Beck modules over a Poisson algebra
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Let $K$ be a commutative ring and $A$ a Lie algebra over $K$. A Lie algebra over $A$ with a cross-section is of the form $A \oplus M$, with projection killing $M$ and cross-section sending $a$ to $(a, 0)$. The Lie algebra structure on $A \oplus M$ is determined by a $K$-linear map $\cdot : A \otimes_K M \to M$, as follows.

\[
[(a, 0), (b, 0)] = ([a, b], 0) \\
[(a, 0), (0, y)] = (0, a \cdot y) \\
[(0, x), (b, 0)] = -(0, b \cdot x) \\
[(0, x), (0, y)] = (0, [x, y]).
\]

We have built in anti-symmetry, as long as we assume anti-symmetry of $[x, y]$.

The Jacobi identity says that the cyclic sum of

\[
a \cdot (b \cdot z) - a \cdot (c \cdot y) + a \cdot [y, z] - [b, c] \cdot x + [x, b \cdot z] - [x, c \cdot y] + [x, [y, z]]
\]

is zero. With $a = b = c = 0$ we get the Jacobi identity for $[x, y]$. With $y = z = 0$ we get

\[
[b, c] \cdot x = b \cdot (c \cdot x) - c \cdot (b \cdot x).
\]

These identities imply the general case.

Now if $A \oplus M$ also has the structure of an abelian object over $A$ (with unit given by the cross-section), then the addition map must be

\[A \oplus M \oplus M = (A \oplus M) \times_A (A \oplus M) \to A \oplus M\]

by $(a, x, y) \mapsto (a, x + y)$, and so this map must be a Lie algebra homomorphism. This is equivalent to requiring $[x, y] = 0$, and assuming this makes $A \oplus M$ into an abelian object.

We have proven:

**Lemma.** Let $A$ be a Lie algebra over $K$. The category of abelian objects over $A$ is equivalent to the category of $K$-modules $M$ equipped with a $K$-linear map $\cdot : A \otimes_K M \to M$ such that

\[
[a, b] \cdot z = a \cdot (b \cdot z) - b \cdot (a \cdot z).
\]
That is to say, it is equivalent to the category of modules over the universal enveloping algebra of $A$,

$$U(A) = \text{Tens}_K(A)/([a, b] - a \otimes b + b \otimes a).$$

Let $i : A \to U(A)$ be the natural map. Let $M$ be a $U(A)$-module. A Lie derivation with values in $M$ is a $K$-linear map $\sigma : A \to M$ such that

$$\sigma([a, b]) = i(a)\sigma(b) - i(b)\sigma(a).$$

This is the same as a section of the abelian object $A \oplus M \downarrow A$.

The $U(A)$-module of “Lie-Kähler differentials” supports the universal Lie derivation out of $A$. It is given by

$$\Omega_{A/K}^{Lie} = \frac{U(A) \otimes_K A}{1 \otimes [a, b] - i(a) \otimes b + i(b) \otimes a}$$

with universal derivation given by $\sigma(a) = 1 \otimes a$.

Following Cartan and Eilenberg, as long as $A$ is free over $K$ there is a resolution of $K$ by $U(A)$ modules given by the Chevalley-Eilenberg complex $U(A) \otimes_K \Lambda^*(A)$, with $d : U(A) \otimes \Lambda^2(A) \to U(A) \otimes A$ extending

$$x \wedge y \mapsto ix \otimes y - iy \otimes x - 1 \otimes [x, y]$$

Thus the indicated quotient is simply

$$I_{A/K} = \ker (\epsilon : U(A) \to K).$$

In these terms, the universal derivation $A \to I_{A/K}$ is given by $a \mapsto i(a)$.

Let $A$ be a Poisson algebra over $K$. A Beck module over $A$ will be a Beck module over $A$ regarded as a commutative $K$-algebra and as a Lie algebra over $K$, separately, so consists of an $A$-module $M$ with a bilinear map $\cdot : A \otimes_K M \to M$ as above, subject to the Poisson identity, which amounts to

$$a \cdot bz + a \cdot cy - bc \cdot x =$$

$$b(a \cdot z) - b(c \cdot x) + [a, c]y + c(a \cdot y) - c(b \cdot x) + [a, b]z.$$

With $y = z = 0$ this gives

$$bc \cdot x = b(c \cdot x) + c(b \cdot x)$$

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With $x = y = 0$ it gives

$$[a, b]z = a \cdot bz - b(a \cdot z).$$

These identities imply the general case.

We have proven:

**Lemma.** Let $A$ be a Poisson algebra over $K$. The category of abelian objects over $A$ is equivalent to the category of $A$-modules $M$ equipped with a $K$-linear map $\cdot : A \otimes_K M \to M$ such that

$$[a, b] \cdot z = a \cdot (b \cdot z) - b \cdot (a \cdot z)
\quad ab \cdot z = a(b \cdot z) - b(a \cdot z)
\quad [a, b]z = a \cdot bz - b(a \cdot z).$$

A section of $A \oplus M \to A$ is precisely a derivation with respect to each of the structures: a $K$-linear map $\sigma : A \to M$ such that

$$\sigma(ab) = a\sigma b + b\sigma a \quad \sigma[a, b] = a \cdot \sigma b - b \cdot \sigma a.$$

Such a map $\sigma$ is a “Poisson derivation.”

To describe the Beck module of Poisson differentials it is useful to discuss the universal enveloping algebra of a Poisson algebra. A Beck module structure over $A$ on a $K$-module $M$ is determined by two $K$-linear maps $A \to \text{End}_K(M)$, or by a $K$-algebra map $\text{Tens}_K(A \oplus A) \to \text{End}_K(M)$ where the first copy of $A$ acts by $(a, x) \mapsto ax$ and the second by $(a, x) \mapsto a \cdot x$. Write $\alpha : A \to \text{Tens}_K(A \oplus A)$ for the inclusion of the first copy and $\lambda : A \to \text{Tens}_K(A \oplus A)$ for the inclusion of the second. The relations on the two “action” maps show that this map factors through the quotient by the ideal $I$ generated by the elements

$$\alpha(1) - 1 \quad \alpha(ab) - \alpha(a)\alpha(b)
\lambda([a, b]) - \lambda(a)\lambda(b) + \lambda(b)\lambda(a)
\lambda(ab) - \alpha(a)\lambda(b) + \alpha(b)\lambda(a) \quad \alpha([a, b]) - \lambda(a)\alpha(b) + \alpha(b)\lambda(a)$$
as $a, b$ range over $A$. 

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Write
\[ U_{\text{Pois}}(A) = \text{Tens}_K(A \oplus A)/I, \]
so that a Beck module over \( A \) is the same thing as a left \( U_{\text{Pois}}(A) \)-module.

There is a universal Poisson derivation \( \sigma : A \to \Omega_{\text{Pois}}^A_K \) where
\[
\Omega_{\text{Pois}}^A_K = \frac{U_{\text{Pois}}(A) \otimes_K A}{\{1 \otimes ab - \alpha(a) \otimes b - \alpha(b) \otimes a, 1 \otimes [a,b] - \lambda(a) \otimes b + \lambda(b) \otimes a\}}
\]
and \( \sigma : a \mapsto 1 \otimes a \). It would be interesting to know more about this Poisson module.

If \( K = \mathbb{F}_2 \) one may wish to require that the Lie algebra \( A \) comes equipped with a restriction, that is, a function \( \xi : A \to A \) such that
\[ [\xi(a), b] = [a, [a, b]] \quad \text{and} \quad \xi(a + b) = \xi(a) + [a, b] + \xi(b). \]
Note that by taking \( a = b \) we find \( \xi(0) = 0 \). A restricted Lie algebra is a Lie algebra together with a restriction. One should think of \( \xi(a) \) as half of \([a, a]\).

A Beck module over the restricted Lie algebra \( A \) is a Beck module over \( A \) as a Lie algebra, together with a restriction, which is given on \((a, x)\) by \((\xi a, \zeta(a, x))\). Compatibility with the section forces \( \zeta(a, 0) = 0 \). The addition formula is equivalent to
\[ \zeta(a + b, x + y) = \zeta(a, x) + a \cdot y + b \cdot x + \zeta(b, y). \]
Taking \( a = 0 \) and \( y = 0 \) gives
\[ \zeta(b, x) = \zeta(0, x) + b \cdot x. \]
Write
\[ \varphi(x) = \zeta(0, x) \]
so that \( \zeta(b, x) = \varphi(x) + b \cdot x \). The addition formula now implies that \( \varphi \) is linear, and this in turn implies the general case.

The bracket formula for the restriction is equivalent to
\[ \xi a \cdot y + b \cdot \varphi(x) + b \cdot (a \cdot x) = a \cdot (a \cdot y) + a \cdot (b \cdot x) + [a, b] \cdot x \]
or
\[ \xi a \cdot y = a \cdot (a \cdot y) \quad , \quad b \cdot \varphi(x) = 0. \]
The addition morphism now automatically commutes with the restriction map.

We have proven:

**Lemma.** A Beck module over a restricted Lie algebra $A$ is an $\mathbb{F}_2$ vector space $M$ together with linear maps

$$\cdot : A \otimes M \to M \quad , \quad \varphi : M \to M$$

such that

$$[a, b] \cdot z = a \cdot (b \cdot z) - b \cdot (a \cdot z)$$

$$a \cdot \varphi(y) = 0 \quad , \quad \xi a \cdot y = a \cdot (a \cdot y).$$

A restriction on a Poisson algebra over $\mathbb{F}_2$ is a restriction $\xi$ on the underlying Lie algebra which satisfies also the Poisson condition

$$\xi(xy) = x^2\xi(y) + x[x, y]y + \xi(x)y^2.$$ 

A Beck module over a restricted Poisson algebra $A$ will be a Beck module over $A$ as a Poisson algebra, which is also a Beck module over $A$ as a restricted Lie algebra, such that the Poisson condition on the restriction is satisfied. This is an $A$-module $M$ together with linear maps

$$\cdot : A \otimes M \to M \quad , \quad \varphi : M \to M$$

such that the above conditions hold; the Poisson condition is then

$$ab \cdot ay + ab \cdot bx + \varphi(ay) + \varphi(bx) =$$

$$a^2\varphi(y) + a^2(b \cdot y) + b^2\varphi(x) + b^2(a \cdot x) + a[a, b]y + ab(a \cdot y) + ab(b \cdot x) + [a, b]bx.$$ 

With $a = 0$ this gives

$$\varphi(bx) = b^2\varphi(x)$$

With $y = 0$ and $a = 1$ it gives

$$b \cdot bx = b(b \cdot x).$$

These (along with the Poisson module identities) imply the general case.

We have proven:
Lemma. A Beck module over the restricted Poisson algebra $A$ consists in a Poisson module $(M, \cdot)$ over $A$ such that

$$a \cdot ay = a(a \cdot y), \quad \xi a \cdot y = a \cdot (a \cdot y)$$

together with an $\mathbb{F}_2$-linear endomorphism $\varphi : M \to M$ such that

$$a \cdot \varphi(y) = 0, \quad \varphi(ay) = a^2 \varphi(y).$$