J. W. Milnor, On the total curvature of knots, Annals of Mathematics 52 (1950) 248-257.

Take a polygonal knot L. At each vertex you bend off straight by some angle, which we may take to be strictly between 0 and π. This angle is independent of orientation but cannot be given a sign. The sum of these angles is the *total curvature* $\kappa(L)$ of the knot. If the knot is actually just a plane convex curve then the total curvature is 2π.

It seems that the total curvature of a notrivial knot must be greater than 4π. To estimate this pick a direction, that is, a unit vector $\hat{u}$. Assume that no segment of L is perpendicular to $\hat{u}$. Then the extrema of $\hat{u}\cdot$ along the knot occur at vertices, and there are as many maxima as minima. Call this common number the *kinkiness with respect to* $\hat{u}$ (although Milnor calls it the *crookedness*). It is a function μ_L on the sphere S of unit vectors minus the union of the great circles perpendicular to the edges of L. Write S_L for this open subset of the sphere.

Theorem 1. $\displaystyle\int_{S_L} \mu_L = 2\kappa(L)$.

Proof. The function μ_L is a sum of functions $\mu_{L,a}$, where a is a vertex of L. $\mu_{L,a}(\hat{u})$ is defined to be one or zero depending upon whether a is or is not a maximum of $\hat{u}\cdot$. Let v and w be unit tangent vectors of the segments meeting at a (oriented arbitrarily but compatibly). a is an extremum exactly when the signs of $\hat{u}\cdot v$ and $\hat{u}\cdot w$ are of opposite sign. Since the great circles removed from S are where these dot products vanish, $\mu_{L,a}$ is the characteristic function of a "lune" bounded by the great circles perpendicular to v and w. The area of this lune divided by the area of the sphere (namely 4π) equals half (since it is only one of the pair of opposite lunes) the angle between v and w divided by π: so it is 2 times that angle. Summing up, the result follows.

Let $\mu(L)$ denote the minimum of $\mu_L(\hat{u})$ as $\hat{u}$ ranges over S_L; the is the *kinkines of* L.

Corollary. $\kappa(L) \geq 2\pi\mu(L)$.

Proof. $\displaystyle 2\kappa(L) = \int_{S_L} \mu_L \geq \int_{S_L} \mu(L) = 4\pi\mu(L)$.

Theorem 2. If $\mu(L) = 1$ then L is the trivial knot.

Proof. Pick $\hat{u}$ so that $\mu_L(\hat{u}) = 1$. There is a single highest vertex and a single lowest vertex. Between these extremes each horizontal plane meets the knot in exactly two points. Draw the horizontal straight line between these points. You have constructed a disk spanning the knot, so the knot is trivial.

Corollary. If $\kappa(L) < 4\pi$, then L is the trivial knot.

Proof. By the Corollary to Theorem 1, the hypothesis implies that $2 > \mu(L)$. But $\mu(L)$ is a positive integer, so it must be 1 and the Proposition carries the day.

Now we can minimize $\mu(L)$ and $\kappa(L)$ over all representatives L of a knot-type $[L]$; write $\mu[L]$ and $\kappa[L]$ for the resulting "infima."

Theorem 3. $\kappa[L] = 2\pi\mu[L]$.

Proof. The Corollary to Theorem 1 shows that $\kappa[L] \geq 2\pi\mu[L]$. Let L be a representative of the knot type $[L]$ for which $\mu(L) = \mu[L]$. Choose a direction $\hat{u} \in S_L$ with $\mu_L(\hat{u}) = \mu(L)$. Now isotop (i.e. deform) the knot by compressing it linearly into a very thin tube with axis $\hat{u}$. The angle at each vertex other than the extrema for $\hat{u}\cdot$ approach 0, while the angles at the extreme vertices approach π. Therefore the total curvature approaches $2\pi\mu(L, \hat{u})$, and this completes the proof.

Theorem 4. If $\kappa(L) = \kappa[L]$ then L is planar and convex. Otherwise $\kappa(L) < \kappa[L]$.

Proof. If L is not planar, there are four consecutive vertices which are not coplanar. The two middle vertices can be moved slightly to make the line though them more nearly coplanar with the line through the other two. The result is to isotop the knot to another with smaller total curvature.

Now assume that L is planar. Straightening out concave angles decreases the total curvature, so if it is minimal there can be no concave angles and the figure is convex.

Notice that this lets you improve the Corollary to Theorem 2 to say that if $\kappa(L) \leq 4\pi$ then L is unknotted.

Theorem 5. Any nontrivial knot meets some plane in at least six points.

Proof. Suppose this is false, and select a counterexample L with the fewest possible number of vertices.

Since $\int_{S_L} \mu_L(\hat{u}) = 2\kappa(L) > 8\pi$, which is twice the area of the sphere, there is some direction $\hat{u}$ for which $\mu_L(\hat{u}) > 2$. Since it is an integer, $\mu_L(\hat{u}) \geq 3$.

Bring a plane perpendicular to $\hat{u}$ down from above and observe when it meets L at extrema for $\hat{u}\cdot$. It first meets a maximum, after which the number of intersections is 2. It must then meet another maximum, after which the number of intersections is 4. If it next meets a third maximum, the number of intersections will shortly be 6, proving the theorem. So suppose that it next meets a minimum. This minimum must lie between the two maxima, with no other intervening extrema. Construct a new knot by joining the two maxima by a straight line and eliminating the portion of the knot originally containing the intervening minimum. The result is a new knot L' isotopic to the original one, with fewer vertices. Since the original knot was a minimal counterexample, some plane meets L' in at least six points. This plane will also meet L in at least six points, contradicting the existence of a counterexample.