 X]_t^s$:*

$$d(\mathrm{DEL}_i(x)) + \mathrm{DEL}_i(dx) = \mathrm{SQ}^{t-i+1,s}(x) = \begin{cases} \mathrm{SQ}^{t-i+1,s}(x), & \text{if } n+1 \leq i \leq t+1; \\ \rho \nabla D^0(x \otimes dx), & \text{if } i = n; \\ 0, & \text{if } i < n. \end{cases}$$

The same equations hold for LAM_i in the extended range $1 \leq i \leq t+1$.

Proof. We may apply Lemma 10.6 to calculate $d\mathbb{D}_{n-i}(x \otimes x)$ and $d\mathbb{D}_{n-i-1}(dx \otimes x)$, since

$$|x \otimes x| = 2n > 2(n-i+1) \quad \text{and} \quad |dx \otimes x| = 2n-1 > 2(n-i-1+1) \quad \text{when } i \geq 2.$$

Note that the first inequality fails when $i = 1$, which will explain the lack of δ_1^{ext}. We can work around this difficulty when defining λ_1^{ext} (the final step of this proof). Returning to DEL_i for $i \geq 2$:

$$\begin{aligned} d(\mathrm{DEL}_i(x)) + \mathrm{DEL}_i(dx) &= \rho d\Big(\mathbb{D}_{n-i}(x \otimes x) + \mathbb{D}_{n-i-1}(dx \otimes x)\Big) + \rho \mathbb{D}_{n-i-1}(dx \otimes dx) \\ &= \rho\Big\{d\mathbb{D}_{n-i}(x \otimes x)\Big\} + \rho\Big\{d\mathbb{D}_{n-i-1}(dx \otimes x)) + \mathbb{D}_{n-i-1}(d(dx \otimes x))\Big\} \\ &= \rho\Big\{\mathbb{D}_{n-i} d(x \otimes x) + (1+\omega)\mathbb{D}_{n-i+1}(x \otimes x) + \nabla \omega D^{i-n-1}(x \otimes x)\Big\} \\ &\quad + \rho\Big\{(1+\omega)\mathbb{D}_{n-i}(dx \otimes x) + \nabla \omega D^{i-n}(dx \otimes x)\Big\}, \end{aligned}$$

where we used braces to indicate the two applications of Lemma 10.6. Everything cancels except for $\rho\nabla(D^{i-n-1}(x\otimes x)+D^{i-n}(x\otimes dx))$ which equals $\mathrm{SQ}^{t-i+1,s}(x)$. We have studied this expression above, explaining the three cases.

If $i=1$, Lemma 10.6 yields an extra term, and if we write x as the sum $\sum_T x_T^{T-n}$ of its homogeneous parts, as $\{D^k\}$ is special, this term is:

$$\rho(\textstyle\sum_\alpha \phi_\alpha\omega^{\alpha+1}D^{\alpha-n}(x\otimes x))=\rho(\sum_T x_T^{T-n}\otimes x_T^{T-n})\in S_2X.$$

Although this term need not vanish, its image in Λ^2X certainly does, so that LAM_1 satisfies the desired equation. □

Suppose now that $x\in[E_rX]_t^s$. In light of the above calculation, when $n<i\le t+1$, the purpose of $\delta_i^{\mathrm{ext}}(x)$ will be to support a d_{t-i+1}-differential to $\mathrm{Sq}_{\mathrm{ext}}^{t-i+1}(x)$. Thus, we would not expect to be able to define $\delta_i^{\mathrm{ext}}(x)$ when $t-i+1<r$; indeed, the following result will construct $\delta_i^{\mathrm{ext}}(x)$ whenever $i\le t-(r-1)$.

Moreover, $\mathrm{Sq}_{\mathrm{ext}}^{t-i+1}(x)$ has indeterminacy vanishing by $[E_{2(r-1)}S_2X]_{2t}^{s+t-i+1}$, and we should expect that whenever $t-i+1<2(r-1)$, $\delta_i^{\mathrm{ext}}(x)$ will be multi-valued, but that the set of values for $\delta_i^{\mathrm{ext}}(x)$ will map onto the set of values for $\mathrm{Sq}_{\mathrm{ext}}^{t-i+1}(x)$ under d_{t-i+1}. We are not saying that we expect the indeterminacy of $\delta_i^{\mathrm{ext}}(x)$ to vanish by a certain page, but rather that we expect the multiple values of $\delta_i^{\mathrm{ext}}(x)$ to all fail to be permanent cycles together. Note that when $r\le 2$, there is no indeterminacy whatsoever in either set of operations.

Proposition 10.8. *Suppose that $r\ge 0$. The chain-level map DEL_i produces a multi-valued operation*

$$\delta_i^{\mathrm{ext}}:[E_rX]_t^s\longrightarrow[E_rS_2X]_{t+i}^s \text{ defined when } 2\le i\le\max\{n,t-(r-1)\}.$$

If $r\ge 1$ and $i<t$ then this function is linear with linear indeterminacy. This operation is single-valued whenever $2\le i\le\max\{n+1,t+1-2(r-1)\}$, and at E_1 may be identified with the operation of §5.2:

$$\pi_t^{\mathrm{v}}(X^s)\xrightarrow{\delta_i^{\mathrm{ext}}}\pi_{t+i}^{\mathrm{v}}S_2(X^s).$$

Suppose that $r\ge 1$ and $x\in[E_rX]_t^s$, and suppose that $\delta_i^{\mathrm{ext}}(x)$ is defined. Then $\delta_i^{\mathrm{ext}}(d_rx)$ is defined and

$$d_r\delta_i^{\mathrm{ext}}(x)+\delta_i^{\mathrm{ext}}(d_rx)=\begin{cases}\mathrm{Sq}_{\mathrm{ext}}^{t-i+1}(x), & \text{if } i>t-s \text{ and } r=t-i+1;\\ \mu^{\mathrm{ext}}(x\otimes d_rx), & \text{if } i=t-s,\ s=0 \text{ and } r\ge 2;\\ 0, & \text{otherwise.}\end{cases}$$

If $i \leq \max\{n+1, t+1-2(r-1)\}$, so that $\delta_i^{\mathrm{ext}} x$ is single-valued, then $\delta_i^{\mathrm{ext}} d_r x$ is also single-valued, and this equation holds exactly. When $i > t-s$ and $r = t-i+1$ the set of values of the left hand side coincides with the set of values of the right hand side. Otherwise, this equation holds modulo the indeterminacy of the left hand side.

For $r \geq 1$, the only potentially nonlinear operations are

$$\delta_t^{\mathrm{ext}} : [E_1 X]_t^s \longrightarrow [E_1 X]_{2t}^s \quad \text{and} \quad \delta_t^{\mathrm{ext}} : [E_r X]_t^0 \longrightarrow [E_r X]_{2t}^0. \tag{10.1}$$

They have no indeterminacy and satisfy $\delta_t^{\mathrm{ext}}(x+y) = \delta_t^{\mathrm{ext}}(x) + \delta_t^{\mathrm{ext}}(y) + \mu^{\mathrm{ext}}(x \otimes y)$.

The same conclusions hold for LAM_i*, producing operations λ_i^{ext}, and the inequality $2 \leq i$ can be replaced with $1 \leq i$ in this case.*

This proposition necessarily contains rather a lot of information. One upshot that we would like to point out is that if $x \in [E_\infty X]_t^s$ and $t-s>0$, then $\mu^{\mathrm{ext}}(x \otimes x) = 0 \in [E_\infty S_2 X]_{2t+1}^{2s+1}$, because of the differential

$$d_s : \delta_{t-s+1}^{\mathrm{ext}} x \longmapsto \mathrm{Sq}_{\mathrm{ext}}^s x = \mu^{\mathrm{ext}}(x \otimes x).$$

That is, although $\mu^{\mathrm{ext}}(x \otimes x)$ need not equal zero on the E_2-page, the E_∞-page mimics an exterior algebra in positive dimension. The fact that this top Steenrod operation has no indeterminacy (c.f. Proposition 10.3) should be compared with the fact that $\delta_{t-s+1}^{\mathrm{ext}} x$ has no indeterminacy.

Proof. Suppose that $x \in [Z_r X]_t^s$. Then Proposition 10.7 shows that $d\mathrm{DEL}_i(x) \in F^{s+r} C(S_2 X)$ as long as $i \leq \max\{n, t-(r-1)\}$, so that $\mathrm{DEL}_i(x) \in [Z_r X]_{t+i}^s$. Proposition 10.8 then provides the formula for $d_r \delta_i^{\mathrm{ext}}(x) + \delta_i^{\mathrm{ext}}(d_r x)$ (modulo whatever indeterminacy we find).

Let us begin with the operations $\delta_i^{\mathrm{ext}} : [E_1 X]_t^s \longrightarrow [E_1 S_2 X]_{t+i}^s$. Due to the assumption that $\{D^k\}$ is special, for any $x, y \in F^s CX$, $\mathbb{D}_k(x \otimes y) \equiv \nabla_k \omega^k D^0(x \otimes y)$ modulo $F^{2s+1} X$, and due to Lemma 10.5, we can ignore the horizontal component of the differential dx appearing in the definition of $\mathrm{DEL}_i(x)$. The resulting operations have leading term which is almost identical to the definition of the operations δ_i^{ext} of §5.2, which we already understand well. The only difference is the ω operator that appears, but this does not affect the resulting operation, by [26, Lemma 4.1]. This calculation at E_1 also demonstrates the expression for $\delta_t^{\mathrm{ext}}(x+y)$ for the operations in (10.1): such an equation is known from §5.2, and as $[E_r X]_t^0 \subseteq [E_1 X]_t^0$ for $r \geq 1$, this equation persists for all of the operations of (10.1).

Next, suppose that $2 \leq i \leq t$, and that $x, x' \in [Z_r X]_t^s$, and define:

$$\mathrm{NL}'(x, x') := \mathrm{DEL}_i(x) + \mathrm{DEL}_i(x') + \mathrm{DEL}_i(x + x')$$

By a calculation using Lemma 10.6 (similar to the calculation of dH that follows shortly):

$$\begin{aligned} d(\mathrm{NL}'(x,x')) = {} & d\rho[\mathbb{D}_{n-i-1}(x\otimes x') + \mathbb{D}_{n-i-2}(x\otimes dx')] \\ & + \rho[\mathbb{D}_{n-i-2}(dx\otimes dx') + \nabla\omega D^{i-n}(x\otimes x') + \nabla\omega D^{i-n+1}(x\otimes dx')] \end{aligned}$$

The terms on the first line are zero in $[Z_rX]^s_{t+i}$, as they are the boundaries of chains of filtration at least s. The three remaining terms lie in filtration exceeding s as long as $i < t$. Thus, whatever indeterminacy these operations are subject to is linear, and the operations themselves are linear.

We will now examine the extent to which DEL_i induces a well defined operation with domain $[E_rX]^s_t$ for $r \geq 1$. We may assume that $s > 0$, as the operations on E_1 were shown earlier to be well defined, and $[E_rX]^0_t \subseteq [E_1X]^0_t$ for $r \geq 1$. This implies that $t - i \geq 1$ whenever $i = n$. To examine the indeterminacy in $\delta_i^{\mathrm{ext}}x$ is to examine the difference $\mathrm{DEL}_i(x) - \mathrm{DEL}_i(x+dy)$ for $y \in [Z_{r-1}X]^{s-r+1}_{t-r+2}$, and by Lemma 10.6, we have the following three equations:

$$\begin{aligned} d\mathbb{D}_{n-i+1}(y\otimes y) &= \mathbb{D}_{n-i+1}d(y\otimes y) + (1+\omega)\mathbb{D}_{n-i+2}(y\otimes y) + \nabla\omega D^{-(n-i+2)}(y\otimes y); \\ d\mathbb{D}_{n-i}(dy\otimes y) &= \mathbb{D}_{n-i}(dy\otimes dy) + (1+\omega)\mathbb{D}_{n-i+1}(dy\otimes y) + \nabla\omega D^{-(n-i+1)}(dy\otimes y); \\ d\mathbb{D}_{n-i-1}(x\otimes dy) &= \mathbb{D}_{n-i-1}(dx\otimes dy) + (1+\omega)\mathbb{D}_{n-i}(x\otimes dy) + \nabla\omega D^{-(n-i)}(x\otimes dy). \end{aligned}$$

(As in the proof of Proposition 10.7, there are extra terms which appear when $i = 1$, but they are annihilated by the application of ρ'.) We define the following chain:

$$H(x,y) := \rho(\mathbb{D}_{n-i-1}(x\otimes dy) + \mathbb{D}_{n-i}(dy\otimes y) + \mathbb{D}_{n-i+1}(y\otimes y)),$$

and note, by Lemma 10.5, that $H(x,y) \in F^{s-r+1}C_{n+i+1}(S_2X)$. The three equations above show that

$$d(H(x,y)) = \mathrm{DEL}_i(x) - \mathrm{DEL}_i(x+dy) + T_1 + T_2 + T_3,$$

where

$$\begin{aligned} &T_1 := \rho\nabla(D^{i-n-2}(y\otimes y)) && \in F^{s+(t-i)-2(r-2)} && \text{equals zero when } i \leq n+1; \\ &T_2 := \rho\nabla(D^{i-n-1}(y\otimes dy)) && \in F^{s+(t-i)-(r-2)} && \text{equals zero when } i \leq n; \\ &T_3 := \rho\nabla(D^{i-n}(dy\otimes x)) && \in F^{s+(t-i)} && \text{equals zero when } i \leq n-1. \end{aligned}$$

As $t - i \geq 1$, $T_3 \in F^{s+1}$ can be ignored. As we have supposed that the operation δ_i^{ext} *can be defined* on x, we must have either $i \leq n$, in which case $T_2 = 0$, or $i \leq t - (r-1)$, in which case $T_2 \in F^{s+1}$ can be ignored. T_3 is assured either to vanish or to lie in F^{s+1} exactly when

$i \leq \max\{n+1, t+1-2(r-1)\}$, in which case, we have shown that $\delta_i^{\mathrm{ext}} x$ is single-valued.

In every case we may summarize the situation as follows. There is some $H(x,y) \in F^{s-r+1}$ such that

$$d(H(x,y)) = \mathrm{DEL}_i(x) - \mathrm{DEL}_i(x+dy) + \mathrm{BC}(x,y),$$

where $\mathrm{BC}(x,y) := T_1 + T_2 + T_3$ is an example of the bounding chain appearing in the proof of Propositions 10.3 and 10.4, so that

$$d(\mathrm{DEL}_i(x)) - d(\mathrm{DEL}_i(x+dy)) \equiv \mathrm{SQ}^{t-i+1,s}(x) - \mathrm{SQ}^{t-i+1,s}(x+dy) \pmod{F^{s+t-i+2}},$$

and the set of values of $\delta_i^{\mathrm{ext}} x$ maps *onto* the set of values $\mathrm{Sq}_{\mathrm{ext}}^{t-i+1} x$ under d_{t-i+1}. $\square$

Proposition 10.9. *Suppose that $X \in (s\mathcal{V})^{\Delta_+}$, i.e. that X admits a coaugmentation from some $X^{-1} \in s\mathcal{V}$. For $2 \leq i \leq t-s$, the operations $\delta_i^{\mathrm{ext}} : [E_\infty X]_t^s \longrightarrow [E_\infty S^2 X]_{t+i+1}^{s+1}$ agree with the homotopy operations $\delta_i^{\mathrm{ext}} : \pi_{t-s}(X^{-1}) \longrightarrow \pi_{t-s+i}(S^2(X^{-1}))$. Similarly, the external pairing at $S_2[E_\infty X] \longrightarrow [E_\infty S_2 X]$ agrees with $\widetilde{\nabla} : S_2\pi_*(X^{-1}) \longrightarrow \pi_*(S_2(X^{-1}))$.*

The same conclusions hold for the λ_i for $1 \leq i \leq t-s$.

Proof. We will only prove the statement about δ_i^{ext}, as the statement about products is easier and more standard. We need to show that the following diagram commutes whenever $2 \leq i \leq n$:

$$\begin{tikzcd} ZC_n(X) \arrow[r, "\mathrm{DEL}_i"] & ZC_n(S_2 X) \\ ZC_n(X^{-1}) \arrow[u, "d_{\mathrm{h}}^0"] \arrow[r, "z \longmapsto \rho(\nabla_{n-i}(z \otimes z))"'] & ZC_n(S_2(X^{-1})) \arrow[u, "d_{\mathrm{h}}^0"'] \end{tikzcd}$$

We calculate

$$\begin{aligned} \mathrm{DEL}_i(d_{\mathrm{h}}^0 z) &:= \rho\mathbb{D}_{n-i}(d_{\mathrm{h}}^0 z \otimes d_{\mathrm{h}}^0 z) + \rho\mathbb{D}_{n-i-1}(d_{\mathrm{h}}^0 z \otimes d(d_{\mathrm{h}}^0 z)) \\ &= \rho\nabla_{n-i}\omega^{n-i} D^0(d_{\mathrm{h}}^0 z \otimes d_{\mathrm{h}}^0 z) + \rho\mathbb{D}_{n-i-1}(d_{\mathrm{h}}^0 z \otimes 0) \\ &= \rho\nabla_{n-i}(d_{\mathrm{h}}^0 z \otimes_{\mathrm{v}} d_{\mathrm{h}}^0 z) = d_{\mathrm{h}}^0(\rho\nabla_{n-i}(z \otimes z)), \end{aligned}$$

where we have used the assumption that $\{D^k\}$ is special in both the second and third equations, and $d(d_{\mathrm{h}}^0 z) = 0$ since $z \in ZC_n(X^{-1})$ is a (vertical) cycle, and d_{h}^0 equalizes d_{h}^0 and d_{h}^1. $\square$

10.6. Internal operations on $[E_rX]$ for $X \in cs\mathscr{C}om$

Suppose that $X \in cs\mathscr{C}om$ is a cosimplicial simplicial commutative non-unital $\mathbb{F}_2$-algebra. We may define operations:

$$\delta_i : \left([E_rX]^s_t \xrightarrow{\delta_i^{\mathrm{ext}}} [E_r\mathrm{q}_2F^{\mathscr{C}om}X]^s_{t+i} \xrightarrow{\mu_*} [E_rX]^s_{t+i}\right),$$
$$\mathrm{Sq}^j : \left([E_rX]^s_t \xrightarrow{\mathrm{Sq}^j_{\mathrm{ext}}} [E_r\mathrm{q}_2F^{\mathscr{C}om}X]^{s+j}_{2t} \xrightarrow{\mu_*} [E_rX]^{s+j}_{2t}\right),$$
$$\mu : \left([E_rX]^s_t \otimes [E_rX]^{s'}_{t'} \xrightarrow{\mu_{\mathrm{ext}}} [E_r\mathrm{q}_2F^{\mathscr{C}om}X]^{s+s'}_{t+t'} \xrightarrow{\mu_*} [E_rX]^{s+s'}_{t+t'}\right),$$

with the δ_i multi-valued functions, defined when $2 \le i \le \max\{n, t-(r-1)\}$, and single-valued whenever $2 \le i \le \min\{n+1, t+1-2(r-1)\}$, and the Sq^j multi-valued functions with indeterminacy vanishing by E_{2r-2}, and which equal zero unless $\min\{t,r\} \le j \le s$.

These operations will not be used in rest of this thesis, as they will equal zero in the case of interest to us, namely when $X \in cs\mathscr{C}om$ is a GEM in each cosimplicial level. Nonetheless, we hope they are of some independent interest.

Numerous properties of these operations follow directly from the earlier results, namely Propositions 10.2, 10.3, 10.4 10.8 and 10.9. In addition, we have

Proposition 10.10. *The operations $\delta_i : [E_1X]^s_t \longrightarrow [E_1X]^s_{t+i}$ are (the restriction of) the homotopy operations of §5.4 applied to the homotopy of the simplicial algebra X^s. Moreover, for each s, $\pi^{\mathrm{v}}_*X^s$ is a graded commutative algebra (again, c.f. §5.4), and the operations μ and Sq^j on E_2 are the standard operations on the cohomotopy of the cosimplicial commutative algebra $\pi^{\mathrm{v}}_*X^s$. As such, the operations Sq^j make $[E_2X]$ is an unstable left module over the homogeneous Steenrod algebra, and satisfy the evident unstableness condition and the Cartan formula.*

If $x \in [E_1X]^s_t$ and $2 \le i \le 2t$ (so that the δ_i^{ext} operation that follows is defined), then $\delta_i\mathrm{Sq}^jx = 0 \in [E_1X]^{s+j}_{2t+i}$. If also $y \in [E_1X]^{s'}_{t'}$, and $2 \le i < t+t'$, then $\delta_i(xy) = 0$.

Proof of Proposition 10.10. Everything here is straightforward, and we will present the calculation $\delta_i\mathrm{Sq}^jx = 0$ as an example. Suppose that $x \in [Z_1X]^s_t$ and $2 \le i \le 2t$. This condition implies that $t > 0$, and for our current purpose we can assume that $x \in X^s_t$, so that $d_{\mathrm{v}}x = 0$. Then $\mathrm{Sq}^j_{\mathrm{ext}}x$ is represented by the image of $D^{s-j}(x\otimes x) + D^{s-j+1}(x \otimes d_{\mathrm{h}}x)$ under the composite

$$N^{s+j}(N_tX \otimes N_tX) \xrightarrow{N^{s+j}(\widetilde{\nabla})} N^{s+j}N_{2t}(X \otimes_{\Sigma_2} X) \xrightarrow{\mu} N^{s+j}N_{2t}(X) \xrightarrow{\delta_i} N^{s+j}N_{2t+i}(X),$$

and Proposition 5.3 states that the final δ-operation annihilates products of positive dimensional classes, so that this composite is zero. $\square$

As a final note here, suppose that $X \in cs\mathscr{L}ie$ (or $cs\mathscr{L}ie^r$). We may define operations:

$$\lambda_i : \left([E_rX]_t^s \xrightarrow{\lambda_i^{\mathrm{ext}}} [E_r\mathrm{q}_2F^{\mathscr{L}ie}X]_{t+i}^s \xrightarrow{[,]_*} [E_rX]_{t+i}^s\right),$$
$$P^{j-1} : \left([E_rX]_t^s \xrightarrow{\mathrm{Sq}_{\mathrm{ext}}^j} [E_r\mathrm{q}_2F^{\mathscr{L}ie}X]_{2t}^{s+j} \xrightarrow{[,]_*} [E_rX]_{2t}^{s+j}\right),$$
$$[\,,] : \left([E_rX]_t^s \otimes [E_rX]_{t'}^{s'} \xrightarrow{\mu_{\mathrm{ext}}} [E_r\mathrm{q}_2F^{\mathscr{L}ie}X]_{t+t'}^{s+s'} \xrightarrow{[,]_*} [E_rX]_{t+t'}^{s+s'}\right),$$

with the λ_i multi-valued functions, defined when $1 \leq i \leq \max\{n, t-(r-1)\}$ and single-valued whenever $1 \leq i \leq \min\{n+1, t+1-2(r-1)\}$. It should be possible to state versions of all of the above results in this case. The author guesses that the operations P^k will form an unstable left action of the P-algebra (the Steenrod algebra for commutative $\mathbb{F}_2$-algebras, as in §6.6) but has not worked out the details.

Chapter 11

Operations in the Bousfield-Kan spectral sequence

In this chapter we will define operations on the BKSS for an object $X \in s\mathcal{C}$ whenever $\mathcal{C}$ is any of the categories $\mathscr{C}om$, $\mathscr{L}ie$ or $\mathscr{L}ie^r$. We will always write $\mathfrak{X}$ for Radulescu-Banu's resolution of $X \in s\mathcal{C}$, the coaugmented cosimplicial simplicial object defined by

$$\mathfrak{X}^s_t = (c(K^{\mathcal{C}}Q^{\mathcal{C}}c)^{s+1}X)_t.$$

11.1. An alternate definition of the Adams tower

We will now give an alternate definition of the Adams tower of §4.2, using the techniques of [11], which is more suited for the definition of spectral sequence operations in our setting.

For $Z \in \mathcal{V}^{\Delta_+}$, the category of coaugmented cosimplicial vector spaces, Bousfield and Kan write VZ for a "path-like construction" [11, §3.1] obtained by shifting Z down and forgetting the 0^{th} coface and codegeneracy. That is, $(VZ)^s := (VZ)^{s+1}$, and:

$$\big((VZ)^s \xrightarrow{d^i} (VZ)^{s+1}\big) := \big(Z^{s+1} \xrightarrow{d^{i+1}} Z^{s+2}\big)$$
$$\big((VZ)^s \xrightarrow{s^i} (VZ)^{s-1}\big) := \big(Z^{s+1} \xrightarrow{s^{i+1}} Z^{s}\big)$$

The unused coface d^0 induces a map $v : Z \longrightarrow VZ$ in $\mathcal{V}^{\Delta_+}$.

For $Y \in s\mathcal{V}$, the standard simplicial path fibration (c.f. [12, p. 82]) produces a contractible simplicial vector space $\Lambda Y \in s\mathcal{V}$ by shifting down and restricting to a kernel:

$$\Lambda Y_s = \ker\big(d_{s+1}\cdots d_1 : Y_{s+1} \longrightarrow Y_0\big).$$

We forget the 0^{th} face and degeneracy as before, and this time, the unused face map d_0

induces a fibration $\lambda : \Lambda Y \longrightarrow Y$.

Each of these constructions can be prolonged to an endofunctor of $(s\mathcal{C})^{\Delta_+}$, endofunctors which are necessary for a key construction of Bousfield and Kan [11]. Define an endofunctor R^1 of the category $(s\mathcal{C})^{\Delta_+}$ of augmented cosimplicial objects in $s\mathcal{C}$, using the pullback (for $W \in (s\mathcal{C})^{\Delta_+}$):

$$\begin{tikzcd} R^1W \arrow[r] \arrow[d, "\delta"] & \Lambda VW \arrow[d, "\lambda"] \\ W \arrow[r, "v"] & VW \end{tikzcd}$$

Then one can form a tower in $(s\mathcal{C})^{\Delta_+}$, (writing $R^n := R^1 \circ \cdots R^1$):

$$\cdots \longrightarrow R^2W \longrightarrow R^1W \longrightarrow R^0W = W.$$

Restricting to augmentations, there is a tower of fiber sequences in $s\mathcal{C}$:

$$\begin{tikzcd} \cdots \arrow[r] & (R^2W)^{-1} \arrow[r] \arrow[d, "d^0"] & (R^1W)^{-1} \arrow[r] \arrow[d, "d^0"] & (R^0W)^{-1} \arrow[r, equal] \arrow[d, "d^0"] & W^{-1} \\ & (R^2W)^0 & (R^1W)^0 & (R^0W)^0 & \end{tikzcd}$$

Bousfield and Kan [11, §3.3 and §4.2] note that this tower *equals* the Adams tower $\{R_nX\}$ when $W = \mathcal{X}$ is Radulescu-Banu's resolution of $X \in s\mathcal{C}$. They also explicitly perform the resulting identification of the E_1-page of the spectral sequence of this tower with $N^s_{\subseteq}\pi_t W$, using iterates of the connecting map

$$\pi_t(W^s) = \pi_t(VW^{s-1}) \xrightarrow{\partial_{\text{conn}}} \pi_{t-1}(R^1W)^{s-1}$$

of the fiber sequence $(R^1W)^{s-1} \longrightarrow W^{s-1} \longrightarrow VW^{s-1}$, which has the property:

Proposition 11.1 [11, Proposition 5.2]**.** *The following composite involving the connecting map ∂_{conn} induces (for each fixed t) an isomorphism of cochain complexes:*

$$N^s_{\subseteq}\pi_t W \subseteq N^{s-1}_{\subseteq}\pi_t VW \xrightarrow{\partial_{\text{conn}}} N^{s-1}_{\subseteq}\pi_{t-1}(R^1W).$$

Note that the inclusion in this theorem can be strict — the subspace $N^s_{\subseteq}\pi_t W$ of $C^s\pi_t W^s$ is defined by the vanishing of the maps $s^0, \ldots, s^{s-1} : \pi_t W^s \longrightarrow \pi_t W^{s-1}$, while $N^{s-1}_{\subseteq}\pi_t(VW)^{s-1}$ is defined by the vanishing only of $s^1, \ldots, s^{s-1} : \pi_t W^s \longrightarrow \pi_t W^{s-1}$, as s^0 is forgotten in passing to VW.

If we declare the spectral sequence an object $W \in cs\mathcal{C}$ to be the spectral sequence of the tower

$$\cdots \longrightarrow (R^2W)^{-1} \longrightarrow (R^1W)^{-1} \longrightarrow (R^0W)^{-1}$$

then the spectral sequence of $R^1\mathcal{X}$ maps to the spectral sequence of $\mathcal{X}$, with a filtration shift, via the map of towers:

$$\begin{tikzcd}
\cdots \arrow[r] & (R^2R^1\mathcal{X})^{-1} \arrow[r] \arrow[d, "="] & (R^1R^1\mathcal{X})^{-1} \arrow[r] \arrow[d, "="] & (R^0R^1\mathcal{X})^{-1} \arrow[d, "="] & \\
\cdots \arrow[r] & (R^3\mathcal{X})^{-1} \arrow[r] & (R^2\mathcal{X})^{-1} \arrow[r] & (R^1\mathcal{X})^{-1} \arrow[r] & (R^0\mathcal{X})^{-1}
\end{tikzcd}$$

That is, there are spectral sequence maps which at E_1 are isomorphisms of the form

$$[E_1R^1\mathcal{X}]^s_t \xrightarrow{\cong} [E_1\mathcal{X}]^{s+1}_{t+1}.$$

Under Bousfield and Kan's identification of E_1, this isomorphism is the inverse of the composite of Proposition 11.1.

A reasonable goal is to create a natural factorization

$$\begin{tikzcd}
& R^1\mathcal{X} \arrow[d, "\delta"] \\
\mathrm{q}_2F^{\mathcal{C}}\mathcal{X} \arrow[r] \arrow[ur, dotted] & \mathcal{X}
\end{tikzcd}$$

of the structure map of $\mathcal{X}$ through δ, as $\mathcal{X}$ is a GEM levelwise. This will be possible up to a natural zig-zag, by a construction which uses the structure of Radulescu-Banu's resolution specifically.

11.2. A modification of the functor R^1

Not only does

$$(V\mathcal{X})^s_t = (c(K^{\mathcal{C}}Q^{\mathcal{C}}c)^{s+2}X)_t \in cs\mathcal{C}$$

have cosimplicial and simplicial structure maps, but there is a cosimplicial simplicial algebra structure on the object $\overline{V}\mathcal{X}$ obtained by omitting the leftmost replacement c:

$$(\overline{V}\mathcal{X})^s_t = ((K^{\mathcal{C}}Q^{\mathcal{C}}c)^{s+2}X)_t \in cs\mathcal{C}.$$

That is, we do not *need* the outermost cofibrant replacement in order to define the cosimplicial structure maps $\overline{V}\mathcal{X}$, as in passing from $\mathcal{X}$ to $V\mathcal{X}$ one discards d^0. There is a $cs\mathcal{C}$-map $\epsilon : V\mathcal{X} \longrightarrow \overline{V}\mathcal{X}$ which is a weak equivalence in each cosimplicial level. Finally, the composite

$$\overline{v} := \epsilon \circ v : \left(\mathcal{X} \xrightarrow{v} V\mathcal{X} \overset{\epsilon}{\twoheadrightarrow} \overline{V}\mathcal{X}\right)$$

is, in each cosimplicial degree, a fibration in $s\mathcal{C}$ since it is defined in cosimplicial degree s by the formula

$$\overline{v} = \eta : c(K^{\mathcal{C}}Q^{\mathcal{C}}c)^{s+2}X \longrightarrow K^{\mathcal{C}}Q^{\mathcal{C}}c(K^{\mathcal{C}}Q^{\mathcal{C}}c)^{s+2}X.$$

The object $\overline{V}\mathcal{X}$ has two key advantages: $\overline{v}$ is a fibration in each cosimplicial level, and $\overline{V}\mathcal{X}$ is a trivial object in $s\mathcal{C}$ (i.e. it is in the image of $K^{\mathcal{C}}$). This second property implies that $\overline{V}\mathcal{X}$ is an abelian group object in $s\mathcal{C}$ in each cosimplicial level, as every vector space is a group object, and $K^{\mathcal{C}}$ is a right adjoint. In other words, since all the structure maps in $\overline{V}\mathcal{X}$ are trivial, they commute with vector space addition. We write

$$\mathrm{add} : \overline{V}\mathcal{X} \times \overline{V}\mathcal{X} \longrightarrow \overline{V}\mathcal{X}$$

for the group operation. Under the identifications arising from Propositions 3.5 and 3.8, the map add induces the expected abelian group and cogroup structures on $H_*^{\mathcal{C}}\overline{V}\mathcal{X}$ and $H^*_{\mathcal{C}}\overline{V}\mathcal{X}$:

$$H_*^{\mathcal{C}}\overline{V}\mathcal{X} \times H_*^{\mathcal{C}}\overline{V}\mathcal{X} \longrightarrow H_*^{\mathcal{C}}\overline{V}\mathcal{X};$$
$$H^*_{\mathcal{C}}\overline{V}\mathcal{X} \sqcup H^*_{\mathcal{C}}\overline{V}\mathcal{X} \longleftarrow H^*_{\mathcal{C}}\overline{V}\mathcal{X}.$$

The observation that $\overline{v}$ is a fibration leads us to define $\overline{R}^1\mathcal{X}$ to be the strict fiber

$$\begin{tikzcd} \overline{R}^1\mathcal{X} \arrow[r] \arrow[d, "\delta"] & 0 \arrow[d] \\ \mathcal{X} \arrow[r, two heads, "\overline{v}"] & \overline{V}\mathcal{X} \end{tikzcd}$$

There is a commuting diagram in $cs\mathcal{C}$ (in which double-headed arrows denote maps which are fibrations in $s\mathcal{C}$ in each cosimplicial level):

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \Lambda\overline{V}\mathcal{X} \arrow[d, two heads, "\lambda"] & \Lambda V\mathcal{X} \arrow[l, "\Lambda(\epsilon)"'] \arrow[d, two heads, "\lambda"] \\ \overline{V}\mathcal{X} \arrow[r, equal] & \overline{V}\mathcal{X} & V\mathcal{X} \arrow[l, two heads, "\epsilon"'] \\ \mathcal{X} \arrow[r, equal] \arrow[u, two heads, "\overline{v}"] & \mathcal{X} \arrow[r, equal] \arrow[u, two heads, "\overline{v}"] & \mathcal{X} \arrow[u, "v"] \end{tikzcd}$$

$$\text{pullbacks:} \qquad \overline{R}^1\mathcal{X} \xrightarrow{E_1\text{-eq}} \widetilde{R}^1\mathcal{X} \xleftarrow{E_1\text{-eq}} R^1\mathcal{X}$$

producing a zig-zag of E_1-equivalences between $\overline{R}^1\mathcal{X}$ and $R^1\mathcal{X}$. In each cosimplicial level, each of the objects in the top row is contractible, yielding homotopy long exact sequences, and the resulting connecting homomorphisms commute:

$$\begin{tikzcd}
\pi_t(\overline{R}^1\mathcal{X}) & \pi_{t+1}(\overline{V}\mathcal{X}) \arrow[l, "\partial_{\mathrm{conn}}"'] \\
\pi_t(R^1\mathcal{X}) \arrow[u, no head, "\text{zig-zag}", "\cong"'] & \pi_{t+1}(V\mathcal{X}) \arrow[l, "\partial_{\mathrm{conn}}"'] \arrow[u, "\cong"']
\end{tikzcd}$$

so that there are isomorphisms of spectral sequences (starting from E_1):

$$\begin{tikzcd}
{[E_r\overline{R}^1\mathcal{X}]^s_t} \arrow[r, "\cong"] \arrow[d, "\cong"] & {[E_r\widetilde{R}^1\mathcal{X}]^s_t} \arrow[d, "\cong"] & {[E_r\widetilde{R}^1\mathcal{X}]^s_t} \arrow[l, "\cong"'] \arrow[d, "\cong"] \\
{[E_r\mathcal{X}]^{s+1}_{t+1}} \arrow[r, equal] & {[E_r\mathcal{X}]^{s+1}_{t+1}} \arrow[r, equal] & {[E_r\mathcal{X}]^{s+1}_{t+1}}
\end{tikzcd}$$

11.3. Definition and properties of the BKSS operations

Whichever of the three categories of interest $\mathcal{C}$ we are working in, there is a factorization

$$\begin{tikzcd}
 & \overline{R}^1\mathcal{X} \arrow[d, "\delta"] \\
\mathrm{q}_2F^{\mathcal{C}}\mathcal{X} \arrow[ur, dotted] \arrow[r] & \mathcal{X}
\end{tikzcd}
\qquad \text{induced by} \qquad
\begin{tikzcd}
 & \overline{R}^1\mathcal{X} \arrow[d, "\delta"] \arrow[r] & 0 \arrow[d] \\
\mathrm{q}_2F^{\mathcal{C}}\mathcal{X} \arrow[ur, dotted] \arrow[r] & \mathcal{X} \arrow[r, two heads, "\overline{v}"] & \overline{V}\mathcal{X}
\end{tikzcd}$$

where the composite $\mathrm{q}_2F^{\mathcal{C}}\mathcal{X} \longrightarrow \overline{V}\mathcal{X}$ must vanish as it factors through the structure map $\mathrm{q}_2F^{\mathcal{C}}\overline{V}\mathcal{X} \longrightarrow \overline{V}\mathcal{X}$, which is zero since $\overline{V}\mathcal{X}$ is a trivial object. We denote by

$$L : [E_r\mathrm{q}_2F^{\mathcal{C}}\mathcal{X}]^s_t \longrightarrow [E_r\mathcal{X}]^{s+1}_{t+1}$$

the resulting map of spectral sequences. Using the isomorphisms

$$S_2V \cong \mathrm{q}_2F^{\mathscr{C}om}V, \qquad \Lambda^2V \cong \mathrm{q}_2F^{\mathscr{L}ie}V, \qquad S^2V \cong \mathrm{q}_2F^{\mathscr{L}ie^r}V,$$

and the various external spectral sequence operations from $[E_rV]$ to each of $[E_rS_2V]$, $[E_r\Lambda^2V]$ and $[E_rS^2V]$, we are now able to define numerous spectral sequence operations on $[E_r\mathcal{X}]^s_t$ in each case. When $\mathcal{C} = \mathscr{C}om$, we define:

$$\begin{aligned}
\delta^{\mathrm{v}}_i &: \left([E_r\mathcal{X}]^s_t \xrightarrow{\delta_i^{\mathrm{ext}}} [E_r\mathrm{q}_2F^{\mathcal{C}}\mathcal{X}]^s_{t+i} \xrightarrow{L} [E_r\mathcal{X}]^{s+1}_{t+i+1}\right), \\
\mathrm{Sq}^j_{\mathrm{h}} &: \left([E_r\mathcal{X}]^s_t \xrightarrow{\mathrm{Sq}^{j-1}_{\mathrm{ext}}} [E_r\mathrm{q}_2F^{\mathcal{C}}\mathcal{X}]^{s+j-1}_{2t} \xrightarrow{L} [E_r\mathcal{X}]^{s+j}_{2t+1}\right), \\
\mu &: \left([E_r\mathcal{X}]^s_t \otimes [E_r\mathcal{X}]^{s'}_{t'} \xrightarrow{\mu_{\mathrm{ext}}} [E_r\mathrm{q}_2F^{\mathcal{C}}\mathcal{X}]^{s+s'}_{t+t'} \xrightarrow{L} [E_r\mathcal{X}]^{s+s'+1}_{t+t'+1}\right),
\end{aligned}$$

with the δ^{v}_i multi-valued functions, defined when $2 \leq i \leq \max\{n, t-(r-1)\}$, and single-valued whenever $i \leq \min\{n+1, t+1-2(r-1)\}$, and the Sq^j multi-valued functions with indeterminacy vanishing by E_{2r-2}, and which equal zero unless $\min\{t, r\} < j \leq s+1$. All of the functions that are defined on E_2 are single-valued, so it makes sense to state

Proposition 11.2. *When $\mathcal{C} = \mathscr{C}om$, under the identification $[E_2\mathfrak{X}]^s_t \cong H^*_{\mathcal{W}(0)} H^*_{\mathscr{C}om} X$, the operations just defined coincide with the $\mathcal{W}(0)$-cohomology operations defined in §8.*

We will prove this result in §11.9. It implies that from the E_2-page onward the operations just defined have the properties cataloged in Propositions 8.2, 8.9 and 8.12 — the δ^{v}_i satisfy the δ-Adem relations, the $\mathrm{Sq}^j_{\mathrm{h}}$ and μ satisfy the properties of such operations on Lie algebra cohomology, and there is a commutation relation between the δ^{v} and the Sq_{h} and μ. These relations persist to relations on the higher pages (modulo appropriate indeterminacy), but evidently do not hold on E_1.

The following results follow from Propositions 10.2, 10.3, 10.4, 10.8 and 10.9 respectively, for $X \in s\mathscr{C}om$ and $\mathfrak{X} \in cs\mathscr{C}om$ its Radulescu-Banu resolution:

Corollary 11.3 (of Proposition 10.2)**.** *The pairing μ satisfies the Leibniz formula. For $r \geq 2$, μ descends to the symmetric quotient $S_2[E_r\mathfrak{X}]$.*

Corollary 11.4 (of Proposition 10.3)**.** *Suppose that $r \geq 2$. The operations*

$$\mathrm{Sq}^i_{\mathrm{h}} : [E_r\mathfrak{X}]^s_t \longrightarrow [E_r S_2\mathfrak{X}]^{s+i}_{2t+1}$$

have indeterminacy vanishing by $[E_{2r-2}\mathfrak{X}]^{s+i}_{2t+1}$ (and thus no indeterminacy at E_2). They are linear maps with linear indeterminacy. Now suppose that and $x \in [E_r\mathfrak{X}]^s_t$. $\mathrm{Sq}^i_{\mathrm{h}}x = 0$ unless $\min\{t,r\} < i \leq s+1$, and this vanishing occurs without indeterminacy. In any case, $\mathrm{Sq}^i_{\mathrm{h}}x$ survives to $[E_{2r-1}\mathfrak{X}]^{s+i}_{2t+1}$, and the following equation in $[E_{2r-1}\mathfrak{X}]^{s+i+2r-1}_{2t+2r-1}$ holds (without indeterminacy):

$$d_{2r-1}(\mathrm{Sq}^i_{\mathrm{h}}x) = \mathrm{Sq}^{i+r-1}_{\mathrm{h}}(d_r x).$$

The notion of top operation *has shifted: $\mathrm{Sq}^{s+1}_{\mathrm{h}}x$ is the top operation, it equals the product-square $x \times x$, and in particular, has no indeterminacy. Finally, $\mathrm{Sq}^0_{\mathrm{h}}x = 0$, $\mathrm{Sq}^1_{\mathrm{h}}x = 0$ when $t > 0$, and $\mathrm{Sq}^2_{\mathrm{h}}x = 0$ when $t > 1$.*

Corollary 11.5 (of Proposition 10.4)**.** *At E_1, the operations $\mathrm{Sq}^i_{\mathrm{h}} : [E_1\mathfrak{X}]^s_t \longrightarrow [E_1 S_2\mathfrak{X}]^{s+i}_{2t+1}$ have no indeterminacy, and d_1 commutes with $\mathrm{Sq}^i_{\mathrm{h}}$ for each i. They need not be linear. Suppose that $x \in [E_1\mathfrak{X}]^s_t$. The top operation $\mathrm{Sq}^{s+1}_{\mathrm{h}}x$ need not equal the product-square $x \times x$ on E_1, and $\mathrm{Sq}^{s+2}_{\mathrm{h}}x$ need not vanish, instead equalling $x \times d_1 x$ on E_1. At least for $i > s+2$, $\mathrm{Sq}^i_{\mathrm{h}}x = 0$. Finally, $\mathrm{Sq}^0_{\mathrm{h}}x = 0$, and $\mathrm{Sq}^1_{\mathrm{h}}x = 0$ when $t > 0$.*

Corollary 11.6 (of Proposition 10.8)**.** *Fix $r \geq 1$. The potentially multi-valued function*

$$\delta^{\mathrm{v}}_i : [E_r\mathfrak{X}]^s_t \longrightarrow [E_r\mathfrak{X}]^{s+1}_{t+i+1}, \text{ defined when } 2 \leq i \leq \max\{n, t-(r-1)\},$$

is linear with linear indeterminacy whenever $i < t$. It is a single-valued operation when $i \leq \max\{n+1, t+1-2(r-1)\}$.

Suppose that $x \in [E_r\mathcal{X}]^s_t$, and suppose that $\delta^{\mathrm{v}}_i(x)$ is defined. Then $\delta^{\mathrm{v}}_i(d_r x)$ is defined and

$$d_r\delta^{\mathrm{v}}_i(x) + \delta^{\mathrm{v}}_i(d_r x) = \begin{cases} \mathrm{Sq}^{t-i+2}_{\mathrm{h}}(x), & \text{if } i > t-s \text{ and } r = t-i+1; \\ \mu(x \otimes d_r x), & \text{if } i = t-s,\ s = 0 \text{ and } r \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

If $i \leq \max\{n, t+1-2(r-1)\}$, so that $\delta^{\mathrm{v}}_i x$ is single-valued, then $\delta^{\mathrm{v}}_i d_r x$ is also single-valued, and this equation holds exactly. When $i > t-s$ and $r = t-i+1$ the set of values of the left hand side coincides with the set of values of the right hand side. Otherwise, this equation holds modulo the indeterminacy of the left hand side.

The only potentially nonlinear operations are

$$\delta^{\mathrm{v}}_t : [E_1\mathcal{X}]^s_t \longrightarrow [E_1\mathcal{X}]^s_{2t} \quad \text{and} \quad \delta^{\mathrm{v}}_t : [E_r\mathcal{X}]^0_t \longrightarrow [E_r\mathcal{X}]^0_{2t}.$$

They have no indeterminacy and satisfy $\delta^{\mathrm{v}}_t(x+y) = \delta^{\mathrm{v}}_t(x) + \delta^{\mathrm{v}}_t(y) + \mu(x \otimes y)$.

Corollary 11.7 (of Proposition 10.9)**.** *For $2 \leq i \leq t-s$, the operations $\delta^{\mathrm{v}}_i : [E_\infty\mathcal{X}]^s_t \longrightarrow [E_\infty\mathcal{X}]^{s+1}_{t+i+1}$ agree with the homotopy operations $\delta_i : \pi_{t-s}X \longrightarrow \pi_{t-s+i}X$ on the target of the spectral sequence. Similarly, the product at $[E_\infty\mathcal{X}]$ agrees with the product on the target.*

It seems likely to the author that this is a complete description of the natural operations on the BKSS in $s\mathscr{C}om$.

Although we do not use the following operations in this thesis (as we do not consider the BKSS for simplicial Lie algebras in detail), we note that when $\mathcal{C} = \mathscr{L}ie$ or $\mathcal{C} = \mathscr{L}ie^r$ there are operations:

$$\lambda^{\mathrm{v}}_i : \left([E_rX]^s_t \xrightarrow{\lambda^{\mathrm{ext}}_i} [E_r\mathrm{q}_2F^{\mathcal{C}}X]^s_{t+i} \xrightarrow{L} [E_rX]^{s+1}_{t+i+1}\right),$$
$$P^j_{\mathrm{h}} : \left([E_rX]^s_t \xrightarrow{\mathrm{Sq}^j_{\mathrm{ext}}} [E_r\mathrm{q}_2F^{\mathcal{C}}X]^{s+j}_{2t} \xrightarrow{L} [E_rX]^{s+j+1}_{2t+1}\right),$$
$$[\,,] : \left([E_rX]^s_t \otimes [E_rX]^{s'}_{t'} \xrightarrow{\mu_{\mathrm{ext}}} [E_r\mathrm{q}_2F^{\mathcal{C}}X]^{s+s'}_{t+t'} \xrightarrow{L} [E_rX]^{s+s'+1}_{t+t'+1}\right),$$

with the λ^{v}_i multi-valued functions, defined when $1 \leq i \leq \max\{n, t-(r-1)\}$, and single-valued whenever $i \leq \min\{n+1, t+1-2(r-1)\}$, and the P^j_{h} multi-valued functions with indeterminacy vanishing by E_{2r-2}, and which equal zero unless $\min\{t, r\} \leq j \leq s$. We will not be able to prove a version of Proposition 11.2 in the present work, since we have not derived a version of §8 for the categories $s\mathscr{L}ie$ and $s\mathscr{L}ie^r$. Nonetheless, these operations will satisfy analogues of Corollaries 11.3-11.7.

The purpose of rest of this chapter is to give the necessary constructions to prove Proposition 11.2, so that in the following, we will work only in the category $\mathcal{C} = \mathscr{C}om$. However, the constructions, including of the following two- and three-cell complexes, generalize to the categories of Lie algebras.

11.4. A chain-level construction $\xi^\star_{\mathrm{res}}$ inducing $\xi_{H\mathcal{C}}$

Let $\mathcal{C} = \mathscr{C}om$. In §6.5, we defined

$$\xi_{H\mathcal{C}} : B^{s+1}_{H\mathcal{C}} H^*_{\mathcal{C}} X \longrightarrow B^s_{H\mathcal{C}} H^*_{\mathcal{C}} X \veebar B^s_{H\mathcal{C}} H^*_{\mathcal{C}} X,$$

and in §8.3 we used this map to define Steenrod operations and a product on $H^*_{H\mathcal{C}} H^*_{\mathcal{C}} X$. We will now construct, at the level of Radulescu-Banu's resolution, a map

$$\xi^\star_{\mathrm{res}} : \mathcal{X}^s \mathbin{\overline{\wedge}^{\mathrm{L}}} \mathcal{X}^s \longrightarrow V\mathcal{X}^s,$$

which, under the isomorphisms of Theorem 4.1 and Proposition 3.8, induces the map $\xi_{H\mathcal{C}}$ on cohomology:

$$\begin{tikzcd}
H^*_{\mathcal{C}}(\mathcal{X}^s \mathbin{\overline{\wedge}^{\mathrm{L}}} \mathcal{X}^s) & H^*_{\mathcal{C}}(V\mathcal{X}^s) \arrow[l, "(\xi^\star_{\mathrm{res}})^*"'] \\
B^s_{\mathcal{C}} H^*_{\mathcal{C}} X \veebar B^s_{\mathcal{C}} H^*_{\mathcal{C}} X \arrow[u, "\cong"] & B^{s+1}_{\mathcal{C}} H^*_{\mathcal{C}} X \arrow[l, "\xi_{H\mathcal{C}}"'] \arrow[u, "\cong"]
\end{tikzcd}$$

We will need to abbreviate a little for the sake of compactness. Fix a cosimplicial degree s. Write $\mathcal{X}$ for $\mathcal{X}^s$, V for $V\mathcal{X}^s$, $\overline{V}$ for $\overline{V}\mathcal{X}^s$, dots for categorical products, and superscripts for categorical self-products. There is a diagram

$$\begin{tikzcd}
\mathcal{X} \mathbin{\overline{\wedge}^{\mathrm{L}}} \mathcal{X} \arrow[rrrrd, dashed, bend left, "\xi^\star_{\mathrm{res}}"] & & & & \\
c(\mathcal{X}^2) \arrow[u] \arrow[r, "c(\epsilon,\mathrm{id})\circ\beta"'] \arrow[rrrr, bend left, "\overline{\xi}^\star_{\mathrm{res}}"] \arrow[dr, dotted, "(\epsilon,\mathrm{id})"'] & c(\mathcal{X}^2 \cdot c(\mathcal{X}^2)) \arrow[r, "c(\overline{v}^2 \cdot c(\mathrm{add}\circ\epsilon^2))"'] \arrow[d, dotted, "\epsilon"] & c(\overline{V}^2 \cdot \mathcal{X}) \arrow[r, "c(\mathrm{add}\cdot\overline{v})"'] \arrow[d, dotted, "\epsilon"] & c(\overline{V} \cdot \overline{V}) \arrow[r, "c(\mathrm{add})"'] \arrow[d, dotted, "\epsilon"] & V \arrow[d, dotted, "\epsilon"] \\
c(\mathcal{X} \sqcup \mathcal{X}) \arrow[u] & \mathcal{X}^2 \cdot c(\mathcal{X}^2) \arrow[r, dotted, "\overline{v}^2 \cdot c(\mathrm{add}\circ\epsilon^2)"] & \overline{V}^2 \cdot \mathcal{X} \arrow[r, dotted, "\mathrm{add}\cdot\overline{v}"] & \overline{V} \cdot \overline{V} \arrow[r, dotted, "\mathrm{add}"] & \overline{V}
\end{tikzcd}$$

in which we define $\overline{\xi}^\star_{\mathrm{res}}$ to be the composite of the horizontal solid arrows. The sub-diagram consisting of solid and dotted arrows strictly commutes, and we will define $\xi^\star_{\mathrm{res}}$ to be the unique map up to homotopy such that the full diagram homotopy commutes, after showing that the composite $c(\mathcal{X} \sqcup \mathcal{X}) \longrightarrow V$ is null. The maps defined here need a little clarification, during which we will resume writing cosimplicial degrees:

$$
\begin{aligned}
c(\epsilon, Id)\circ\beta : \big(c(\mathcal{X}^s)^2 \xrightarrow{\beta} cc(\mathcal{X}^s)^2 \xrightarrow{c(\mathrm{id},\mathrm{id})} c(c(\mathcal{X}^s)^2\cdot c(\mathcal{X}^s)^2) \xrightarrow{c(\epsilon\cdot\mathrm{id})} c((\mathcal{X}^s)^2\cdot c(\mathcal{X}^s)^2)\big)\\
c(\mathrm{add}\circ\epsilon^2) : \big(c((\mathcal{X}^s)^2) \xrightarrow{c(\epsilon^2)} c((\overline{V}\mathcal{X}^{s-1})^2) \xrightarrow{c(\mathrm{add})} c(\overline{V}\mathcal{X}^{s-1}) = \mathcal{X}^s\big).
\end{aligned}
$$

Fortunately, the fact that the diagram (without the dashed arrow) commutes is obvious: the small triangle commutes by counitality of β, and the three squares commute by naturality of $\epsilon : c \longrightarrow \mathrm{id}$.

Proposition 11.8. *The map $\overline{\xi}^{\star}_{\mathrm{res}}$ induces the map $\overline{\xi}_{H\mathcal{C}}$ on cohomology, and descends to a map $\xi^{\star}_{\mathrm{res}} : \mathcal{X} \,\overline{\wedge}^{\mathrm{L}}\, \mathcal{X} \longrightarrow V$ as suggested by the dashed arrow above. This map induces the map $\xi^{\star}_{H\mathcal{C}}$ on homology.*

Proof. Under the isomorphisms of Propositions 3.5 and 3.8, if we apply $\pi^*(\mathbf{D}Q^{\mathcal{C}}(-))$ to the solid maps in this diagram, we obtain (abbreviating $H^*_{\mathcal{C}}$ to H):

$$
\begin{tikzcd}
(H\mathcal{X})^{\vee 2} \arrow[d, tail] & & & & \\
(H\mathcal{X})^{\sqcup 2} \arrow[d, tail] & (H\mathcal{X})^{\sqcup 2}\sqcup(H\mathcal{X})^{\sqcup 2} \arrow[l, "(\mathrm{id},\mathrm{id})"'] & (HV)^{\sqcup 2}\sqcup H\mathcal{X} \arrow[l, "d_0\sqcup d_0\sqcup\varphi_s"'] & HV\sqcup HV \arrow[l, "\varphi_{s+1}\sqcup d_0"'] & HV \arrow[l, "\varphi_{s+1}"'] \arrow[llll, bend right, "\overline{\xi}_{H\mathcal{C}}"'] \arrow[llllu, bend right, "\xi_{H\mathcal{C}}"'] \\
H\mathcal{X}\times H\mathcal{X} & & & &
\end{tikzcd}
$$

One observes that the horizontal composite is the very definition of $\overline{\xi}_{H\mathcal{C}}$. We know from §6.5 that $\overline{\xi}_{H\mathcal{C}}$ factors through the smash coproduct, which is how we were able to fill in the dashed arrow on cohomology.

In order to obtain a map $\xi^{\star}_{\mathrm{res}}$, it is enough to check that the composite $c(\mathcal{X}\sqcup\mathcal{X}) \longrightarrow V$ is null. However, as V is a GEM, a map into V is null if and only if it is zero on cohomology. We have just stated that the map $\overline{\xi}_{H\mathcal{C}}$ factors through $(H\mathcal{X})^{\vee 2}$, which is to say that the composite $HV \longrightarrow H\mathcal{X}\times H\mathcal{X}$ is zero. □

This map $\overline{\xi}^{\star}_{\mathrm{res}}$ is very rich, but it will be important to note that postcomposition with ϵ destroys much of that richness. That is, reading off the dotted portion of the above commuting diagram:

Lemma 11.9. *The map $\epsilon\circ\overline{\xi}^{\star}_{\mathrm{res}}$ equals the following sum in $\hom_{s\mathcal{V}}(c(\mathcal{X}^2), \overline{V})$:*

$$
\big(\overline{v}\circ c(\mathrm{add})\circ c(\epsilon^2)\big) + \big(\overline{v}\circ\pi_1\circ\epsilon\big) + \big(\overline{v}\circ\pi_2\circ\epsilon\big),
$$

where the π_i are the two projections $\mathcal{X}^{\times 2} \longrightarrow \mathcal{X}$.

11.5. A three-cell complex with non-trivial bracket

Let $\mathcal{C} = \mathscr{C}om$, and fix $t, t' \geq 1$. There is a map $\mathbb{S}^{\mathcal{C}}_{t+t'} \longrightarrow \mathbb{S}^{\mathcal{C}}_t \sqcup \mathbb{S}^{\mathcal{C}}_{t'}$ sending the fundamental class $z_{t+t'}$ to the shuffle product of the two fundamental classes in the codomain:

$$z_{t+t'} \longmapsto \mu(\nabla(z_t \otimes z_{t'})),$$

where μ is the structural pairing in $\mathscr{C}om$. Consider the complex $J_{t,t'}$ formed as the pushout:

$$\begin{tikzcd}
\mathbb{S}^{\mathcal{C}}_{t+t'} \arrow[r] \arrow[d, tail] & \mathbb{S}^{\mathcal{C}}_t \sqcup \mathbb{S}^{\mathcal{C}}_{t'} \arrow[d, tail] \\
C\mathbb{S}^{\mathcal{C}}_{t+t'} \arrow[r] & J_{t,t'}
\end{tikzcd}$$

The left vertical is evidently almost free (and thus a cofibration), and thus its pushout, the map $\mathbb{S}^{\mathcal{C}}_t \sqcup \mathbb{S}^{\mathcal{C}}_{t'} \longrightarrow J_{t,t'}$, is almost free. The generating subspace $V_{t+t'+1} \subseteq (J_{t,t'})_{t+t'+1}$ has a $(t+t'+1)$-dimensional generator $h_{t,t'}$, the image of the cone class h in $(\mathbb{S}^{\mathcal{C}}_{t+t'})_{t+t'+1}$ (c.f. §2.5). Moreover, the object $J_{t,t'}$ is cofibrant, and $h_{t,t'}$ becomes a cycle in $Q^{\mathcal{C}} J_{t,t'}$, since $d_i h_{t,t'} = 0$ for $i \geq 1$, and $d_0 h_{t,t'} := z_{t+t'}$, which we have identified with the decomposable element $\mu(\nabla(z_t \otimes z_{t'}))$ in passing to the pushout.

The homology long exact sequence shows that $H_* J_{t,t'}$ is three-dimensional, containing classes z_t, $z_{t'}$ and $h_{t,t'}$. Moreover, there is a co-operation Δ on $H^{\mathcal{C}}_*$ dual to the $S(\mathscr{L})$ structure map on cohomology, and we prove:

Proposition 11.10. *Under $\Delta : H^{\mathcal{C}}_* J_{t,t'} \longrightarrow (S^2 H^{\mathcal{C}}_* J_{t,t'})_{*-1}$, $h_{t,t'} \longmapsto z_t \otimes z_{t'} + z_{t'} \otimes z_t$. All other co-operations on $H^{\mathcal{C}}_* J_{t,t'}$ are zero.*

Proof. The representative g has the property that $d_0(g) = \mu(\nabla(z_t \otimes z_{t'}))$ and $d_i(g) = 0$ for $i > 0$. By Lemma 6.3 and the description of $\mathrm{qu}_{\mathcal{C}}$ in §3.10:

$$\psi_{\mathcal{C}}(g) = \mathrm{qu}_{\mathcal{C}}(\mu(\nabla(z_t \otimes z_{t'}))) = \mathrm{tr}(\nabla(z_t \otimes z_{t'}))) \in (S^2 Q^{\mathcal{C}} J_{t,t'})_{t+t'}. \qquad \square$$

11.6. A chain level construction of $j^{\star}_{H\mathcal{C}}$

Let $\mathcal{C} = \mathscr{C}om$. We can use the cofibration just defined to construct, at the chain level, the image under

$$j^{\star}_{H\mathcal{C}} : \mathrm{Pr}^{H\mathcal{C}-\mathrm{coalg}}_t(HY) \otimes \mathrm{Pr}^{H\mathcal{C}-\mathrm{coalg}}_{t'}(HZ) \longrightarrow \mathrm{Pr}^{H\mathcal{C}-\mathrm{coalg}}_{t+t'+1}(HY \barwedge HZ)$$

of a tensor product $\alpha \otimes \beta$ of *spherical* homology classes. Abbreviating $H^{\mathcal{C}}_*$ to H and $\mathrm{Pr}^{H\mathcal{C}-\mathrm{coalg}}$ to Pr:

Proposition 11.11. *There is a function*

$$\overline{F} : \hom_{s\mathfrak{C}}(\mathbb{S}^{\mathfrak{C}}_{t}, Y) \times \hom_{s\mathfrak{C}}(\mathbb{S}^{\mathfrak{C}}_{t'}, Z) \longrightarrow \hom_{s\mathfrak{C}}(J_{t,t'}, c(Y \times Z)),$$

natural in $Y, Z \in s\mathfrak{C}$*, such that the function*

$$F : \hom_{s\mathfrak{C}}(\mathbb{S}^{\mathfrak{C}}_{t}, Y) \times \hom_{s\mathfrak{C}}(\mathbb{S}^{\mathfrak{C}}_{t'}, Z) \longrightarrow \pi_{t+t'+1}(Q^{\mathfrak{C}} c(Y \times Z)) =: H^{\mathfrak{C}}_{t+t'+1}(Y \times Z)$$

defined by $F(\alpha, \beta) := H^{\mathfrak{C}}_{*}(\overline{F}(\alpha, \beta))(h_{t,t'})$ *makes the following diagram commute:*

$$\begin{tikzcd}
\begin{smallmatrix} \hom_{s\mathfrak{C}}(\mathbb{S}^{\mathfrak{C}}_{t}, Y) \times \\ \hom_{s\mathfrak{C}}(\mathbb{S}^{\mathfrak{C}}_{t'}, Z) \end{smallmatrix} \arrow[r, "\mathrm{hur}^{\otimes 2}"] \arrow[d, "F"] & \mathrm{Pr}(HY)_t \otimes \mathrm{Pr}(HZ)_{t'} \arrow[r, "j^{\star}_{H\mathfrak{C}}"] \arrow[ddl, "\mathrm{id}+T"'] & \mathrm{Pr}(HY \,\overline{\wedge}\, HZ)_{t+t'+1} \arrow[d, tail] \\
H_{t+t'+1}(Y \times Z) \arrow[r] \arrow[d, "\Delta"] & (HY \times HZ)_{t+t'+1} \arrow[r, two heads] \arrow[d, "\Delta"] & (HY \,\overline{\wedge}\, HZ)_{t+t'+1} \\
(S^2 H(Y \times Z))_{t+t'} \arrow[r] & (S^2(HY \times HZ))_{t+t'} &
\end{tikzcd}$$

The south-westerly arrow in this diagram is composite of the tensor product of the maps

$$\mathrm{Pr}(H^{\mathfrak{C}}_{*} Y)_t \subseteq H^{\mathfrak{C}}_{t} Y \longrightarrow H^{\mathfrak{C}}_{t}(Y \times Z) \text{ and } \mathrm{Pr}(H^{\mathfrak{C}}_{*} Z)_t \subseteq H^{\mathfrak{C}}_{t} Z \longrightarrow H^{\mathfrak{C}}_{t}(Y \times Z)$$

followed by $(\mathrm{id} + T) : (H^{\mathfrak{C}}_{*}(Y \times Z))^{\otimes 2} \longrightarrow S^2 H^{\mathfrak{C}}_{*}(Y \times Z)$.

Proof. The value of $\overline{F}$ on (α, β) is defined as follows. Construct canonical lifts (c.f. §3.6):

$$\begin{tikzcd}
 & c_1(Y \times Z) \arrow[d] \\
\mathbb{S}^{\mathfrak{C}}_{t} \arrow[ur, "\widetilde{(\alpha,0)}"] \arrow[r, "(\alpha,0)"'] & Y \times Z
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & c_1(Y \times Z) \arrow[d] \\
\mathbb{S}^{\mathfrak{C}}_{t} \arrow[ur, "\widetilde{(0,\beta)}"] \arrow[r, "(0,\beta)"'] & Y \times Z
\end{tikzcd}$$

and then form the commuting diagram

$$\begin{tikzcd}
\mathbb{S}^{\mathfrak{C}}_{t+t'} \arrow[r, "\mu(\nabla(z_t \otimes z_{t'}))"] \arrow[d, tail] & \mathbb{S}^{\mathfrak{C}}_{t} \sqcup \mathbb{S}^{\mathfrak{C}}_{t'} \arrow[r, "\widetilde{(\alpha,0)} \sqcup \widetilde{(0,\beta)}"] \arrow[d, tail] & c_1(Y \times Z) \arrow[d] \\
C\mathbb{S}^{\mathfrak{C}}_{t+t'} \arrow[r] \arrow[rr, bend right, "0"'] & J_{t,t'} \arrow[r, dashed] & Y \times Z
\end{tikzcd}$$

The reason that the zero map $C\mathbb{S}^{\mathfrak{C}}_{t+t'} \longrightarrow Y \times Z$ makes the outer square commute is that the composite $\mathbb{S}^{\mathfrak{C}}_{t+t'} \longrightarrow Y \times Z$ vanishes, as it sends $z_{t+t'}$ to $\mu(\nabla((\alpha, 0) \otimes (0, \beta))) = 0 \in Y \times Z$.

Corresponding to the right square is a map $J_{t,t'} \longrightarrow c_2(Y \times Z)$, and the composite with the cofibration $c_2(Y \times Z) \longrightarrow c(Y \times Z)$ is $\overline{F}(\alpha, \beta)$. This function $\overline{F}$ is evidently natural in Y and Z, and so then is F.

The required commuting diagram consists of a square, a triangle and a hexagon. The square commutes as the horizontal arrows are maps in $H\mathcal{C}-$coalg, and we can see that the triangle commutes because we understand the $H\mathcal{C}-$coalg structure of $H_*(J_{t,t'})$ (and $H_*^{\mathcal{C}}(\overline{F}(\alpha,\beta))$ is a map of $\mathcal{C}$-H_*-coalgebras). As all of the maps in the hexagon are natural, we may check that it commutes on the universal example alone:

$$(z_t, z_{t'}) \in \hom_{s\mathcal{C}}(\mathbb{S}_t^{\mathcal{C}}, \mathbb{S}_t^{\mathcal{C}}) \times \hom_{s\mathcal{C}}(\mathbb{S}_{t'}^{\mathcal{C}}, \mathbb{S}_{t'}^{\mathcal{C}}).$$

That is, it is enough to check that the following hexagon, with a one element set at the top left entry, commutes:

$$\begin{tikzcd}
\{(z_t, z_{t'})\} \arrow[r, "\mathrm{hur}^{\otimes 2}"] \arrow[d, "F"] & \mathrm{Pr}_t(H\mathbb{S}_t^{\mathcal{C}}) \otimes \mathrm{Pr}_{t'}(H\mathbb{S}_{t'}^{\mathcal{C}}) \arrow[r, "j^\star_{H\mathcal{C}}"] & \mathrm{Pr}_{t+t'+1}(H\mathbb{S}_t^{\mathcal{C}} \barwedge H\mathbb{S}_{t'}^{\mathcal{C}}) \arrow[d, tail, "\mathrm{inc}"] \\
H_{t+t'+1}(\mathbb{S}_t^{\mathcal{C}} \times \mathbb{S}_{t'}^{\mathcal{C}}) \arrow[r, "r"] & (H\mathbb{S}_t^{\mathcal{C}} \times H\mathbb{S}_{t'}^{\mathcal{C}})_{t+t'+1} \arrow[r, two heads, "\mathrm{proj}"] & (H\mathbb{S}_t^{\mathcal{C}} \barwedge H\mathbb{S}_{t'}^{\mathcal{C}})_{t+t'+1}
\end{tikzcd}$$

In this diagram, $j^\star_{H\mathcal{C}}$ and inc are isomorphisms of 1-dimensional vector spaces, so it is enough to check that $r(F(z_t, z_{t'}))$ does not lie in the kernel of proj, i.e.:

$$r(F(z_t, z_{t'})) \notin (H\mathbb{S}_t^{\mathcal{C}} \sqcup H\mathbb{S}_{t'}^{\mathcal{C}})_{t+t'+1} = H_{t+t'+1}\mathbb{S}_t^{\mathcal{C}} \oplus H_{t+t'+1}\mathbb{S}_{t'}^{\mathcal{C}} = 0,$$

yet $\Delta(r(F(z_t, z_{t'}))) = z_t \otimes z_{t'} + z_{t'} \otimes z_t \neq 0$, using the commuting square and triangle already established. $\square$

We record here a useful calculation:

Lemma 11.12. *For $\alpha : \mathbb{S}_t^{\mathcal{C}} \longrightarrow \mathcal{X}^s$ and $\beta : \mathbb{S}_{t'}^{\mathcal{C}} \longrightarrow \mathcal{X}^s$, the composite*

$$\mathbb{S}_t^{\mathcal{C}} \sqcup \mathbb{S}_{t'}^{\mathcal{C}} \longrightarrow J_{t,t'} \overset{\overline{F}(\alpha,\beta)}{\longrightarrow} c(\mathcal{X}^s \times \mathcal{X}^s) \overset{c(\mathrm{add}\circ(\epsilon^2))}{\longrightarrow} \mathcal{X}^s$$

equals $\widetilde{\epsilon\alpha} \sqcup \widetilde{\epsilon\beta}$. In particular, $(c(\mathrm{add} \circ (\epsilon^2)) \circ \overline{F}(\alpha, \beta))(\mu\nabla(z_t \otimes z_{t'})) = \overline{\mu}(\nabla(\widetilde{\epsilon\alpha} \otimes \widetilde{\epsilon\beta}))$.

Proof. We may calculate the restrictions to the two summands individually, and by symmetry, we need only consider:

$$\mathbb{S}_t^{\mathcal{C}} \rightarrowtail J_{t,t'} \xrightarrow{\overline{F}(\alpha,\beta)} c(\mathcal{X}^s \cdot \mathcal{X}^s) \xrightarrow{c(\epsilon\times\epsilon)} c(\overline{V}\mathcal{X}^{s-1} \cdot \overline{V}\mathcal{X}^{s-1}) \xrightarrow{c(\mathrm{add})} c(\overline{V}\mathcal{X}^{s-1}) = \mathcal{X}^s.$$

The composite $J_{t,t'} \longrightarrow c(\overline{V}\mathcal{X}^{s-1} \cdot \overline{V}\mathcal{X}^{s-1})$ equals $\overline{F}(\epsilon\alpha, \epsilon\beta)$, due to the naturality of $\overline{F}$. By definition of $\overline{F}$, the composite $\mathbb{S}_t^{\mathcal{C}} \longrightarrow c(\overline{V}\mathcal{X}^{s-1} \cdot \overline{V}\mathcal{X}^{s-1})$ equals $\widetilde{(\epsilon\alpha, 0)}$. The naturality of the operation $\alpha \longmapsto \widetilde{\alpha}$ finishes the proof. $\square$

11.7. A two-cell complex with non-trivial P^i operation

Let $\mathcal{C} = \mathscr{C}om$. In this section, we give a construction of a two-cell complex whose cohomology has a P^i connecting the two cells. Fix t, i with $2 \leq i \leq t$. There is a map $\mathbb{S}^{\mathcal{C}}_{t+i} \longrightarrow \mathbb{S}^{\mathcal{C}}_t$ defined by

$$z_{t+i} \longmapsto \mu(\nabla_{t-i}(z_t \otimes z_t)),$$

where μ is the structural pairing in $\mathcal{C}$. Consider the complex $\Theta_{t,i}$ formed as the pushout:

$$\begin{tikzcd} \mathbb{S}^{\mathcal{C}}_{t+i} \arrow[r] \arrow[d, tail] & \mathbb{S}^{\mathcal{C}}_t \arrow[d, tail] \\ C\mathbb{S}^{\mathcal{C}}_{t+i} \arrow[r] & \Theta_{t,i} \end{tikzcd}$$

By the same observations as made in §11.5, this map is a cofibration, and $H^{\mathcal{C}}_*\Theta_{t,i}$ has cohomology spanned by z_t and $h_{t,i}$ in dimension $t+i+1$. For dimension reasons, z_t is primitive. On the other hand:

Proposition 11.13. *In $H^*_{\mathcal{C}}\Theta_{t,i}$, $P^i z^*_t = h^*_{t,i}$.*

Proof. We will calculate the action of $(P^j)^*$ and Δ on $h_{t,i}$. By the same methods as in the proof of Proposition 11.10:

$$\psi_{\mathcal{C}}(g) = \mathrm{tr}(\nabla_{t-i}(z_t \otimes z_t))) \in (S^2 Q^{\mathcal{C}}\Theta_{t,i})_{t+i},$$

which represents $\sigma^{\mathrm{ext}}_i z_t$. so that the defining equation

$$(\psi_{\mathcal{C}})_*(h_{t,i}) = \textstyle\sum_j \pi_*(1+T)(y_j \otimes z_j) + \sum_k \sigma_k((P^k)^* h_{t,i})$$

degenerates to $\sigma_i((P^k)^* h_{t,i}) = \sigma^{\mathrm{ext}}_i z_t$. □

11.8. A chain level construction of $\theta^\star_i$

Let $\mathcal{C} = \mathscr{C}om$, and recall the linear maps

$$\theta^\star_i : V_t \longrightarrow (C^{H\mathscr{C}om-\mathrm{coalg}} V)_{t+i+1} \text{ defined when } 2 \leq i < t$$

of Proposition 8.1. After stating Proposition 8.1, we explained that we would define a non-linear function

$$\theta^\star_t : V_t \longrightarrow (C^{H\mathscr{C}om-\mathrm{coalg}} V)_{2t+1} \text{ defined when } 2 \leq t$$

using the Proposition 11.14. Thus, in the following proposition, the final statement holds by definition when $i = t$.

Proposition 11.14. *For $2 \leq i \leq t$, there is a function*

$$\overline{G} : \hom_{s\mathcal{V}}(\mathbb{K}_t, W) \longrightarrow \hom_{s\mathcal{C}}(\Theta_{t,i}, cK^{\mathcal{C}}W),$$

natural in $W \in s\mathcal{V}$, and satisfying $\overline{G}(\alpha)(z_t) = \widetilde{\alpha}$, such that the function

$$G : \hom_{s\mathcal{V}}(\mathbb{K}_t, W) \longrightarrow \pi_{t+i+1}(Q^{\mathcal{C}}cK^{\mathcal{C}}W) =: H_{t+i+1}(K^{\mathcal{C}}W)$$

defined by $G(\alpha) := H_*(\overline{G}(\alpha))(h)$ descends to a function

$$G : \pi_t W \longrightarrow H_{t+i+1}(K^{\mathcal{C}}W),$$

and, whenever $2 \leq i \leq t$, G equals the composite

$$\pi_t W \xrightarrow{\theta_i^\star} C^{H\mathcal{C}-\mathrm{coalg}}(\pi_* W)_{t+i+1} \cong H_{t+i+1}(K^{\mathcal{C}}W).$$

Proof. The value of $\overline{G}$ on α is defined as follows. There is a commuting diagram

$$\begin{tikzcd}
\mathbb{S}^{\mathcal{C}}_{t+i} \arrow[r, "\mu\nabla_{t-i}(z_t\otimes z_t)"'] \arrow[d, tail] & \mathbb{S}^{\mathcal{C}}_t \arrow[r, "\widetilde{\alpha}"] \arrow[d, tail] & c_1(KW) \arrow[d] \\
C\mathbb{S}^{\mathcal{C}}_{t+i} \arrow[r] \arrow[rr, bend right, "0"'] & \Theta_{t,i} \arrow[r, dashed] & KW
\end{tikzcd}$$

Corresponding to the right square is a map $\Theta_{t,i} \longrightarrow c_2(KW)$, and the composite with the cofibration $c_2(KW) \longrightarrow c(KW)$ is $\overline{G}(\alpha)$. This function $\overline{G}$ is evidently natural in W, and so then is G. In order to check that the resulting function G descends to $\pi_t W$, suppose that $\alpha_1, \alpha_2 \in \hom_{s\mathcal{V}}(\mathbb{K}_t, W)$ are homotopic. Choose a homotopy $a : \Delta^1 \otimes \mathbb{K}_t \longrightarrow W$ between α_1 and α_2. Using the generating cofibrations included in §3.6 and the same technique as used to define $\overline{G}(\alpha)$ produces a homotopy $\Delta^1 \otimes \Theta_{t,i} \longrightarrow cK^{\mathcal{C}}W$ between $\overline{G}(\alpha_1)$ and $\overline{G}(\alpha_2)$.

The calculation of G is vacuous when $i = t$, since G was used to *define* $\theta_t^\star$. when $2 \leq i < t$ is natural in $W \in s\mathcal{V}$, so may be checked on the universal example $z_t \in \hom_{s\mathcal{V}}(\mathbb{K}_t, \mathbb{K}_t)$. As $\overline{G}(z)$ is a map of $\mathcal{C}$-H_*-coalgebras:

$$(P^j)^* G(z_t) = \begin{cases} 0, & \text{if } j \neq i,\ 2 \leq j \leq (t+i)/2; \\ \mathrm{hur}(z_t), & \text{if } j = i. \end{cases}$$

Moreover, $\Delta G(z_t)$ vanishes since $i < t$. These conditions suffice to identify $G(z_t)$, as, by construction, $G(z_t)$ lies in quadratic grading 2 of the cofree construction:

$$G(z_t) \in \mathrm{q}_2 H_{t+i+1}(\mathbb{K}_t^{\mathcal{C}}) = \mathrm{q}_2 C^{H\mathcal{C}-\mathrm{coalg}}\{z_t\}. \qquad \square$$

11.9. Proof of Proposition 11.2

Let $\mathcal{C} = \mathscr{C}om$. Proposition 11.2 follows immediately from the following two commutative diagrams. In each, the bottom row is that used to define the cohomology operations on the derived functors with which the E_2-page can be identified, and the top composite is that used to define the spectral sequence operations (after applying N_{h}^* and using the inverse of the composite of Proposition 11.1).

The commutative diagrams that follow are necessarily large, and at various points throughout the following two Propositions and their proofs we will use the following abbreviations: $\mathcal{X}$ for $\mathcal{X}^s$, $V\mathcal{X}$ for $V\mathcal{X}^s$, $\overline{V}\mathcal{X}$ for $\overline{V}\mathcal{X}^s$, $\overline{R}^1\mathcal{X}$ for $\overline{R}^1\mathcal{X}^s$, H for $H_*^{\mathcal{C}}$, π_* for π_*^{v}, Q for $Q^{\mathcal{C}}$ and Pr for $\mathrm{Pr}^{H\mathcal{C}-\mathrm{coalg}}$.

Proposition 11.15. *There is a commuting diagram (writing $\mathfrak{t} = t + t'$):*

$$\begin{tikzcd}
\pi_t^{\mathrm{v}}(\mathcal{X}^s) \otimes \pi_{t'}^{\mathrm{v}}(\mathcal{X}^s) \arrow[r, "\widetilde{\nabla}"] \arrow[dd, "\mathrm{hur}^{\otimes 2}", "\cong"'] & \pi_{\mathfrak{t}}^{\mathrm{v}}(\mathrm{q}_2 F^{\mathcal{C}}\mathcal{X}^s) \arrow[r, "\overline{\mu}"] & \pi_{\mathfrak{t}}^{\mathrm{v}}(\overline{R}^1\mathcal{X}^s) \arrow[d, dash, "\cong", "\text{zig-zag}"'] & \pi_{\mathfrak{t}+1}^{\mathrm{v}}(\overline{V}\mathcal{X}^s) \arrow[l, "\partial_{\mathrm{conn}}"'] \\
 & & \pi_{\mathfrak{t}}^{\mathrm{v}}(R^1\mathcal{X}^s) & \pi_{\mathfrak{t}+1}^{\mathrm{v}}(V\mathcal{X}^s) \arrow[l, "\partial_{\mathrm{conn}}"'] \arrow[u, "\cong"'] \arrow[d, "\mathrm{hur}^{\otimes 2}", "\cong"'] \\
\mathrm{Pr}_t(H\mathcal{X}^s) \otimes \mathrm{Pr}_{t'}(H\mathcal{X}^s) \arrow[r, "j_{H\mathcal{C}}^{\star}"] & \mathrm{Pr}_{\mathfrak{t}+1}(H\mathcal{X}^s \,\overline{\wedge}\, H\mathcal{X}^s) \arrow[rr, "\xi_{H\mathcal{C}}^{\star}"] & & \mathrm{Pr}_{\mathfrak{t}+1}(HV\mathcal{X}^s)
\end{tikzcd}$$

Proposition 11.16. *Whenever $2 \leq i \leq t$ there is a commuting diagram*

$$\begin{tikzcd}
\pi_t^{\mathrm{v}}\mathcal{X}^s \arrow[r, "\delta_i^{\mathrm{ext}}"] \arrow[d, "\mathrm{hur}^{\otimes 2}", "\cong"'] & \pi_{t+i}^{\mathrm{v}}(\mathrm{q}_2 F^{\mathcal{C}}\mathcal{X}^s) \arrow[r, "\overline{\mu}"] & \pi_{t+i}^{\mathrm{v}}(\overline{R}^1\mathcal{X}^s) & \pi_{t+i+1}^{\mathrm{v}}(\overline{V}\mathcal{X}^s) \arrow[l, "\partial_{\mathrm{conn}}"] \arrow[d, "\mathrm{hur}^{\otimes 2}", "\cong"'] \\
\mathrm{Pr}_t(H\mathcal{X}^s) \arrow[r, "\theta_i^{\star}"] & (C^{H\mathcal{C}-\mathrm{coalg}}(\mathrm{Pr}(H\mathcal{X}^s)))_{t+i+1} \arrow[r, equal] & H_{t+i+1}^{\mathcal{C}}\mathcal{X}^s \arrow[r, equal] & \mathrm{Pr}_{t+i+1}(HV\mathcal{X}^s)
\end{tikzcd}$$

Proof of Proposition 11.15. It will help to modify and augment this diagram a little. Indeed, for each cardinality one subset $\{(\alpha,\beta)\} \subseteq \hom_{s\mathcal{C}}(\mathbb{S}^{\mathcal{C}}_t, \mathcal{X}) \times \hom_{s\mathcal{C}}(\mathbb{S}^{\mathcal{C}}_{t'}, \mathcal{X})$, there is a diagram:

$$\begin{tikzcd}
\{(\alpha,\beta)\} \arrow[r, "\widetilde{\nabla}"] \arrow[d, squiggly, "j^\star_{H\mathcal{C}} \circ (\mathrm{hur}^{\otimes 2})"'] \arrow[ddddr, dotted, "F"] & \pi_{\mathrm{t}}(\mathrm{q}_2 F^{\mathcal{C}}\mathcal{X}) \arrow[r, "\overline{\mu}_*"] & \pi_{\mathrm{t}}(\overline{R}^1\mathcal{X}) & \pi_{\mathrm{t}+1}(\overline{V}\mathcal{X}) \arrow[l, "\partial_{\mathrm{conn}}"'] \\
 & & \pi_{\mathrm{t}+1}(Q\overline{V}\mathcal{X}) \arrow[ur, dotted, "="] & \\
\mathrm{Pr}_{\mathrm{t}+1}(H\mathcal{X} \barwedge H\mathcal{X}) \arrow[rrr, squiggly, "\xi^\star_{H\mathcal{C}}"] \arrow[d] & & & \mathrm{Pr}_{\mathrm{t}+1}(HV\mathcal{X}) \arrow[uu, squiggly, "\cong"', "\text{zig-zag}"] \arrow[d] \\
(H\mathcal{X} \barwedge H\mathcal{X})_{\mathrm{t}+1} \arrow[r, equal] & \pi_{\mathrm{t}+1}Q(\mathrm{cof}) \arrow[r, "\pi_*(Q\xi_{\mathrm{res}})"] & \pi_{\mathrm{t}+1}QV\mathcal{X} \arrow[r, equal] \arrow[uu, dotted, "\pi_*(Q\epsilon)"'] & H_{\mathrm{t}+1}V\mathcal{X} \\
(H\mathcal{X} \times H\mathcal{X})_{\mathrm{t}+1} \arrow[r, equal] \arrow[u, two heads] & \pi_{\mathrm{t}+1}Qc(\mathcal{X} \times \mathcal{X}) \arrow[u, two heads] \arrow[ur, dotted, "\pi_*(Q\overline{\xi}_{\mathrm{res}})"'] & &
\end{tikzcd}$$

Although all of the arrows in this modified diagram have already been defined, we've decorated some of them for emphasis. It will be enough to check that for each (α, β), this modified diagram commutes, since the collection of such (α, β) will exhaust all of the pure tensors in $\pi_t(\mathcal{X}) \otimes \pi_{t'}(\mathcal{X})$. What we need to prove is that the large rectangle consisting of wavy and solid arrows commutes.

The composite of the dotted maps equals the composite of the wavy maps, by results above. That is, Proposition 11.11 states that the two composites $\{(\alpha,\beta)\} \longrightarrow (H\mathcal{X} \barwedge H\mathcal{X})_{\mathrm{t}+1}$ are equal. The content of Proposition 11.8 is that the small triangle and square at the bottom of the diagram each commute, and the two composites $\mathrm{Pr}_{\mathrm{t}+1}(H\mathcal{X} \barwedge H\mathcal{X}) \longrightarrow H_{\mathrm{t}+1}V\mathcal{X}$ are equal. Finally, the two composites $\mathrm{Pr}_{\mathrm{t}+1}(HV\mathcal{X}) \longrightarrow \pi_{\mathrm{t}+1}(\overline{V}\mathcal{X})$ are equal, by Lemma 3.6.

Thus the image of (α, β) under either the wavy or the dotted composite equals the image of $h_{t,t'} \in \pi_{\mathrm{t}+1}(QJ_{t,t'})$ under the composite

$$QJ_{t,t'} \xrightarrow{Q\overline{F}(\alpha,\beta)} Qc(\mathcal{X} \times \mathcal{X}) \xrightarrow{Q\overline{\xi}_{\mathrm{res}}} QV\mathcal{X} \xrightarrow{Q\epsilon} Q\overline{V}\mathcal{X} = \overline{V}\mathcal{X},$$

which, by Lemma 11.9, decomposes as the sum of the three maps $\overline{v} \circ c(\mathrm{add}) \circ c(\epsilon^2) \circ \overline{F}(\alpha,\beta)$ and $\overline{v} \circ \pi_i \circ \epsilon \circ \overline{F}(\alpha, \beta)$ for $i = 1$ and 2. The composite $\pi_1 \circ \epsilon \circ \overline{F}(\alpha,\beta) : J_{t,t'} \longrightarrow \mathcal{X}$, by construction of $\overline{F}$, is the (dashed) map out of the pushout in the diagram:

$$\begin{tikzcd}
\mathbb{S}^{\mathcal{C}}_{\mathrm{t}} \arrow[r, "\mu\nabla(z_t \otimes z_{t'})"'] \arrow[d] & \mathbb{S}^{\mathcal{C}}_t \sqcup \mathbb{S}^{\mathcal{C}}_{t'} \arrow[d] \arrow[dr, "\alpha \sqcup 0"] & \\
C\mathbb{S}^{\mathcal{C}}_{\mathrm{t}} \arrow[r] \arrow[rr, bend right, "0"'] & J_{t,t'} \arrow[r, dashed] & \mathcal{X}
\end{tikzcd}$$

Now $h_{t,t'}$ is in the image of the map $C\mathbb{S}^{\mathcal{C}}_{\mathrm{t}} \longrightarrow J_{t,t'}$, and so maps to zero under the dashed map to $\mathcal{X}$. Similarly, the composite $\pi_2 \circ \epsilon \circ \overline{F}(\alpha,\beta)$ vanishes on $h_{t,t'}$. Thus, the image of

(α, β) under the dotted composite is represented by

$$A := (\overline{v} \circ c(\mathrm{add} \circ \epsilon^2) \circ \overline{F}(\alpha, \beta))(h_{t,t'}).$$

Consider the following commuting diagram:

$$\begin{tikzcd}
h_{t,t'} \in QJ_{t,t'} \arrow[r, "Q\overline{F}(\alpha,\beta)"] & Qc(\mathcal{X} \times \mathcal{X}) \arrow[r, "Qc(\mathrm{add}\circ\epsilon^2)"] & Q\mathcal{X} \arrow[r, "Q\overline{v}"] & Q\overline{V}\mathcal{X} \ni A \arrow[d, equal] \\
h_{t,t'} \in J_{t,t'} \arrow[u, two heads] \arrow[d, "d_0"'] \arrow[r, "\overline{F}(\alpha,\beta)"] & c(\mathcal{X} \times \mathcal{X}) \arrow[u, two heads] \arrow[d, "d_0"'] \arrow[r, "c(\mathrm{add}\circ\epsilon^2)"] & \mathcal{X} \arrow[u, two heads] \arrow[d, "d_0"] \arrow[r, two heads, "\overline{v}"] & \overline{V}\mathcal{X} \ni A \\
\mu\nabla(z_t \otimes z_{t'}) \in J_{t,t'} \arrow[r, "\overline{F}(\alpha,\beta)"] & c(\mathcal{X} \times \mathcal{X}) \arrow[r, "c(\mathrm{add}\circ\epsilon^2)"] & \mathcal{X} \ni \overline{\mu}(\nabla(\widetilde{\epsilon\alpha} \otimes \widetilde{\epsilon\beta})) &
\end{tikzcd}$$

The element $h_{t,t'} \in N^{\mathrm{v}}_{t+1}(J_{t,t'})$ may been used to populate the whole diagram as shown. To understand the images of $h_{t,t'}$ at either end of the bottom row, note that $d_0 h_{t,t'} = \mu\nabla(z_t \otimes z_{t'})$ by construction, and Lemma 11.12 states that under the maps of bottom row, $\mu\nabla(z_t \otimes z_{t'})$ maps to $\overline{\mu}(\nabla(\widetilde{\epsilon\alpha} \otimes \widetilde{\epsilon\beta}))$.

The data in the bottom right corner of this diagram demonstrates that $\partial_{\mathrm{conn}}\overline{A} \in \pi_{\mathrm{t}}(\overline{R}^1\mathcal{X})$ is represented by $\overline{\mu}(\nabla(\widetilde{\epsilon\alpha} \otimes \widetilde{\epsilon\beta}))$, which suffices, as $\alpha \sim \widetilde{\epsilon\alpha}$ and $\beta \sim \widetilde{\epsilon\beta}$. $\square$

Proof of Proposition 11.16. Choose a representative $\alpha \in \hom_{s\mathcal{C}}(\mathbb{S}^{\mathcal{C}}_t, \mathcal{X})$. Then, setting $W = Q\mathcal{X}^{s-1}$ in Proposition 11.14, we obtain a map $\overline{G}(\epsilon\alpha) : \Theta_{t,i} \longrightarrow \mathcal{X}$ such that $(\theta_i^\star \circ \mathrm{hur})(\alpha)$ is represented by

$$(\overline{v} \circ \overline{G}(\epsilon\alpha))(h_{t,i}) \in N^{\mathrm{v}}_{t+i+1}(\overline{V}\mathcal{X}).$$

We populate the following commuting diagram using the element $h_{t,i} \in N^{\mathrm{v}}_{t+i+1}(\Theta_{t,i})$:

$$\begin{tikzcd}
h_{t,i} \in N^{\mathrm{v}}_{t+i+1}\Theta_{t,i} \arrow[r, "\overline{G}(\epsilon\alpha)"] \arrow[d, "d_0"'] & N^{\mathrm{v}}_{t+i+1}\mathcal{X} \arrow[r, "\overline{v}"] \arrow[d, "d_0"'] & N^{\mathrm{v}}_{t+i+1}(\overline{V}\mathcal{X}) \ni (\eta \circ \overline{G}(\epsilon\alpha))(h_{t,i}) \\
\mu\nabla_{t-i}(z_t \otimes z_t) \in ZN^{\mathrm{v}}_{t+i}\Theta_{t,i} \arrow[r, "\overline{G}(\epsilon\alpha)"] & ZN^{\mathrm{v}}_{t+i}\mathcal{X} \ni \mu\nabla_{t-i}(\widetilde{\epsilon\alpha} \otimes \widetilde{\epsilon\alpha}) &
\end{tikzcd}$$

Here, the value of $d_0 h_{t,i}$ is known by definition of $\Theta_{t,i}$, and the fact that $\overline{G}(\epsilon\alpha)(z_t) = \widetilde{\epsilon\alpha}$ allows us to calculate $(\overline{G}(\epsilon\alpha) \circ d_0)(h_{t,i})$. Finally, in order to calculate $\partial_{\mathrm{conn}}(\theta_i^\star \circ \mathrm{hur})(\alpha)$, we find a preimage under $N^{\mathrm{v}}_{t+i+1}\mathcal{X} \xrightarrow{\overline{v}} N^{\mathrm{v}}_{t+i+1}\overline{V}\mathcal{X}$ of the representative $(\eta \circ \overline{G}(\epsilon\alpha))(h_{t,i})$, and then apply the differential d_0. We may use the preimage $\overline{G}(\epsilon\alpha)$, which maps to $\mu\nabla_{t-i}(\widetilde{\epsilon\alpha} \otimes \widetilde{\epsilon\alpha}) \in N^{\mathrm{v}}_{t+i}\overline{R}^1\mathcal{X}$ under d_0. This is homotopic to $\mu\nabla_{t-i}(\alpha \otimes \alpha)$, which represents $\overline{\mu}\delta_i^{\mathrm{ext}}(\alpha)$. $\square$

Chapter 12

Composite functor spectral sequences

It will be important for us to identify the derived functors $H^*_{\mathcal{W}(0)}X := \mathbf{D}(\mathbb{L}_*Q^{\mathcal{W}(0)}X)$ for $X \in \mathcal{W}(0)$, in order to determine the E_2-page of the BKSS for a connected simplicial commutative algebra. More generally, we will now present a spectral sequence whose goal is to calculate $H^*_{\mathcal{W}(n)}X$ for $X \in \mathcal{W}(n)$. This will be a CFSS analogous to Miller's spectral sequence in [42, §2]. The factorization of $Q^{\mathcal{W}(n)}$ we will use is of course

$$Q^{\mathcal{W}(n)} = \left(\mathcal{W}(n) \xrightarrow{Q^{\mathcal{U}(n)}} \mathcal{L}(n) \xrightarrow{Q^{\mathcal{L}(n)}} \mathcal{V}^+_n\right)$$

There is an added challenge in this context — indeed, the available factorization of $Q^{\mathcal{W}(n)}$ is through a non-abelian category. Thus, the standard technology for CFSSs does not apply, and we must use Blanc and Stover's methods [3]. They observe that the left derived functors $\mathbb{L}_*Q^{\mathcal{U}(n)}X$ are calculated as the homotopy groups of a simplicial object in $\mathcal{L}(n)$, namely $Q^{\mathcal{U}(n)}B^{\mathcal{W}(n)}X$, and as such, they have the structure of a $\mathcal{L}(n)$-Π-algebra. That is, they form an object of $\mathcal{W}(n+1)$. After verifying that the functor $Q^{\mathcal{U}(n)}$ satisfies the requisite acyclicity condition (indeed it preserves free objects), one may apply [3, Theorem 4.4]: there is a spectral sequence, with $E_r \in \mathcal{V}^+_{n+2}$,

$$[E^2_{\mathscr{G}}X]^t_{s_{n+2},\ldots,s_1} = ((H^{\mathcal{W}(n+1)}_*)(\mathbb{L}_*Q^{\mathcal{U}(n)})X)^t_{s_{n+2},\ldots,s_1} \implies ((H^{\mathcal{W}(n)}_*)X)^t_{s_{n+2}+s_{n+1},s_n,\ldots,s_1}$$

If $U^{\mathcal{W},\mathcal{U}} : \mathcal{W} \longrightarrow \mathcal{U}$ is the forgetful functor, resulting from the fact that an object of $\mathcal{W}(n)$ is in particular an object of $\mathcal{U}(n)$:

Proposition 12.1. *For $X \in s\mathcal{W}(n)$, the groups $\mathbb{L}_*Q^{\mathcal{U}(n)}X$ are isomorphic to $H^{\mathcal{U}(n)}_*U^{\mathcal{W}}_{\mathcal{U}}X$, the $\mathcal{U}(n)$-homology of the object of $s\mathcal{U}(n)$ underlying X.*

Proof. We may take X to be almost free in $s\mathcal{W}(n)$, and calculate $\mathbb{L}_*Q^{\mathcal{U}(n)}X$ simply as $\pi_*Q^{\mathcal{U}(n)}X$. Then X, viewed as an object of $s\mathcal{U}(n)$, is levelwise free, but potentially not almost free. We need to show then that $\pi_* Q^{\mathcal{U}(n)} X$ does indeed calculate $H_*^{\mathcal{U}(n)} X$ whenever $X \in s\mathcal{U}(n)$ is *levelwise* free, which is to say that the map $Q^{\mathcal{U}(n)} B^{\mathcal{U}(n)} X \longrightarrow Q^{\mathcal{U}(n)} X$ is a weak equivalence in $s\mathcal{V}$. For this, $Q^{\mathcal{U}(n)} B^{\mathcal{U}(n)} X$ is the diagonal of the bisimplicial vector space $Q^{\mathcal{U}(n)} B_q^{\mathcal{U}(n)} X_p$, and we use the spectral sequence arising from filtering by p. As X is levelwise free, the E^1-page is isomorphic to the chain complex $N_p(Q^{\mathcal{U}(n)} X)$, concentrated in $q = 0$. $\square$

We will prefer to work with the dual spectral sequence, which has $E_r \in \mathcal{V}_+^{n+2}$:

$$[E_2^{\mathscr{G}} X]_t^{s_{n+2},\dots,s_1} = ((H^*_{\mathcal{W}(n+1)})(H_*^{\mathcal{U}(n)})X)_t^{s_{n+2},\dots,s_1} \implies [E_0 H^*_{\mathcal{W}(n)} X]_t^{s_{n+2},\dots,s_1}$$

These two spectral sequences are respectively the homotopy and cohomotopy spectral sequences of a certain object of $ss\mathcal{V}_n^+$, with which we will need to work directly. Indeed, in §12.1, we will define a comonad $\mathscr{G}$ on $s\mathcal{L}(n)$, and, for $X \in \mathcal{W}(n)$, we will use the object

$$Q^{\mathcal{L}(n)} B^{\mathscr{G}} L \in ss\mathcal{V}_n^+ \text{ where } L := Q^{\mathcal{U}(n)} B^{\mathcal{W}(n)} X \in s\mathcal{L}(n).$$

The identification of $E_2^{\mathscr{G}}$ follows from Lemma 3.1 and Propositions 12.1 and 12.2.

Before we do, we will recall Blanc and Stover's constructions, and imbue them with certain extra structure that will be reflected in the spectral sequence.

12.1. The Blanc-Stover comonad in categories monadic over $\mathbb{F}_2$-vector spaces

Fix an algebraic category $\mathcal{C}$, monadic over a category of graded $\mathbb{F}_2$-vector spaces $\mathcal{V}$. As we are working over a category of vector spaces, rather than a category of graded sets, we can find further structure on the following comonad on $s\mathcal{C}$ defined by Blanc and Stover. While they use the notation 'W' in [3] and '$\mathscr{V}$' in [54], we will use the symbol '$\mathscr{G}$' to avoid notational confusion. In our context, Blanc-Stover's comonad $\mathscr{G}$, applied to $L \in s\mathcal{C}$, is the pushout

$$\begin{array}{ccc} \coprod_{\substack{S \in \mathrm{sph}(\mathcal{C}) \\ y: CS \to L}} S_{y \circ \imath n} & \longrightarrow & \coprod_{\substack{S \in \mathrm{sph}(\mathcal{C}) \\ x: S \to L}} S_x \\ \downarrow & & \downarrow \\ \coprod_{\substack{S \in \mathrm{sph}(\mathcal{C}) \\ y: CS \to L}} CS_y & \longrightarrow & \mathscr{G} L \end{array}$$

The subscripts are just notation to distinguish multiple copies of S and CS for each sphere $S \in \mathrm{sph}(\mathcal{C})$. The top horizontal map sends the sphere $S_{y \circ \imath n}$ isomorphically onto *itself*. The left vertical map is the coproduct of copies of the inclusion $\imath n : S \longrightarrow CS$. The effect of taking this pushout is to modify the coproduct S_x of spheres by attaching the cone on S_x

once for each nullhomotopy of $x \in L$.

It will be useful to write h_y for the image in $N_*\mathscr{G}L$ of $h \in N_*CS_y$, and similarly, z_x for image in $ZN_*\mathscr{G}L$ of $z \in ZN_*S_x$. Indeed, recalling the discussion in §3.1, the data of $S \in \mathrm{sph}(\mathcal{C})$ with a map $S \longrightarrow L$ is equivalent to the data of a homogeneous normalized cycle of L, and similarly, $S \in \mathrm{sph}(\mathcal{C})$ with a map $CS \longrightarrow L$ is equivalent to a homogeneous normalized chain of L which is *not in dimension zero*. From this viewpoint, if we write $\mathrm{hg}(ZN_*L)$ for the homogeneous normalized cycles and $\mathrm{hg}(N_{\geq 1}L)$ for the homogeneous normalized chains of L not in dimension zero, the pushout may be written as

$$\begin{tikzcd}
\coprod_{y\in \mathrm{hg}(N_{\geq 1}L)} S_{dy} \arrow[r] \arrow[d] & \coprod_{x\in \mathrm{hg}(ZN_*L)} S_x \arrow[d] \\
\coprod_{y\in \mathrm{hg}(N_{\geq 1}L)} CS_y \arrow[r] & \mathscr{G}L
\end{tikzcd}$$

We will now show that $\mathscr{G}L$ is homotopy equivalent to a coproduct of spheres. Indeed, let

$$\mathrm{hg}(BN_*L) = \mathrm{im}\big(d : \mathrm{hg}(N_{\geq 1}L) \longrightarrow \mathrm{hg}(ZN_*L)\big),$$

and choose a section f of the surjection $d : \mathrm{hg}(N_{\geq 1}L) \twoheadrightarrow \mathrm{hg}(BN_*L)$. Then $\mathscr{G}L$ contains a contractible subobject, the pushout

$$\begin{tikzcd}
\coprod_{x\in \mathrm{hg}(BN_*L)} S_x \arrow[r] \arrow[d] & \coprod_{x\in \mathrm{hg}(BN_*L)} S_x \arrow[d] \\
\coprod_{x\in \mathrm{hg}(BN_*L)} CS_{f(x)} \arrow[r] & C_0
\end{tikzcd}$$

whose inclusion is a cofibration. Then

$$\mathscr{G}L/C_0 \cong \left(\coprod_{y\in \mathrm{hg}(N_{\geq 1}L)\backslash \mathrm{im}(f)} CS_y/S \right) \sqcup \left(\coprod_{x\in \mathrm{hg}(ZN_*L)\backslash \mathrm{hg}(BN_*L)} S_x \right)$$

where we have written 'A/B' for the pushout of a cofibration $B \longrightarrow A$ along the map $B \longrightarrow 0$, using the cofibrations $C_0 \longrightarrow \mathscr{G}L$ and $\imath n : S \longrightarrow CS_y$. As CS/S is *isomorphic* to the sphere of one dimension higher than S (consider the construction of §2.5), this shows that $\mathscr{G}L$ is homotopic to a coproduct of spheres.

The promised comonad structure maps $\epsilon : \mathscr{G}L \longrightarrow L$ and $\Delta : \mathscr{G}L \longrightarrow \mathscr{G}^2L$ are determined by:

$$\epsilon(h_x) = x,\ \epsilon(z_y) = y,\ \Delta(h_x) = h_{h_x},\ \text{and } \Delta(z_y) = z_{z_y} \text{ for } x \in N_{n+1}L \text{ and } y \in ZN_nL.$$

We would like to find a sub*space* of $\pi_*(\mathscr{G}L)$ which freely generates it as a $\mathcal{C}$-Π-algebra. Even better, we have the following rendition of an observation used in [3, Proof of Theorem 4.2]. We give the proof since we will need to be explicit about some parts of it in what follows.

Proposition 12.2. *For $L \in s\mathcal{C}$, $\pi_*(B^{\mathscr{G}}L)$ is an almost free (monadic over $\mathcal{V}$) simplicial $\mathcal{C}$-Π-algebra weakly equivalent to π_*L.*

This differs from the observation in [3, Proof of Theorem 4.2], in that we show that all the structure maps of $\pi_*(B^{\mathscr{G}}L) \in s\pi\mathcal{C}$ except for d_0 preserve *vector spaces* of generators, rather than *sets* of generators.

Proof. That the augmentation to π_*L is a weak equivalence follows from Stover's result [54, 2.7]. The only change from Blanc-Stover is that $\pi_*(B^{\mathscr{G}}L)$ is almost free over the category $\mathcal{V}$, rather than the category of pointed sets.

During this proof, for any set A we will write $\mathbb{F}_2\{A\}$ for the vector space generated by the symbols $\underline{a}$ for $a \in A$. Suppose that $M \in s\mathcal{L}(n)$. There is a natural map $d_* : \mathbb{F}_2\{\mathrm{hg}(N_{\geq 1}M)\} \longrightarrow \mathbb{F}_2\{\mathrm{hg}(ZN_*M)\}$, and a natural monomorphism $\alpha : \ker(d_*) \longrightarrow \pi_*(\mathscr{G}M)$, defined by

$$\alpha(\underline{x_1} - \underline{x_0}) = \overline{h_{x_1} - h_{x_0}}, \text{ for } x_1, x_2 \in N_{\geq 1}M \text{ with } dx_1 = dx_2.$$

Moreover, there is a natural map $\beta : \mathbb{F}_2\{\mathrm{hg}(ZN_*M)\} \longrightarrow \pi_*(\mathscr{G}M)$ (which is not monomorphic) defined by

$$\beta(\underline{x}) = \overline{z_x}, \text{ for } x \in \mathrm{hg}(ZN_*M).$$

From the above expression for $\mathscr{G}M/C_0$, one sees that $\mathrm{im}(\alpha)$ and $\mathrm{im}(\beta)$ are linearly independent subspaces of $\pi_*(\mathscr{G}M)$, and that $\pi_*(\mathscr{G}M)$ is free on $\mathrm{im}(\alpha) \oplus \mathrm{im}(\beta)$. Moreover, if $M \longrightarrow M'$ is a map in $s\mathcal{L}(n)$, then the generating subspaces are preserved by the induced map $\pi_*\mathscr{G}M \longrightarrow \pi_*\mathscr{G}M'$.

Applying this analysis to $\pi_*B^{\mathscr{G}}L \in s\pi\mathcal{C}$, every face and degeneracy map except for s_0 and d_0 preserves the generators. In order to check that s_0 preserves generators, we must see that the comonad diagonal of $\mathscr{G}$ sends the subspaces $\mathrm{im}(\alpha_L)$ and $\mathrm{im}(\beta_L)$ into the subspaces $\mathrm{im}(\alpha_{\mathscr{G}L})$ and $\mathrm{im}(\beta_{\mathscr{G}L})$. That $\mathrm{im}(\beta_L)$ maps into $\mathrm{im}(\beta_{\mathscr{G}L})$ is immediate. For $\mathrm{im}(\alpha_L)$, the image of $h_{x_1} - h_{x_0}$ under the diagonal is $h_{h_{x_1}} - h_{h_{x_0}}$, which is in $\mathrm{im}(\alpha_{\mathscr{G}L})$, since $dh_{x_1} = z_{dx_1} = z_{dx_0} = dh_{x_0}$. □

12.2. A chain-level diagonal on the $\mathscr{G}$ construction

We have seen, for $M \in s\mathcal{C}$, that $\pi_*(\mathscr{G}M)$ is a free object in $\pi\mathcal{C}$. As such, there is a diagonal

$$\varphi_{\pi\mathcal{C}} : \pi_*(\mathscr{G}M) \longrightarrow \pi_*(\mathscr{G}M) \sqcup \pi_*(\mathscr{G}M).$$

In this section, we will describe how $\varphi_{\pi\mathcal{C}}$ is the map on homotopy induced by a morphism $\varphi_{\mathscr{G}} : \mathscr{G}M \longrightarrow \mathscr{G}M \sqcup \mathscr{G}M$ in $s\mathcal{C}$, and construct a map $\xi_{\mathscr{G}}$ related to the map $\xi_{\pi\mathcal{C}}$ of §6.5.

In order to construct a map $\varphi_{\mathscr{G}}$, each $S \in \operatorname{sph}(\mathcal{C})$ equals $S = F^{\mathcal{C}}\mathbb{K}$ for some $\mathbb{K}$ as in §2.5 (with indices omitted), and we construct a commuting diagram:

$$\begin{tikzcd} S \arrow[r, "\varphi_1"] \arrow[d, "\imath n"] & S \sqcup S \arrow[d, "\imath n \sqcup \imath n"] \\ CS \arrow[r, "\varphi_2"] & CS \sqcup CS \end{tikzcd} \quad \text{by applying } F^{\mathcal{C}} \text{ to} \quad \begin{tikzcd} \mathbb{K} \arrow[r, "\Delta"] \arrow[d, "\imath n"] & \mathbb{K} \oplus \mathbb{K} \arrow[d, "\imath n \sqcup \imath n"] \\ C\mathbb{K} \arrow[r, "\Delta"] & C\mathbb{K} \oplus C\mathbb{K} \end{tikzcd}$$

The maps φ_1 and φ_2 can then be applied respectively to all of the sphere and cone classes appearing in $\mathscr{G}M$. To understand the effect of $\varphi_{\mathscr{G}}$ on homotopy, it is enough to identify where the generators of $\pi_*(\mathscr{G}M)$ are sent in $\pi_*(\mathscr{G}M) \sqcup \pi_*(\mathscr{G}M)$, which is easy. The theory of this map mimics that presented in §6.5, as intended, and we list some of its properties here, with proofs omitted.

Lemma 12.3. *$\mathscr{G}M$ is naturally a (strict) commutative cogroup object, having comultiplication map $\varphi_{\mathscr{G}}$, counit map $0 : \mathscr{G}M \longrightarrow 0$, and inverse map $\mathrm{id} : \mathscr{G}M \longrightarrow \mathscr{G}M$. In particular, $\hom(\mathscr{G}M, -)$ takes values in $\mathbb{F}_2$-vector spaces.*

Writing $\boxplus$ for the group operation on $\hom_{s\mathcal{C}}(\mathscr{G}M, M')$, we have the following:

Lemma 12.4. *For maps $f, g : \mathscr{G}M \longrightarrow M'$ we have*

$$Q^{\mathcal{C}}(f \boxplus g) = (Q^{\mathcal{C}}f + Q^{\mathcal{C}}g) : Q^{\mathcal{C}}(\mathscr{G}M) \longrightarrow Q^{\mathcal{C}}M'.$$

Proof. It is enough to check that $Q^{\mathcal{C}}(\varphi_{\mathscr{G}}) : Q^{\mathcal{C}}(\mathscr{G}M) \longrightarrow Q^{\mathcal{C}}(\mathscr{G}M \sqcup \mathscr{G}M)$ equals the diagonal map $Q^{\mathcal{C}}(\mathscr{G}M) \longrightarrow Q^{\mathcal{C}}(\mathscr{G}M) \oplus Q^{\mathcal{C}}(\mathscr{G}M)$. For this, $Q^{\mathcal{C}}$ converts all the colimits involved in the construction of $\mathscr{G}M$ to direct sums of simplicial vector spaces, and $Q^{\mathcal{C}}\varphi_1$ and $Q^{\mathcal{C}}\varphi_2$ are both precisely the diagonal map. $\square$

Now let $\overline{\xi}_{\mathscr{G}}$ denote the following composite:

$$\overline{\xi}_{\mathscr{G}} : \mathscr{G}^2 M \xrightarrow{\varphi_{\mathscr{G}}} (\mathscr{G}^2 M)^{\sqcup 2} \xrightarrow{a \sqcup b} (\mathscr{G}M)^{\sqcup 2}$$

where $a, b : \mathscr{G}^2 M \longrightarrow (\mathscr{G}M)^{\sqcup 2}$ are the composites

$$a : \mathscr{G}^2 M \xrightarrow{\varphi_{\mathscr{G}}} (\mathscr{G}^2 M)^{\sqcup 2} \xrightarrow{\epsilon^{\sqcup 2}} (\mathscr{G}M)^{\sqcup 2}$$
$$b : \mathscr{G}^2 M \xrightarrow{\epsilon} (\mathscr{G}M) \xrightarrow{\varphi_{\mathscr{G}}} (\mathscr{G}M)^{\sqcup 2}$$

Thanks to Lemma 12.5, $\overline{\xi}_{\mathscr{G}}$ factors through the smash coproduct, defining a natural map

$$\xi_{\mathscr{G}} : \mathscr{G}^2 M \longrightarrow (\mathscr{G}M)^{\veebar 2}.$$

Lemma 12.5. *The composite* $\mathscr{G}^2 M \xrightarrow{\overline{\xi}_{\mathscr{G}}} (\mathscr{G}M)^{\sqcup 2} \longrightarrow (\mathscr{G}M)^{\times 2}$ *is zero.*

Proof. This follows from the observation that both composites $(\mathrm{id} \sqcup 0)\overline{\xi}_{\mathscr{G}}$ and $(0 \sqcup \mathrm{id})\overline{\xi}_{\mathscr{G}}$ equal $\epsilon : \mathscr{G}^2 M \longrightarrow \mathscr{G}M$. □

The desired property for $\xi_{\mathscr{G}}$ is then the following lemma (involving the natural isomorphism i of Proposition 3.2, and the almost free structure given in Proposition 12.2).

Lemma 12.6. *For* $L \in s\mathcal{C}$*, we have* $(i \circ \xi_{\pi\mathcal{C}}) = \pi_*(\xi_{\mathscr{G}})$*, i.e. a commuting diagram:*

$$\begin{tikzcd} \pi_*(B^{\mathscr{G}}_s L) \arrow[r, "\pi_*(\xi_{\mathscr{G}})"] \arrow[dr, "\xi_{\pi\mathcal{C}}"'] & \pi_*((B^{\mathscr{G}}_{s-1} L)^{\veebar 2}) \\ & (\pi_*(B^{\mathscr{G}}_{s-1} L))^{\veebar 2} \arrow[u, "i"', "\cong"] \end{tikzcd}$$

Proof. In view of the short exact sequences of Proposition 3.2, this is equivalent to $(i \circ \overline{\xi}_{\pi\mathcal{C}}) = \pi_*(\overline{\xi}_{\mathscr{G}}) : \pi_*(\mathscr{G}^{s+1}L) \longrightarrow \pi_*((\mathscr{G}^s L)^{\sqcup 2})$, which holds as $(i \circ \varphi_{\pi\mathcal{C}}) = \pi_*(\varphi_{\mathscr{G}})$. □

Lemma 12.7. $(d_i)^{\veebar 2}\xi_{\mathscr{G}} = \xi_{\mathscr{G}} d_{i+1}$ *for* $i \geq 1$*, and* $(d_0)^{\veebar 2}\xi_{\mathscr{G}} = (\xi_{\mathscr{G}} d_0) \boxplus (\xi_{\mathscr{G}} d_1)$*, so that the map* $Q^{\mathcal{C}}\xi_{\mathscr{G}}$ *induces a degree* $(-1, 0)$ *bicomplex map:*

$$N_*N_*(Q^{\mathcal{C}} B^{\mathscr{G}}_{\bullet} L)_{s_{n+2}, s_{n+1}} \longrightarrow N_*N_*(Q^{\mathcal{C}}((B^{\mathscr{G}}_{\bullet} L)^{\veebar 2}))_{s_2 - 1, s_1}.$$

As in §6.5, we will use the composite double complex map

$$\psi_{\mathscr{G}} := j_{\mathcal{L}(n)} \circ Q^{\mathcal{L}(n)}\xi_{\mathscr{G}} : N^{\mathrm{h}}_* N^{\mathrm{v}}_* (Q^{\mathcal{L}(n)} B^{\mathscr{G}} L)^{t+1}_{s_{n+2}, \ldots, s_1} \longrightarrow N^{\mathrm{h}}_* N^{\mathrm{v}}_* (S^2(Q^{\mathcal{L}(n)} B^{\mathscr{G}} L))^{t}_{s_{n+2}-1, s_{n+1}, \ldots, s_1}$$

in what follows.

12.3. Quadratic grading

We will say that an object $X \in \mathcal{C}$, where $\mathcal{C}$ is any of $\mathcal{W}(n)$, $\mathcal{U}(n)$ or $\mathcal{L}(n)$, is *quadratically graded* if the underlying vector space of X is equipped with a quadratic grading such that the action map $F^{\mathcal{C}}X \longrightarrow X$ preserves quadratic gradings (i.e. is a map in $\mathrm{q}\mathcal{V}^+_r$). Recall that $F^{\mathcal{C}}$ is in fact a monad on $\mathrm{q}\mathcal{V}^+_n$, by Lemmas 6.10 and 7.2. There are evident categories of quadratically graded objects in these three categories, which we write as $\mathrm{q}\mathcal{W}(n)$, $\mathrm{q}\mathcal{U}(n)$ or $\mathrm{q}\mathcal{L}(n)$, and the various homology and cohomology functors can be enriched to functors

$$H^{\mathcal{C}}_* : s(\mathrm{q}\mathcal{C}) \longrightarrow \mathrm{q}\mathcal{V}^+_{n+1} \quad \text{and} \quad H^*_{\mathcal{C}} : s(\mathrm{q}\mathcal{C}) \longrightarrow \mathrm{q}\mathcal{V}^{n+1}_+.$$

Similarly, the categories $\mathcal{M}_{\mathrm{v}}(n+1)$ and $\mathcal{M}_{\mathrm{h}}(n+1)$, in which $H^*_{\mathcal{W}(n)}$ takes values, can both be enriched in this way, and if $X \in \mathrm{q}\mathcal{W}(n)$ then $H^*_{\mathcal{W}(n)}X$ is an object of $\mathrm{q}\mathcal{M}_{\mathrm{v}}(n+1)$ and

$q\mathcal{M}_h(n+1)$, and $H_*^{\mathcal{U}(n)}X$ is an object of $q\mathcal{W}(n+1)$.

Thus, if $X \in q\mathcal{W}(n)$ then the CFSS

$$[E_2^{\mathscr{G}}X]_t^{s_{n+2},\dots,s_1} = ((H^*_{\mathcal{W}(n+1)})(H_*^{\mathcal{U}(n)})X)_t^{s_{n+2},\dots,s_1} \implies [E_0H^*_{\mathcal{W}(n)}X]_t^{s_{n+2},\dots,s_1}$$

has both E_2 and target quadratically graded. Because all of the cohomology and homotopy operations constructed in §§5-8 are formed at the chain level using quadratic operations, it is not hard to check

Proposition 12.8. *If $X \in q\mathcal{W}(n)$ then the CFSS is quadratically graded:*

$$q_k[E_2^{\mathscr{G}}X]_t^{s_{n+2},\dots,s_1} = q_k(H^*_{\mathcal{W}(n+1)})(H_*^{\mathcal{U}(n)})X)_t^{s_{n+2},\dots,s_1} \implies q_k[E_0H^*_{\mathcal{W}(n)}X]_t^{s_{n+2},\dots,s_1}.$$

12.4. The edge homomorphism and edge composite

For $X \in s\mathcal{W}(n)$, the spectral sequence

$$[E_2^{\mathscr{G}}X]_t^{s_{n+2},\dots,s_1} = ((H^*_{\mathcal{W}(n+1)})(H_*^{\mathcal{U}(n)})X)_t^{s_{n+2},\dots,s_1} \implies [E_0H^*_{\mathcal{W}(n)}X]_t^{s_{n+2},\dots,s_1}$$

has *edge homomorphism*

$$(H^*_{\mathcal{W}(n)}X)_t^{s_{n+1},\dots,s_1} \twoheadrightarrow [E_0H^*_{\mathcal{W}(n)}X]_t^{0,s_{n+1},\dots,s_1} \cong [E_\infty^{\mathscr{G}}X]_t^{0,s_{n+1},\dots,s_1} \subseteq [E_2^{\mathscr{G}}X]_t^{0,s_{n+1},\dots,s_1}$$

which we may compose with the inclusion

$$[E_2^{\mathscr{G}}X]_t^{0,s_{n+1},\dots,s_1} = (\mathbf{D}(Q^{\mathcal{W}(n+1)}H_*^{\mathcal{U}(n)}X))_t^{s_{n+1},\dots,s_1} \subseteq (H^*_{\mathcal{U}(n)}X)_t^{s_{n+1},\dots,s_1}.$$

to form the *edge composite*:

$$(H^*_{\mathcal{W}(n)}X)_t^{s_{n+1},\dots,s_1} \longrightarrow (H^*_{\mathcal{U}(n)}X)_t^{s_{n+1},\dots,s_1}$$

Proposition 12.9. *Suppose that $n \geq 1$. Then the edge composite commutes with the vertical Steenrod operations of Proposition 8.6:*

$$\begin{tikzcd}
(H^*_{\mathcal{W}(n)}X)_t^{s_{n+1},\dots,s_1} \arrow[r, "\mathrm{Sq}^i_{\mathrm{v}}"] \arrow[d, "\text{edge comp.}"] & (H^*_{\mathcal{W}(n)}X)_{2t+1}^{s_{n+1}+1,s_n+i-1,2s_{n-1},\dots,2s_1} \arrow[d, "\text{edge comp.}"] \\
(H^*_{\mathcal{U}(n)}X)_t^{s_{n+1},\dots,s_1} \arrow[r, "\mathrm{Sq}^i_{\mathrm{v}}"] & (H^*_{\mathcal{U}(n)}X)_{2t+1}^{s_{n+1}+1,s_n+i-1,2s_{n-1},\dots,2s_1}
\end{tikzcd}$$

Setting $n = 0$, suppose that $2 \leq i < t$. The same composite commutes with the δ^{v}-operations *of Propositions 8.2 and 8.3:*

$$\begin{array}{ccc} (H^*_{\mathcal{W}(0)}X)^s_t & \xrightarrow{\delta^{\mathrm{v}}_i} & (H^*_{\mathcal{W}(0)}X)^{s+1}_{t+i+1} \\ \downarrow{\scriptstyle \text{edge comp.}} & & \downarrow{\scriptstyle \text{edge comp.}} \\ (H^*_{\mathcal{U}(0)}X)^s_t & \xrightarrow{\delta^{\mathrm{v}}_i} & (H^*_{\mathcal{U}(0)}X)^{s+1}_{t+i+1} \end{array}$$

Proof. For this proof, we will suppress the '(n)' notation, as the proof is the same for all $n \geq 0$. We will also suppress all internal gradings, and write $*$ for the grading s_{n+1}. The edge composite is dual to

$$H^{\mathcal{W}}_* X := \pi_*(Q^{\mathcal{W}}|B^{\mathcal{W}}X|) \xleftarrow{d^{\mathrm{h}}_0} \pi^{\mathrm{h}}_0\pi^{\mathrm{v}}_*(Q^{\mathcal{L}}B^{\mathscr{G}}Q^{\mathcal{U}}|B^{\mathcal{W}}X|) \xleftarrow{z_-} \pi_*(Q^{\mathcal{U}}|B^{\mathcal{W}}X|) \cong H^{\mathcal{U}}_* X.$$

Abbreviating further by setting $D := Q^{\mathcal{U}}|B^{\mathcal{W}}X|$ and $C := Q^{\mathcal{L}}B^{\mathscr{G}}Q^{\mathcal{U}}|B^{\mathcal{W}}X|$, the map z_- sends the class of $x \in ZN_*D$ to $\overline{z_x} \in \pi^{\mathrm{h}}_0\pi^{\mathrm{v}}_*C$. This assignment does not produce a well defined map $\pi_*D \longrightarrow N^{\mathrm{h}}_0\pi^{\mathrm{v}}_*C$, as if $y \in ZN_*D$ represents the same class as x, $\overline{z_y}$ need not equal $\overline{z_x}$ in $N^{\mathrm{h}}_0\pi^{\mathrm{v}}_*C$: we only know that $\overline{z_{x-y}} = 0 \in N^{\mathrm{h}}_0\pi^{\mathrm{v}}_*C$. Fortunately, the element $\overline{z_{z_x - z_y} - z_{z_{x-y}}} \in N^{\mathrm{h}}_1\pi^{\mathrm{v}}_*C$ provides a homotopy between $\overline{z_{x-y}}$ and $\overline{z_x - z_y}$ in $N^{\mathrm{h}}_0\pi^{\mathrm{v}}_*C$:

$$d^{\mathrm{h}}_0\big(\overline{z_{z_x-z_y} - z_{z_{x-y}}}\big) = \overline{z_x - z_y} - \overline{z_{x-y}}, \text{ and } d^{\mathrm{h}}_1\big(\overline{z_{z_x-z_y} - z_{z_{x-y}}}\big) = \overline{z_{x-y} - z_{x-y}} = 0,$$

so that the map z_- is well defined.

We may model the final isomorphism as follows. Write $U^{\mathcal{W}}_{\mathcal{U}} : \mathcal{W} \longrightarrow \mathcal{U}$ for the forgetful functor. For any $V \in \mathcal{V}^+_n$, there is a natural inclusion $F^{\mathcal{U}}V \longrightarrow U^{\mathcal{W}}_{\mathcal{U}}F^{\mathcal{W}}V$ in the category $\mathcal{U}$, adjoint to the inclusion $V \longrightarrow F^{\mathcal{W}}V$. This morphism yields an inclusion of bar constructions, a weak equivalence $|B^{\mathcal{U}}U^{\mathcal{W}}_{\mathcal{U}}X| \longrightarrow U^{\mathcal{W}}_{\mathcal{U}}|B^{\mathcal{W}}X|$ in $s\mathcal{U}$. Suppressing the forgetful functors, for $X \in \mathcal{W}$, we have a weak equivalence $Q^{\mathcal{U}}|B^{\mathcal{U}}X| \longrightarrow Q^{\mathcal{U}}|B^{\mathcal{W}}X|$ inducing the isomorphism. Our conclusion is then that the entire composite $H^{\mathcal{U}}_*X \longrightarrow H^{\mathcal{W}}_*X$ is the map on homotopy induced by the composite

$$Q^{\mathcal{U}}|B^{\mathcal{U}}X| \longrightarrow Q^{\mathcal{U}}|B^{\mathcal{W}}X| \twoheadrightarrow Q^{\mathcal{W}}|B^{\mathcal{W}}X|,$$

and the operations we are considering are easily understood in relation to this map. □

12.5. An equivalent reverse Adams spectral sequence

It happens that the CFSS recently defined actually coincides with an instance of Miller's reverse Adams spectral sequence used Goerss [33, Chapter V] (c.f. §3.4). This seems to the author to be somewhat of a coincidence, as in [33], the reverse Adams spectral sequence appears for quite different reasons than in the present work. We continue using Blanc and Stover's resolution, for two reasons. Firstly, that resolution more closely reflects out intention in constructing the spectral sequence in question, and secondly, the techniques we use here may be generalizable to other contexts in which the Blanc-Stover resolution is used.

Proposition 12.10. *The CFSS applied to $X \in s\mathcal{W}(n)$ coincides with the reverse Adams spectral sequence applied to $L := Q^{\mathcal{U}(n)}B^{\mathcal{W}(n)}X \in s\mathcal{L}(n)$.*

Before proving this fact, we should remove any confusion about the convergence targets of these spectral sequences. Indeed, the reverse Adams spectral sequence has target

$$\pi^*\mathbf{D}Q^{\mathcal{L}(n)}B^{\mathcal{L}(n)}L \cong \pi^*\mathbf{D}Q^{\mathcal{L}(n)}L = \pi^*\mathbf{D}Q^{\mathcal{W}(n)}B^{\mathcal{W}(n)}X =: H^*_{\mathcal{W}(n)}X,$$

where the isomorphism follows from the same acyclicity condition needed to define the CFSS. Thus the targets coincide, as hoped.

Proof. We will use the Dwyer-Kan-Stover E^2 model structure on the category $ss\mathcal{C}$, which originated in [27] for bisimplicial sets, and is reinterpreted for objects of $ss\mathcal{C}$ in [3, §4.1.1].

Viewing L as a constant object in $ss\mathcal{L}(n)$, each of $B_p^{\mathcal{L}(n)}L_q$ and $B^{\mathscr{G}}L$ admits an E^2-weak equivalence to L. Moreover, each is cofibrant. Indeed, $B^{\mathscr{G}}L$ is cofibrant by construction, while we must check that $B_p^{\mathcal{L}(n)}L_q$ is M-free, in the sense of [3, §4.1.1].

For this, we use Lemma 2.3. That is, for each q, the horizontal simplicial object $(B_p^{\mathcal{L}(n)}L_q)_{p,q} := B_p^{\mathcal{L}(n)}L_q$ has an obvious structure of almost free simplicial (in p) object, and the generating subspaces are preserved by the vertical simplicial maps. Thus, Lemma 2.3 yields decompositions

$$V_p = \mathrm{im}\Big(V_{p-1} \xrightarrow{s_0^{\mathrm{h}}} V_p\Big) \oplus \cdots \oplus \mathrm{im}\Big(V_{p-1} \xrightarrow{s_{p-1}^{\mathrm{h}}} V_p\Big) \oplus \Big(V_p \cap N_p^{\mathrm{h}} B_p^{\mathcal{L}(n)}L_q\Big).$$

To show that V_p is M-free, we need to decompose each V_p into a coproduct of objects $\mathbb{K}^t_{s_{n+1},\ldots,s_1} \in s\mathcal{V}_n^+$, up to homotopy, and ensure that the degeneracies are induced up to homotopy by sphere inclusions. The decompositions of V_p just provided make this a simple task. Suppose that V_{p-1} already has chosen decomposition as a sum of objects $\mathbb{K}^t_{s_{n+1},\ldots,s_1}$ up to homotopy. Then if we choose such a decomposition of $V_p \cap N_p^{\mathrm{h}} B_p^{\mathcal{L}(n)}L_q$, and use the p inclusions $s_i^{\mathrm{h}} : V_{p-1} \longrightarrow V_p$ to induce decompositions of the other summands of V_p using the decomposition of V_{p-1}, we have the decomposition up to homotopy that we need.

Now, by factoring the map $0 \longrightarrow L$ by a cofibration followed by an acyclic fibration $B \longrightarrow L$ in the E_2 model structure, we can form the solid maps in a diagram in which each object 'B' is cofibrant:

$$\begin{tikzcd} B \arrow[dr, "\sim"] & B_p^{\mathcal{L}(n)} L_q \arrow[l, dotted, "\sim"'] \arrow[d, "\sim"] \\ B^{\mathscr{G}} L \arrow[u, dotted, "\sim"'] \arrow[r, "\sim"'] & L \end{tikzcd}$$

By the lifting axiom (of cofibrations against acyclic fibrations) we can find the dotted maps, weak equivalences making the diagram commute. The theory presented in [27] then explains that the three resulting spectral sequences coincide. The spectral sequence arising from $B_p^{\mathcal{L}(n)} L_q$ is the reverse Adams spectral sequence of L in $s\mathcal{L}(n)$, and that arising from $B^{\mathscr{G}} L$ is the CFSS of $X \in s\mathcal{W}(n)$. $\square$

Chapter 13

Operations in composite functor spectral sequences

Singer [52] developed a useful theory of products and Steenrod operations in the first quadrant cohomology spectral sequence arising from a bisimplicial cocommutative coalgebra. Goerss used this theory in [33, §14] in his calculation of the category $H\mathscr{C}om$. In the applications we have in mind, the bisimplicial object

$$Q^{\mathcal{L}(n)} B^{\mathscr{G}} Q^{\mathcal{U}(n)} B^{\mathcal{W}(n)} X$$

will *not* be a coalgebra. Instead the situation will resemble more the situation of §6.5, where there was a linear map $\psi_{\mathcal{C}} : Q^{\mathcal{C}} X^s \longrightarrow S^2(Q^{\mathcal{C}} X^{s-1})$ for any almost free object $X \in s\mathcal{C}$, but certainly not a coalgebra map.

The lack of an underlying coalgebra structure will not stop us from applying Singer's techniques after we make the appropriate modifications. The idea is to externalize Singer's operations, so that for every bisimplicial vector space V, there are various external operations of type:

$$[E_r V] \longrightarrow [E_{r'} S^2 V] \ (r' \geq r) \ \text{ and } \ S_2[E_r V] \longrightarrow [E_r S^2 V],$$

(which we will discuss shortly) compatible at E_∞ with external operations of type:

$$H^*(\mathbf{D}(TV)) \longrightarrow H^*(\mathbf{D}(TS^2V)) \text{ and } S_2 H^*(\mathbf{D}(TV)) \longrightarrow H^*(\mathbf{D}(TS^2V)).$$

When V is in fact a bisimplicial cocommutative coalgebra, one recovers Singer's theory by composing with the map of spectral sequences induced by the coproduct:

$$[E_r S^2 V] \longrightarrow [E_r V].$$

In §10.1 we discussed spectral sequences with indeterminacy, and multi-valued functions. They reappear in Singer's theory, as some of the operations are constructed as (actual) linear functions $[E_rV] \longrightarrow [E_{r'}S^2V]$ between different spectral sequence pages. Such an operation is equivalent to an external operation $[E_rV] \longrightarrow [E_rS^2V]$ with indeterminacy r' which also satisfies a survival property.

13.1. External spectral sequence operations of Singer

We now summarize some key aspects of Singer's work in [52], in particular Theorems 2.15, 2.16, 2.17 and 2.22, and Proposition 2.21. Fix $V \in ss\mathcal{V}$ with a (horizontal) augmentation $d_{\mathrm{h}}^0 : V \longrightarrow V_{-1}$. The key construction is that of chain level operations:

$$S^k : \mathbf{D}(TV) \longrightarrow \mathbf{D}(TS^2V)$$

inducing external operations as in the bottom row of the following diagrams:

$$\begin{tikzcd} \pi^m(\mathbf{D}(V_{-1})) \arrow[r, "\mathrm{Sq}^k_{\mathrm{ext}}"] \arrow[d] & \pi^{m+k}(\mathbf{D}(S^2V_{-1})) \arrow[d] \\ H^m(\mathbf{D}(TV)) \arrow[r, "\mathrm{Sq}^k_{\mathrm{ext}}"] & H^{m+k}(\mathbf{D}(TS^2V)) \end{tikzcd} \qquad \begin{tikzcd} S_2\pi^*(\mathbf{D}(V_{-1})) \arrow[r, "\mu_{\mathrm{ext}}"] \arrow[d] & \pi^*(\mathbf{D}(S^2V_{-1})) \arrow[d] \\ S_2H^*(\mathbf{D}(TV)) \arrow[r, "\mu_{\mathrm{ext}}"] & H^*(\mathbf{D}(TS^2V)) \end{tikzcd}$$

The top rows are the operations arising from the singly (vertically) simplicial object V_{-1}, as in §6.2. Singer studies the effect of S^k on filtration in detail, determining that it induces the following operations. For all $p, q \geq 0$ and all $r \geq 2$, there are well-defined vector space homomorphisms:

$$\begin{aligned} &\mathrm{Sq}^k_{\mathrm{ext}} : [E_rV]^{p,q} \longrightarrow [E_rS^2V]^{p,q+k}, && \text{if } 0 \leq k \leq q; \\ &\mathrm{Sq}^k_{\mathrm{ext}} : [E_rV]^{p,q} \longrightarrow [E_{r+k-2}S^2V]^{p+k-q,2q}, && \text{if } q \leq k \leq q+r-2; \\ &\mathrm{Sq}^k_{\mathrm{ext}} : [E_rV]^{p,q} \longrightarrow [E_{2r-2}S^2V]^{p+k-q,2q}, && \text{if } q+r-2 \leq k; \end{aligned}$$

which commute with the differentials (in the appropriate, somewhat complicated sense, c.f. [52, Theorem 2.17]), and an external (not 'exterior') commutative product operation which satisfies the Leibniz rule:

$$\mu_{\mathrm{ext}} : [E_rV]^{p_1,q_1} \otimes [E_rV]^{p_2,q_2} \longrightarrow [E_rS^2V]^{p_1+p_2,q_1+q_2}.$$

Note that the second and third operations are from $E_r \longrightarrow E_{r'}$, sometimes with $r' \geq r$, which is to say that these operations have indeterminacy disappearing by $E_{r'}$, and the implied survival property.

Those operations with domain $[E_2V]$ have no indeterminacy, and we reindex them as follows:

$$\begin{aligned}
&\mathrm{Sq}^k_{\mathrm{v,ext}} = \mathrm{Sq}^k_{\mathrm{ext}} : [E_2V]^{p,q} \longrightarrow [E_2S^2V]^{p,q+k}, && \text{if } 0 \le k \le q,\\
&\mathrm{Sq}^k_{\mathrm{v,ext}} = 0, && \text{if } k > q,\\
&\mathrm{Sq}^k_{\mathrm{h,ext}} = \mathrm{Sq}^{q+k}_{\mathrm{ext}} : [E_2V]^{p,q} \longrightarrow [E_2S^2V]^{p+k,2q}, && \text{if } 0 \le k.
\end{aligned}$$

Under the identification $[E_2V]^{p,q} = \pi^p_{\mathrm{h}}\pi^q_{\mathrm{v}}(\mathbf{D}V)$, the operations $\mathrm{Sq}^k_{\mathrm{v,ext}}$ are obtained by applying π^p_{h} to the linear maps of §6.2:

$$\pi^q_{\mathrm{v}}(\mathbf{D}V) \xrightarrow{\mathrm{Sq}^k_{\mathrm{ext}}} \pi^{q+k}_{\mathrm{v}}(S_2\mathbf{D}V) \longrightarrow \pi^{q+k}_{\mathrm{v}}(\mathbf{D}S^2V).$$

On the other hand, the operation $\mathrm{Sq}^k_{\mathrm{h,ext}}$ equals the composite:

$$\pi^p_{\mathrm{h}}\pi^q_{\mathrm{v}}\mathbf{D}V \xrightarrow{\mathrm{Sq}^k_{\mathrm{ext}}} \pi^{p+k}_{\mathrm{h}}(S_2\pi^*_{\mathrm{v}}\mathbf{D}V)^{2q} \xrightarrow{\pi^{p+k}_{\mathrm{h}}(\mu_{\mathrm{ext}})} \pi^{p+k}_{\mathrm{h}}\pi^{2q}_{\mathrm{v}}S_2\mathbf{D}V \longrightarrow \pi^{p+k}_{\mathrm{h}}\pi^{2q}_{\mathrm{v}}\mathbf{D}S^2V,$$

and the pairing $\mu_{\mathrm{ext}} : S_2([E_2V]) \longrightarrow [E_2S^2V]$ equals:

$$S^2\pi^*_{\mathrm{h}}\pi^*_{\mathrm{v}}\mathbf{D}V \xrightarrow{\mu_{\mathrm{ext}}} \pi^*_{\mathrm{h}}(S_2\pi^*_{\mathrm{v}}\mathbf{D}V) \xrightarrow{\pi^*_{\mathrm{h}}(\mu_{\mathrm{ext}})} \pi^*_{\mathrm{h}}\pi^*_{\mathrm{v}}S_2\mathbf{D}V \longrightarrow \pi^*_{\mathrm{h}}\pi^*_{\mathrm{v}}\mathbf{D}S^2V.$$

These operations on E_2 determine the operations at each E_r, $r > 2$. The operations $\mathrm{Sq}^k_{\mathrm{ext}}$ commute with differentials as appropriate. Finally, the $\mathrm{Sq}^k_{\mathrm{ext}}$ stabilize to well defined maps on E_∞, and there is a commuting diagram

$$\begin{tikzcd}
{[E_\infty V]^{p,q}} \arrow[r, "\mathrm{Sq}^k_{\mathrm{ext}}"] \arrow[d, "\cong"] & {[E_\infty S^2V]^{p,q+k}} \arrow[d, "\cong"] \\
{[E_0H^*(\mathbf{D}(TV))]^{p,q}} \arrow[r, "\mathrm{Sq}^k_{\mathrm{ext}}"] & {[E_0H^*(\mathbf{D}(TS^2V))]^{p,q+k}}
\end{tikzcd}$$

whenever $0 \le k \le q$, and a commuting diagram

$$\begin{tikzcd}
{[E_\infty V]^{p,q}} \arrow[r, "\mathrm{Sq}^k_{\mathrm{ext}}"] \arrow[d, "\cong"] & {[E_\infty S^2V]^{p+k-q,2q}} \arrow[d, "\cong"] \\
{[E_0H^*(\mathbf{D}(TV))]^{p,q}} \arrow[r, "\mathrm{Sq}^k_{\mathrm{ext}}"] & {[E_0H^*(\mathbf{D}(TS^2V))]^{p+k-q,2q}}
\end{tikzcd}$$

whenever $q \le k$ (which summarizes also Singer's computation of how the $\mathrm{Sq}^k_{\mathrm{ext}}$ interact with the filtration on cohomology).

13.2. Application to composite functor spectral sequences

In order to use Singer's constructions in the present work, we will use the map of double complexes:

$$\psi_{\mathscr{G}} = j_{\mathcal{L}(n)} \circ Q^{\mathcal{L}(n)}\xi_{\mathscr{G}} : N_{p+1}N_q(Q^{\mathcal{L}(n)}B^{\mathscr{G}}L)^{t+1}_{s_n,\ldots,s_1} \longrightarrow N_pN_q(S^2(Q^{\mathcal{L}(n)}B^{\mathscr{G}}L))^{t}_{s_n,\ldots,s_1}$$

to define a spectral sequence map

$$[E_2S^2(Q^{\mathcal{L}(n)}B^{\mathscr{G}}L)]^{p,q,s_n,\ldots,s_1}_t \xrightarrow{(\psi_{\mathscr{G}})^*} [E_2^{\mathscr{G}}X]^{p+1,q,s_n,\ldots,s_1}_{t+1}.$$

We then define internal spectral sequence operations

$$\begin{aligned}
\mathrm{Sq}^k &= \psi^*_{\mathscr{G}} \circ \mathrm{Sq}^{k-1}_{\mathrm{ext}} : [E_r^{\mathscr{G}}X]^{p,q,s_n,\ldots,s_1}_t \longrightarrow [E_r^{\mathscr{G}}X]^{p+1,q+k-1,2s_n,\ldots,2s_1}_{2t+1} && (0 \le k-1 \le q),\\
\mathrm{Sq}^k &= \psi^*_{\mathscr{G}} \circ \mathrm{Sq}^{k-1}_{\mathrm{ext}} : [E_r^{\mathscr{G}}X]^{p,q,s_n,\ldots,s_1}_t \longrightarrow [E_{r+k-q-1}^{\mathscr{G}}X]^{p+k-q,2q,2s_n,\ldots,2s_1}_{2t+1} && (q \le k-1 \le q+r-2),\\
\mathrm{Sq}^k &= \psi^*_{\mathscr{G}} \circ \mathrm{Sq}^{k-1}_{\mathrm{ext}} : [E_r^{\mathscr{G}}X]^{p,q,s_n,\ldots,s_1}_t \longrightarrow [E_{2r-2}^{\mathscr{G}}X]^{p+k-q,2q,2s_n,\ldots,2s_1}_{2t+1} && (q+r-2 \le k-1).
\end{aligned}$$

which at E_2 we may write (dropping internal degrees) as:

$$\begin{aligned}
\mathrm{Sq}^k_{\mathrm{v}} &= \psi^*_{\mathscr{G}} \circ \mathrm{Sq}^{k-1}_{\mathrm{v,ext}} = \psi^*_{\mathscr{G}} \circ \mathrm{Sq}^{k-1}_{\mathrm{ext}} : E_2^{p,q} \longrightarrow E_2^{p+1,q+k-1} && \text{if } 0 \le k-1 \le q,\\
\mathrm{Sq}^k_{\mathrm{h}} &= \psi^*_{\mathscr{G}} \circ \mathrm{Sq}^{k-1}_{\mathrm{h,ext}} = \psi^*_{\mathscr{G}} \circ \mathrm{Sq}^{q+k-1}_{\mathrm{ext}} : E_2^{p,q} \longrightarrow E_2^{p+k,2q} && \text{if } 0 \le k-1 \le p.
\end{aligned}$$

Similarly, we define a pairing:

$$\mu = \psi^*_{\mathscr{G}} \circ \mu_{\mathrm{ext}} : [E_r^{\mathscr{G}}X]^{p,q,s_n,\ldots,s_1}_t \otimes [E_r^{\mathscr{G}}X]^{p',q',s'_n,\ldots,s'_1}_{t'} \longrightarrow [E_r^{\mathscr{G}}X]^{p+p'+1,q+q',s_n+s'_n,\ldots,s_1+s'_1}_{t+t'+1}$$

The reader might now guess the key results:

Theorem 13.1. *At $E_2 \cong H^*_{\mathcal{W}(n+1)}H^{\mathcal{U}(n)}_*X$, the operations $\mathrm{Sq}^k_{\mathrm{h}}$ and μ defined here are equal to the $\mathcal{M}_{\mathrm{h}}(n+2)$-operations of the same name defined on $\mathcal{W}(n+1)$-cohomology in §8.3.*

Theorem 13.2. *At $E_2 \cong H^*_{\mathcal{W}(n+1)}H^{\mathcal{U}(n)}_*X$, the operations $\mathrm{Sq}^k_{\mathrm{v}}$ defined here are equal to the $\mathcal{M}_{\mathrm{v}}(n+2)$-operations of the same name defined on $\mathcal{W}(n+1)$-cohomology in §8.2.*

Theorem 13.3. *At $E_\infty \cong [E_0H^*_{\mathcal{W}(n)}X]$, the operations Sq^k are compatible with the $\mathcal{M}_{\mathrm{h}}(n+1)$-operations of the same name defined on $\mathcal{W}(n)$-cohomology in §8.3.*

13.3. Proofs of Theorems 13.1-13.3

Proof of Theorem 13.1. This proof relies on a commuting diagram, in which we employ the notation $L = Q^{\mathcal{U}(n)} B^{\mathcal{W}(n)} X \in s\mathcal{L}(n)$, and abbreviate using $\mathcal{L} = \mathcal{L}(n)$ and $\mathcal{W} = \mathcal{W}(n+1)$.

$$\begin{tikzcd}
(N^{\mathrm{h}}_p \pi^{\mathrm{v}}_* Q^{\mathcal{L}} B^{\mathscr{G}} L)^{\otimes 2} & & (N^{\mathrm{h}}_p Q^{\mathcal{W}} \pi^{\mathrm{v}}_* B^{\mathscr{G}} L)^{\otimes 2} \arrow[ll, "N^{\mathrm{h}}_*(\gamma)^{\otimes 2}"'] \\
N^{\mathrm{h}}_{p+k-1}((\pi^{\mathrm{v}}_* Q^{\mathcal{L}} B^{\mathscr{G}} L)^{\otimes 2}) \arrow[u, "(D^{p-k+1}_{\mathrm{h}})^\star"] & & N^{\mathrm{h}}_{p+k-1}((Q^{\mathcal{W}} \pi^{\mathrm{v}}_* B^{\mathscr{G}} L)^{\otimes 2}) \arrow[u, "(D^{p-k+1}_{\mathrm{h}})^\star"'] \arrow[ll, "N^{\mathrm{h}}_*(\gamma^{\otimes 2})"'] \\
N^{\mathrm{h}}_{p+k-1} \pi^{\mathrm{v}}_*((Q^{\mathcal{L}} B^{\mathscr{G}} L)^{\otimes 2}) \arrow[u, "N^{\mathrm{h}}_*(D^0_{\mathrm{v}})^\star"] & & \\
N^{\mathrm{h}}_{p+k} \pi^{\mathrm{v}}_* Q^{\mathcal{L}} B^{\mathscr{G}} L \arrow[u, "\pi^{\mathrm{v}}_*(\psi_{\mathscr{G}})"] & & N^{\mathrm{h}}_{p+k} Q^{\mathcal{W}} \pi^{\mathrm{v}}_* B^{\mathscr{G}} L \arrow[ll, "N^{\mathrm{h}}_*(\gamma)"'] \arrow[uu, "\psi_{\mathcal{W}}"']
\end{tikzcd}$$

All of the horizontal maps are the isomorphisms of Lemma 3.1. By [52, Theorem 2.23] (summarized in §13.1), the left hand vertical composite is that used to define the horizontal operations $\mathrm{Sq}^k_{\mathrm{h}}$ on E_2. On the other hand, the right vertical was used in §8.3 to define the $\mathcal{M}_{\mathrm{h}}(n+2)$-operations on the $\mathcal{W}(n+1)$-cohomology groups with which the E_2-page can be identified. Thus, if the diagram commutes, we are done. If we replace the maps $(D^j_{\mathrm{h}})^\star$ in the top square with $(D^0_{\mathrm{h}})^\star$, the same proof applies for μ.

What remains is to prove that the bottom square commutes. It may be expanded into the eight maps in the outer square of the following larger commuting diagram:

$$\begin{tikzcd}
\pi^{\mathrm{v}}_*((Q^{\mathcal{L}} \mathscr{G}^{p+k} L)^{\otimes 2}) \arrow[r, "(D^0_{\mathrm{v}})^\star"'] & (\pi^{\mathrm{v}}_* Q^{\mathcal{L}} \mathscr{G}^{p+k} L)^{\otimes 2} & (Q^{\mathcal{W}} \pi^{\mathrm{v}}_* \mathscr{G}^{p+k} L)^{\otimes 2} \arrow[l, "\gamma^{\otimes 2}"] \\
\pi^{\mathrm{v}}_* Q^{\mathcal{L}}((\mathscr{G}^{p+k} L)^{\veebar 2}) \arrow[u, "\pi^{\mathrm{v}}_*(j_{\mathcal{L}})"] & Q^{\mathcal{W}} \pi^{\mathrm{v}}_*((\mathscr{G}^{p+k} L)^{\veebar 2}) \arrow[l, "\gamma"'] & Q^{\mathcal{W}}((\pi^{\mathrm{v}}_* \mathscr{G}^{p+k} L)^{\veebar 2}) \arrow[l, "Q^{\mathcal{W}}(i)"'] \arrow[u, "j_{\mathcal{W}}"'] \\
\pi^{\mathrm{v}}_* Q^{\mathcal{L}} \mathscr{G}^{p+k+1} L \arrow[u, "\pi^{\mathrm{v}}_* Q^{\mathcal{L}}(\xi_{\mathscr{G}})"] & Q^{\mathcal{W}} \pi^{\mathrm{v}}_* \mathscr{G}^{p+k+1} L \arrow[l, "\gamma"'] \arrow[u, "Q^{\mathcal{W}} \pi^{\mathrm{v}}_*(\xi_{\mathscr{G}})"'] & Q^{\mathcal{W}} \pi^{\mathrm{v}}_* \mathscr{G}^{p+k+1} L \arrow[l, "="'] \arrow[u, "Q^{\mathcal{W}}(\xi_{\mathcal{W}})"']
\end{tikzcd}$$

The bottom left square commutes by naturality of γ, while the bottom right square is an instance of Lemma 12.6. What remains is to check that the hexagon commutes. For notational convenience, write $A = \mathscr{G}^{p+k} L \in s\mathcal{L}$, $\mathrm{br} : A^{\otimes 2} \longrightarrow A^{\veebar 2}$ for the $\mathcal{L}$-bracket, and $\mathrm{br} : (\pi^{\mathrm{v}}_* A)^{\otimes 2} \longrightarrow (\pi^{\mathrm{v}}_* A)^{\veebar 2}$ for the $\mathcal{W}$-bracket on homotopy.

The source in the hexagon is then $Q^{\mathcal{W}}((\pi^{\mathrm{v}}_* A)^{\veebar 2})$, the smash product being the coproduct in $\mathcal{W} = \pi\mathcal{L}$ of two copies of $\pi^{\mathrm{v}}_* A$. Any element of $Q^{\mathcal{W}}((\pi^{\mathrm{v}}_* A)^{\veebar 2})$ can be represented by a sum $\sum_k \mathrm{br}(\overline{x_k} \otimes \overline{y_k}) + E$, with the x_k (resp. y_k) representatives of elements $\overline{x_k}$ in the first (resp. second) copy of $\pi^{\mathrm{v}}_* A$ and, E a sum of at least three-fold brackets of elements in the two copies. This extra term E is annihilated by both $j_{\mathcal{W}}$ and $\pi^{\mathrm{v}}_*(j) \circ \gamma \circ Q^{\mathcal{W}}(i)$, so can be ignored. One calculates:

$$(\gamma^{\otimes 2}\circ j_{\mathcal{W}})(\textstyle\sum_k \mathrm{br}(\overline{x_k}\otimes\overline{y_k})+E)=\sum_k \overline{x_k}\otimes\overline{y_k}.$$

On the other hand, the map $Q^{\mathcal{W}}(i)$ is induced by the Eilenberg-Mac Lane map shuffle map ∇_{v} as in Proposition 5.2, and

$$\textstyle\sum_k \mathrm{br}(\overline{x_k}\otimes\overline{y_k}) \overset{\gamma\circ Q^{\mathcal{W}}(i)}{\longmapsto} \overline{\sum_k \mathrm{br}(\nabla_{\mathrm{v}}(x_k\otimes y_k))} \overset{\pi^{\mathrm{v}}_*(j_{\mathcal{L}})}{\longmapsto} \overline{\sum_k(\nabla_{\mathrm{v}}(x_k\otimes y_k))} \overset{(D^0_{\mathrm{v}})^\star}{\longmapsto} \sum_k \overline{x_k}\otimes\overline{y_k}.$$

The last mapping follows from the fact that $(D^0_{\mathrm{v}})^\star\circ\nabla_{\mathrm{v}}=\mathrm{id}$, as $\{D^k\}$ is special. □

Proof of Theorem 13.2. We again employ the notation $L=Q^{\mathcal{U}(n)}B^{\mathcal{W}(n)}X\in s\mathcal{L}(n)$, and abbreviate using $\mathcal{L}=\mathcal{L}(n)$ and $\mathcal{W}=\mathcal{W}(n{+}1)$. Further, write $\mathbb{B}$ for the object $B^{\mathcal{W}}_m\pi^{\mathrm{v}}_*L\in s\mathcal{W}$. Write V_m for the subspace $(F^{\mathcal{W}})^m\subseteq\mathbb{B}$ of generators, and $V'_m:=V_m\cap N^{\mathrm{h}}_m\mathbb{B}$. For each $m\geq 0$, write $F_m\mathbb{B}$ for the m-skeleton of $\mathbb{B}$ (c.f. §2.6), which is almost free on subspaces $F_mV_m\subseteq V_m$.

We must identify the operations $\mathrm{Sq}^i_{\mathrm{v}}=\psi^*_{\mathscr{G}}\circ\mathrm{Sq}^{i-1}_{\mathrm{v,ext}}$ with the $\mathcal{W}$-cohomology operations $\mathrm{Sq}^i_{\mathrm{v}}$ defined in §8.2 using the maps θ^i. However, the θ^i are defined on the bar construction, while $\psi^*_{\mathscr{G}}$ is defined on the Blanc-Stover resolution. In order to make the comparison, we will need to choose a sufficiently explicit weak equivalence of resolutions of π^{v}_*L in $s\mathcal{W}$

$$\chi:\mathbb{B}\longrightarrow\pi^{\mathrm{v}}_*(B^{\mathscr{G}}L).$$

In order to define χ, we recursively define its restriction to the skeleta $F_m\mathbb{B}$. Lemma 2.3 implies that in order to extend a (horizontal simplicial) map $\chi_{m-1}:F_{m-1}\mathbb{B}\longrightarrow\pi^{\mathrm{v}}_*(B^{\mathscr{G}}L)$ to a map $\chi_m:F_m\mathbb{B}\longrightarrow\pi^{\mathrm{v}}_*(B^{\mathscr{G}}L)$, we need only to specify the values of χ_m on V'_m. That is, we only need to choose a lift in the diagram

$$\begin{tikzcd} V'_m \arrow[r, dashed, "\chi_m"] \arrow[d, "d^{\mathrm{h}}_0"] & N^{\mathrm{h}}_m\pi^{\mathrm{v}}_*(B^{\mathscr{G}}L) \arrow[d, "d^{\mathrm{h}}_0"] \\ ZN^{\mathrm{h}}_{m-1}\mathbb{B} \arrow[r, "\chi_{m-1}"] & ZN^{\mathrm{h}}_{m-1}\pi^{\mathrm{v}}_*(B^{\mathscr{G}}L) \end{tikzcd}$$

However, in order to actually carry out this process, we will need to record some chain level information, and we will construct maps into $N^{\mathrm{v}}_*B^{\mathscr{G}}L$, rather than just $\pi^{\mathrm{v}}_*B^{\mathscr{G}}L$.

It is best to view the domain and codomain of the proposed map χ as augmented (horizontal) simplicial objects, and start by defining χ_{-1} to be the identity of π^{v}_*L. Then for $m\geq 0$, we will recursively construct functions $\overline{\chi}_m:V'_m\longrightarrow ZN^{\mathrm{v}}_*B^{\mathscr{G}}_mL$, with the property that $\mathrm{im}(\overline{\chi}_m)$ is contained in the span of the classes z_w for $w\in ZN^{\mathrm{v}}_*B^{\mathscr{G}}_{m-1}L$, so that there is a commuting diagram:

$$\begin{tikzcd} V'_m \arrow[r, "\overline{\chi}_m"] \arrow[d, "\chi_m"'] & ZN^{\mathrm{v}}_* B^{\mathscr{G}}_m L \arrow[d, two heads] \\ N^{\mathrm{h}}_m \pi^{\mathrm{v}}_* B^{\mathscr{G}} L \arrow[r, tail] & \pi^{\mathrm{v}}_* B^{\mathscr{G}}_m L \end{tikzcd}$$

In order to do this, one may choose a basis of V'_m, and then for each basis element $v \in V'_m$, choose a $\mathcal{W}$-expression e for $d^{\mathrm{h}}_0 v$, so that

$$d^{\mathrm{h}}_0 v = e(s^{\mathrm{h}}_{\alpha_j} w_j) \in ZN^{\mathrm{h}}_{m-1}\mathbb{B} \text{ is a } \mathcal{W}\text{-expression in various } s^{\mathrm{h}}_{\alpha_j} w_j \in V_{m-1},$$

with $w_j \in V'_{n_j}$ for integers $n_j \leq m-1$ and degeneracy operators $s_{\alpha_j} : V'_{n_j} \longrightarrow V_{m-1}$. Then, from the cycles $s^{\mathrm{h}}_{\alpha_j} \overline{\chi}_{n_j}(w_j) \in ZN^{\mathrm{v}}_* B^{\mathscr{G}}_{m-1} L$, form a cycle

$$e^{\mathrm{rep}}(s^{\mathrm{h}}_{\alpha_j} \overline{\chi}_{n_j}(w_j)) \in ZN^{\mathrm{v}}_*(B^{\mathscr{G}}_{m-1} L),$$

using the explicit formulae of [20, §8] (which is a *normalized* cycle, as these formulae preserve the normalized subcomplex), so that

$$\begin{aligned} \overline{e^{\mathrm{rep}}(s^{\mathrm{h}}_{\alpha_j} \overline{\chi}_{n_j}(w_j))} &= e\left(\overline{s^{\mathrm{h}}_{\alpha_j} \overline{\chi}_{n_j}(w_j)}\right) \in \pi^{\mathrm{v}}_*(B^{\mathscr{G}}_{m-1} L) \\ &= e\left(s^{\mathrm{h}}_{\alpha_j} \chi_{n_j}(w_j)\right) \\ &= \chi_{m-1}(d^{\mathrm{h}}_0 v) \in ZN^{\mathrm{h}}_{m-1} \pi^{\mathrm{v}}_*(B^{\mathscr{G}} L). \end{aligned}$$

Our definition of $\overline{\chi}_m(v)$ is

$$\overline{\chi}_m(v) := z_{e^{\mathrm{rep}}(s^{\mathrm{h}}_{\alpha_j} \overline{\chi}_{n_j}(w_j))} \in ZN^{\mathrm{v}}_* B^{\mathscr{G}}_m L.$$

To check that the class of $\overline{\chi}_m(v)$ in $\pi^{\mathrm{v}}_*(B^{\mathscr{G}}_m L)$ is in fact in $N^{\mathrm{h}}_m \pi^{\mathrm{v}}_*(B^{\mathscr{G}} L)$, for $1 \leq i \leq m$ (c.f. [54, Lemma 2.7]):

$$d^{\mathrm{h}}_i \overline{\chi}_m(v) = z_{d^{\mathrm{h}}_{i-1} e^{\mathrm{rep}}(s^{\mathrm{h}}_{\alpha_j} \overline{\chi}_{n_j}(w_j))}, \text{ and } \overline{d^{\mathrm{h}}_{i-1} e^{\mathrm{rep}}(s^{\mathrm{h}}_{\alpha_j} \overline{\chi}_{n_j}(w_j))} = d^{\mathrm{h}}_{i-1} \chi_{m-1}(d^{\mathrm{h}}_0 v) = 0.$$

By construction of the comonad $\mathscr{G}$, $d^{\mathrm{h}}_i \overline{\chi}_m(v)$ must itself be null. Thus $\overline{\chi}_m$ does induce a map $\chi_m : V'_m \longrightarrow N^{\mathrm{h}}_m \pi^{\mathrm{v}}_*(B^{\mathscr{G}} L)$, completing the construction of χ.

Recall that the operations of §8.2 are the maps induced on cohomology by the degree -1 endomorphism θ^i of the chain complex $N^{\mathrm{h}}_*(Q^{\mathcal{W}} B^{\mathcal{W}} \pi^{\mathrm{v}}_* L)$:

$$\theta^i : N^{\mathrm{h}}_{p+1}(Q^{\mathcal{W}} B^{\mathcal{W}} \pi^{\mathrm{v}}_* L)^{2t+1}_{q+i-1,2s_n,\ldots,2s_1} \longrightarrow N^{\mathrm{h}}_p(Q^{\mathcal{W}} B^{\mathcal{W}} \pi^{\mathrm{v}}_* L)^t_{q,s_n,\ldots,s_1}.$$

If we write $V = Q^{\mathcal{L}} B^{\mathscr{G}} L$ for the double complex yielding the spectral sequence, our goal is to identify these operations with the spectral sequence operations

$$\psi_{\mathscr{G}}^* \circ \mathrm{Sq}^{i-1}_{\mathrm{v,ext}} : \Big([E_2 V]^{p,q} \xrightarrow{\mathrm{Sq}^{i-1}_{\mathrm{v,ext}}} [E_2 S^2 V]^{p,q+i-1} \xrightarrow{\psi^*_{\mathscr{G}}} [E_2 V]^{p+1,q+i-1} \Big)$$

using the equivalence $Q^{\mathcal{W}}\chi$ in $s\mathcal{V}$ induced by χ and the isomorphism γ:

$$Q^{\mathcal{W}}\chi : \Big(Q^{\mathcal{W}} B^{\mathcal{W}} \pi_*^{\mathrm{v}} L \xrightarrow{Q^{\mathcal{W}}\chi} Q^{\mathcal{W}} \pi_*^{\mathrm{v}}(B^{\mathscr{G}}) \xrightarrow{\gamma} \pi_*^{\mathrm{v}}(V) \Big).$$

The composite $\psi_{\mathscr{G}}^* \circ \mathrm{Sq}^{i-1}_{\mathrm{v,ext}}$ has been identified as the dual of the composite in the bottom row of

$$\begin{tikzcd}
N^{\mathrm{h}}_{p+1}(Q^{\mathcal{W}} B^{\mathcal{W}} \pi_*^{\mathrm{v}} L)_{q+i-1} \arrow[rr, "\theta^i"] \arrow[d, "Q^{\mathcal{W}}\chi"] & & N^{\mathrm{h}}_p(Q^{\mathcal{W}} B^{\mathcal{W}} \pi_*^{\mathrm{v}} L)_q \arrow[d, "Q^{\mathcal{W}}\chi"] \\
N^{\mathrm{h}}_{p+1}(\pi_*^{\mathrm{v}} V)_{q+i-1} \arrow[r, "\psi_{\mathscr{G}}"] & N^{\mathrm{h}}_p(\pi_*^{\mathrm{v}} S^2 V)_{q+i-1} \arrow[r, "(\mathrm{Sq}^{i-1}_{\mathrm{v,ext}})^\star"] & N^{\mathrm{h}}_p(\pi_*^{\mathrm{v}} V)_q
\end{tikzcd}$$

so that it is enough to prove that this diagram commutes for $1 \leq i \leq q$.

Given the equations in §6.3 defining the operations $(\mathrm{Sq}^{i-1}_{\mathrm{v,ext}})^\star$, it will suffice to show that the composite

$$(Q^{\mathcal{W}} B^{\mathcal{W}}_{p+1} \pi_*^{\mathrm{v}} L)_{q+i-1} \xrightarrow{Q^{\mathcal{W}}\chi} (\pi_*^{\mathrm{v}} Q^{\mathcal{L}} B^{\mathscr{G}}_{p+1} L)_{q+i-1} \xrightarrow{\psi_{\mathscr{G}}} (\pi_*^{\mathrm{v}} S^2 Q^{\mathcal{L}} B^{\mathscr{G}}_p L)_{q+i-1}$$

equals the sum of the composite

$$\begin{aligned} &(Q^{\mathcal{W}} B^{\mathcal{W}}_{p+1} \pi_*^{\mathrm{v}} L)_{q+i-1} \xrightarrow{\psi_{\mathcal{W}}} (S^2 Q^{\mathcal{W}} B^{\mathcal{W}}_p \pi_*^{\mathrm{v}} L)_{q+i-1} \\ &\qquad \xrightarrow{S^2(Q^{\mathcal{W}}\chi)} (S^2 \pi_*^{\mathrm{v}} Q^{\mathcal{L}} B^{\mathscr{G}}_p L)_{q+i-1} \xrightarrow{\widetilde{\nabla}} (\pi_*^{\mathrm{v}} S^2 Q^{\mathcal{L}} B^{\mathscr{G}}_p L)_{q+i-1} \end{aligned}$$

and those composites, for $1 \leq i \leq q$,

$$(Q^{\mathcal{W}} B^{\mathcal{W}}_{p+1} \pi_*^{\mathrm{v}} L)_{q+i-1} \xrightarrow{\theta^i} (Q^{\mathcal{W}} B^{\mathcal{W}}_p \pi_*^{\mathrm{v}} L)_q \xrightarrow{Q^{\mathcal{W}}\chi} (\pi_*^{\mathrm{v}} Q^{\mathcal{L}} B^{\mathscr{G}}_p L)_q \xrightarrow{\sigma_{i-1}} (\pi_*^{\mathrm{v}} S^2 Q^{\mathcal{L}} B^{\mathscr{G}}_p L)_{q+i-1},$$

that are actually defined (these fail to be defined when $i = 1$ in internal degrees satisfying
ero when $i = 1$'? $s_n = \cdots = s_1 = 0$).

By Lemma 2.5, we may represent any homology class of interest by an element $E = \sum_k v_k$, where the $v_k \in V'_{p+1}$ are elements of the basis chosen while defining χ. We wrote each v_k as a $\mathcal{W}$-expression e_k in various $u_{kj} \in V_p$:

$$v_k := e_k(u_{kj}) \in (V'_{p+1})_{q+i-1} \subseteq F^{\mathcal{W}} V_p,$$

so that $d_0^{\mathrm{h}} v = e_k(u_{kj})$, and defined $\chi(v_k)$ by the formula

$$\chi(v_k) = z_{e_k^{\mathrm{rep}} \chi(u_{kj})}.$$

That each $\chi(u_{kj})$ is a sum of the classes z_a implies that

$$\psi_{\mathscr{G}}(\chi(v_k)) = \mathrm{qu}_{\mathcal{L}}(e_k^{\mathrm{rep}})(\chi(u_{kj})).$$

Taking $\mathrm{qu}_{\mathcal{L}}(e_k^{\mathrm{rep}})$ extracts the part of e_k^{rep} corresponding to the quadratic grading 2 part of e_k, in $\mathrm{q}_2 F^{\mathcal{W}}$. That is, we may write $e_k \in F^{\mathcal{W}} V_p$ as

$$e_k = \mathrm{qu}_{\mathcal{W}}(e_k)(u_{kj}) + \textstyle\sum_{1 \leq i \leq q} \lambda_{i-1}(\theta^i e_k)(u_{kj}) + w \in F^{\mathcal{W}} V_p,$$

where $w \in F^{\mathcal{W}} V_p$ is the quadratic grading $\neq 2$ part of e_k, if we view $\mathrm{qu}_{\mathcal{W}}(e_k) \in S^2 V_p$ as an element of $F^{\mathcal{W}} V_p$ via the inclusion $F^{\mathcal{L}(n+1)} V_p \longrightarrow F^{\mathcal{W}(n+1)} V_p$, and then

$$\begin{aligned}
\psi_{\mathscr{G}}(\chi(v_k)) &= \mathrm{qu}_{\mathcal{L}} \left(\widetilde{\nabla}(\mathrm{qu}_{\mathcal{W}}(e_k))(\chi(u_{kj})) + \sum \sigma_{i-1}(\theta^i e_k)(\chi(u_{kj})) \right) \\
&= \widetilde{\nabla}(\mathrm{qu}_{\mathcal{W}}(e_k))(\chi(u_{kj})) + \sum \sigma_{i-1}(\theta^i e_k)(\chi(u_{kj})) \\
&= \left(\widetilde{\nabla} \circ S^2(Q^{\mathcal{W}} \chi) \circ \psi_{\mathcal{W}} + \sum \sigma_{i-1} \circ Q^{\mathcal{W}} \chi \circ \theta^i \right) (v_k).
\end{aligned}$$

We were able to discard the application of $\mathrm{qu}_{\mathcal{L}}$ as its argument already has quadratic grading 2. This formula is exactly what we needed to check in order to use the equations in §6.3. □

Proof of Theorem 13.3. Write $L = Q^{\mathcal{U}(n)} B^{\mathcal{W}(n)} X \in s\mathcal{L}(n)$, $\mathcal{L} = \mathcal{L}(n)$ and $\mathcal{W} = \mathcal{W}(n)$ (not $\mathcal{W}(n+1)$). We only need to show that the diagram of chain complexes

$$\begin{tikzcd}
T_m(Q^{\mathcal{L}} B^{\mathscr{G}} L) \arrow[r, "\psi_{\mathscr{G}}"] \arrow[d, "d_0^{\mathrm{h}}"', "\epsilon"] & T_{m-1}(S^2(Q^{\mathcal{L}} B^{\mathscr{G}} L)) \arrow[d, "d_0^{\mathrm{h}}"', "\epsilon"] \\
N_m(Q^{\mathcal{L}} L) \arrow[r, "\psi_{\mathcal{W}}"] & N_{m-1}(S^2(Q^{\mathcal{L}} L))
\end{tikzcd}$$

commutes up to homotopy (recall that $\psi_{\mathscr{G}}$ reduces filtration by one). The augmentation maps d_0^{h} are induced by the augmentation of $\mathscr{G}$:

$$\epsilon : \left(N_0^{\mathrm{h}} N_*^{\mathrm{v}} (Q^{\mathcal{L}} B^{\mathscr{G}} L) = N_*^{\mathrm{v}} Q^{\mathcal{L}} \mathscr{G} L \xrightarrow{\epsilon} N_*^{\mathrm{v}} Q^{\mathcal{L}} L \right).$$

We may understand $N_*^{\mathrm{v}} Q^{\mathcal{L}} \mathscr{G} L$ using the pushout square of chain complexes (obtained by applying $N_*^{\mathrm{v}} \circ Q^{\mathcal{L}}$ to that defining $\mathscr{G}$):

$$\begin{tikzcd}
\bigoplus_{y\in \mathrm{hg}(N_{\geq 1}L)} \mathbb{F}_2\{z_{dy}\} \arrow[r] \arrow[d] & \bigoplus_{x\in \mathrm{hg}(ZN_*L)} \mathbb{F}_2\{z_x\} \arrow[d] \\
\bigoplus_{y\in \mathrm{hg}(N_{\geq 1}L)} \mathbb{F}_2\{h_y, z_{dy}\} \arrow[r] & N_*^{\mathrm{v}} Q^{\mathcal{L}} \mathscr{G} L
\end{tikzcd}$$

which shows that $N_*^{\mathrm{v}} Q^{\mathcal{L}} \mathscr{G} L$ is the following complex (with differential $h_y \longmapsto z_{d_0^{\mathrm{v}} y}$):

$$N_*^{\mathrm{v}} Q^{\mathcal{L}} \mathscr{G} L = \bigoplus_{y\in \mathrm{hg}(N_{\geq 1}L)} \mathbb{F}_2\{h_y\} \oplus \bigoplus_{x\in \mathrm{hg}(ZN_*L)} \mathbb{F}_2\{z_x\}$$

We will use the notation

$$L_t := Q^{\mathfrak{U}} B_t^{\mathcal{W}} X \cong F^{\mathcal{L}}_{\langle -1\rangle} F^{\mathcal{W}}_{\langle 0\rangle} \cdots F^{\mathcal{W}}_{\langle t-1\rangle} X_t$$

so that we may write the basis elements of $N_m Q^{\mathcal{L}} \mathscr{G} Q^{\mathfrak{U}} B^{\mathcal{W}} X$ in the form

$$z_{f^{\langle -1\rangle}(g_{i_1}^{\langle 0\rangle}(h_{i_1 i_2}^{\langle 1\rangle}))} \text{ and } h_{f^{\langle -1\rangle}(g_{i_1}^{\langle 0\rangle}(h_{i_1 i_2}^{\langle 1\rangle}))},$$

where the $h_{i_1 i_2}$ are various elements of $F^{\mathcal{W}}_{\langle 1\rangle} \cdots F^{\mathcal{W}}_{\langle m-1\rangle} X_m$, each g_{i_1} is a $\mathcal{W}$-expression $g_{i_1}(h_{i_1 i_2})$ in certain of the $h_{i_1 i_2}$, and finally, f is some $\mathcal{L}$-expression in the various g_{i_1}. For brevity we will write $k_{f^{\langle -1\rangle}(g_{i_1}^{\langle 0\rangle}(h_{i_1 i_2}^{\langle 1\rangle}))}$ for either of $z_{f^{\langle -1\rangle}(g_{i_1}^{\langle 0\rangle}(h_{i_1 i_2}^{\langle 1\rangle}))}$ and $h_{f^{\langle -1\rangle}(g_{i_1}^{\langle 0\rangle}(h_{i_1 i_2}^{\langle 1\rangle}))}$.

A chain homotopy $\Phi : T_m(Q^{\mathcal{L}} B^{\mathscr{G}} L) \longrightarrow N_m(S^2(Q^{\mathcal{W}} B^{\mathcal{W}} X))$ is constructed as follows. Let Φ be zero except on $N_0^{\mathrm{h}} N_m^{\mathrm{v}}(Q^{\mathcal{L}} B^{\mathscr{G}} L) = N_m^{\mathrm{v}} Q^{\mathcal{L}} \mathscr{G} Q^{\mathfrak{U}} B^{\mathcal{W}} X$, where it is defined by

$$k_{f^{\langle -1\rangle}(g_{i_1}^{\langle 0\rangle}(h_{i_1 i_2}^{\langle 1\rangle}))} \longmapsto \mathrm{qu}_{\mathcal{L}}(f)(g_{i_1}^{\langle 0\rangle}(h_{i_1 i_2}^{\langle 1\rangle})).$$

This definition makes sense (and yields a non-trivial map) because f is an operator in $Q^{\mathfrak{U}} F^{\mathcal{W}} = F^{\mathcal{L}}$. The chain map $d\Phi + \Phi d$ is a sum of three terms:

(a) $d \circ \Phi : N_0^{\mathrm{h}} N_m^{\mathrm{v}}(Q^{\mathcal{L}} B^{\mathscr{G}} L) \xrightarrow{\Phi} N_m(S^2(Q^{\mathcal{L}} L)) \xrightarrow{d} N_{m-1}(S^2(Q^{\mathcal{L}} L))$

(b) $\Phi \circ d^{\mathrm{v}} : N_0^{\mathrm{h}} N_m^{\mathrm{v}}(Q^{\mathcal{L}} B^{\mathscr{G}} L) \xrightarrow{d^{\mathrm{v}}} N_0^{\mathrm{h}} N_{m-1}^{\mathrm{v}}(Q^{\mathcal{L}} B^{\mathscr{G}} L) \xrightarrow{\Phi} N_{m-1}(S^2(Q^{\mathcal{L}} L))$

(c) $\Phi \circ d^{\mathrm{h}} : N_1^{\mathrm{h}} N_{m-1}^{\mathrm{v}}(Q^{\mathcal{L}} B^{\mathscr{G}} L) \xrightarrow{d^{\mathrm{h}}} N_0^{\mathrm{h}} N_{m-1}^{\mathrm{v}}(Q^{\mathcal{L}} B^{\mathscr{G}} L) \xrightarrow{\Phi} N_{m-1}(S^2(Q^{\mathcal{L}} L))$

We calculate

$$\begin{aligned}
(d \circ \Phi)(k_{f(g_{i_1}^{\langle 0\rangle}(h_{i_1 i_2}^{\langle 1\rangle}))}) &= d(\mathrm{qu}_{\mathcal{L}}(f)(g_{i_1}^{\langle 0\rangle}(h_{i_1 i_2}^{\langle 1\rangle}))) \\
&= \mathrm{qu}_{\mathcal{L}}(f)(g_{i_1}(h_{i_1 i_2}^{\langle 0\rangle})) \\
&= \mathrm{qu}_{\mathcal{L}}(f)(\epsilon(g_{i_1})(h_{i_1 i_2}^{\langle 0\rangle})),
\end{aligned}$$

(the last equation holds as we calculate in $S^2(Q^{\mathcal{L}}L)$), and

$$(\Phi \circ d^{\mathrm{v}})(k_{f(g^{\langle 0\rangle}_{i_1}(h^{\langle 1\rangle}_{i_1 i_2}))}) = \begin{cases} \Phi(k_{f(g_{i_1})(h^{\langle 0\rangle}_{i_1 i_2})}), & \text{if `$k$' stands for `h'},\\ \Phi(0), & \text{if `$k$' stands for `z'}, \end{cases}$$
$$= \mathrm{qu}_{\mathcal{L}}(f(g_{i_1}))(h^{\langle 0\rangle}_{i_1 i_2}) \quad \text{(in either case).}$$

By the equation of §3.10, the sum of these two terms is $\mathrm{qu}_{\mathcal{L}}(\epsilon f(g_{i_1}))(h^{\langle 0\rangle}_{i_1 i_2})$, which is exactly the formula for $(\psi_{\mathcal{W}} \circ \epsilon)(k_{f(g^{\langle 0\rangle}_{i_1}(h^{\langle 1\rangle}_{i_1 i_2}))})$.

It remains to show that $\Phi \circ d^{\mathrm{h}}$ coincides with $\epsilon^{\otimes 2} \circ \psi_{\mathcal{W}}$. These two maps are only non-zero on the graded part $N^{\mathrm{h}}_1 N^{\mathrm{v}}_{m-1}(Q^{\mathcal{L}} B^{\mathscr{G}} L) \subseteq Q^{\mathcal{L}} \mathscr{G}^2 L$ of $T_m(Q^{\mathcal{L}} B^{\mathscr{G}} L)$, and an element therein is a linear combination

$$K := \textstyle\sum_j k_{e_j\left(s_{\alpha_{ji_0}} k_{f_{ji_0} g^{\langle 0\rangle}_{ji_0 i_1} h^{\langle 1\rangle}_{ji_0 i_1 i_2}}\right)}$$

which satisfies the equation $d^{\mathrm{h}}_1(K) = 0$, i.e.:

$$d^{\mathrm{h}}_1(K) = \textstyle\sum_j k_{e_j(f_{ji_0})\left(s_{\alpha_{ji_0}} g^{\langle 0\rangle}_{ji_0 i_1} h^{\langle 1\rangle}_{ji_0 i_1 i_2}\right)} = 0 \text{ in } N_{m-1} Q^{\mathcal{L}} \mathscr{G} L.$$

There is a map

$$N_{m-1} Q^{\mathcal{L}} \mathscr{G} L \subseteq \mathbb{F}_2\{\mathrm{hg}(N_{m-1}L)\} \oplus \mathbb{F}_2\{\mathrm{hg}(Z N_{m-1} L)\} \longrightarrow N_{m-1} S^2 Q^{\mathcal{L}} L$$

defined on generators using the function

$$N_{m-1} L \subseteq F^{\mathcal{L}}(F^{\mathcal{W}})^{m-1} X \xrightarrow{\mathrm{qu}_{\mathcal{L}}} S^2 (F^{\mathcal{W}})^{m-1} X \cong S^2 Q^{\mathcal{L}} L.$$

This map sends $d^{\mathrm{h}}_1(K) = 0$ to

$$\textstyle\sum_j \mathrm{qu}_{\mathcal{L}}(e_j(f_{ji_0}))\left(s_{\alpha_{ji_0}} g^{\langle 0\rangle}_{ji_0 i_1} h^{\langle 1\rangle}_{ji_0 i_1 i_2}\right) = 0,$$

which by the equation of §3.10, gives an equation in $N_{m-1} S^2 Q^{\mathcal{L}} L$:

$$\textstyle\sum_j \mathrm{qu}_{\mathcal{L}}(e_j(\epsilon(f_{ji_0})))\left(s_{\alpha_{ji_0}} g^{\langle 0\rangle}_{ji_0 i_1} h^{\langle 1\rangle}_{ji_0 i_1 i_2}\right) = \textstyle\sum_j \mathrm{qu}_{\mathcal{L}}(\epsilon(e_j)(f_{ji_0}))\left(s_{\alpha_{ji_0}} g^{\langle 0\rangle}_{ji_0 i_1} h^{\langle 1\rangle}_{ji_0 i_1 i_2}\right).$$

The proof is completed upon noting that the left hand side of this equation equals $(\epsilon^{\otimes 2} \circ$

$\psi_{\mathscr{G}})(K)$, while the right hand side equals $(\Phi \circ d^{\mathrm{h}})(K)$. We calculate:

$$\begin{aligned}
(\epsilon^{\otimes 2} \circ \psi_{\mathscr{G}})(K) &= \epsilon^{\otimes 2} \left(\textstyle\sum_j \mathrm{qu}_{\mathcal{L}}(e_j) \left(s_{\alpha_{ji_0}} k_{f_{ji_0} g^{\langle 0 \rangle}_{ji_0 i_1} h^{\langle 1 \rangle}_{ji_0 i_1 i_2}} \right) \right) \\
&= \textstyle\sum_j \mathrm{qu}_{\mathcal{L}}(e_j) \left(s_{\alpha_{ji_0}} \epsilon(f_{ji_0}) g^{\langle 0 \rangle}_{ji_0 i_1} h^{\langle 1 \rangle}_{ji_0 i_1 i_2} \right) \\
&= \textstyle\sum_j \mathrm{qu}_{\mathcal{L}}(e_j(\epsilon(f_{ji_0}))) \left(s_{\alpha_{ji_0}} g^{\langle 0 \rangle}_{ji_0 i_1} h^{\langle 1 \rangle}_{ji_0 i_1 i_2} \right) = \mathrm{LHS},
\end{aligned}$$

and

$$\begin{aligned}
d^{\mathrm{h}}(K) &= \textstyle\sum_j e_j \left(s_{\alpha_{ji_0}} k_{f_{ji_0} g^{\langle 0 \rangle}_{ji_0 i_1} h^{\langle 1 \rangle}_{ji_0 i_1 i_2}} \right) \\
&= \textstyle\sum_j \epsilon(e_j) \left(s_{\alpha_{ji_0}} k_{f_{ji_0} g^{\langle 0 \rangle}_{ji_0 i_1} h^{\langle 1 \rangle}_{ji_0 i_1 i_2}} \right) && (\text{in } N^{\mathrm{v}}_{m-1} Q^{\mathcal{L}} \mathscr{G} L) \\
\Phi(d^{\mathrm{h}}(K)) &= \textstyle\sum_j \epsilon(e_j) \left(\mathrm{qu}_{\mathcal{L}}(f_{ji_0})(s_{\alpha_{ji_0}} g^{\langle 0 \rangle}_{ji_0 i_1} h^{\langle 1 \rangle}_{ji_0 i_1 i_2}) \right) && (\text{relevant } s_{\alpha_{ji_0}} \text{ are id}) \\
&= \textstyle\sum_j \mathrm{qu}_{\mathcal{L}}(\epsilon(e_j)(f_{ji_0})) \left(s_{\alpha_{ji_0}} g^{\langle 0 \rangle}_{ji_0 i_1} h^{\langle 1 \rangle}_{ji_0 i_1 i_2} \right) = \mathrm{RHS}
\end{aligned}$$

To explain further the third equation, note that $N^{\mathrm{v}}_{m-1} Q^{\mathcal{L}} \mathscr{G} L$ is spanned by the classes $k_{\dots}$, and none of their degeneracies. Thus, all of the degeneracies $s_{\alpha_{ji_0}}$ appearing in the second line that have not already been annihilated during the application of ϵ must be the identity. Thus, they can be carried harmlessly through to the end of the calculation, as shown. $\square$

Chapter 14

Calculations of $\mathcal{W}(n)$-cohomology and the BKSS E_2-page

In this section, we will calculate the value of $H^*_{\mathcal{W}(n)}X$ for certain objects X of $\mathcal{W}(n)$ of finite type. In each subsection, we will write $V_{(n)} = \mathbf{D}X$, so that X has underlying vector space dual to $V_{(n)} \in \mathcal{V}^n_+$. In fact, we will reinstate the upper asterisk for linear dualization, writing $\mathbf{D}V_{(n)} := V^*_{(n)}$, and recursively define:

$$V_{(k+1)} := H^*_{\mathcal{U}(k)} V^*_{(k)} \text{ and } V^*_{(k+1)} := H^{\mathcal{U}(k)}_* V^*_{(k)} \text{ for } k \geq n.$$

In this way, for each $k \geq n$, $V^*_{(k+1)}$ is an object of $\mathcal{W}(k+1)$, vector space dual to $V_{(k+1)} \in \mathcal{V}^{k+1}_+$, which itself has the structure of an object of $\mathcal{M}_{\mathrm{v}}(k+1)$. Having all of this data will allow us to draw conclusions about $H^*_{\mathcal{W}(n)} V^*_{(n)}$, using, for each $k \geq n$, the $(k+1)^{\mathrm{st}}$ composite functor spectral sequence:

$$[E_2^{(k+1)}]_t^{s_{k+2},\dots,s_1} := (H^*_{\mathcal{W}(k+1)} V^*_{(k+1)})_t^{s_{k+2},\dots,s_1} \implies (H^*_{\mathcal{W}(k)} V^*_{(k)})_t^{s_{k+2}+s_{k+1},s_k,\dots,s_1}.$$

The first CFSS, which calculates $H^*_{\mathcal{W}(0)}$ from $H^*_{\mathcal{W}(1)}$, will appear in §14.5.

14.1. When $X \in \mathcal{W}(n)$ is one-dimensional and $n \geq 1$

Let $X = V^*_{(n)} \in \mathcal{W}(n)$ be a one dimensional object of $\mathcal{W}(n)$, dual to a one-dimensional vector space $V_{(n)} \in \mathcal{V}^n_+$, with non-zero element $v \in (V_{(n)})_T^{S_n,\dots,S_1}$. Write $v^* \in X^T_{S_n,\dots,S_1}$ for the non-zero element of X. As every $\mathcal{W}(n)$-operation changes degrees, X is necessarily trivial. We distinguish two cases: when v is *restrictable* and when v is *not restrictable*. Recall that v is said to be restrictable when $v^{[2]}$ is defined, i.e. when $S_n, \dots, S_1$ are not all zero.

Proposition 14.1. *For each $k \geq n$,*

$$V_{(k)} = F^{\mathcal{M}_{\mathrm{v}}(k)} F^{\mathcal{M}_{\mathrm{v}}(k-1)} \cdots F^{\mathcal{M}_{\mathrm{v}}(n+1)} V_{(n)},$$

and $V^*_{(k)}$ is a trivial object of $\mathcal{W}(k)$.

Proof. The proof is by induction, with the case $k = n$ simply our standing assumptions. If the statement holds for $V_{(k)}$, then Proposition 9.4 shows that the Koszul complex calculating $V_{(k+1)}$ has zero differentials, as $V_{(k)}$ has trivial $\mathcal{W}(k)$-structure, so that $V_{(k+1)} = F^{\mathcal{M}_{\mathrm{v}}(k+1)} V_{(k)}$. This has trivial $\mathcal{W}(k+1)$-structure, by the results of §9.2. $\square$

Our next step is to calculate, for $k \geq n$, the groups:

$$[E_2^{(k+1)}]_t^{s_{k+2},0,s_k,\dots,s_1} := (H^*_{\mathcal{W}(k+1)} V^*_{(k+1)})_t^{s_{k+2},0,s_k,\dots,s_1} \cong (H^{s_{k+2}}_{\mathcal{L}(k)} V^*_{(k)})_t^{s_k,\dots,s_1}.$$

The isomorphism shown here follows from the observation that in dimension $s_{k+1} = 0$, an object of $\mathcal{W}(k+1)$ is nothing more than an object of $\mathcal{L}(k)$. More precisely, consider the functor $-_{\mathbf{0}} : \mathcal{V}^+_{k+1} \longrightarrow \mathcal{V}^+_k$ given by

$$(Y_{\mathbf{0}})^t_{s_k,\dots,s_1} := Y^t_{0,s_k,\dots,s_1}.$$

Then $-_{\mathbf{0}}$ induces a functor $-_{\mathbf{0}} : \mathcal{W}(k+1) \longrightarrow \mathcal{L}(k)$, such that, for all $Y \in \mathcal{W}(k+1)$:

$$(F^{\mathcal{W}(k+1)}(Y))_{\mathbf{0}} \cong F^{\mathcal{L}(k)}(Y_{\mathbf{0}}) \text{ and } (Q^{\mathcal{W}(k+1)} Y)_{\mathbf{0}} \cong Q^{\mathcal{L}(k)}(Y_{\mathbf{0}}),$$

so that $(Q^{\mathcal{W}(k+1)} B^{\mathcal{W}(k+1)} Y)_{\mathbf{0}} \cong (Q^{\mathcal{L}(k)} B^{\mathcal{L}(k)} Y_{\mathbf{0}})$ for any $Y \in s\mathcal{W}(k+1)$, and thus:

Proposition 14.2. *Suppose that $Y \in s\mathcal{W}(k+1)$, where $k \geq 0$. Then*

$$(H^*_{\mathcal{W}(k+1)} Y)_t^{s_{k+2},0,s_k,\dots,s_1} \cong (H^{s_{k+2}}_{\mathcal{L}(k)} Y_{\mathbf{0}})_t^{s_k,\dots,s_1}.$$

Returning to the calculation at hand, we may identify a part of the E_2-page with the Chevalley-Eilenberg-May complex of Appendix A.3:

Proposition 14.3. *For each $k \geq n$, there is an isomorphism of commutative algebras:*

$$[E_2^{(k+1)}]_t^{s_{k+2},0,s_k,\dots,s_1} \cong (\mathbf{D}\bar{X}'(V^*_{(k)}))_t^{s_{k+2},s_k,\dots,s_1}.$$

When $v \in V_{(n)}$ is restrictable, $\mathbf{D}\bar{X}'(V^*_{(k)}) = S(\mathscr{C})[V_{(k)}]$, the free non-unital commutative algebra. When $v \in V_{(n)}$ is not restrictable, $V_{(k)} = \mathbb{F}_2\{v\}$ is one-dimensional, and $\mathbf{D}\bar{X}'(V^*_{(k)})$ is the one-dimensional exterior algebra $\Lambda(\mathscr{C})[v]$. In either case, for each individual value of *the grading t, the group*

$$\bigoplus_{s_{k+2},s_k,\dots,s_1} [E_2^{(k+1)}]_t^{s_{k+2},0,s_k,\dots,s_1}$$

is finite-dimensional.

Proof. The only further observation necessary to prove this isomorphism is that if $v \in V_{(n)}$ is restrictable, every element of the trivial partially restricted Lie algebra $V_{(k)}$ is in restrictable degree, and that if $v \in V_{(n)}$ is not restrictable, each $V_{(k)}$ is one-dimensional, concentrated in non-restrictable degree. For the finiteness property, one simply notes that the $V_{(k)}$ have such a property, and that there is a degree shift in the algebra structure. $\square$

Consider the diagram:

$$\begin{tikzcd}
{} & H^*_{\mathcal{W}(n+1)}H^{\mathcal{U}(n)}_*V^*_{(n)} \arrow[dl, Rightarrow, "g_{n+1}"] \arrow[d, equal] & H^*_{\mathcal{W}(n+2)}H^{\mathcal{U}(n+1)}_*V^*_{(n+1)} \arrow[dl, Rightarrow, "g_{n+2}"] \arrow[d, equal] & \\
H^*_{\mathcal{W}(n)}V^*_{(n)} & H^*_{\mathcal{W}(n+1)}V^*_{(n+1)} & H^*_{\mathcal{W}(n+2)}V^*_{(n+2)} & \cdots \\
F^{\mathcal{M}_{\mathrm{hv}}(n+1)}V_{(n)} \arrow[u, "\rho_n"] \arrow[d, equal] & F^{\mathcal{M}_{\mathrm{hv}}(n+2)}V_{(n+1)} \arrow[u, "\rho_{n+1}"] \arrow[d, equal] \arrow[dl, Rightarrow, "f_{n+1}"] & F^{\mathcal{M}_{\mathrm{hv}}(n+3)}V_{(n+2)} \arrow[u, "\rho_{n+2}"] \arrow[d, equal] \arrow[dl, Rightarrow, "f_{n+2}"] & \\
F^{\mathcal{M}_{\mathrm{h}}(n+1)}V_{(n+1)} & F^{\mathcal{M}_{\mathrm{h}}(n+2)}V_{(n+2)} & F^{\mathcal{M}_{\mathrm{h}}(n+3)}V_{(n+3)} &
\end{tikzcd}$$

For each $k \geq n$, the map ρ_k is induced by the inclusion $V_{(k)} \cong H^0_{\mathcal{W}(k)}V^*_{(k)} \subseteq H^*_{\mathcal{W}(k)}V^*_{(k)}$ (which exists as $V_{(k)}$ is trivial) and the $F^{\mathcal{M}_{\mathrm{hv}}(k+1)}$-operations defined on $H^*_{\mathcal{W}(k)}V_{(k)}$. (Note that ρ_k is a graded map, since the effect of these operations on dimensions is the same in its domain and codomain.)

The double arrow g_{k+1}, representing the convergence of the $(k+1)^{\text{st}}$ CFSS $[E_2^{(k+1)}] \implies H^*_{\mathcal{W}(k)}V^*_{(k)}$, is in truth shorthand for the function

$$[E_\infty^{(k+1)}]_t^{s_{k+2},\dots,s_1} \longrightarrow [E_0 H^*_{\mathcal{W}(k)}V^*_{(k)}]_t^{s_{k+2},\dots,s_1}$$

so that g_{k+1} may only be defined on the permanent cycles within $[E_2^{(k+1)}]$, and lands in the associated graded of $H^*_{\mathcal{W}(k)}V^*_{(k)}$.

Similarly, we employ the double arrow f_{k+1} as shorthand for the function of Theorem 8.15, which is defined on the entirety of $F^{\mathcal{M}_{\mathrm{h}}(k+2)}F^{\mathcal{M}_{\mathrm{v}}(k+2)}V_{(k+1)}$, but whose true codomain is the graded object $E_0(F^{\mathcal{M}_{\mathrm{h}}(k+1)}V_{(k+1)})$ associated with the target filtration defined in Theorem 8.15.

Theorem 14.4. *For each $k \geq n$, $\mathrm{im}(\rho_{k+1})$ consists of permanent cycles and ρ_k preserves the target filtrations, so that it is possible to form the composites $g_{k+1} \circ \rho_{k+1}$ and $E_0(\rho_k) \circ f_{k+1}$. These composites are equal, and moreover, ρ_k is an isomorphism. In particular, for $k \geq n$, the $(k+1)^{\text{st}}$ CFSS collapses at E_2.*

Before giving the proof, we remark that in some dimensions, ρ_k is already known to be an isomorphism:

Proposition 14.5. *For $k \geq n$, ρ_k is an isomorphism in dimension $s_k = 0$:*

$$\rho_k : (F^{\mathcal{M}_{\mathrm{h}}(k+1)}F^{\mathcal{M}_{\mathrm{v}}(k+1)}V_{(k)})_t^{s_{k+1},0,s_{k-1},\ldots,s_1} \xrightarrow{\cong} (H^*_{\mathcal{W}(k)}V^*_{(k)})_t^{s_{k+1},0,s_{k-1},\ldots,s_1}.$$

Proof. In this dimension, ρ_k factors as

$$\begin{aligned}
(F^{\mathcal{M}_{\mathrm{h}}(k+1)}F^{\mathcal{M}_{\mathrm{v}}(k+1)}V_{(k)})_t^{s_{k+1},0,s_{k-1},\ldots,s_1} &= (F^{\mathcal{M}_{\mathrm{h}}(k+1)}F^{\mathcal{M}_{\mathrm{v}}(k+1)}V^{\mathbf{0}}_{(k)})_t^{s_{k+1},0,s_{k-1},\ldots,s_1} \\
&= (F^{\mathcal{M}_{\mathrm{h}}(k+1)}V^{\mathbf{0}}_{(k)})_t^{s_{k+1},0,s_{k-1},\ldots,s_1} \\
&\cong (\mathbf{D}\bar{X}'((V^*_{(k)})_{\mathbf{0}}))_t^{s_{k+1},0,s_{k-1},\ldots,s_1} \\
&\cong (H^*_{\mathcal{W}(k)}V^*_{(k)})_t^{s_{k+1},0,s_{k-1},\ldots,s_1}
\end{aligned}$$

Here, we are viewing $V^{\mathbf{0}}_{(k)}$, the subspace of $V_{(k)}$ in degree $s_k = 0$, as an object of $\mathcal{V}^{k+1}_+$ in order to apply $F^{\mathcal{M}_{\mathrm{v}}(k+1)}$. The inclusion $F^{\mathcal{M}_{\mathrm{h}}(k+1)}F^{\mathcal{M}_{\mathrm{v}}(k+1)}V^{\mathbf{0}}_{(k)} \subseteq F^{\mathcal{M}_{\mathrm{h}}(k+1)}F^{\mathcal{M}_{\mathrm{v}}(k+1)}V_{(k)}$ restricts to the identity in degree $s_k = 0$, explaining the first equation. The second equation is similar: any non-trivial $\mathcal{M}_{\mathrm{v}}(k+1)$-operation lands outside degree $s_k = 0$. The first isomorphism follows from Corollary 8.11, which ensures that $F^{\mathcal{M}_{\mathrm{h}}(k+1)}V^{\mathbf{0}}_{(k)}$ is a quotient of the polynomial algebra on $V^{\mathbf{0}}_{(k)}$, and indeed, the same quotient as $\mathbf{D}\bar{X}'(\mathbf{D}(V^{\mathbf{0}}_{(k)}))$. The second isomorphism is Proposition 14.3, since $(V^*_{(k)})_{\mathbf{0}} = V^*_{(k-1)}$. $\square$

Proof of Theorem 14.4. For each $k \geq n$, we will use the diagram

$$\begin{tikzcd}
H^*_{\mathcal{W}(k)}V^*_{(k)} & H^*_{\mathcal{W}(k+1)}H_*^{\mathcal{U}(k)}V^*_{(k)} \arrow[l, Leftarrow, "g_{k+1}"'] & H^*_{\mathcal{U}(k)}V^*_{(k)} \arrow[l, dotted, "i_1"] \arrow[ll, dotted, <->, bend right, "\text{edge composite}"'] & V_{(k)} \arrow[l, dotted, "i_2"'] \arrow[lll, dotted, bend right, "i_0"'] \arrow[dl, dotted, "j_2"] \\
F^{\mathcal{M}_{\mathrm{h}}(k+1)}F^{\mathcal{M}_{\mathrm{v}}(k+1)}V_{(k)} \arrow[u, "\rho_k"] & W(F^{\mathcal{M}_{\mathrm{v}}(k+1)}V_{(k)}) \arrow[l, ->>, "\overline{f}_{k+1}"'] \arrow[u, "\rho_{k+1}"'] & F^{\mathcal{M}_{\mathrm{v}}(k+1)}V_{(k)} \arrow[l, dotted, "j_1"'] \arrow[u, equal] &
\end{tikzcd}$$

where $W(F^{\mathcal{M}_{\mathrm{v}}(k+1)}V_{(k)})$ is the object introduced in the proof of Theorem 8.15, so that there is a quotient map

$$W(F^{\mathcal{M}_{\mathrm{v}}(k+1)}V_{(k)}) \twoheadrightarrow F^{\mathcal{M}_{\mathrm{h}}(k+2)}F^{\mathcal{M}_{\mathrm{v}}(k+2)}F^{\mathcal{M}_{\mathrm{v}}(k+1)}V_{(k)}.$$

Here, the maps j_1, j_2 are the evident inclusions of generators, while the maps i_0, i_1, i_2 are the inclusions arising because $V_{(k)}$ is trivial.

We may define

$$c := (\rho_k \circ \overline{f}_{k+1} \circ j_1) : F^{\mathcal{M}_{\mathrm{v}}(k+1)}V_{(k)} \longrightarrow H^*_{\mathcal{W}(k)}V^*_{(k)},$$

without the need to pass to any associated graded objects. By construction of $\overline{f}_{k+1}$, c is induced by the inclusion i_0 and the $\mathcal{M}_{\mathrm{v}}(k+1)$-structure of $H^*_{\mathcal{W}(k)}V^*_{(k)}$.

The edge composite is the composite of a surjection, a monomorphism m_1, and an isomorphism m_2 (with inverse i_1):

$$H^*_{\mathcal{W}(k)}V^*_{(k)} \twoheadrightarrow [E_0 H^*_{\mathcal{W}(k)}V^*_{(k)}]^{\mathbf{0}} \cong [E_\infty^{(k+1)}]^{\mathbf{0}} \xrightarrow{m_1} [E_2^{(k+1)}]^{\mathbf{0}} \xrightarrow{m_2} H^*_{\mathcal{U}(k)}V^*_{(k)}.$$

Moreover, c is a section of the edge composite, since both maps are compatible with $\mathcal{M}_{\mathrm{v}}(k+1)$-structures (Proposition 12.9), and their composite is the identity on $V_{(k)} \subseteq F^{\mathcal{M}_{\mathrm{v}}(k+1)}V_{(k)}$. In particular, the edge composite is a surjection, so that m_1 is an isomorphism. That is, every class in $\mathrm{im}(i_1)$ is a permanent cycle. Singer's work (c.f. §13.1) then shows that $\mathrm{im}(\rho_{k+1})$ consists of permanent cycles, as permanent cycles are preserved by the $F^{\mathcal{M}_{\mathrm{hv}}(k+2)}$-operations on $[E_2^{(k+1)}]$.

Any section of $H^*_{\mathcal{W}(k)}V^*_{(k)} \twoheadrightarrow [E_\infty^{(k+1)}]^{\mathbf{0}} \cong [E_2^{(k+1)}]^{\mathbf{0}}$ will realize, up to filtration, the restriction of g_{k+1} to $[E_\infty^{(k+1)}]^{\mathbf{0}} \subset [E_\infty^{(k+1)}]$, so we choose

$$E^{0,*}_{2,(k+1)} \xrightarrow{m_2} H^*_{\mathcal{U}(k)}V^*_{(k)} \cong F^{\mathcal{M}_{\mathrm{v}}(k+1)}V_{(k)} \xrightarrow{c} H^*_{\mathcal{W}(k)}V^*_{(k)}.$$

In particular, $g_{k+1} \circ \rho_{k+1} \circ j_1 = g_{k+1} \circ i_1 = c \circ m_2 \circ i_1 = c$, up to filtration. More precisely, $g_{k+1} \circ \rho_{k+1} \circ j_1$ equals the composite

$$F^{\mathcal{M}_{\mathrm{v}}(k+1)}V_{(k)} \xrightarrow{c} H^*_{\mathcal{W}(k)}V^*_{(k)} \twoheadrightarrow [E_0 H^*_{\mathcal{W}(k)}V^*_{(k)}]^{\mathbf{0}}.$$

Now the target filtrations on the domain and codomain of ρ_k are induced by the filtrations on the domain and codomain of ρ_{k+1} by cohomological dimension s_{k+2}, and ρ_{k+1} is a graded map. Thus, for any $w \in F^p W(F^{\mathcal{M}_{\mathrm{v}}(k+1)}V_{(k)})$, we must see that $\rho_k(\overline{f}_{k+1}(w))$ coincides with $g_{k+1}(\rho_{k+1}(w))$ modulo $F^{p+1}H^*_{\mathcal{W}(k)}V^*_{(k)}$, as this will prove both that ρ_k preserves target filtrations and that $g_{k+1} \circ \rho_{k+1} = E_0(\rho_k) \circ f_{k+1}$. However, this coincidence follows from the fact that $c = g_{k+1} \circ \rho_{k+1} \circ j_1$, as $W(F^{\mathcal{M}_{\mathrm{v}}(k+1)}V_{(k)})$ is generated by $\mathrm{im}(j_1)$ under $F^{\mathcal{M}_{\mathrm{hv}}(k+2)}$-operations, and the definition of $\overline{f}_{k+1}$ is modelled on the interaction of g_{k+1} with these operations, as studied by Singer (c.f. §13.1).

What remains is to show that the maps ρ_k are isomorphisms. Suppose that

$$x_{(k)} \in [E_2^{(k)}]_t^{s^k_{k+1},\ldots,s^k_1} = (H^*_{\mathcal{W}(k)}V^*_{(k)})_t^{s^k_{k+1},\ldots,s^k_1}.$$

Now $x_{(k)}$ is detected by some permanent cycle $x_{(k+1)} \in [E_2^{(k+1)}]$, which is detected by some permanent cycle $x_{(k+2)} \in [E_2^{(k+2)}]$, and so on, giving a sequence of elements

$$x_{(r)} \in [E_2^{(r)}]_t^{s^r_{r+1},\ldots,s^r_1} = (H^*_{\mathcal{W}(r)} V^*_{(r)})_t^{s^r_{r+1},\ldots,s^r_1} \text{ for } r \geq k,$$
$$\text{where } s^r_{r+1} + s^r_r = s^{r-1}_r \text{ and } s^r_i = s^{r-1}_i \text{ for } 1 \leq i \leq r-1 \text{ and } r > k.$$

We will say that $x_{(k)}$ has iterated filtration at least $(s^{k+1}_{k+2}, s^{k+2}_{k+3}, s^{k+3}_{k+4}, \ldots)$ whenever a sequence of such classes $x_{(r)}$ exists, and partially order the set of possible iterated filtrations lexicographically. Then $x_{(r)}$ only determines $x_{(k)}$ modulo elements of $E_{2,(k)}$ of higher iterated filtration.

Simply because these gradings are always non-negative, it is inevitable that $s^r_r = 0$ for some $r \geq k$, so that by Proposition 14.5, $x_{(r)} = \rho_r y_{(r)}$ for some $y_{(r)} \in F^{\mathcal{M}_{\mathrm{h}}(r+1)} F^{\mathcal{M}_{\mathrm{v}}(r+1)} V_{(r)}$. Moreover, one only needs to examine finitely many sequences of gradings, each of the form

$$(s^r_{r+1}, 0, s^r_{r-1}, \ldots, s^r_{k+1}, s^k_k, \ldots, s^k_1) \text{ where } s^k_{k+1} = s^r_{r+1} + s^r_{r-1} + s^r_{r-2} + \cdots + s^r_{k+1}.$$

This, along with Proposition 14.3, shows that $(H^*_{\mathcal{W}(k)} V^*_{(k)})_t^{s^k_{k+1},\ldots,s^k_1}$ is finite dimensional for each given value of t.

By the commutativity established above, $x_{(k)} \equiv \rho_k f_{k+1} \cdots f_{r-1} f_r(y_{(r)})$, modulo higher iterated filtration. As this congruence holds in a group which is finite dimensional for each given t, this establishes the surjectivity of ρ_k, and that every one of the spectral sequences is degenerate. Thus, we have shown that all of the maps g_k are in fact isomorphisms, or rather that in the following commuting square, for any $k \geq n$, g_{k+1} is an isomorphism:

$$\begin{tikzcd}
{[E_0 H^*_{\mathcal{W}(k)} V^*_{(k)}]} & H^*_{\mathcal{W}(k+1)} H_*^{\mathcal{U}(k)} V^*_{(k)} \arrow[l, "g_{k+1}"', "\cong"] \\
{[E_0 F^{\mathcal{M}_{\mathrm{h}}(k+1)} V_{(k+1)}]} \arrow[u, two heads, "E_0(\rho_k)"] & F^{\mathcal{M}_{\mathrm{h}}(k+2)} F^{\mathcal{M}_{\mathrm{v}}(k+2)} F^{\mathcal{M}_{\mathrm{v}}(k+1)} V_{(k)} \arrow[l, "f_{k+1}"', "\cong"] \arrow[u, two heads, "\rho_{k+1}"']
\end{tikzcd}$$

For each k, ρ_k is injective if and only if $E_0(\rho_k)$ is injective. This holds by repeated application of the snake lemma, using the fact that ρ_k is surjective, and the observation that for any given value of the grading t, the group $(F^{\mathcal{M}_{\mathrm{h}}(k+1)} V_{(k+1)})_t$ is finite dimensional, so that the filtrations of both the domain and codomain of ρ_k are eventually zero in each degree t. More specifically,

$$\rho_k : (F^{\mathcal{M}_{\mathrm{h}}(k+1)} V_{(k+1)})_t^{s_{k+1},\ldots,s_1} \longrightarrow (H^*_{\mathcal{W}(k)} V^*_{(k)})_t^{s_{k+1},\ldots,s_1}$$

is injective if and only if

$$E_0(\rho_k) : [E_0 F^{\mathcal{M}_{\mathrm{h}}(k+1)} V_{(k+1)}]_t^{s'_{k+2},s'_{k+1},s_k,\ldots,s_1} \longrightarrow [E_0 H^*_{\mathcal{W}(k)} V^*_{(k)}]_t^{s'_{k+2},s'_{k+1},s_k,\ldots,s_1}$$

is injective whenever $s'_{k+2} + s'_{k+1} = s_{k+1}$. As in the argument for surjectivity, in order to check that all the ρ_k are injective, we now only need to check that every map

$$(F^{\mathcal{M}_{\mathrm{h}}(r+1)}F^{\mathcal{M}_{\mathrm{v}}(r+1)}V_{(r)})_t^{s^r_{r+1},0,s^r_{r-1},\ldots,s^r_1} \xrightarrow{\rho_r} (H^*_{\mathcal{W}(r)}V^*_{(r)})_t^{s^r_{r+1},0,s^r_{r-1},\ldots,s^r_1}$$

is injective, which is part of Proposition 14.5. □

14.2. A Künneth Theorem for $\mathcal{W}(n)$-cohomology

This is an opportune moment to prove:

Theorem 14.6. *Suppose that $X, Y \in \mathcal{W}(n)$ are of finite type, with $n \geq 0$. Then*

$$H^*_{\mathcal{W}(n)}(X \times Y) \cong H^*_{\mathcal{W}(n)}(X) \sqcup H^*_{\mathcal{W}(n)}(Y),$$

where the coproduct is of non-unital commutative algebras.

Proof. This follows from the Künneth Theorem (6.15) adapted to $s\mathcal{L}(k)$, and the observation that $H_*^{\mathcal{U}(k)}(Z \times Z') \cong H_*^{\mathcal{U}(k)}Z \times H_*^{\mathcal{U}(k)}Z'$, using the techniques of the proof of Theorem 14.4. □

Theorems 14.4 and 14.6 together imply:

Corollary 14.7. *For $n \geq 1$, the category $\mathcal{M}_{\mathrm{hv}}(n+1)$ is the category $H\mathcal{W}(n)$ of $\mathcal{W}(n)$-H^*-algebras.*

14.3. A two-dimensional example in $\mathcal{W}(2)$

In this section, we suppose that $T \geq 1$, and let $X = V^*_{(2)} \in \mathcal{W}(2)$ be the two-dimensional object of $\mathcal{W}(2)$ spanned by non-zero classes

$$v_0^* \in (V^*_{(2)})^T_{0,1} \text{ and } v_1^* \in (V^*_{(2)})^{2T+1}_{0,2}$$

such that $v_1^* = v_0^*\lambda_0 = (v_0^*)^{[2]}$, and with all other operations trivial.

Proposition 14.8. *For all $k \geq 2$, $V^*_{(k)}$ is two-dimensional, spanned by*

$$v_0^* \in (V^*_{(k)})^T_{0,\ldots,0,1} \text{ and } v_1^* \in (V^*_{(k)})^{2T+1}_{0,\ldots,0,2},$$

with $v_1^* = (v_0^*)^{[2]}$ the only non-trivial operation.

Proof. An induction as in the proof of Proposition 14.1, using the fact that at each stage, the only non-trivial λ-operation is a *top* operation, and thus does not yield a differential in

$K_*^{\mathcal{U}(k)}V_{(k)}$. One also uses Propositions 9.9, 9.11 and 9.12 to calculate the $\mathcal{W}(k+1)$-structure of $V_{(k+1)}$ at each stage. □

Proposition 14.9. *For each $k \geq 2$,*

$$(H^*_{\mathcal{W}(k+1)}V^*_{(k+1)})_t^{s_{k+2},0,s_k,\ldots,s_1} \cong (H^{s_{k+2}}_{\mathcal{L}(k)}V^*_{(k)})_t^{s_k,\ldots,s_1} \cong (S(\mathscr{C})[v_1^2] \sqcup \Lambda(\mathscr{C})[v_0])_t^{s_{k+2},s_k,\ldots,s_1}.$$

These groups are zero unless $s_k = \cdots = s_2 = 0$.

Proof. One performs this calculation in the Chevalley-Eilenberg-May complex $\mathbf{D}\bar{X}'(V^*_{(k)})$, which by Proposition A.9 is the differential graded algebra $\mathbb{F}_2[v_0, v_1]$ with differential

$$d(v_0) = (\sqrt[[2]]{v_0})^2 = 0, \quad d(v_1) = (\sqrt[[2]]{v_1})^2 = v_0^2. \qquad \square$$

By a greatly simplified version of the proof of Theorem 14.4:

Corollary 14.10. *For each $k \geq 2$, $(E_{2,(k+1)})_t^{s_{k+2},s_{k+1},\ldots,s_1}$ is zero unless $s_{k+1} = \cdots s_2 = 0$, so that the spectral sequence $E_{2,(k+1)} \implies H^*_{\mathcal{W}(k)}V^*_{(k)}$ collapses, and in particular,*

$$H^*_{\mathcal{W}(2)}V^*_{(2)} \cong S(\mathscr{C})[v_1^2] \sqcup \Lambda(\mathscr{C})[v_0].$$

14.4. An infinite-dimensional example in $\mathcal{W}(1)$

In this section, we suppose that $S, T \geq 1$, and let $X = V^*_{(1)} \in \mathcal{W}(1)$ be the infinite dimensional object of $\mathcal{W}(1)$ spanned by non-zero classes

$$v_j^* \in (V^*_{(1)})_{S+j}^{2^j(T+1)-1} \text{ for } j \geq 0,$$

such that $v^*_{j+1} = v_j^*\lambda_1$ for $j \geq 0$, and all other operations are trivial.

Proposition 14.11. *The Koszul complex $K_*^{\mathcal{U}(1)}V^*_{(1)}$ has basis*

$$\left\{\mathrm{Sq}_{\mathrm{v}}^{J\star}(v_j^*) \;\middle|\; j \geq 0,\ J \text{ is Sq-admissible, } \underline{m}(J) \leq S + j \text{ and } 1 \notin J\right\}$$

and all differentials zero except for:

$$\mathrm{Sq}_{\mathrm{v}}^{(i_\ell,\ldots,i_2,2)\star}(v_j^*) \longmapsto \mathrm{Sq}_{\mathrm{v}}^{(i_\ell,\ldots,i_2)\star}(v^*_{j+1}).$$

Proof. The basis given for the Koszul complex is just a reading of Proposition 9.4, but we must think a little about the differentials. As λ_1 is the only non-zero operation:

$$d(\mathrm{Sq}_{\mathrm{v}}^{J\star}(v_j^*)) = \sum_{\substack{(k_\ell,\ldots,k_2,2)\overset{\mathrm{Sq}}{\to}J \\ (k_\ell,\ldots,k_2)\ \text{Sq-admis.}}} \mathrm{Sq}_{\mathrm{v}}^{(k_\ell,\ldots,k_2)\star}(v_{j+1}^*).$$

Consider a sequence $(k_\ell,\ldots,k_2,2)$ corresponding to a summand of this formula. Supposing that $\ell \geq 2$ and $\mathrm{Sq}_{\mathrm{v}}^{k_2}\mathrm{Sq}_{\mathrm{v}}^{2}$ is not Sq-admissible, it follows that k_2 is either 3 or 2, so that $\mathrm{Sq}_{\mathrm{v}}^{k_2}\mathrm{Sq}_{\mathrm{v}}^{2}$ is either zero or $\mathrm{Sq}_{\mathrm{v}}^{3}\mathrm{Sq}_{\mathrm{v}}^{1}$. As J does not contain 1, and the two-sided ideal in $\mathcal{A}$ generated by $\mathrm{Sq}_{\mathrm{h}}^{1}$ is spanned by those admissible sequences ending in $\mathrm{Sq}_{\mathrm{h}}^{1}$, it cannot happen that $(k_\ell,\ldots,k_2,2) \overset{\mathrm{Sq}}{\to} J$. Thus, the only summand appearing is that in which $(k_\ell,\ldots,k_2,2) = J$, confirming our description of the differential. □

Proposition 14.12. *When $S \geq 2$, $V_{(2)}^* := H_*^{\mathcal{U}(1)} V_{(1)}^*$ is the subquotient*

$$\frac{\mathbb{F}_2\left\{\mathrm{Sq}_{\mathrm{v}}^{J\star}(v_j^*)\ \middle|\ j \geq 0,\ J \text{ is Sq-admissible, } \underline{m}(J) \leq S+j \text{ and } \ 1,2 \notin J\ \right\}}{\mathbb{F}_2\left\{\mathrm{Sq}_{\mathrm{v}}^{J\star}(v_j^*)\ \middle|\ j \geq 1,\ J \text{ is Sq-admissible, } \underline{m}(J) \leq S+j \text{ and } 1,2,3 \notin J\right\}}$$

of $K_*^{\mathcal{U}(1)} V_{(1)}^*$. Equivalently, $V_{(2)}$ is the subquotient of $F^{\mathcal{M}_{\mathrm{v}}(2)} V_{(1)}$ in which we restrict to the sub-$\mathcal{M}_{\mathrm{v}}(2)$-object generated by the elements

$$\{v_0, \mathrm{Sq}_{\mathrm{v}}^2 v_1, \mathrm{Sq}_{\mathrm{v}}^3 v_1, \mathrm{Sq}_{\mathrm{v}}^2 v_2, \mathrm{Sq}_{\mathrm{v}}^3 v_2, \mathrm{Sq}_{\mathrm{v}}^2 v_3, \mathrm{Sq}_{\mathrm{v}}^3 v_3, \ldots\}$$

and in which we set $\mathrm{Sq}_{\mathrm{v}}^2 v_j$ to zero for all $j \geq 0$. As an object of $\mathcal{W}(2)$, $V_{(2)}^*$ is trivial.

Proposition 14.13. *When $S = 1$, $V_{(2)}^* := H_*^{\mathcal{U}(1)} V_{(1)}^*$ is the subquotient*

$$\frac{\mathbb{F}_2\left\{\mathrm{Sq}_{\mathrm{v}}^{J\star}(v_j^*)\ \middle|\ j \geq 0,\ J \text{ is Sq-admissible, } \underline{m}(J) \leq S+j \text{ and } \ 1,2 \notin J\ \right\}}{\mathbb{F}_2\left\{\mathrm{Sq}_{\mathrm{v}}^{J\star}(v_j^*)\ \middle|\ j \geq 2,\ J \text{ is Sq-admissible, } \underline{m}(J) \leq S+j \text{ and } 1,2,3 \notin J\right\}}$$

of $K_*^{\mathcal{U}(1)} V_{(1)}^*$. Equivalently, $V_{(2)}$ is the subquotient of $F^{\mathcal{M}_{\mathrm{v}}(2)} V_{(1)}$ in which we restrict to the sub-$\mathcal{M}_{\mathrm{v}}(2)$-object generated by the elements

$$\{v_0, v_1, \mathrm{Sq}_{\mathrm{v}}^2 v_2, \mathrm{Sq}_{\mathrm{v}}^3 v_2, \mathrm{Sq}_{\mathrm{v}}^2 v_3, \mathrm{Sq}_{\mathrm{v}}^3 v_3, \ldots\}$$

and in which we set $\mathrm{Sq}_{\mathrm{v}}^2 v_j$ to zero for all $j \geq 1$. As an object of $\mathcal{W}(2)$, $V_{(2)}^*$ admits a single non-zero operation, $\lambda_0 : v_0^* \longmapsto v_1^*$, and so decomposes as the direct sum of $\mathbb{F}_2\{v_0^*, v_1^*\}$ with a trivial object $\mathbf{D}(V'_{(2)})$, dual to $V'_{(2)}$, the subquotient of $F^{\mathcal{M}_{\mathrm{v}}(2)} V_{(1)}$ in which we restrict to the sub-$\mathcal{M}_{\mathrm{v}}(2)$-object generated by $\{\mathrm{Sq}_{\mathrm{v}}^2 v_2, \mathrm{Sq}_{\mathrm{v}}^3 v_2, \mathrm{Sq}_{\mathrm{v}}^2 v_3, \mathrm{Sq}_{\mathrm{v}}^3 v_3, \ldots\}$ and set $\mathrm{Sq}_{\mathrm{v}}^2 v_j$ to zero for all $j \geq 2$.

Proof of Propositions 14.12 and 14.13. For any $S \geq 1$, taking the homology of this differential provides the formula for $V^*_{(2)}$, and dualizing provides that for $V_{(2)}$. In order to determine $V^*_{(2)}$ as an object of $\mathcal{W}(2)$, note first that 9.9 and 9.11 show that all operations are zero except perhaps for λ_0. Consider the operation λ_0 applied to a cycle of the form $\mathrm{Sq}_{\mathrm{v}}^{J\star}(v_j^*) \in K_*^{\mathcal{U}(1)} V^*_{(1)}$ with $J \neq \emptyset$ (so that $1, 2 \notin J$). As J ends in an integer no less than 3, and as λ_1 is the only non-zero operation in $V^*_{(1)}$, the second part of Proposition 9.12 implies that $\mathrm{Sq}_{\mathrm{v}}^{J\star}(v_j^*)\lambda_0 = 0$.

In the case $J = \emptyset$, Proposition 9.12 states that $(\mathrm{Sq}_{\mathrm{v}}^{\emptyset\star}(v_j^*))\lambda_0 \in V^*_{(2)}$ is represented by $(\mathrm{Sq}_{\mathrm{v}}^{\emptyset\star}(v_j^*\lambda_{S+j}))$, which is zero unless $j = 0$ and $S = 1$. Thus the only non-zero operation on $V^*_{(2)}$ is $v_0^*\lambda_0 = v_1^*$ in the case $S = 1$. □

Theorem 14.14. *The spectral sequence $H^*_{\mathcal{W}(2)} V^*_{(2)} \implies H^*_{\mathcal{W}(1)} V^*_{(1)}$ collapses, with*

$$[E_2^{(2)}] = H^*_{\mathcal{W}(2)} V^*_{(2)} \cong \begin{cases} F^{\mathcal{M}_{\mathrm{h}}(3)} F^{\mathcal{M}_{\mathrm{v}}(3)} V_{(2)}, & \text{if } S \geq 2; \\ F^{\mathcal{M}_{\mathrm{h}}(3)} F^{\mathcal{M}_{\mathrm{v}}(3)} V'_{(2)} \sqcup S(\mathscr{C})[v_1^2] \sqcup \Lambda(\mathscr{C})[v_0], & \text{if } S = 1. \end{cases}$$

Proof. The calculations of $[E_2^{(2)}]$ follow from Theorems 14.4 and 14.6, Propositions 14.8, 14.12 and 14.13 and Corollary 14.10. What remains is to prove the collapsing result in each case.

Suppose that $S \geq 2$. The first point is to observe that the generators v_0 and $\mathrm{Sq}_{\mathrm{v}}^3 v_j$ ($j \geq 1$) of $V_{(2)}$ under $\mathcal{M}_{\mathrm{v}}(2)$-operations are all permanent cycles in $(H^*_{\mathcal{W}(2)} V_{(2)})^{0**}_*$. For $v_0 \in [E_2^{(2)}]^{00S}_T$, this is obvious. It is less obvious for $\mathrm{Sq}_{\mathrm{v}}^3 v_j$ ($j \geq 1$), which has only one opportunity to support a differential:

$$\mathrm{Sq}_{\mathrm{v}}^3 v_j \in [E_2^{(2)}]^{0,1,2+S+j}_{2^{j+1}(T+1)-1} \xrightarrow{d_2} [E_2^{(2)}]^{2,0,2+S+j}_{2^{j+1}(T+1)-1}.$$

Fortunately, this target group is zero, due to the constraint that $s_2 = 0$. To see this, note that this group is spanned by three-fold products of classes in $[E_2^{(2)}]^{00*}_*$, namely:

$$v_{j_1} v_{j_2} v_{j_3} \in [E_2^{(2)}]^{2,0,3S+j_1+j_2+j_3}_{(2^{j_1}+2^{j_2}+2^{j_3})(T+1)-1},$$

and if this target group is non-zero, these indices must coincide. In order that 2^{j+1} equals $2^{j_1} + 2^{j_2} + 2^{j_3}$ it must happen that j_1, j_2, j_3 equal $j, j-1, j-1$ (in some order), but then $2 + S + j = 3S + j_1 + j_2 + j_3$ implies that $S + j = 2$. This is impossible, as $S \geq 2$ and $j \geq 1$.

Next, we can derive that $\mathrm{Sq}_{\mathrm{v}}^J v_j$ is a permanent cycle for all Sq-admissible J and $j \geq 0$ such that J has final entry 3 when $j > 0$. For this, we will use Proposition 12.9, that there is a commuting diagram:

$$\begin{tikzcd}
(H^*_{\mathcal{W}(1)}V^*_{(1)})^{s_2,s_1}_t \arrow[r, "\mathrm{Sq}^i_{\mathrm{v}}"] \arrow[d, "\text{edge hom}"] & (H^*_{\mathcal{W}(1)}V^*_{(1)})^{s_2+1,s_1+i-1}_{2t+1} \arrow[d, "\text{edge hom}"] \\
{[E^{(2)}_2]^{0,s_2,s_1}_t} \arrow[d, equal] & {[E^{(2)}_2]^{0,s_2+1,s_1+i-1}_{2t+1}} \arrow[d, equal] \\
(V_{(2)})^{s_2,s_1}_t \arrow[r, "\mathrm{Sq}^i_{\mathrm{v}}"] & (V_{(2)})^{s_2+1,s_1+i-1}_{2t+1}
\end{tikzcd}$$

As we have shown that the classes v_0 and $\mathrm{Sq}^3_{\mathrm{v}}v_j$ ($j \geq 1$) are all permanent cycles, they are in the image of the edge homomorphism. Then this diagram shows that all of $V_{(2)}$ is in the image of the edge homomorphism, so that every element of $V_{(2)}$ is a permanent cycle. Finally, as E_2 is (freely) generated by $V_{(2)}$ under the $\mathcal{M}_{\mathrm{hv}}(3)$-operations, and we understand how these operations interact with the differential, this shows that the spectral sequence collapses.

Suppose instead that $S = 1$. Then rather that having generators v_0 and $\mathrm{Sq}^3_{\mathrm{v}}v_j$ ($j \geq 1$) as before, E_2 has generators v_0, v_1^2 and $\mathrm{Sq}^3_{\mathrm{v}}v_j$ ($j \geq 2$). Note that $\mathrm{Sq}^3_{\mathrm{v}}v_1 = 0$ when $S = 1$. That $v_1^2 \in [E^{(2)}_2]^{1,0,4}_{4T-3}$ cannot support differentials is obvious, while for $j \geq 2$, the same degree argument as before shows that $\mathrm{Sq}^3_{\mathrm{v}}v_j$ is also a permanent cycle. The same argument with the edge homomorphism shows that every element of $V'_{(2)}$ is a permanent cycle, so E_2 is again generated by permanent cycles under the $\mathcal{M}_{\mathrm{hv}}(3)$-operations, completing the proof. □

Corollary 14.15. *If $S \geq 2$, then $H^*_{\mathcal{W}(1)}V^*_{(1)}$ is isomorphic, as a vector space in $\mathcal{V}^2_+$, to the subquotient of $F^{\mathcal{M}_{\mathrm{h}}(2)}F^{\mathcal{M}_{\mathrm{v}}(2)}V_{(1)}$ generated by the elements*

$$\{v_0, \mathrm{Sq}^2_{\mathrm{v}}v_1, \mathrm{Sq}^3_{\mathrm{v}}v_1, \mathrm{Sq}^2_{\mathrm{v}}v_2, \mathrm{Sq}^3_{\mathrm{v}}v_2, \mathrm{Sq}^2_{\mathrm{v}}v_3, \mathrm{Sq}^3_{\mathrm{v}}v_3, \ldots\}$$

and subject to relations generated by $\mathrm{Sq}^2_{\mathrm{v}}v_j = 0$ for all $j \geq 0$. Under $\mathcal{M}_{\mathrm{hv}}(2)$-operations, $H^*_{\mathcal{W}(1)}V^*_{(1)}$ is generated by v_0, $\mathrm{Sq}^3_{\mathrm{v}}v_1$, $\mathrm{Sq}^3_{\mathrm{v}}v_2$, etc.

If $S = 1$, then $H^*_{\mathcal{W}(1)}V^*_{(1)} \in \mathcal{V}^2_+$ is isomorphic, as a vector space in $\mathcal{V}^2_+$, to the commutative algebra coproduct

$$\mathrm{subquo} \sqcup S(\mathscr{C})[v_1^2] \sqcup \Lambda(\mathscr{C})[v_0],$$

where $v_1^2 \in (H^*_{\mathcal{W}(1)}V^*_{(1)})^{1,2^2}_{2^2(T+1)-1}$, and subquo *is the subquotient of $F^{\mathcal{M}_{\mathrm{h}}(2)}F^{\mathcal{M}_{\mathrm{v}}(2)}V_{(1)}$ generated by the elements*

$$\{\mathrm{Sq}^2_{\mathrm{v}}v_2, \mathrm{Sq}^3_{\mathrm{v}}v_2, \mathrm{Sq}^2_{\mathrm{v}}v_3, \mathrm{Sq}^3_{\mathrm{v}}v_3, \ldots\}$$

and subject to relations generated by $\mathrm{Sq}^2_{\mathrm{v}}v_j = 0$ for all $j \geq 2$. Under this isomorphism, $H^*_{\mathcal{W}(1)}V^*_{(1)}$ is generated by v_0, v_1^2, $\mathrm{Sq}^3_{\mathrm{v}}v_2$, $\mathrm{Sq}^3_{\mathrm{v}}v_3$, et cetera, under the $\mathcal{M}_{\mathrm{hv}}(2)$-operations.

Proof. Suppose first that $S \geq 2$. Consider the elements

$$v_0 \in (V_{(2)})_T^{0,S}, \quad \mathrm{Sq}_{\mathrm{v}}^j v_0 \in (V_{(2)})_{2(T+1)-1}^{1,S+j-1} \ (j \geq 2), \quad \text{and} \quad \mathrm{Sq}_{\mathrm{v}}^3 v_i \in (V_{(2)})_{2^{i+1}(T+1)-1}^{1,S+i+2} \ (i \geq 1).$$

These elements span $(V_{(2)})_*^{0,*}$ and $(V_{(2)})_*^{1,*}$, and can all be distinguished by their internal degrees, so the restrictions

$$(H^*_{\mathcal{W}(1)}V^*_{(1)})_*^{0,*} \twoheadrightarrow [E_2^{(2)}]_*^{0,0,*} = (V_{(2)})_*^{0,*}, \quad (H^*_{\mathcal{W}(1)}V^*_{(1)})_*^{1,*} \twoheadrightarrow [E_2^{(2)}]_*^{0,1,*} = (V_{(2)})_*^{1,*}$$

of the edge composite (c.f. Proposition 12.9) are isomorphisms. We write

$$h : (V_{(2)})_*^{0,*} \oplus (V_{(2)})_*^{1,*} \longrightarrow H^*_{\mathcal{W}(1)}V^*_{(1)}$$

for the injection obtained by adding their inverse maps. Use the basis of $V_{(2)}$ arising from Propositions 14.12 and 8.8 to extend h to a vector space map $H : V_{(2)} \longrightarrow H^*_{\mathcal{W}(1)}V^*_{(1)}$ by the rule $H(\mathrm{Sq}_{\mathrm{v}}^j x) = \mathrm{Sq}_{\mathrm{v}}^j H(x)$. Although H is not a map in $\mathcal{M}_{\mathrm{v}}(2)$, it does induce the vector space isomorphism required for the proposition.

Suppose instead that $S = 1$. The same argument produces a map subquo $\longrightarrow H^*_{\mathcal{W}(1)}V^*_{(1)}$. The difference is that we must find candidates for v_1^2 and v_0 in $H^*_{\mathcal{W}(1)}V^*_{(1)}$. We send v_0 to the unique non-zero element of $(H^*_{\mathcal{W}(1)}V^*_{(1)})_T^{0,1}$ and v_1^2 to the unique non-zero element of $(H^*_{\mathcal{W}(1)}V^*_{(1)})_{4(T+1)-1}^{1,4}$. $\square$

14.5. The Bousfield-Kan E_2-page for a sphere

Let $X = V^*_{(0)} \in \mathcal{W}(0)$ be a one dimensional object of $\mathcal{W}(0)$, dual to a one-dimensional vector space $V_{(0)} \in \mathcal{V}^0_+$, with non-zero element $\imath \in (V_{(0)})_T$. Write $\imath^* \in X^T$ for the non-zero element of X.

As every $\mathcal{W}(0)$-operation changes degrees, X is necessarily trivial. Moreover, it is quadratically graded, by setting $\imath^* \in \mathrm{q}_1 X^T$. By Proposition 12.8, the first CFSS will admit a quadratic grading.

Recall the function $\mathfrak{T}_T : \mathrm{adm}_+(\Delta, T) \longrightarrow \mathrm{adm}_+(\Delta, T)$ of §9.2. In view of the strict inequality derived during the proof of Lemma 9.10, it need not be true that $I = \mathfrak{T}_T^{i-1-j}(i_{j+1}, \ldots, i_1)$ whenever $I = (i_\ell, \ldots, i_1) \in \mathrm{adm}_+(\Delta, T)$ satisfies $(i_{j+1}, \ldots, i_1) = \mathfrak{T}_T(i_j, \ldots, i_1)$. Nevertheless, we may use $\mathfrak{T}_T$ to decompose $\mathrm{adm}_+(\Delta, T)$. Define:

$$\mathrm{adm}_+^{\mathrm{irr}}(\Delta, T) := \mathrm{adm}_+(\Delta, T) \setminus \mathrm{im}(\mathfrak{T}_T : \mathrm{adm}_+(\Delta, T) \longrightarrow \mathrm{adm}_+(\Delta, T)),$$

the set of sequences in $\mathrm{adm}_+(\Delta, T)$ not in the image of $\mathfrak{T}_T$, so that we may decompose

$\mathrm{adm}_+(\Delta, T)$ as the disjoint union

$$\mathrm{adm}_+(\Delta, T) = \bigsqcup_{I \in \mathrm{adm}_+^{\mathrm{irr}}(\Delta,T)} \left\{ I, \mathfrak{T}_T I, \mathfrak{T}_T^2 I, \ldots \right\}.$$

Proposition 14.16. *$V_{(1)}^* := H_*^{\mathcal{U}(0)} V_{(0)}^*$ has basis $\{\imath^*\} \sqcup \{\delta_I^{\mathrm{v}\star}\imath^* \mid I \in \mathrm{adm}_+(\Delta, T)\}$ and all $\mathcal{W}(1)$-operations trivial except for λ_1, which is defined (only when $\ell(I) \geq 1$) by*

$$\delta_I^{\mathrm{v}\star}\imath^* \overset{\lambda_1}{\longmapsto} \delta_{\mathfrak{T}_T I}^{\mathrm{v}\star}\imath^*.$$

Thus, as an object of $\mathcal{W}(1)$, $V_{(1)}^*$ decomposes as a direct sum

$$\mathbb{F}_2\{\imath^*\} \oplus \bigoplus_{I \in \mathrm{adm}_+^{\mathrm{irr}}(\Delta,T)} \mathbb{F}_2 \left\{ \delta_I^{\mathrm{v}\star}(\imath^*\lambda_1^j) \mid j \geq 0 \right\}.$$

Proof. The basis of the Koszul complex was described in Proposition 9.2, and the Koszul differential is zero as X is trivial. The λ-operations were calculated in Proposition 9.11. □

Now we have put considerable effort into calculating $H_{\mathcal{W}(1)}^*$ of each summand in this decomposition: Theorem 14.4 proves that

$$H_{\mathcal{W}(1)}^*(\mathbb{F}_2\{\imath^*\}) \cong F^{\mathcal{M}_{\mathrm{h}}(2)} F^{\mathcal{M}_{\mathrm{v}}(2)}(\mathbb{F}_2\{\imath\}) \cong \Lambda(\mathscr{C})(\imath),$$

while Propositions 14.12 and 14.13 calculate

$$H_{\mathcal{W}(1)}^* \left(\mathbb{F}_2 \left\{ (\delta_I^{\mathrm{v}\star}\imath^*)\lambda_1^j \mid j \geq 0 \right\} \right) \text{ for } I \in \mathrm{adm}_+^{\mathrm{irr}}(\Delta, T).$$

With a view to calculating the first CFSS, we catalogue a collection of generators of $[E_2^{(1)}]$ under the $\mathcal{M}_{\mathrm{hv}}(2)$-operations. The *fundamental class* $\imath \in \mathrm{q}_1[E_2^{(1)}]_T^{0,0}$ is an exterior generator (arising in Theorem 14.4). Moreover, for all $I \in \mathrm{adm}_+^{\mathrm{irr}}(\Delta, T)$, there are further generators, arising in Corollary 14.15:

$$\begin{aligned}
&\delta_I^{\mathrm{v}}\imath \in \mathrm{q}_{2^{\ell I}}[E_2^{(1)}]_{T+nI+\ell I}^{0,\ell I} && && (14.1)\\
&\mathrm{Sq}_{\mathrm{v}}^3 \delta_{\mathfrak{T}_T^j I}^{\mathrm{v}}\imath \in \mathrm{q}_{2^{1+\ell I+j}}[E_2^{(1)}]_{2^{j+1}(T+nI+\ell I+1)-1}^{1,2+\ell I+j} && \text{(when } j \geq 1\text{, but not } j = \ell I = 1), && (14.2)\\
&(\delta_{\mathfrak{T}_T I}^{\mathrm{v}}\imath)^2 \in \mathrm{q}_{2^3}[E_2^{(1)}]_{2^2(T+nI+\ell I+1)-1}^{1,4} && \text{(when } \ell I = 1), && (14.3)
\end{aligned}$$

where they are referred to as v_0, $\mathrm{Sq}_{\mathrm{v}}^3 v_j$ and v_1^2 respectively. Note that this final generator, $(\delta_{\mathfrak{T}_T I}^{\mathrm{v}}\imath)^2$, has the same degrees as the generator $\mathrm{Sq}_{\mathrm{v}}^3 \delta_{\mathfrak{T}_T^j I}^{\mathrm{v}}\imath$ that is *missing* when $j = \ell I = 1$.

Theorem 14.17. *The first CFSS collapses at E_2:*

$$[E_2^{(1)}] = H^*_{\mathcal{W}(1)} V^*_{(1)} \implies H^*_{\mathcal{W}(0)} V^*_{(0)}.$$

Proof. The fundamental class is a permanent cycle, so to prove that the spectral sequence collapses, it is enough to show that no classes

$$x \in \mathrm{q}_{2^{\ell I}} [E_2^{(1)}]^{0,\ell I}_{T+nI+\ell I} \text{ or } y \in \mathrm{q}_{2^{1+\ell I+j}} [E_2^{(1)}]^{1,2+\ell I+j}_{2^{j+1}(T+nI+\ell I+1)-1}$$

can support a differential, I a non-empty δ-admissible sequence.

To see this, all one needs to have learned about the entire E_2-page is that it is a sub-quotient (in which $\imath^2 = 0$) of the polynomial algebra on symbols

$$\mathrm{Sq}_\mathrm{h}^A \mathrm{Sq}_\mathrm{v}^B \delta^\mathrm{v}_C \imath \in \mathrm{q}_{2^{\ell A+\ell B+\ell C}} [E_2^{(1)}]^{\ell B+nA, 2^{\ell A}(nB-\ell B+\ell C)}_{2^{\ell A+\ell B}(T+nC+\ell C+1)-1},$$

in which B is Sq-admissible, B does not contain 1 or 2, if C is empty then so is B, and if B is empty then so is A. These conditions imply that $nB - 2\ell B \geq 2^{\ell B} - 1$.

If for $r \geq 2$ there is a differential d_r supported by y, then $d_r y$ must be a sum of products of $N \geq 1$ such classes. The generic such monomial may be written as:

$$\prod_{k=1}^N \mathrm{Sq}_\mathrm{h}^{A_k} \mathrm{Sq}_\mathrm{v}^{B_k} \delta^\mathrm{v}_{C_k} \imath \in \mathrm{q}_{\sum 2^{\ell A_k+\ell B_k+\ell C_k}} [E_2^{(1)}]^{\sum(\ell B_k+nA_k)+N-1, \sum 2^{\ell A_k}(nB_k-\ell B_k+\ell C_k)}_{-1+\sum 2^{\ell A_k+\ell B_k}(T+nC_k+\ell C_k+1)}$$

in which $\ell C_k = 0$ for at most one k. We derive the following constraints:

$$\textstyle\sum(\ell B_k + nA_k) \geq 4 - N, \tag{14.4}$$

$$\log_2(N) + \tfrac{1}{N}\textstyle\sum_k [\ell A_k + \ell B_k + \ell C_k] \geq 1 + \ell I + j, \tag{14.5}$$

$$4 + \ell I + j = \textstyle\sum_k [\ell B_k + nA_k] + N - 1 + \sum_k \left[2^{\ell A_k}(nB_k - \ell B_k + \ell C_k)\right], \tag{14.6}$$

$$\log_2(N) \geq \textstyle\sum_k \left((2^{\ell A_k} - \tfrac{1}{N})\ell C_k + \left[\left(2^{\ell A_k}(nB_k - \ell B_k) - \tfrac{1}{N}\ell B_k\right) - \tfrac{1}{N}(\ell A_k)\right]\right). \tag{14.7}$$

The inequality (14.4) is just the requirement that $r \geq 2$, while (14.5) results from the observation that d_r preserves the quadratic grading and the convexity of the exponential function. Equation (14.6) holds since the total degree of the differential is one, and (14.7) is derived by rearranging the sum of (14.4), (14.5) and (14.6). (14.7) is a very strong inequality, since the expression $2^{\ell A_k}(nB_k - \ell B_k) - \frac{1}{N}\ell B_k$ is at least $2^{\ell B_k} - \frac{1}{N}$, and $nB_k - \ell B_k \geq 2$ if $\ell B_k \neq 0$. Thus, in (14.7), each expression in square brackets is always non-negative, is at least $2 - \frac{1}{N}$ when $\ell B_k \neq 0$, and exceeds $2 - \frac{1}{N}$ if $\ell B_k \geq 2$ or $\ell A_k \neq 0$.

When $N = 1$ or $N = 3$, $\log_2(N) < 2 - \frac{1}{N}$, so that (14.7) implies that $\ell B_k = 0$ for all k, violating (14.4). When $N \leq 2$, $\log_2(N) \leq 2 - \frac{1}{N}$, so that (14.7) implies that $\ell B_k \neq 0$

for at most one k, with $\ell B_k = 1$, violating (14.4). When $N \geq 4$, all but at most one of the summands $(2^{\ell A_k} - \frac{1}{N})\ell C_k$ in (14.7) is at least $\frac{3}{4}$, and as $\frac{3}{4}(N-1) \geq \log_2(N)$ when $N \geq 4$, (14.7) is violated. Thus $y \in E_2$ is a permanent cycle.

Performing the same calculations for $d_r x$, we find that the inequality (14.7) is unchanged, while (14.4) is replaced by

$$\sum(\ell B_k + nA_k) \geq 3 - N. \tag{14.8}$$

The argument is unchanged when $N = 1$ or $N \geq 4$, while if $2 \leq N \leq 3$ we may still draw the same conclusions from (14.7). When $N = 2$, we may assume that $\ell B_1 = 1$ and $\ell B_2 = 0$, and although (14.8) is not violated, (14.7) is violated as $\ell C_1 \neq 0$. When $N = 3$, we must have $\ell C_k = 0$ for each k, and the following equations must be satisfied

$$\ell I - 1 = \ell C_1 + \ell C_2 + \ell C_3, \quad 2^{\ell I} = 2^{\ell C_1} + 2^{\ell C_2} + 2^{\ell C_3}.$$

As in the proof of Theorem 14.14, these equations imply that $\ell C_1, \ell C_2, \ell C_3$ equal $\ell I - 1, \ell I - 2, \ell I - 2$, in some order. The first equation then implies that $\ell I = 2$, implying that $\ell C_k = 0$ for more than one k, which we have prohibited. Thus $x \in E_2$ is a permanent cycle. $\square$

This theorem has the following corollary, stated in this form due to potential hidden extensions:

Corollary 14.18. *Suppose that $X = \mathbb{S}_T^{\mathscr{C}om}$ for $T \geq 1$. Then the BKSS E_2-page $[E_2\mathfrak{X}] \cong H^*_{\mathcal{W}(0)}(H^*_{\mathscr{C}om}X)$ is isomorphic, as a vector space in $\mathcal{V}^1_+$, to the $\mathcal{M}_{\mathrm{h}}(1)$-subquotient of $F^{\mathcal{M}_{\mathrm{h}}(1)}F^{\mathcal{M}_{\mathrm{v}}(1)}\{\imath\}$ generated by the fundamental class $\imath$ and the elements*

$$\{\delta^{\mathrm{v}}_I\imath, \mathrm{Sq}^2_{\mathrm{h}}\delta^{\mathrm{v}}_{\mathfrak{T}^1_T I}\imath, \mathrm{Sq}^3_{\mathrm{h}}\delta^{\mathrm{v}}_{\mathfrak{T}^1_T I}\imath, \mathrm{Sq}^2_{\mathrm{h}}\delta^{\mathrm{v}}_{\mathfrak{T}^2_T I}\imath, \mathrm{Sq}^3_{\mathrm{h}}\delta^{\mathrm{v}}_{\mathfrak{T}^2_T I}\imath, \ldots\} \text{ for } I \in \mathrm{adm}^{\mathrm{irr}}_+(\Delta, T),$$

and subject to relations generated under $\mathcal{M}_{\mathrm{h}}(1)$-operations by

$$\{\mathrm{Sq}^2_{\mathrm{h}}\delta^{\mathrm{v}}_I\imath, \mathrm{Sq}^2_{\mathrm{h}}\delta^{\mathrm{v}}_{\mathfrak{T}^1_T I}\imath, \mathrm{Sq}^2_{\mathrm{h}}\delta^{\mathrm{v}}_{\mathfrak{T}^2_T I}\imath, \mathrm{Sq}^2_{\mathrm{h}}\delta^{\mathrm{v}}_{\mathfrak{T}^3_T I}\imath, \ldots\} \text{ for } I \in \mathrm{adm}^{\mathrm{irr}}_+(\Delta, T).$$

Proof. This follows from the collapsing of the first CFSS, our knowledge of the generators $\imath$ and (14.1)-(14.3) of $[E_2^{(1)}]$, and a few observations in the low-dimensional cases.

When $\ell I = 0$: in $F^{\mathcal{M}_{\mathrm{h}}(1)}F^{\mathcal{M}_{\mathrm{v}}(1)}\{\imath\}$, by unstableness of the horizontal Steenrod operations, $\mathrm{Sq}^2_{\mathrm{h}}\imath = 0$, $\mathrm{Sq}^3_{\mathrm{h}}\imath = 0$ and $\imath^2 = \mathrm{Sq}^1_{\mathrm{h}}\imath = 0$, so that $\imath$ contributes no more to this subquotient than it did as an exterior generator of $[E_2^{(1)}]$.

When $\ell I = 1$: in $F^{\mathcal{M}_{\mathrm{h}}(1)}F^{\mathcal{M}_{\mathrm{v}}(1)}\{\imath\}$, the generators (14.3) satisfy $(\delta^{\mathrm{v}}_{\mathfrak{T}_T I}\imath)^2 = \mathrm{Sq}^3_{\mathrm{h}}\delta^{\mathrm{v}}_{\mathfrak{T}_T I}\imath$, and taking the quotient by $\mathrm{Sq}^2_{\mathrm{h}}\delta^{\mathrm{v}}_{\mathfrak{T}_T I}\imath$ ensures that these generators produce no more material in $F^{\mathcal{M}_{\mathrm{h}}(1)}F^{\mathcal{M}_{\mathrm{v}}(1)}\{\imath\}$ than the polynomial algebras arising in the $S = 1$ case of Corollary 14.15. $\square$

14.6. An alternative Bousfield-Kan E_1-page

We will now suggest a somewhat artificial E_1-page for the BKSS for a sphere $X = \mathbb{S}_T^{\mathscr{C}om}$ for $T \geq 1$, but one that will be motivated by the conjectures and calculations of §16. Define:

$$\begin{aligned} \mathrm{adm}(\mathcal{A}_{>1}, s) &:= \{J \mid J \text{ a Sq-admissible sequence with } m(I) \leq s+1,\ 1 \notin J\}\,; \\ \mathrm{adm}^{\mathrm{irr}}(\mathcal{A}_{>1}, s) &:= \{J \mid J \text{ a Sq-admissible sequence with } e(I) \leq s,\ 1 \notin J\}\,; \\ \mathrm{adm}(\Delta, T) &:= \{I \mid I \text{ a } \delta\text{-admissible sequence with } \overline{m}(I) \leq T\}\,. \end{aligned}$$

The difference between $\mathrm{adm}(\Delta, T)$ and $\mathrm{adm}_+(\Delta, T)$ is just that we have removed the requirement that I be non-empty. The following lemma explains the sense in which $\mathrm{adm}^{\mathrm{irr}}(\mathcal{A}_{>1}, s)$ is the subset of irreducible sequences in $\mathrm{adm}(\mathcal{A}_{>1}, s)$.

Lemma 14.19. *There is an injective function* $\mathfrak{S}_t : \mathrm{adm}(\mathcal{A}_{>1}, s) \longrightarrow \mathrm{adm}(\mathcal{A}_{>1}, s)$ *given by*

$$J = (j_\ell, \ldots, j_1) \overset{\mathfrak{S}_s}{\longmapsto} (s + nJ + 1, j_\ell, \ldots, j_1).$$

Moreover, $\mathrm{adm}^{\mathrm{irr}}(\mathcal{A}_{>1}, s) = \mathrm{adm}(\mathcal{A}_{>1}, s) \setminus \mathrm{im}(\mathfrak{S}_s)$*, and*

$$\mathrm{adm}(\mathcal{A}_{>1}, s) = \textstyle\bigsqcup_{J \in \mathrm{adm}^{\mathrm{irr}}(\mathcal{A}_{>1}, s)} \{J, \mathfrak{S}_s J, \mathfrak{S}_s^2 J, \ldots\}.$$

The proof is similar to that of Lemma 9.10, but the outcome is a little different. Indeed, Lemma 14.19 shows that if a $\mathcal{M}_{\mathrm{h}}(1)$-expression $\mathrm{Sq}_{\mathrm{h}}^J x$ contains a top Steenrod operation, then all of the Steenrod operations following it are also top operations.

Define

$$[E_1'\mathcal{X}] := \mathbb{F}_2 \left\{ \textstyle\prod_{k=1}^N \mathrm{Sq}_{\mathrm{h}}^{J_k} \delta_{I_k}^{\mathrm{v}} \imath \;\middle|\; \begin{array}{c} I_k \in \mathrm{adm}(\Delta, T),\ J_k \in \mathrm{adm}(\mathcal{A}_{>1}, \ell I_k) \\ (J_k, I_k) \neq (J_{k'}, I_{k'}) \text{ unless } k' = k \end{array} \right\}, \tag{14.9}$$

and define a differential on $[E_1'\mathcal{X}]$ by:

(14.10) setting $d_1 \imath = 0$;

(14.11) requiring that d_1 distributes across the monomials in (14.9) according to the Leibniz rule;

(14.12) requiring, for $x \in [E_1'\mathcal{X}]_t^s$, that $\mathrm{Sq}_{\mathrm{h}}^{s+2} x = x d_1 x$ and $\mathrm{Sq}_{\mathrm{h}}^j x = 0$ for $j > s+2$;

(14.13) requiring, for $x \in [E_1'\mathcal{X}]_t^s$, that $d_1 \mathrm{Sq}_{\mathrm{h}}^j x = \mathrm{Sq}_{\mathrm{h}}^j d_1 x$;

(14.14) requiring, for $x \in [E_1'\mathfrak{X}]^s_t$, that $d_1\delta^{\mathrm{v}}_i x = \begin{cases} \delta^{\mathrm{v}}_i d_1 x, & \text{if } 2 \le i < t; \\ \delta^{\mathrm{v}}_i d_1 x + \mathrm{Sq}^2_{\mathrm{h}} x, & \text{if } 2 \le i = t; \end{cases}$

(14.15) enforcing the equation $\delta^{\mathrm{v}}_i \mathrm{Sq}^j_{\mathrm{h}} = 0$;

(14.16) enforcing the Sq-Adem relations and the identity $\mathrm{Sq}^1_{\mathrm{h}} = 0$;

(14.17) whenever a summand in the image of d_1 violates the requirement that the factors $\mathrm{Sq}^{J_k}_{\mathrm{h}} \delta^{\mathrm{v}}_{I_k} \imath$ be unique, applying the unstableness condition

$$(\mathrm{Sq}^{J_k}_{\mathrm{h}} \delta^{\mathrm{v}}_{I_k} \imath)^2 = \mathrm{Sq}^{\mathfrak{S}_{\ell I_k} J_k}_{\mathrm{h}} \delta^{\mathrm{v}}_{I_k} \imath.$$

Note that (14.12), (14.16) and (14.17) imply that $\imath^2 = 0$ and $\mathrm{Sq}^2_{\mathrm{h}}\imath = 0$. The key point is that we do not want the differential to be determined by manipulations such as:

$$d_1(\mathrm{Sq}^5_{\mathrm{h}} \delta^{\mathrm{v}}_{(22,10,5,2)}\imath) \text{ ``=''}\, d_1((\delta^{\mathrm{v}}_{(22,10,5,2)}\imath)^2) \text{ ``=''}\, 2(\delta^{\mathrm{v}}_{(22,10,5,2)}\imath)(d_1\delta^{\mathrm{v}}_{(22,10,5,2)}\imath) = 0,$$

which is why the phrasing of (14.11) and (14.17) is so restrictive. Indeed, when we define in §11 operations on the Bousfield-Kan spectral sequence, the top Steenrod operation will *not* equal the product square at E_1, but only at E_2, and we are mimicking this behaviour in our definition of $[E_1'\mathfrak{X}]$.

Let us calculate the proposed differential applied to a generator $\mathrm{Sq}^J_{\mathrm{h}} \delta^{\mathrm{v}}_I \imath$ of $[E_1'\mathfrak{X}]$ with $I \neq \emptyset$. Suppose that $I = (i_{\ell I}, \ldots, i_1)$, with $\delta^{\mathrm{v}}_{i_a}$ acting as a top operation at precisely the indices $a = a_n, \ldots, a_1$. Then we calculate,

$$\begin{aligned} d_1 \mathrm{Sq}^J_{\mathrm{h}} \delta^{\mathrm{v}}_I \imath &= \mathrm{Sq}^J_{\mathrm{h}} d_1 \delta^{\mathrm{v}}_I \imath \\ &= \textstyle\sum_{m=1}^n \mathrm{Sq}^J_{\mathrm{h}} \delta^{\mathrm{v}}_{(i_{\ell I}, \ldots, i_{a_m+1})} \mathrm{Sq}^2_{\mathrm{h}} \delta^{\mathrm{v}}_{(i_{a_m-1}, \ldots, i_1)} \imath \\ &= \begin{cases} \mathrm{Sq}^J_{\mathrm{h}} \mathrm{Sq}^2_{\mathrm{h}} \delta^{\mathrm{v}}_{(i_{\ell I-1}, \ldots, i_1)} \imath, & \delta^{\mathrm{v}}_{i_{\ell I}} \text{ a top operation, } 2, 3 \notin J,\ \ell I \ge 2; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The first equation holds by (14.13), and the second holds by (14.10) and (14.14). To explain the third equation, all of the n summands vanish by (14.15), except perhaps for the $m = n$ summand, which need not vanish when $a_n = i_{\ell I}$. Even this summand may still vanish, as (14.16) implies that $\mathrm{Sq}^J_{\mathrm{h}} \mathrm{Sq}^2_{\mathrm{h}}$ vanishes unless it is already Sq-admissible.

Although the definition of this complex seemed complicated, the differential ends up being quite simple. Indeed, one deduces that, writing $J = (j_{\ell J}, \ldots, j_1)$:

$$\text{if } J \in \mathrm{adm}(\mathcal{A}_{>1}, s) \text{ is non-empty and } 2, 3 \notin J, \text{ then } (j_{\ell J}, \ldots, j_1, 2) \in \mathrm{adm}^{\mathrm{irr}}(\mathcal{A}_{>1}, s-1).$$

From this, we conclude that if J is non-empty:

$$\mathrm{Sq}_{\mathrm{h}}^{J}\delta_{I}^{\mathrm{v}}\imath \text{ is a cycle if and only if } I \in \mathrm{adm}_{+}^{\mathrm{irr}}(\Delta, T) \text{ or } J \text{ contains 2 or 3,} \tag{14.18}$$

and if J is empty but I is non-empty:

$$\delta_{I}^{\mathrm{v}}\imath \text{ is a cycle if and only if } I \in \mathrm{adm}_{+}^{\mathrm{irr}}(\Delta, T). \tag{14.19}$$

(Recall that $\mathrm{adm}_{+}^{\mathrm{irr}}(\Delta, T)$ contains all of the length one sequences (i) for $2 \leq i \leq T$). We can combine all of this information into the following observation, valid for any I, J:

$$\mathrm{Sq}_{\mathrm{h}}^{J}\delta_{I}^{\mathrm{v}}\imath \text{ is a cycle if and only if } I = \emptyset \text{ or } I \in \mathrm{adm}_{+}^{\mathrm{irr}}(\Delta, T) \text{ or } J \text{ contains 2 or 3.} \tag{14.20}$$

The determination of the homology of $[E_1'\mathcal{X}]$ will follow from a generalization of this calculation made in §16.2, in particular Proposition 16.3. While the calculations in §16.2 are contingent on Conjectures 1 and 2, the statements are independent of these conjectures insofar as they apply to $[E_1'\mathcal{X}]$. As a result, we can state the following:

Corollary 14.20 (of Proposition 16.3)**.** *The homology of $[E_1'\mathcal{X}]$ is isomorphic, as a vector space, to $[E_2\mathcal{X}]$ as calculated in Corollary 14.18.*

Proof. The isomorphism of vector spaces $H_*[E_1'\mathcal{X}] \longrightarrow [E_2\mathcal{X}]$ sends the class of one of the cycles $\mathrm{Sq}_{\mathrm{h}}^{J}\delta_{I}^{\mathrm{v}}\imath$ of (14.20) to the element $\mathrm{Sq}_{\mathrm{h}}^{J}\delta_{I}^{\mathrm{v}}\imath$ of the subquotient of $F^{\mathcal{M}_{\mathrm{h}}(1)}F^{\mathcal{M}_{\mathrm{v}}(1)}\{\imath\}$ identified in Corollary 14.18. Proposition 16.3 provides a basis of $H_*[E_1'\mathcal{X}]$ which can be compared directly with that of the subquotient. $\square$

Chapter 15

A May-Koszul spectral sequence for $\mathcal{W}(0)$-cohomology

15.1. The quadratic filtration and resulting spectral sequence

Suppose that $X \in s\mathcal{W}(n)$ for $n \geq 0$, and write $\mathrm{QBX} \in s\mathcal{V}_n^+$ for the simplicial bar construction calculating $H_*^{\mathcal{W}(n)}X$:

$$(Q^{\mathcal{W}(n)}B^{\mathcal{W}(n)}X)_s \cong (F^{\mathcal{W}(n)})^s X_s.$$

We may view the vector space $\mathcal{U}^{\mathcal{W}(n)}X$ as being quadratically graded, concentrated in quadratic grading 1, and as explained in §12.3, the monad $F^{\mathcal{W}(n)}$ may be promoted to a monad on $\mathrm{q}\mathcal{V}_n^+$, so that QBX is quadratically graded in each simplicial degree individually.

We derive from these gradings the *quadratic filtration*, the following increasing filtration of $N_*\mathrm{QBX} \in \mathrm{ch}_+\mathcal{V}_n^+$:

$$F_m N_*\mathrm{QBX} = \bigoplus\nolimits_{k \leq m} \mathrm{q}_k N_*\mathrm{QBX}.$$

This definition is the direct analogue of Priddy's definition [46]. It appears difficult to use his techniques to calculate, say, $H^*_{\mathcal{W}(0)} H^*_{\mathscr{C}om} \mathbb{S}_T^{\mathscr{C}om}$ directly, as the bar construction in $\mathcal{W}(0)$ grows so much faster than the bar construction in a category of modules, and the resulting spectral sequence is not degenerate. Nonetheless, the quadratic filtration is in finite in each internal degree:

Lemma 15.1. *Suppose that $n \geq 0$, $X \in s\mathcal{W}(n)$, and $k \geq 0$. Then for any $s_k, \ldots, s_1 \geq 0$ and $t \geq 1$,*

$$(F_{2^{t-1}} N_*\mathrm{QBX})^t_{s_k,\ldots,s_1} = (N_*\mathrm{QBX})^t_{s_k,\ldots,s_1}.$$

Proof. This follows from the observation that every possible unary (resp. quadratic) operation increases t by at least one and doubles quadratic gradings (resp. adds quadratic gradings). It is obvious in dimension $t = 1$, as there can have been no non-trivial operations applied in this dimension (the grading t is always non-negative). The full statement follows by induction on t. $\square$

Moreover, there is an isomorphism

$$[E^0 N_* \mathrm{QBX}] \cong N_* Q^{\mathcal{W}(n)} B^{\mathcal{W}(n)} K^{\mathcal{W}(n)} U^{\mathcal{W}(n)} X$$

of chain complexes, so that:

Proposition 15.2. *The cohomotopy spectral sequence of the quadratic filtration is a strongly convergent spectral sequence, the* May-Koszul *spectral sequence:*

$$[E_1^{\mathrm{MK}} N_* \mathrm{QBX}]_t^{m,s_n,\ldots,s_1} \cong \mathrm{q}_m (H^*_{\mathcal{W}(n)} K^{\mathcal{W}(n)} U^{\mathcal{W}(n)} X)_t^{s_n,\ldots,s_1} \implies (H^*_{\mathcal{W}(n)} X)_t^{s_n,\ldots,s_1}.$$

If $\pi_* X$ is of finite type, the E_1-page may be rewritten as:

$$[E_1^{\mathrm{MK}} N_* \mathrm{QBX}]_t^{m,s_n,\ldots,s_1} \cong \mathrm{q}_m (F^{H\mathcal{W}(n)} \mathbf{D}(\pi_* X))_t^{s_n,\ldots,s_1},$$

which reduces when $n \geq 1$ to:

$$[E_1^{\mathrm{MK}} N_* \mathrm{QBX}]_t^{m,s_n,\ldots,s_1} \cong \mathrm{q}_m (F^{\mathcal{M}_{\mathrm{hv}}(n+1)} \mathbf{D}(\pi_* X))_t^{s_n,\ldots,s_1}.$$

Notes that all of the spectral sequence operations defined in §8 respect the quadratic filtration — the unary operations double quadratic filtrations while the pairing operations sum then. We leave it to the interested reader to derive the resulting theory of operations in the May-Koszul spectral sequence from this fact, for any $n \geq 0$.

15.2. A vanishing line on the Bousfield-Kan E_2-page

It is possible to obtain by the following method a vanishing line of slope $4/5$ whenever $\pi_1 X$ is of finite type. In the interest of brevity however, we prove only the following:

Theorem 15.3. *If $X \in s\mathscr{C}om$ is connected (with $\pi_* X$ not necessarily of finite type) then the BKSS admits a vanishing line on E_2 of slope 1 and intercept 0:*

$$[E_2 \mathcal{X}]_t^s = 0 \quad \text{whenever} \quad s \geq 1 \cdot (t - s).$$

Proof. We will prove that the right derived functors

$$((\mathbb{R}^s \operatorname{Pr}^{H\mathscr{C}om-\mathrm{coalg}}) W)_t$$

have such a vanishing line for any $W \in H\mathscr{C}om$−coalg with $W_0 = 0$. Any such W is the union of its finite-dimensional subobjects, as all of the structure maps in $\mathcal{W}(0)$ increase the degree t, so it is enough to prove this Proposition for finite-dimensional W. Then, by passing to duals, it is enough to produce a vanishing line in the isomorphic vector space

$$H^*_{\mathcal{W}(0)}\mathbf{D}W.$$

This group is calculated by the May-Koszul spectral sequence whose E_1-page is given by

$$[E_1^{\mathrm{MK}}]_t^{m,s} \cong \mathrm{q}_m(H^*_{\mathcal{W}(0)}K^{\mathcal{W}(0)}U^{\mathcal{W}(0)}\mathbf{D}W)^s_t.$$

Now $K^{\mathcal{W}(0)}U^{\mathcal{W}(0)}\mathbf{D}W$ decomposes as a product (for various $T_i \geq 1$):

$$K_0^{\mathcal{W}(0),T_1} \times \cdots \times K_0^{\mathcal{W}(0),T_N},$$

so if we can prove that

$$(H^*)^s_t := (H^*_{\mathcal{W}(0)}K_0^{\mathcal{W}(0),T})^s_t = 0 \quad \text{whenever} \quad s \geq t - s,$$

the same will be true for $H^*_{\mathcal{W}(0)}\mathbf{D}W$ by Theorem 14.6. However, we have already calculated these groups in Corollary 14.18, and found that $(H^*)^s_t$ is spanned by the image of $\imath \in (H^*)^0_T$ under various $\mathcal{M}_{\mathrm{v}}(1)$- and $\mathcal{M}_{\mathrm{h}}(1)$-operations. All of these operations preserve the half-plane specified by $s < t - s$. $\square$

Chapter 16

The Bousfield-Kan spectral sequence for $\mathbb{S}_T^{\mathscr{C}om}$

For any $T \geq 1$, let $X = \mathbb{S}_T^{\mathscr{C}om}$, so that we may write $[E_r\mathcal{X}]$ for the Bousfield-Kan spectral sequence of the sphere $\mathbb{S}_T^{\mathscr{C}om}$. In this chapter we will give conjectures which will allows us to construct a complete system of differentials in $[E_r\mathcal{X}]$, that would explain the convergence of $[E_2\mathcal{X}]$ (whose underlying vector space was calculated in Corollary 14.18) to

$$\pi_*(\mathbb{S}_T^{\mathscr{C}om}) \cong \Lambda(\mathscr{C})[\delta_I\imath \mid I \in \mathrm{adm}^e(\Delta, T)].$$

Here $\imath \in \pi_T(\mathbb{S}_T^{\mathscr{C}om})$ is the fundamental class (c.f. Proposition 5.6), and we write

$$\mathrm{adm}^e(\Delta, T) := \{I \mid I \text{ is } \delta\text{-admissible, } e(I) \leq T\}.$$

16.1. Some conjectures on the E_1-level structure

In order to construct all of the differentials needed, we will *assume* from this point on:

Conjecture 1. *It is possible to modify the definitions of the spectral sequence operations μ, Sq_h^j and δ_i^v defined in §11.3 in order that the* Sq-*Adem relations and the relations $\delta_i^\mathrm{v}\mathrm{Sq}_\mathrm{h}^j = 0$ hold on E_1 (without compromising the existing properties of these operations summarized in Proposition 11.2 and Corollaries 11.3-11.7).*

That is, we will replace the operations defined in §11.3 with their conjectural counterpart (without no change of notation).

Recall the alternative Bousfield-Kan E_1-page defined in §14.6, written $[E_1'\mathcal{X}]_t^s$. There was already a map of vector spaces $[E_1'\mathcal{X}] \longrightarrow [E_1\mathcal{X}]$ in $\mathcal{V}_+^1$ defined by

$$[E_1'\mathcal{X}] \ni \prod_{k=1}^N \mathrm{Sq}_\mathrm{h}^{J_k}\delta_{I_k}^\mathrm{v}\imath \longmapsto \prod_{k=1}^N \mathrm{Sq}_\mathrm{h}^{J_k}\delta_{I_k}^\mathrm{v}\imath \in [E_1\mathcal{X}],$$

and using the conjectural definitions of the operations on $[E_1\mathcal{X}]$, it is a map of chain complexes. Indeed, we may calculate the differential in $[E_1\mathcal{X}]$ exactly as we calculated in §14.6. Thus, there is an induced map $[E'_2\mathcal{X}] \longrightarrow [E_2\mathcal{X}]$ (where we write $[E'_2\mathcal{X}]$ for the homology of the chain complex $[E'\mathcal{X}]$). From now on, we will *also* assume:

Conjecture 2. *The induced map $[E'_2\mathcal{X}] \longrightarrow [E_2\mathcal{X}]$ is an isomorphism (of vector spaces).*

This conjecture is not so unreasonable, since by Corollary 14.18 there is an isomorphism of vector spaces $[E'_2\mathcal{X}] \longrightarrow [E_2\mathcal{X}]$ given by mapping an element of $[E'_2\mathcal{X}]$ to the element of $[E_2\mathcal{X}]$ of the same name, under the calculation of $[E_2\mathcal{X}]$ given by Corollary 14.18. In any case, we assume no more than the stated conjectures.

16.2. The resulting differentials

We will now analyze the differentials d_r applied to the various terms $\mathrm{Sq}_{\mathrm{h}}^J \delta_I^{\mathrm{v}} \imath$. Define functions

$$\ell, n, e : \mathrm{adm}(\Delta, T) \longrightarrow \{0, 1, 2, \ldots\}$$

which evaluate on a sequence $I = (i_l, \ldots, i_1)$ as follows:

$$\ell(I) := l; \quad n(I) := i_1 + \cdots + i_l; \quad e(I) := i_l - i_{l-1} - \cdots - i_1 = 2i_l - n(I).$$

Define a function

$$a : \mathrm{adm}(\Delta, T) \longrightarrow \mathbb{Z}$$

by $a(I) := \ell(I) - 1 - (e(I) - T)$. Now write $\mathcal{G}$ for the set of (J, I) involved in the definition of $[E'_1\mathcal{X}]$:

$$\mathcal{G} := \{(J, I) \mid I \in \mathrm{adm}(\Delta, T),\ J \in \mathrm{adm}(\mathcal{A}_{>1}, \ell(I))\}.$$

We may decompose $\mathcal{G}$ into three subsets:

$$\begin{aligned}
\mathcal{G}^e &:= \{(\emptyset, I) \mid I \in \mathrm{adm}^e(\Delta, T)\}, \\
\mathcal{G}' &:= \{(J, I) \in \mathcal{G} \setminus \mathcal{G}^e \mid J = (j_{\ell(J)}, \ldots, j_1),\ \ell(J) = 0 \ \text{ or } \ a(I) + 3 - j_1 < 0\}, \\
\mathcal{G}'' &:= \{(J, I) \in \mathcal{G} \setminus \mathcal{G}^e \mid J = (j_{\ell(J)}, \ldots, j_1),\ \ell(J) > 0 \text{ and } a(I) + 3 - j_1 \geq 0\}.
\end{aligned}$$

Every class $\delta_I^{\mathrm{v}} \imath$ for $(\emptyset, I) \in \mathcal{G}^e$ is a permanent cycle. On the other hand, we will prove:

Proposition 16.1. *Assuming Conjectures 1 and 2, there is a bijective map $g : \mathcal{G}' \longrightarrow \mathcal{G}''$ such that if $g(J, I) = (J', I')$ then there is a differential $d_r : \mathrm{Sq}_{\mathrm{h}}^J \delta_I^{\mathrm{v}} \imath \longmapsto \mathrm{Sq}_{\mathrm{h}}^{J'} \delta_{I'}^{\mathrm{v}} \imath$.*

The class $\delta_I \imath$ is a permanent cycle if $I \in \mathrm{adm}^e(\Delta, T)$. On the other hand, using Conjecture 1 we may mimic the calculation of $d_1 \mathrm{Sq}_{\mathrm{h}}^J \delta_I^{\mathrm{v}} \imath$ made in §14.6. We find that if

$I \in \mathrm{adm}(\Delta, T) \setminus \mathrm{adm}^e(\Delta, T)$, $\delta_I \imath$ survives to $E_{a(I)+1}$, at which point

$$d_{a(I)+1} : \delta_I \imath \longmapsto \mathrm{Sq}_{\mathrm{h}}^{a(I)+2} \delta_{I^-} \imath, \tag{16.1}$$

where we write I^- for the sequence $(i_{\ell(I)-1}, \dots, i_1)$ obtained by removing the outermost entry of I.

For an element $J \in \mathrm{adm}(\mathcal{A}_{>1}, s)$ with $J = (j_{\ell(J)}, \dots, j_1)$, and any $n \geq 2 - j_1$ we will write $\Phi_n J$ for the sequence

$$\Phi_n J := (j_{\ell(J)} + 2^{\ell(J)-1} n, \dots, j_2 + 2n, j_1 + n) \in \mathrm{adm}(\mathcal{A}_{>1}, s + n).$$

Then there is a differential, obtained by applying $\mathrm{Sq}_{\mathrm{h}}^J$ to the $d_{a(I)+1}$-differential (16.1):

$$d_{2^{\ell(J)} a(I)+1} : \mathrm{Sq}_{\mathrm{h}}^J \delta_I \imath \longmapsto \mathrm{Sq}_{\mathrm{h}}^{\Phi_{a(I)} J} \mathrm{Sq}_{\mathrm{h}}^{a(I)+2} \delta_{I^-} \imath = \mathrm{Sq}_{\mathrm{h}}^{J^+} \delta_{I^-} \imath,$$

where $J^+ := (j_{\ell(J)} + 2^{\ell(J)-1} a(I), \dots, j_2 + 2a(I), j_1 + a(I), a(I) + 2)$. We define the map g to send this (J, I) to (J^+, I^-) whenever $(J, I) \in \mathcal{G}'$.

Proof of Proposition 16.1. Firstly, we should check that g is well defined. Suppose first that $J = \emptyset$. Then we must have $e(I) > T$, so that

$$a(I) + 2 = \ell(I) - 1 - (e(I) - T) + 2 \leq \ell(I^-) + 1,$$

a condition required for J^+ to have any chance of lying in $\mathrm{adm}(\mathcal{A}_{>1}, \ell(I^-))$. After this initial check, it is easy to check the condition required of $\overline{m}(J^+)$. Thus, $(J, I) \in \mathcal{G} \setminus \mathcal{G}^e$. We must also check that $a(I^-) + 3 - (a(I) + 2) \geq 0$, i.e. that $a(I^-) - a(I) \geq -1$, which reduces to the tautological condition $e(I) \geq e(I^-)$. Thus g is well defined.

The injectivity of g is obvious, but we must check its surjectivity. Suppose for this purpose that $(J, I) \in \mathcal{G}''$, so that $a(I) + 3 - j_1 \geq 0$. We will begin by producing a differential

$$d_{j_1 - 1} : \delta^{\mathrm{v}}_{I^+} \imath \longmapsto \mathrm{Sq}_{\mathrm{h}}^{j_1} \delta^{\mathrm{v}}_I \imath$$

with I^+ a δ-admissible sequence $(i_{\ell(I)+1}, i_{\ell(I)}, \dots, i_1)$. For this, we need $e(I^+) \geq e(I)$ (to ensure admissibility of I^+) and $a(I^+) = j_1 - 2$, but we are otherwise unconstrained, as $j_1 \geq 2$, and the demand $a(I^+) \geq 0$ will ensure that the additional $\delta_{i_{\ell(I)+1}}$ is defined. Focusing on the constraint $a(I^+) = j_1 - 2$:

$$\ell(I^+) - 1 - (e(I^+) - T) = j_1 - 2 \iff e(I^+) - e(I) = a(I) + 3 - j_1,$$

but we have assumed that $a(I) + 3 - j_1$ is non-negative, so we have no difficulty satisfying this constraint.

Now we use the sequence $\Phi_{-a(I^+)}J^-$ where $J^- := (j_{\ell(J)}, \ldots, j_2)$, producing the required differential

$$d_{2^{\ell(J)-1}a(I^+)+1} : \mathrm{Sq}_{\mathrm{h}}^{\Phi_{-a(I^+)}J^-}\delta^{\mathrm{v}}_{I^+}\imath \longrightarrow \mathrm{Sq}^J_{\mathrm{h}}\delta_I\imath,$$

as long as either J^- is empty or $a(I^+) + 3 - (j_2 - a(I^+)) < 0$. If J^- is non-empty, then the second condition reduces to the condition that the concatenation $\Phi_{a(I^+)}\Phi_{-a(I^+)}J^- \star (a(I^+) + 2)$ be Sq-admissible, but this concatenation is J itself. □

Proposition 16.2. *Assuming Conjectures 1 and 2, the differentials given in Proposition 16.1, along with those arising from them by taking products and applying the Leibniz formula, are a complete set of differentials for the BKSS for this sphere.*

Proof. Although the E_r-page of the spectral sequence is not an exterior algebra for any finite r, we are working in a spectral sequence of commutative $\mathbb{F}_2$-algebras. As such, the differential is not sensitive to the difference between the polynomial algebra $S(\mathscr{C})[x]$ and the exterior algebra $\Lambda(\mathscr{C})[x, x_2, x_4, x_8 \ldots]$ where x_{2^i} is placed in the dimension of x^{2^i}. In this setting, the upshot is that the E_r-page is isomorphic as a chain complex to an infinite coproduct of exterior algebras, starting with

$$[E'_1] \cong \coprod_{(J,I)\in\mathcal{G}} \Lambda(\mathscr{C})[\mathrm{Sq}^{J_k}_{\mathrm{h}}\delta^{\mathrm{v}}_{I_k}\imath].$$

We rely on the properties of the Steenrod operations to allow us to deal with terms of the form $x \times x$. Whatever the explanation, the differentials given in Proposition 16.1 are enough to eliminate all summands except for

$$[E_\infty] \cong \coprod_{(\emptyset,I)\in\mathcal{G}^e} \Lambda(\mathscr{C})[\delta^{\mathrm{v}}_{I_k}\imath],$$

which really is isomorphic as an algebra to the target $\pi_*(\mathbb{S}^{\mathscr{C}om}_T)$. □

Filtering the sets $\mathcal{G}'$ and $\mathcal{G}''$ by the length of the differentials associated with their elements, so that

$$\mathcal{G}'_r := \{(J, I) \in \mathcal{G}' \mid \text{if } g(J, I) = (J', I') \text{ then } n(J') + \ell(I') \geq n(J) + \ell(I) + r\},$$

and $\mathcal{G}''_r := \mathrm{im}(g|_{\mathcal{G}'_r})$, the proofs of Propositions 16.1 and 16.2 also prove:

Proposition 16.3. *Assuming Conjectures 1 and 2, there is an isomorphism of chain com-*

plexes, for $r \geq 2$:

$$[E_r] \cong \coprod_{(J,I)\in\mathcal{G}} \Lambda(\mathscr{C})[\mathrm{Sq}_{\mathrm{h}}^{J_k}\delta_{I_k}^{\mathrm{v}}\imath] \sqcup \coprod_{(J,I)\in\mathcal{G}'_r} \Lambda(\mathscr{C})[\mathrm{Sq}_{\mathrm{h}}^{J_k}\delta_{I_k}^{\mathrm{v}}\imath] \sqcup \coprod_{(J,I)\in\mathcal{G}''_r} \Lambda(\mathscr{C})[\mathrm{Sq}_{\mathrm{h}}^{J_k}\delta_{I_k}^{\mathrm{v}}\imath].$$

Moreover, the complete calculation of the BKSS for a finite connected model in $s\mathscr{C}om$ now follows simply by taking the coproduct of non-unital differential graded algebras at each page, with the appropriate grading shifts, for example:

$$[E_r(\mathbb{S}_{T_1}^{\mathscr{C}om} \sqcup \mathbb{S}_{T_2}^{\mathscr{C}om})] \cong [E_r\mathbb{S}_{T_1}^{\mathscr{C}om}] \sqcup [E_r\mathbb{S}_{T_2}^{\mathscr{C}om}].$$

Appendix A

Cohomology operations for Lie algebras

In this appendix, we will prove that Priddy's definitions of cohomology operations for simplicial (restricted) Lie algebras coincides with our own. There are three settings which we are interested in: the categories $s\mathscr{L}ie$, $s\mathscr{L}ie^r$ and $s\mathcal{L}(n)$ for $n \geq 0$. We will work in the third setting in this appendix, as the proofs in the other two cases are strictly simpler.

A.1. The partially restricted universal enveloping algebra

For the following discussion, we will need one last category of graded vector spaces, $\mathcal{V}_n^-$, an object of which is simply the direct sum of an object V of $\mathcal{V}_n^+$ and a vector space $V_{0,\dots,0}^{-1}$:

$$V = V_{0,\dots,0}^{-1} \oplus \bigoplus_{t\geq 1} \bigoplus_{s_n,\dots,s_1\geq 0} V_{s_n,\dots,s_1}^t \in \mathcal{V}_n^-.$$

Denote by $\mathcal{A}(n)$ the following category of graded augmented associative algebras. An object of $\mathcal{A}(n)$ is a graded vector space $A \in \mathcal{V}_n^-$ such that $A_{0,\dots,0}^{-1} = \mathbb{F}_2\langle 1\rangle$ is one-dimensional, spanned by the unit of an associative unital pairing

$$A_{s_n,\dots,s_1}^t \otimes A_{p_n,\dots,p_1}^q \longrightarrow A_{s_n+p_n,\dots,s_1+p_1}^{t+q+1}.$$

That is, $A_{0,\dots,0}^{-1}$ is not part of the data of A, but only a graded piece added to hold the unit. Such an algebra is certainly augmented, and the augmentation ideal may be viewed as a forgetful functor $I : \mathcal{A}(n) \longrightarrow \mathcal{L}(n)$, which sends A to the partially restricted Lie algebra

$$\bigoplus_{t\geq 1} \bigoplus_{s_n,\dots,s_1\geq 0} A_{s_n,\dots,s_1}^t,$$

with bracket $[x,y] := xy - yx$, and restriction operation $x^{[2]} := x^2$ whenever $x \in A^t_{s_n,\ldots,s_1}$ and not all of $s_n, \ldots, s_1$ zero.

The composite forgetful functor $\mathcal{A}(n) \xrightarrow{I} \mathcal{L}(n) \longrightarrow \mathcal{V}_n^+$ has a left adjoint, none other than the *free associative algebra functor* $F^{\mathcal{A}(n)}$ (also known as the *tensor algebra functor*). The multiplicative unit 1 is placed in $A^{-1}_{0,\ldots,0}$, as is appropriate given the grading shift. Moreover, the functor I has a left adjoint, U', *the partially restricted universal enveloping algebra* functor, with $U'L$ obtained as the quotient of $F^{\mathcal{A}(n)}L$ by the two-sided ideal generated by any $[x,y] - xy - yx$ and by $x^{[2]} - x^2$ with x of restrictable degree. Indeed, there is a composite of adjunctions

$$\mathcal{V}_n^+ \underset{\text{forget}}{\overset{F^{\mathcal{L}(n)}}{\rightleftarrows}} \mathcal{L}(n) \underset{I}{\overset{U'}{\rightleftarrows}} \mathcal{A}(n),$$

showing that $U' \circ F^{\mathcal{L}(n)} \cong F^{\mathcal{A}(n)}$. As in the non-restricted and fully restricted case, $U'L$ is naturally a Hopf algebra, having diagonal defined by the requirement $\Delta x = 1 \otimes x + x \otimes 1$ for $x \in L \subseteq U'L$, and:

Lemma A.1 (PBW Theorem)**.** *If $L \in \mathcal{L}(n)$, then there is a natural increasing filtration of $U'L$, the Lie filtration (by powers of $\langle 1 \rangle \oplus \mathrm{im}(L \longrightarrow U'(L))$), and the associated graded algebra is naturally isomorphic to $\mathbb{F}_2[L_{\mathbf{0}}] \otimes E[L_{\neq \mathbf{0}}]$, where $L = L_{\mathbf{0}} \oplus L_{\neq\mathbf{0}}$ is the decomposition of L into the sum of its subspaces of in non-restrictable and restrictable degrees respectively.*

Here, $\mathbb{F}_2[-]$ and $E[-]$ denote the (shifted, unital) polynomial and exterior algebra functors respectively, which differ from $S(\mathscr{C})$ and $\Lambda(\mathscr{C})$ only by the addition of the unit in $(\mathbb{F}_2[-])^{-1}_{0,\ldots,0}$ and $(E[-])^{-1}_{0,\ldots,0}$. The unit $1 \otimes 1$ of this tensor product is in $(\mathbb{F}_2[-] \otimes E[-])^{-1}_{0,\ldots,0}$, as the product has a $+1$-shift in the cohomological dimension.

Lemma A.2. *The prolonged functor $U' : s\mathcal{L}(n) \longrightarrow s\mathcal{A}(n)$ preserves weak equivalences.*

Proof. Suppose that $L \longrightarrow L'$ is a weak equivalence in $s\mathcal{L}(n)$. The Lie filtration makes $C_*(U'L) \longrightarrow C_*(U'L')$ a map of filtered commutative differential graded algebras, so there is an induced map of the resulting spectral sequences. By Lemma A.1, the E^0-page of the spectral sequence for $U'L$ is the differential graded algebra $C_*(\mathbb{F}_2[L_{\mathbf{0}}] \otimes E[L_{\neq\mathbf{0}}])$. By Dold's Theorem (2.4), the E^1-page is a functor (determined by the results of §5.4) of $\pi_*(L_{\mathbf{0}})$ and $\pi_*(L_{\neq\mathbf{0}})$. As the induced maps $\pi_*(L_{\mathbf{0}}) \longrightarrow \pi_*(L'_{\mathbf{0}})$ and $\pi_*(L_{\neq\mathbf{0}}) \longrightarrow \pi_*(L'_{\neq\mathbf{0}})$ are isomorphisms, the map of spectral sequences is an isomorphism from E^1. ☐

A.2. The proof of Proposition 6.12

In this section we will demonstrate Proposition A.3, which is stated for *partially restricted* Lie algebras $L \in s\mathcal{L}(n)$, but can be reinterpreted for objects of $s\mathscr{L}ie$ or $s\mathscr{L}ie^r$ as necessary. From this result, Propositions 6.12 and 8.9 follow.

Let $L \in s\mathcal{L}(n)$ be almost free on a fixed choice of subspaces $V_p \subseteq L_p$. We will use a bisimplicial model for $\bar{W}U'L$:

$$\mathbf{B}_{pq} := \bar{B}_q U' L_p = (U'L_p)^{\otimes q} \in ss\mathcal{V}_n^-,$$

which in each simplicial level p is the standard simplicial bar construction for calculation of $\mathrm{Tor}^{U'L_p}(\mathbb{F}_2, \mathbb{F}_2)$ (c.f. [46, §1]). There are natural equivalences

$$C_*|\mathbf{B}| \simeq \mathrm{Tot}(C_*C_*\mathbf{B}) = \mathrm{Bar}(C_*U'L) \simeq C_*\bar{W}U'L,$$

so that $\pi^*\mathbf{D}|\mathbf{B}| \cong H^*_{\bar{W}}L$. Here, we have written Bar for the bar construction of [28, §7], and the final equivalence is the homomorphism of [28, Theorem 20.1]. What is a little less well known is that there is a natural weak equivalence of simplicial coalgebras underlying this equivalence of chain complexes, given in [15, Theorem 1.1]. A simple construction of such a map $|\mathbf{B}| \longrightarrow \bar{W}U'L$ is, in simplicial level n:

$$d_0 \otimes d_0^{\circ 2} \otimes \cdots \otimes d_0^{\circ n} : (U'X_n)^{\otimes n} \longrightarrow U'X_{n-1} \otimes \cdots \otimes U'X_0,$$

where we use the conventions of [42, §5] to define $\bar{W}$.

As such, the operations defined by Priddy on $H^*_{\bar{W}}$ correspond, under this equivalence, to those that we define on $\pi^*\mathbf{D}|\mathbf{B}|$ by the formulae

$$\mathrm{Sq}^k : \left(\pi^n\mathbf{D}|\mathbf{B}| \xrightarrow{\mathrm{Sq}^k_{\mathrm{ext}}} \pi^{n+k}\mathbf{D}S^2|\mathbf{B}| \xrightarrow{\Delta^*_{\mathbf{B}}} \pi^{n+k}\mathbf{D}|\mathbf{B}|\right);$$
$$\mu : \left(S_2(\pi^*\mathbf{D}|\mathbf{B}|) \xrightarrow{\mu_{\mathrm{ext}}} \pi^*S_2\mathbf{D}|\mathbf{B}| \longrightarrow \pi^*\mathbf{D}S^2|\mathbf{B}| \xrightarrow{\Delta^*_{\mathbf{B}}} \pi^*\mathbf{D}|\mathbf{B}|\right).$$

where $\Delta_{\mathbf{B}}$ is the bisimplicial cocommutative coalgebra diagonal:

$$\Delta_{\mathbf{B}} : \left(\bar{B}(U'L) \xrightarrow{\bar{B}(\Delta)} \bar{B}(U'L \otimes U'L) \cong \bar{B}(U'L) \otimes \bar{B}(U'L)\right).$$

Thus, we may forget the functor $\bar{W}$, and restrict our attention to the object $\mathbf{B}$ with this coalgebra map. We are also going to use the simplicial chain complex $\mathbf{Q} \in s\,\mathrm{ch}_+\mathcal{V}_n^-$:

$$\mathbf{Q}_{\bullet *} := \begin{cases} Q^{\mathcal{L}(n)}L_\bullet, & \text{if } * = 1; \\ \mathbb{F}_2\{1\}, & \text{if } * = 0; \\ 0, & \text{otherwise.} \end{cases}$$

with zero differentials in each simplicial level. Of course, we mean that $1 \in (\mathbf{Q}_{0,0})^{-1}_{0,\ldots,0}$. There is a map of simplicial chain complexes $r : N^{\mathrm{v}}_*\mathbf{B}_\bullet \longrightarrow \mathbf{Q}_{\bullet *}$, defined in level p by the identification $N_0^{\mathrm{v}}\mathbf{B}_p = \mathbb{F}_2\{1\} = \mathbf{Q}_{p0}$ and the composite:

$$N_1^{\mathrm{v}}\mathbf{B}_p = IU'L_p \twoheadrightarrow IU'L_p/(IU'L_p)^2 \cong Q^{\mathcal{L}(n)}L_p.$$

Proposition A.3. *The composite*

$$N_*|\mathbf{B}| \simeq \mathrm{Tot}(N_*^{\mathrm{h}}N_*^{\mathrm{v}}\mathbf{B}) \xrightarrow{r} \mathrm{Tot}(N_*^{\mathrm{h}}\mathbf{Q}_{\bullet *}) = \mathbb{F}_2 \oplus \Sigma N_* Q^{\mathcal{L}(n)}L$$

is a weak equivalence of chain complexes under which the operations on $\pi^*\mathbf{D}|\mathbf{B}|$ defined using $\Delta_{\boldsymbol{B}}$ correspond to the operations $\mathrm{Sq}^k := \psi_{\mathcal{L}(n)} \circ \mathrm{Sq}_{\mathrm{ext}}^{k-1}$ and $\mu := \psi_{\mathcal{L}(n)} \circ \mu_{\mathrm{ext}}$ on $\pi^*(\mathbf{D}(Q^{\mathcal{L}(n)}L)) =: H^*_{\mathcal{L}(n)}L$.

We will prove this proposition using the external spectral sequence operations of §13.1 in the spectral sequence of $\mathbf{B}$. By E_2, the only interesting non-zero entries of this spectral sequence lie on the horizontal line $q = 1$, so that Singer's operations will prove very uninteresting without modification. Our method will be to perform such a modification by using the chain homotopy h (defined shortly) to shift the horizontal operations one higher in filtration. The shifted homotopy operations will preserve the line $q = 1$, and will abut to operations on E_∞ that satisfy the same relations as those on $|\mathbf{B}|$. As the abutment filtration is trivial, they must satisfy the same relations at E_2. Finally, we will note that what we have produced at E_2 is the definition of the Steenrod operations from §6.8.

As L is levelwise free, the evident map $F^{\mathcal{A}(n)}V_p \longrightarrow U'L_p$ is an isomorphism for each p, and we define a vertical homotopy $h : N_*^{\mathrm{v}}\mathbf{B}_p \longrightarrow N_{*+1}^{\mathrm{v}}\mathbf{B}_p$ by the following formulae (in which the v_{i_j} are taken to be in $V_p \subseteq L_p \subseteq U'L_p$):

$$\begin{aligned} h_q : N_q^{\mathrm{v}}\bar{B}U'L_p &\longrightarrow N_{q+1}^{\mathrm{v}}\bar{B}U'L_p \\ [\underbrace{v_{i_1}|\cdots|v_{i_{k-1}}}_{\text{length 1 bars}}|v_{i_k}v_{i_{k+1}}\cdots|\cdots] &\longmapsto [v_{i_1}|\cdots|v_{i_k}|v_{i_{k+1}}\cdots|\cdots] \\ [v_{i_1}|\cdots|v_{i_q}] &\longmapsto 0. \end{aligned}$$

This homotopy is of the same type as that used in §8, §9 and [46, Proof of Theorem 5.3], and commutes with all of the horizontal simplicial structure except d_0^{h}, so that $d^{\mathrm{h}}h_q + h_q d^{\mathrm{h}} = d_0^{\mathrm{h}}h_q + h_q d_0^{\mathrm{h}}$.

Lemma A.4. *Under the map* $(\mathrm{Id} + h_{q-1}d^{\mathrm{v}} + d^{\mathrm{v}}h_q) : N_q^{\mathrm{v}}\bar{B}U'L_p \longrightarrow N_q^{\mathrm{v}}\bar{B}U'L_p$,

$$[v_{i_1}\cdots|\cdots|\cdots v_{i_r}] \longmapsto 0 \textit{ unless } r = q = 1, \textit{ in which case}$$
$$[v_{i_1}] \longmapsto [v_{i_1}].$$

Lemma A.5. *The composite*

$$N_p^{\mathrm{h}} N_2^{\mathrm{v}} \mathbf{B} \xrightarrow{\Delta_{\mathbf{B}}} N_p^{\mathrm{h}} N_2^{\mathrm{v}} (\mathbf{B} \otimes \mathbf{B}) \xrightarrow{(D_{\mathrm{v}}^0)^\star} N_p^{\mathrm{h}} (N_1^{\mathrm{v}} \mathbf{B} \otimes N_1^{\mathrm{v}} \mathbf{B}) \xrightarrow{r \otimes r} N_p^{\mathrm{h}} (\mathbf{Q}_{\bullet 1} \otimes \mathbf{Q}_{\bullet 1})$$

vanishes except on terms $[x|y]$ with x and y generators of L_p, which have image $x \otimes y$.

Proof. A generic element of the domain is a sum of terms $[x_1 \cdots x_I | y_1 \cdots y_J]$, with $x_1, \ldots, x_I$ and $y_1, \ldots, y_J$ in $V_p \subseteq L_p$. This element maps under $\Delta_{\mathbf{B}}$ to the following sum, taken over all sequences of exponents $a_1, \ldots, a_I, b_1, \ldots, b_J \in \{0, 1\}$:

$$\sum \Big[x_1^{a_1} \cdots x_I^{a_I} \;\Big|\; y_1^{b_1} \cdots y_J^{b_J} \Big] \otimes \Big[x_1^{1-a_1} \cdots x_I^{1-a_I} \;\Big|\; y_1^{1-b_1} \cdots y_J^{1-b_J} \Big] \in N_p^{\mathrm{h}} N_2^{\mathrm{v}} (\mathbf{B} \otimes \mathbf{B}),$$

and $(D_{\mathrm{v}}^0)^\star$ annihilates all terms except for those in which all a_i are 1 and all b_j are 0, leaving

$$\big[x_1 \cdots x_I \big] \otimes \big[y_1 \cdots y_I \big] \in N_p^{\mathrm{h}} (N_1^{\mathrm{v}} \mathbf{B} \otimes N_1^{\mathrm{v}} \mathbf{B}).$$

Finally, $r \otimes r$ annihilates this term unless $I = J = 1$. □

Lemma A.6. *The composite*

$$N_{p+1}^{\mathrm{h}} N_1^{\mathrm{v}} \mathbf{B} \xrightarrow{\Delta_{\mathbf{B}}} N_{p+1}^{\mathrm{h}} N_1^{\mathrm{v}} (\mathbf{B} \otimes \mathbf{B}) \xrightarrow{(D_{\mathrm{v}}^1)^\star} N_{p+1}^{\mathrm{h}} (N_1^{\mathrm{v}} \mathbf{B} \otimes N_1^{\mathrm{v}} \mathbf{B}) \xrightarrow{r \otimes r} N_{p+1}^{\mathrm{h}} (\mathbf{Q}_{\bullet 1} \otimes \mathbf{Q}_{\bullet 1})$$

vanishes except on terms $[xy]$ with x and y generators of L_{p+1}, which have image $x \otimes y + y \otimes x$.

Proof. A generic element of the domain is a sum of terms $[x_1 \cdots x_I]$, with $x_1, \ldots, x_I$ in $V_{p+1} \subseteq L_{p+1}$. This element maps under $\Delta_{\mathbf{B}}$ to

$$\sum \Big[x_1^{a_1} \cdots x_I^{a_I} \Big] \otimes \Big[x_1^{1-a_1} \cdots x_I^{1-a_I} \Big].$$

As $\{D^k\}$ was chosen to be a special k-cup product, $(D_{\mathrm{v}}^1)^\star$ acts as the identity in this case. Finally, $r \otimes r$ annihilates this term unless $I = 2$ and $a_1 \neq a_2$. □

Lemma A.7. *There is a commuting diagram:*

$$\begin{tikzcd}
N_{n+k}^{\mathrm{h}} N_1^{\mathrm{v}} \mathbf{B} \arrow[r, "d_0^{\mathrm{h}} h_{n+k} + h_{n+k-1} d_0^{\mathrm{h}}"] \arrow[d, "r"] & N_{n+k-1}^{\mathrm{h}} N_2^{\mathrm{v}} \mathbf{B} \arrow[r, "(D_{\mathrm{v}}^0)^\star \circ \Delta_{\mathbf{B}}"] & N_{n+k-1}^{\mathrm{h}} (N_1^{\mathrm{v}} \mathbf{B} \otimes N_1^{\mathrm{v}} \mathbf{B}) \arrow[d, "r \otimes r"] \\
N_{n+k}^{\mathrm{h}} Q^{\mathcal{L}(n)} L \arrow[r, "Q^{\mathcal{L}(n)}(\xi_{\mathcal{L}(n)})"] & N_{n+k-1}^{\mathrm{h}} Q^{\mathcal{L}(n)} (L \veebar L) \arrow[r, "j_{\mathcal{L}(n)}"] & N_{n+k-1}^{\mathrm{h}} (Q^{\mathcal{L}(n)} L \otimes Q^{\mathcal{L}(n)} L)
\end{tikzcd}$$

Proof. Write LHS $= (r \otimes r) \circ (D_{\mathrm{v}}^0)^\star \circ \Delta_{\mathbf{B}} \circ (d_0^{\mathrm{h}} h + h d_0^{\mathrm{h}})$ and RHS $= \psi_{\mathcal{L}(n)} \circ r$. Consider first an element $e = [v_1 v_2 \cdots v_b]$ of $N_{n+k}^{\mathrm{h}} N_1^{\mathrm{v}} \mathbf{B}$ with $b \geq 2$. By definition, r vanishes on such an element, so that RHS$(e) = 0$. Lemma A.5 states that the map $(r \otimes r) \circ (D_{\mathrm{v}}^0)^\star \circ \Delta_{\mathbf{B}}$ vanishes except on expressions of the form $[u|w]$ for $u, w \in V_{n+k-1}$. However, the expressions of this form appearing in $d_0^{\mathrm{h}} h(e)$ coincide with such expressions in $h d_0^{\mathrm{h}}(e)$, so that there is a cancellation, and $\mathrm{LHS}(e) = 0$ as hoped.

Next, consider an element $[v]$ of $N_{n+k}^{\mathrm{h}} N_1^{\mathrm{v}} \mathbf{B}$. As $h[v] = 0$, and in light of Lemma A.5, $\mathrm{LHS}([v])$ equals the quadratic part of $d_0^{\mathrm{h}} v$, after writing $d_0^{\mathrm{h}} v$ as an expression in elements of V_{n+k-1}. This is exactly the description given in Lemma 6.3 of $\mathrm{RHS}([v]) = \psi_{\mathcal{L}(n)}(v)$. $\square$

Proof of Proposition A.3. Fix a cocycle $\alpha \in \mathbf{D}(N_n Q^{\mathcal{L}(n)} L)$. Then α may be viewed an a permanent cocycle in $[Z_\infty \mathbf{D}(\mathbf{Q}_{\bullet *})]^{n,1}$ in the spectral sequence obtained by dualizing $\mathbf{Q}_{\bullet *}$.

Singer [52, (2.14)] defines an operator S^k on the total cochain complex of a bisimplicial coalgebra which induces the cohomology operation $\mathrm{Sq}^k_{\mathrm{ext}}$. We will apply the chain-level operator S^k to the class $r^*\alpha \in [Z_\infty \mathbf{B}]^{n,1}$. As α is a permanent cycle, $d(r^*\alpha) = 0$, and Singer's expression simplifies to:

$$\begin{aligned} S^k(r^*\alpha) &:= \Delta^*_{\mathbf{B}} K^*_{n+1-k} \phi(r^*\alpha \otimes r^*\alpha) = T_1 + T_2, \text{ where:} \\ T_1 &:= \Delta^*_{\mathbf{B}} D^0_{\mathrm{v}} D^{n+1-k}_{\mathrm{h}} \phi(r^*\alpha \otimes r^*\alpha) \in \mathbf{D}(N^{\mathrm{h}}_{n+k-1} N^{\mathrm{v}}_2 \mathbf{B}) \\ T_2 &:= \Delta^*_{\mathbf{B}} D^1_{\mathrm{v}} (T D^{n-k}_{\mathrm{h}} T) \phi(r^*\alpha \otimes r^*\alpha) \in \mathbf{D}(N^{\mathrm{h}}_{n+k} N^{\mathrm{v}}_1 \mathbf{B}). \end{aligned}$$

Our method will be to compress each of these terms into filtration one higher, using the cochain homotopy $h^* : \mathbf{D}(N^{\mathrm{h}}_* N^{\mathrm{v}}_* \mathbf{B}) \longrightarrow \mathbf{D}(N^{\mathrm{h}}_* N^{\mathrm{v}}_{*-1} \mathbf{B})$. Using Lemma A.4:

$$(\mathrm{Id} + d^{\mathrm{v}} h^* + h^* d^{\mathrm{v}}) T_1 = 0 \text{ and } (\mathrm{Id} + d^{\mathrm{v}} h^* + h^* d^{\mathrm{v}}) T_2 = 0.$$

The first equation holds as $(\mathrm{Id} + h d^{\mathrm{v}} + d^{\mathrm{v}} h)$ is zero on $N^{\mathrm{v}}_2 \mathbf{B}$. For the second equation, on $N^{\mathrm{v}}_1 \mathbf{B}$, $(\mathrm{Id} + h d^{\mathrm{v}} + d^{\mathrm{v}} h)$ is the projection onto terms of the form $[v]$, yet Lemma A.6 shows that the composite

$$((r \otimes r) \circ (T (D^{n-k}_{\mathrm{h}})^\star T) \circ (D^1_{\mathrm{v}})^\star \circ \Delta_{\mathbf{B}}) : N^{\mathrm{h}}_{n+k} N^{\mathrm{v}}_1 \mathbf{B} \longrightarrow N^{\mathrm{v}}_{n+k} (\mathbf{Q}_{\bullet 1} \otimes \mathbf{Q}_{\bullet 1})$$

vanishes except on terms of the form $[vw]$ (recall that r commutes with the horizontal simplicial structure).

As $d^{\mathrm{h}} h^* + h^* d^{\mathrm{h}}$ increases filtration, we have compressed $S^k(r^*\alpha)$ to the filtration $n+k$ expression $(d^{\mathrm{h}} h^* + h^* d^{\mathrm{h}}) T_1$, modulo even higher filtration. The commuting diagram of Lemma A.7 is the left square in a larger commuting diagram:

$$\begin{tikzcd} N^{\mathrm{h}}_{n+k} N^{\mathrm{v}}_1 \mathbf{B} \arrow[r, "(D^0_{\mathrm{v}})^\star \circ \Delta_{\mathbf{B}} \circ (d^{\mathrm{h}} h + h d^{\mathrm{h}})"] \arrow[d, "r"] & N^{\mathrm{h}}_{n+k-1} (N^{\mathrm{v}}_1 \mathbf{B} \otimes N^{\mathrm{v}}_1 \mathbf{B}) \arrow[r, "(D^{n+1-k}_{\mathrm{h}})^\star"] \arrow[d, "r \otimes r"] & N^{\mathrm{h}}_n N^{\mathrm{v}}_1 \mathbf{B} \otimes N^{\mathrm{h}}_n N^{\mathrm{v}}_1 \mathbf{B} \arrow[d, "r \otimes r"] \\ N^{\mathrm{h}}_{n+k} Q^{\mathcal{L}(n)} L \arrow[r, "\psi_{\mathcal{L}(n)}"] & N^{\mathrm{h}}_{n+k-1} (Q^{\mathcal{L}(n)} L \otimes Q^{\mathcal{L}(n)} L) \arrow[r, "(D^{n+1-k}_{\mathrm{h}})^\star"] & N^{\mathrm{h}}_n Q^{\mathcal{L}(n)} L \otimes N^{\mathrm{h}}_n Q^{\mathcal{L}(n)} L \end{tikzcd}$$

Now $\mathbf{D}(N_n^{\mathrm{h}}Q^{\mathcal{L}(n)}L \otimes N_n^{\mathrm{h}}Q^{\mathcal{L}(n)}L)$ contains the cocycle $\phi(\alpha\otimes\alpha)$, and pulling $\phi(\alpha\otimes\alpha)$ back to $\mathbf{D}(N_{n+k}^{\mathrm{h}}N_1^{\mathrm{v}}\mathbf{B})$ along the lower composite yields $r^*\psi_{\mathcal{L}(n)}^*\mathrm{Sq}_{\mathrm{ext}}^{k-1}(\alpha)$. Pulling back along the upper composite yields the E_2 representative of the shifted version of Singer's operations. Both spectral sequences collapse at E_2 and induce trivial filtrations on their shared target, so that understanding the shifted operations at E_2 is equivalent to understanding the operations on $\pi^*\mathbf{D}|\mathbf{B}|$, which we do: they equal Priddy's operations on $\pi^*\bar{W}U'L$ [47, §5]. As r^* is an E_2-equivalence, this proves the result. A simple modification proves the result for pairings. $\square$

A.3. The Chevalley-Eilenberg-May complex

Suppose that $M \in \mathcal{L}(n)$ is a partially restricted Lie algebra of finite type (not simplicial). One can define a differential coalgebra, the *Chevalley-Eilenberg-May complex*, to be the subcoalgebra $\bar{X}'(M) := E[M_{\mathbf{0}}]\otimes\Gamma[M_{\neq\mathbf{0}}]$ of the divided power Hopf algebra $\Gamma[M]$ with its usual coalgebra structure (c.f. [39, p. 141]), graded as follows. The Hopf algebra $\Gamma[M]$ is to be $\mathcal{V}_{n+1}^+$-graded, with product and divided square operations

$$\Gamma[M]^t_{p,s_n,\ldots,s_1}\otimes\Gamma[M]^{t'}_{p',s_n',\ldots,s_1'} \longrightarrow \Gamma[M]^{t+t'+1}_{p+p'+1,s_n+s_n',\ldots,s_1+s_1'},$$
$$\gamma_2:\Gamma[M]^t_{p,s_n,\ldots,s_1}\longrightarrow\Gamma[M]^{2t+1}_{2p+1,2s_n,\ldots,2s_1},$$

generated by the subspace

$$\Gamma[M]^t_{0,s_n,\ldots,s_1} = M^t_{s_n,\ldots,s_1},$$

and we define $\bar{X}'(M)$ to be the coaugmentation coideal of the subcoalgebra spanned by those expressions

$$\gamma_{r_1}(y_1)\cdots\gamma_{r_m}(y_m)\quad(\text{with } y_1,\ldots,y_m\in M \text{ homogeneous})$$

for which $r_i\leq 1$ when $y_i\in M_{\mathbf{0}}$ (i.e. $y_i\in M^t_{0,\ldots,0}$). The coalgebra structure map and differential are the restriction to $E[M_{\mathbf{0}}]\otimes\Gamma[M_{\neq\mathbf{0}}]$ of those given in [39, p. 141] (after tensoring the formula [39, (6.19)] down to a formula on $\overline{X}(M)$, which kills the first term $\Sigma_{i=1}^n f_iy_i$).

This differs by a shift from the standard definitions, given in [16] in the unrestricted setting, and given in [39] in the restricted setting. It also differs from those definitions in that we have taken the coaugmentation coideal. Correspondingly, $\bar{X}'(M)$ is a *shift* of the homology (in the sense of [46]) of the associated graded algebra appearing in the partially restricted PBW Theorem, Lemma A.1.

Now let $L = B^{\mathcal{L}(n)}M\in s\mathcal{L}(n)$. Using the equation $|\mathbf{B}| = \bar{B}_\bullet(U'M)$ of simplicial coalgebras and May's injection [39, Theorem 18 and (7.8)] of $\bar{X}'(M)$ into the bar construction, there are maps:

$$\Sigma\bar{X}'(M) \longrightarrow N_*\bar{B}_\bullet(U'M) \simeq \mathrm{Tot}(N_*^{\mathrm{h}}N_*^{\mathrm{v}}\mathbf{B}) \xrightarrow{r} \mathrm{Tot}(N_*^{\mathrm{h}}\mathbf{Q}_{\bullet *}) = \mathbb{F}_2\{1\} \oplus \Sigma N_* Q^{\mathcal{L}(n)}L.$$

Now the first map, after the suspension Σ shift homological degree, is a degree preserving map of differential coalgebras, both of which only have a shift in the cohomological degree t. In light of this discussion, Proposition A.3 implies Proposition A.8 (in which we move back into the notation $S(\mathscr{C})$ and $\Lambda(\mathscr{C})$ for the non-unital commutative and exterior algebra monads).

Proposition A.8. *$H^*_{\mathcal{L}(n)}M$ may be calculated, as a* non-unital commutative algebra*, as the cohomology algebra of the differential graded algebra $\mathbf{D}(\bar{X}'(M))$, where $\mathbf{D}(\bar{X}'(M))$ is the non-unital commutative algebra $\mathbf{D}(\bar{X}'(M)) = \Lambda(\mathscr{C})[\mathbf{D}M_{\mathbf{0}}] \sqcup S(\mathscr{C})[\mathbf{D}M_{\neq\mathbf{0}}]$. This algebra is $\mathcal{V}_+^{n+1}$-graded, generated by its subspace*

$$(\mathbf{D}(\bar{X}'(M)))_t^{0,s_n,\dots,s_1} = (\mathbf{D}M)_t^{s_n,\dots,s_1},$$

and has grading shifted product

$$\mathbf{D}(\bar{X}'(M))_t^{p,s_n,\dots,s_1} \otimes \mathbf{D}(\bar{X}'(M))_{t'}^{p',s_n',\dots,s_1'} \longrightarrow \mathbf{D}(\bar{X}'(M))_{t+t'+1}^{p+p'+1,s_n+s_n',\dots,s_1+s_1'}.$$

Recall that the coproduct $A \sqcup B$ of non-unital commutative algebras is the direct sum $A \oplus (A \otimes B) \oplus B$.

We are particularly interested in the case that M is *trivial as a Lie algebra*, but may still have non-zero restriction. In this case, the restriction is in fact a *linear* map, and we may write $\sqrt[[2]]{-} : \mathbf{D}M \longrightarrow \mathbf{D}M$ for its dual (a map which we consider to be everywhere defined, but necessarily equal to zero on $M_{\mathbf{0}}$). Examination of [39, (6.19)] shows:

Proposition A.9. *If M has bracket zero, then the differential on the cohomological differential graded algebra $\mathbf{D}(\bar{X}'(M))$ is defined on generators $\alpha \in \mathbf{D}M \subseteq \mathbf{D}(\bar{X}'(M))$ by the formula*

$$\alpha \longmapsto (\sqrt[[2]]{\alpha}\,)^2.$$

Bibliography

[1] Michel André. *Méthode simpliciale en algèbre homologique et algèbre commutative*. Lecture Notes in Mathematics, Vol. 32. Springer-Verlag, Berlin-New York, 1967, pp. iii+122.

[2] Jon Beck. “Distributive laws”. In: *Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67)*. Springer, Berlin, 1969, pp. 119–140.

[3] David Blanc and Christopher Stover. “A generalized Grothendieck spectral sequence”. In: *Adams Memorial Symposium on Algebraic Topology, 1 (Manchester, 1990)*. Vol. 175. London Math. Soc. Lecture Note Ser. Cambridge: Cambridge Univ. Press, 1992, pp. 145–161.

[4] A. J. Blumberg and E. Riehl. “Homotopical resolutions associated to deformable adjunctions”. In: *ArXiv e-prints* (Aug. 2012).

[5] A. K. Bousfield. “A vanishing theorem for the unstable Adams spectral sequence”. In: *Topology* 9 (1970), pp. 337–344.

[6] A. K. Bousfield. *Homogeneous functors and their derived functors*. 1967.

[7] A. K. Bousfield. “On the homology spectral sequence of a cosimplicial space”. In: *Amer. J. Math.* 109.2 (1987), pp. 361–394.

[8] A. K. Bousfield. *Operations on derived functors of non-additive functors*. 1967.

[9] A. K. Bousfield and D. M. Kan. “A second quadrant homotopy spectral sequence”. In: *Trans. Amer. Math. Soc.* 177 (1973), pp. 305–318.

[10] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Lecture Notes in Mathematics, Vol. 304. Berlin: Springer-Verlag, 1972, pp. v+348.

[11] A. K. Bousfield and D. M. Kan. “Pairings and products in the homotopy spectral sequence”. In: *Trans. Amer. Math. Soc.* 177 (1973), pp. 319–343.

[12] A. K. Bousfield and D. M. Kan. “The homotopy spectral sequence of a space with coefficients in a ring.” In: *Topology* 11 (1972), pp. 79–106.

[13] A. K. Bousfield et al. “The mod $-\,p$ lower central series and the Adams spectral sequence”. In: *Topology* 5 (1966), pp. 331–342.

[14] H. Cartan. “Puissances divisées”. fre. In: *Séminaire Henri Cartan* 7 no. 1 (1954-1955), pp. 1–12.

[15] A. M. Cegarra and Josué Remedios. “The relationship between the diagonal and the bar constructions on a bisimplicial set”. In: *Topology Appl.* 153.1 (2005), pp. 21–51.

[16] Claude Chevalley and Samuel Eilenberg. “Cohomology theory of Lie groups and Lie algebras”. In: *Trans. Amer. Math. Soc.* 63 (1948), pp. 85–124.

[17] M. Ching. “Homotopy Operations on Simplicial Algebras over an Operad”. Available at `http://www3.amherst.edu/~mching/Work/homotopy_operations.pdf`.

[18] M. Ching and J. E. Harper. “Derived Koszul duality and TQ-homology completion of structured ring spectra”. In: *ArXiv e-prints* (Feb. 2015).

[19] E. B. Curtis. “Lower central series of semi-simplicial complexes”. In: *Topology* 2 (1963), pp. 159–171.

[20] E. B. Curtis. “Simplicial homotopy theory”. In: *Advances in Math.* 6 (1971), 107–209 (1971).

[21] Ioannis Dokas. “Quillen-Barr-Beck (co-) homology for restricted Lie algebras”. In: *J. Pure Appl. Algebra* 186.1 (2004), pp. 33–42.

[22] Albrecht Dold. “Homology of symmetric products and other functors of complexes”. In: *Ann. of Math. (2)* 68 (1958), pp. 54–80.

[23] Albrecht Dold. “Über die Steenrodschen Kohomologieoperationen”. In: *Ann. of Math. (2)* 73 (1961), pp. 258–294.

[24] Albrecht Dold and Dieter Puppe. “Homologie nicht-additiver Funktoren. Anwendungen”. In: *Ann. Inst. Fourier Grenoble* 11 (1961), pp. 201–312.

[25] W. G. Dwyer. “Higher divided squares in second-quadrant spectral sequences”. In: *Trans. Amer. Math. Soc.* 260.2 (1980), pp. 437–447.

[26] W. G. Dwyer. “Homotopy operations for simplicial commutative algebras”. In: *Trans. Amer. Math. Soc.* 260.2 (1980), pp. 421–435.

[27] W. G. Dwyer, D. M. Kan, and C. R. Stover. “An E^2 model category structure for pointed simplicial spaces”. In: *J. Pure Appl. Algebra* 90.2 (1993), pp. 137–152.

[28] Samuel Eilenberg and Saunders Mac Lane. “On the groups of $H(\Pi, n)$. I”. In: *Ann. of Math. (2)* 58 (1953), pp. 55–106.

[29] Samuel Eilenberg and John C. Moore. “Limits and spectral sequences”. In: *Topology* 1 (1962), pp. 1–23.

[30] B. Fresse. “Koszul duality of operads and homology of partition posets”. In: *ArXiv Mathematics e-prints* (Jan. 2003).

[31] B. Fresse. “On the homotopy of simplicial algebras over an operad”. In: *Trans. Amer. Math. Soc.* 352.9 (2000), pp. 4113–4141.

[32] Paul G. Goerss. “A Hilton-Milnor theorem for categories of simplicial algebras”. In: *Amer. J. Math.* 111.6 (1989), pp. 927–971.

[33] Paul G. Goerss. “On the André-Quillen cohomology of commutative $\mathbf{F}_2$-algebras”. In: *Astérisque* 186 (1990), p. 169.

[34] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory.* Vol. 174. Progress in Mathematics. Basel: Birkhäuser Verlag, 1999, pp. xvi+510.

[35] Thomas G. Goodwillie. “Calculus. II. Analytic functors”. In: *K-Theory* 5.4 (1991/92), pp. 295–332.

[36] Philip Hackney. “Homology operations and cosimplicial iterated loop spaces”. In: *Homology Homotopy Appl.* 16.1 (2014), pp. 1–25.

[37] Philip Hackney. “Operations in the homology spectral sequence of a cosimplicial infinite loop space”. In: *J. Pure Appl. Algebra* 217.7 (2013), pp. 1350–1377.

[38] Mark Hovey, John H. Palmieri, and Neil P. Strickland. “Axiomatic stable homotopy theory”. In: *Mem. Amer. Math. Soc.* 128.610 (1997), pp. x+114.

[39] J. P. May. “The cohomology of restricted Lie algebras and of Hopf algebras”. In: *J. Algebra* 3 (1966), pp. 123–146.

[40] J. Peter May. *Simplicial objects in algebraic topology.* Van Nostrand Mathematical Studies, No. 11. D. Van Nostrand Co., 1967, pp. vi+161.

[41] John McCleary. *A user's guide to spectral sequences.* Second. Vol. 58. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2001, pp. xvi+561.

[42] Haynes Miller. “The Sullivan conjecture on maps from classifying spaces”. In: *Ann. of Math. (2)* 120.1 (1984), pp. 39–87.

[43] Brian Munson and Ismar Volić. “Cubical Homotopy Theory”. In preparation to be published by Cambridge University Press. 2014.

[44] Joseph Neisendorfer. *Algebraic methods in unstable homotopy theory.* Vol. 12. New Mathematical Monographs. Cambridge: Cambridge University Press, 2010, pp. xx+554.

[45] L. Pereira. “A general context for Goodwillie Calculus”. In: *ArXiv e-prints* (Jan. 2013).

[46] Stewart B. Priddy. “Koszul resolutions”. In: *Trans. Amer. Math. Soc.* 152 (1970), pp. 39–60.

[47] Stewart B. Priddy. “Primary cohomology operations for simplicial Lie algebras”. In: *Illinois J. Math.* 14 (1970), pp. 585–612.

[48] Daniel G. Quillen. *Homotopical algebra.* Lecture Notes in Mathematics, No. 43. Berlin: Springer-Verlag, 1967.

[49] A. Radulescu-Banu. “Cofibrance and Completion”. PhD thesis. MIT, 2006.

[50] James W. Schlesinger. “The semi-simplicial free Lie ring”. In: *Trans. Amer. Math. Soc.* 122 (1966), pp. 436–442.

[51] Lionel Schwartz. *Unstable modules over the Steenrod algebra and Sullivan’s fixed point set conjecture.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1994, pp. x+229.

[52] William M. Singer. *Steenrod squares in spectral sequences.* Vol. 129. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2006, pp. xii+155.

[53] Dev P. Sinha. “The topology of spaces of knots: cosimplicial models”. In: *Amer. J. Math.* 131.4 (2009), pp. 945–980.

[54] Christopher R. Stover. “A van Kampen spectral sequence for higher homotopy groups”. In: *Topology* 29.1 (1990), pp. 9–26.

[55] James M. Turner. “Operations and spectral sequences. I”. In: *Trans. Amer. Math. Soc.* 350.9 (1998), pp. 3815–3835.

[56] J. H. C. Whitehead. “On adding relations to homotopy groups”. In: *Ann. of Math. (2)* 42 (1941), pp. 409–428.