Crossed products of S categories

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I want to axiomatize the process by which the cyclic category is constructed from the simplicial category and the family of cyclic groups.

Let $\mathbf{A}$ and $\mathbf{B}$ be two categories with common object set S; we will call them S-categories. Write $A(s,t)$ for the set of morphisms from s to t in $\mathbf{A}$. Write A for the set of all morphisms in $\mathbf{A}$. A admits source and target maps to S. When we write fibered products, we will think of the source as on the right and the target as on the left, so morphisms are directed leftwards. Thus composition is a map $A \times_S A \to A$ which sends (a, a') (so that the source of a is the target of a') to $a \circ a'$.

We will construct a new category $\mathbf{C}$ in which

$$C = A \times_S B$$

or

$$C(s,t) = \coprod_u A(u,t) \times B(s,u)\,.$$

This means that the normal form of a morphism will have a morphism from $\mathbf{B}$ applied first, and then a morphism from $\mathbf{A}$. To define composition, we need to now how to commute a morphism in $\mathbf{A}$ across a morphism in $\mathbf{B}$. We'll do this using a map

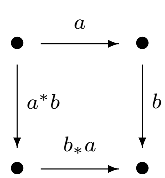
$$c : B \times_S A \to A \times_S B$$

fibered over $S \times S$. We will write $\mathbf{C} = \mathbf{AB}$ for the resulting S category, leaving the structure map c undenoted.

We will write

$$c(b,a) = (b_* a, a^* b)$$

so that



Then the composition in $\mathbf{C}$ is determined by the equations

$$(1,b) \circ (a,1) = (b_* a, a^* b)\,, \quad (a,1) \circ (1,b) = (a,b)\,,$$
$$(a',1) \circ (a,1) = (a' \circ a, 1)\,, \quad (1,b') \circ (1,b) = (1, b' \circ b)\,,$$

which imply the general composition law

(1) $$(a', b') \circ (a,b) = (a' \circ b'_* a, a^* b' \circ b)$$

The identity map on an object s is given by $(1_s, 1_s)$. The category axioms for $\mathbf{C}$ require some properties of c which we collect in the following definition.

Definition. A *crossed structure* on a pair of S categories $\mathbf{A}$ and $\mathbf{B}$ is a bijection

$$c : B \times_S A \to A \times_S B\,, \quad c(b,a) = (b_*a, a^*b)\,,$$

over $S \times S$ which satisfies the following identities.

$$\begin{gathered}(a \circ a')^*b = a'^*(a^*b)\,, \quad 1^*b = b\,,\\ (b \circ b')_*a = b_*(b'_*a)\,, \quad 1_*a = a\,,\\ a^*(b \circ b') = (b'_*a)^*b \circ a^*b'\,, \quad a^*1 = 1\,,\\ b_*(a \circ a') = b_*a \circ (a^*b)_*a'\,, \quad b_*1 = 1\,.\end{gathered}$$

Lemma. If c is a crossed structure on $(\mathbf{A}, \mathbf{B})$, then the composition law (1) given above renders $\mathbf{C} = \mathbf{AB}$ an S-category. The functions $A \to C$ and $B \to C$ given by $a \mapsto (a, 1)$ and $b \mapsto (1, b)$ define functors $\mathbf{A} \to \mathbf{C}$ and $\mathbf{B} \to \mathbf{C}$ which are the identity on objects.

Remark. Recall that a double category consists of "vertical" and "horizontal" S categories, along with "bimorphisms" having horizontal source and target and vertical source and target, with the property that they form the edges of a square; and both horizontal and vertical composition laws for bimorphisms, such that the horizontal composite of two vertical composites is the vertical composite of two horizontal composites. (See Fiedorowicz and Loday.) A crossed structure on $(\mathbf{A}, \mathbf{B})$ is then exactly a double category with horizontal category $\mathbf{A}$ and vertical category $\mathbf{B}$ with the property that for any $a \in A(s', s)$ and any $b \in B(s, s'')$ there is a unique bimorphism with horizontal source a and vertical target b. The vertical source is a^*b and the horizontal target is b_*a.

Example. Suppose that all morphisms in $\mathbf{B}$ are endomorphisms, so that c is given by maps $B(s,s) \times A(s', s) \to A(s', s) \times B(s', s')$. Suppose also that $b_*a = a$ for every $a \in A(s, s')$, $b \in B(s, s)$. Such a crossed structure is determined by a functor $\tilde{B} : \mathbf{A}^{\mathrm{op}} \to \mathbf{Mon}$ such that on objects, $\tilde{B}(s) = B(s, s)$.

If instead all morphisms in $\mathbf{A}$ are endomorphisms, and $a^*b = b$ for all a and b, then we have a functor $\tilde{A} : \mathbf{B} \to \mathbf{Mon}$.

Example. In [3] Fiedorowicz and Loday concern themselves with the case in which $S = \mathbb{N}$ with elements $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$; all morphisms of $\mathbf{B}$ are automorphisms; and $\mathbf{A}$ is the simplicial category $\mathbf{\Delta}$. They call such a structure a *crossed simplicial group*. Compare the definition of a crossed structure above with their Proposition 1.6. These authors give a classification of the categories $\mathbf{B}$ with only automorphisms which admit a crossing structure with $\mathbf{\Delta}$.

Their first example is that given by a simplicial group, as above: a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Gp}$, as in our example above.

Other examples, with nontrivial action on the simplicial category, include case in which $B(n, n)$ is the cyclic group of order $n + 1$. This had been considered long ago by Connes [2] and subsequently by many others. The example in which $B(n, n)$ is the symmetric group on $n+1$ letters was considered later by Kapranov and Manin [4] and by Pirashvili and Richter [5], who called the resulting object the category of "noncommutative sets."

Here is how this works out for the cyclic example. Let $t : [n] \to [n]$ be the cyclic permutation $i \mapsto i+1$ for $i < n$ and $n \mapsto 0$. Let $\phi : [m] \to [n]$ be an order preserving map, and let $0 \le i < n$. We have to rewrite the composite $t^i \circ \phi$ in the form $\phi' \circ t^j$. Define j to be the unique integer such that

$$\phi(m - j) \le n - i\,, \quad \phi(m - j + 1) \ge n - i + 1\,.$$

Define ϕ' by

$$\phi'(k) = \begin{cases} \phi(m - j + k - 1) + i & \text{for } k < j\,, \\ \phi(k - j) - n + i - 1 & \text{for } k \ge j\,. \end{cases}$$

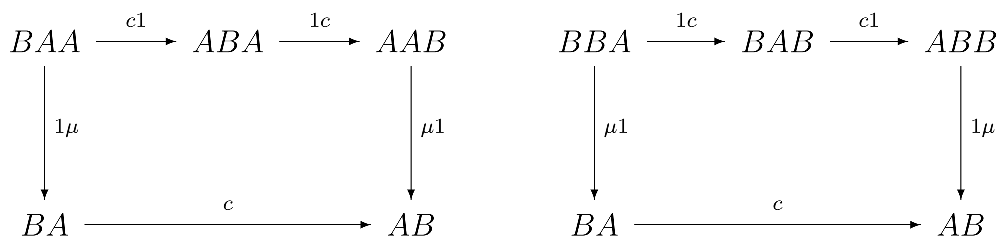
This mode of construction makes look as though the natural functor to finite nonempty sets is faithful, but this is not the case. For example, $s^0 \circ 1$ and $s^0 \circ t : [1] \to [0]$ are different morphisms in the cyclic category.

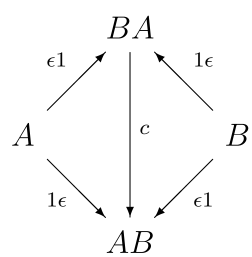
The symmetric case is similar: for any ordered map $\phi : [m] \to [n]$ and any permutation $\sigma : [n] \to [n]$, there is a unique pair (ϕ', σ') with ϕ' ordered and σ' a permutation such that

$$\begin{array}{ccc} [m] & \xrightarrow{\phi} & [n] \\ \downarrow{\scriptstyle \sigma'} & & \downarrow{\scriptstyle \sigma} \\ [m] & \xrightarrow{\phi'} & [n] \end{array}$$

commutes.

Example. This is interesting even if S is a singleton, so we are speaking of monoids. The crossed structure is diagrammatic:





These diagrams have simple braid descriptions; the first pair give a form of Type III move invariance, and the unit condition assures that the empty incoming strand can be put anywhere.

So we can consider monoids more generally, with respect to a monoidal structure on a category.

For example, the category might be the category of endofunctors of some category; this is then a crossed structure on a pair of triples. This is precisely what Beck [1] calls a "distributive law." Or it could be the category of symmetric sequences in a cartesian closed category; then this is a crossed structure on a pair of operads. (Thanks to Muriel Livernet and Jacob Lurie.)

Loday pointed out to me that there is a crossed structure on the pair Σ_n, C_{n+1} for which $\Sigma_n C_{n+1} = \Sigma_{n+1}$.

We might say that a "commutative S-category" is an S-category $\mathbf{A}$ with a crossed structure on the pair $(\mathbf{A}, \mathbf{A})$ such that

$$b \circ a = b_* a \circ a^* b\,.$$

For example, if S is a singleton, so we are speaking of monoids, any pair of S categories has a canonical crossed structure, given by $a^* b = b$, $b_* a = a$. With this crossed structure, an S category is commutative exactly when it is a commutative monoind in the ordinary sense. If we are working in a symmetric monoidal category, a monoid object has a canonical crossed structure with itself, given by the symmetry.

A monoid in a monoidal category can have a self-crossing structure $c : AA \to AA$. If c is an isomorphism (e.g. if $c^2 = 1$), then just one of each pair of diagrams suffices. I guess that the free R-module triple on sets or on abelian groups are examples, with $c(s[tx]) = t[sx]$. This satisfies the axioms (and has $c^2 = 1$).

References

[1] J. Beck, Distributive laws, Seminar on Triples and Categorical Homology Theory, Springer Lect. Notes in Math. 80 (1969) 119–140.

[2] A. Connes, Cohomologie cyclique et foncteurs Ext^n, C.R.A.S. 296 (1983) 953–958.

[3] Z. Fiedorowicz and J.-L. Loday, Crossed simplicial groups and their associated homology, Trans. Amer. Math. Soc. 326 (1991) 57–87.

[4] M. Kapranov and Yu. Manin, Modules and Morita theory for operads, Amer. J. Math. 123 (2001) 811–838.

[5] T. Pirashvili and B. Richter, Hochschild and cyclic homology via functor homology, K-Theory 25 (2002) 39–49.