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A localization theorem in homological algebra

BY HAYNES R. MILLER
Harvard University

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Introduction. In (1), J. F. Adams showed that for p odd, the Adams E_2-term for a sphere, $\mathrm{Ext}_A^{s,t}(\mathbb{F}_p, \mathbb{F}_p)$, is zero for $s < t < (2p-1)s-1$, while $\mathrm{Ext}_A^{s,s}(\mathbb{F}_p, \mathbb{F}_p)$ is the one-dimensional vector space generated by q_0^s, where $q_0 \in \mathrm{Ext}_A^{1,1}(\mathbb{F}_p, \mathbb{F}_p)$ corresponds to the Bockstein.

We can express part of this result as a statement about the q_0-localization of the cohomology of the Steenrod algebra: namely, that $q_0^{-1}\,\mathrm{Ext}_A(\mathbb{F}_p, \mathbb{F}_p)$ is one-dimensional over $\mathbb{F}_p[q_0, q_0^{-1}]$. The purpose of this paper is to prove a generalization of this theorem. It turns out that under fairly general circumstances, the localization of the cohomology of an algebra obtained by inverting a polynomial generator in homological degree one can be expressed as the *differential* Ext of an associated *differential* algebra. Then standard homological methods can be brought to bear on its calculation.

This result should be regarded as a first step in a program to reinterpret results on vanishing lines (e.g. (3)) as degenerate cases of localization theorems.

The procedure of localization has been rather highly developed ((11), (12)) as a computational tool in the study of the Novikov E_2-term, motivated by the fundamental results of Jack Morava (14). Morava expresses certain localizations of the Novikov E_2-term as the continuous cohomology of certain p-adic Lie groups. In as sense the differential algebras encountered here are a replacement for (or approximation to) Morava's Lie groups. We shall explore this connexion in future work.

Our proof of the localization theorem relies on the construction of a certain multiplicative resolution for semi-tensor product algebras. This construction is carried out in section 1 in considerable generality, and is perhaps of independent interest. Then we prove the localization theorem (Theorem 2·2) and in section 3 give examples of its application to the Steenrod algebra. In section 4 we reinterpret these examples slightly and notice that often the localization map is an isomorphism in a range. In particular we compute $\mathrm{Ext}_A^{s,t}(H^*(S^0 \cup_p e^1), \mathbb{F}_p)$ for $t \leqslant p(p-1)s - p^2$ when $p > 2$; this is the odd-primary analogue of a result of M. Mahowald (9). Finally, in section 5 we indicate briefly how these results can be fed into a systematic computational program analogous to (12). This is the 'chromatic' spectral sequence, and it shows how the cohomology of the Steenrod algebra is built up out of periodic constituents.

Most of this work was carried out in the spring of 1974. It began as an attempt to understand certain parts of Joe Neisendorfer's thesis (Princeton University, unpublished). The influence of my thesis advisor John Moore is quite evident in section 1 and is gratefully acknowledged. The stimulus to complete this work was provided by conversations with Mark Mahowald in 1976–7. This research has been supported in part by the NSF.

Remark. In the body of this paper we shall work with coalgebras and Cotor rather than with algebras and Ext. This substitution simplifies consideration of the algebra structure in Ext, it is more closely related to actual computational procedures, and it can be related to geometry in more generality. We remind the reader that if A is an algebra of finite type over a field k and M is an A-module of finite type, then

$$A^* = \mathrm{Hom}_k(A, k)$$

is a coalgebra, $M^* = \mathrm{Hom}_k(M, k)$ is an A^*-comodule, and

$$\mathrm{Ext}_A(M, k) = \mathrm{Cotor}_{A^*}(k, M^*).$$

1. *A loop construction for semi-tensor products.* Suppose a group G acts by automorphisms on another group S. Then we may form the semi-direct product group $A = S \tilde{\times} G$; and a G-equivariant S-space X is exactly an A-space. If EH denotes a free contractible H-space for a group H, then there is a well-known homotopy equivalence

$$EA \times_A X \simeq EG \times_G (ES \times_S X). \tag{1·1}$$

Our object in this section is to prove an analogue of (1·1) in an algebraic setting. A very special case of this construction will be used in Section 2 to prove the localization theorem. A by-product of this work is a very rigid construction of a change-of-rings spectral sequence for certain extensions of coalgebras.

We shall work in the differential-graded setting, although for the application it suffices to suppose $d = 0$ in all the coalgebras involved.

Let R be a commutative ring and let G be a DG R-Hopf algebra with commutative multiplication. The category of (left) G-comodules then has an internal tensor product $\otimes^\Delta$; if M and N are G-comodules then $M \otimes^\Delta N$ is the G-comodule with underlying DG R-module $M \otimes N$ and with G-coaction

$$\psi_{M\otimes^\Delta N}: M \otimes N \xrightarrow{\psi_M \otimes \psi_N} G \otimes M \otimes G \otimes N \xrightarrow{G\otimes T \otimes N} G \otimes G \otimes M \otimes N \xrightarrow{\mu \otimes M \otimes N} G \otimes M \otimes N,$$

where ψ_L is the coaction map for the comodule L, T is the signed switch map, and μ is multiplication in G. The augmentation coaction on R provides a unit for this tensor product. A coalgebra S with respect to $\otimes^\Delta$ will be called a coalgebra *over* G; this is sometimes called a G-comodule-coalgebra (16). Thus S is a G-comodule and a DG coalgebra, with diagonal $\Delta x = \Sigma x' \otimes x''$, counit $\epsilon x = \bar{x}$, and G-coaction $\psi x = \Sigma x_{,} \otimes x_{,,}$, related by the equations

$$\Sigma(-1)^{|x''_{,}||x'_{,,}|} x'_{,} x''_{,} \otimes x'_{,,} \otimes x''_{,,} = \Sigma x_{,} \otimes x'_{,,} \otimes x''_{,,},$$

$$\Sigma \bar{x}_{,} \bar{x}_{,,} = \bar{x}.$$

A (left) S-comodule, still with respect to $\otimes^\Delta$, will be called an S-comodule *over* G.

Let M be a right S-comodule over G and N a left S-comodule over G.

Then

$$M \square_S N = \ker(\psi_M \otimes^{\Delta} N - M \otimes^{\Delta} \psi_N \colon M \otimes^{\Delta} N \to M \otimes^{\Delta} S \otimes^{\Delta} N)$$

is the kernel of a G-comodule-map and hence, if G is R-flat, it supports a G-comodule structure.

Given such G and S we may form their semi-tensor product (10) coalgebra

$$A = S \tilde{\otimes} G.$$

As a DG R-module $A = S \otimes G$. The diagonal is the composition

$$\Delta \colon SG \xrightarrow{\Delta\Delta} SSGG \xrightarrow{S\psi GG} SGSGG \xrightarrow{SGTG} SGGSG \xrightarrow{S\mu SG} SGSG,$$

where we have omitted '$\otimes$'. Thus, again using upper primes to denote S-coactions and lower primes to denote G-coactions,

$$\Delta(s \otimes g) = \Sigma(-1)^{|g_{\prime}||s''_{\prime\prime}|} s' \otimes s''_{\prime} g_{\prime} \otimes s''_{\prime\prime} \otimes g_{\prime\prime}.$$

There are natural coalgebra maps

$$\pi = S \otimes \epsilon \colon A \to S \quad \text{and} \quad \sigma = \epsilon \otimes G \colon A \to G;$$

and if S is supplemented by $\eta \colon R \to S$ over G then σ is split by $\iota \colon G \to A \colon \iota(g) = \eta(1) \otimes g$. Write $JS = \operatorname{coker}(\eta \colon R \to S)$.

Let M be an S-comodule over G. Define an A-coaction on M by means of the composite

$$M \xrightarrow{\psi_S} S \otimes M \xrightarrow{S \otimes \psi_G} S \otimes G \otimes M.$$

Conversely, given an A-comodule M with coaction map ψ, define S- and G-coactions ψ_S and ψ_G by means of the composites

It is easy to check that M then becomes an S-comodule over G, and that these constructions establish an equivalence between the category of A-comodules and the category of S-comodules over G.

Let S be a supplemented DG R-coalgebra. Then the relative differential right-derived functor of $R\square_S \colon (S\text{-comod}) \to (R\text{-mod})$, denoted by $\operatorname{Cotor}_S(R, \text{-})$, is the homology of the *cobar construction* $\Omega(S; \text{-})$. For a description of this functor see (8), II, §3, where $\Omega(S; M)$ is written as $\Omega S \otimes_{\tau} M$. We will use the standard notation $[s_1| \dots |s_p]x$, with $s_i \in S$ and $x \in M$, for a decomposable tensor in $\Omega(S; M)$.

We must also recall the theory of twisted tensor products. Our reference remains (8). Let S be a DG supplemented R-coalgebra and B a DG supplemented R-algebra. A *twisting morphism* is an R-linear map $\theta \colon S \to B$ of degree -1 satisfying the 'differential equation' $d\theta + \theta d = \theta \cup \theta$ and the 'initial conditions' $\theta\eta = 0 = \epsilon\theta$. Here given

$$\alpha, \beta \colon S \to B, \quad \alpha \cup \beta = \mu \circ (\alpha \otimes \beta) \circ \Delta \colon S \to B.$$

For example, $[\ ]: S \to \Omega S$ by $s \mapsto [s]$ is a twisting morphism; indeed, it is *universal* in the sense that there is a bijection between twisting morphisms $\theta: S \to B$ and supplemented *DG* algebra homomorphisms $f: \Omega S \to B$ so that f corresponds to the twisting morphism $s \mapsto f[s]$. Write $f = \bar{\theta}$.

Given a twisting morphism $\theta: S \to B$ and a B-module N, the *total space* $S \otimes_\theta N$ is the S-comodule with underlying R-module $S \otimes N$ and differential

$$d = d \otimes N + S \otimes d - d_\theta,$$

where $d_\theta = (S \otimes \phi) \circ (S \otimes \theta \otimes N) \circ (\Delta \otimes N)$. Then for example the total space of the universal twisting morphism, $S \otimes_{[\,]} \Omega(S; M)$, is contractible for any S-comodule M.

If S is a supplemented *DG* coalgebra over a *DG* Hopf algebra G, and M is an S-comodule over G, then in the diagonal G-coaction ΩS is a supplemented *DG* algebra over G and $\Omega(S; M)$ is a ΩS-module over G. Thus $\mathrm{Cotor}_S(R, R)$ is naturally an algebra over the Hopf algebra $H(G)$, and $\mathrm{Cotor}_S(R, M)$ is a $\mathrm{Cotor}_S(R, R)$-module over $H(G)$.

With these preliminaries, we can now state the main result of this section.

PROPOSITION 1·2. *Let G be a DG Hopf algebra with commutative multiplication and let S be a supplemented DG coalgebra over G. Then there is a twisting morphism*

$$\theta: S \tilde{\otimes} G \to \Omega(G; \Omega S)$$

such that the total space $(S \tilde{\otimes} G) \otimes_\theta \Omega(G; \Omega(S; M))$ is contractible for any S-comodule M over G.

To describe the algebra structure on $\Omega(G; \Omega S)$ intended here, we back off slightly. Let S and T be supplemented *DG* R-coalgebras, let M be an S-comodule, and let N be a T-comodule. Then $S \otimes T$ is naturally a coalgebra, $M \otimes N$ is naturally an $S \otimes T$-comodule, and there is a differential pairing

$$\Omega(S; M) \otimes \Omega(T; N) \to \Omega(S \otimes T; M \otimes N)$$

determined by the equation

$$[s_1|\dots|s_p]x \,.\, [t_1|\dots|t_q]y = \Sigma \pm [s_1 \otimes 1|\dots|s_p \otimes 1|x_{(1)} \otimes t_1|\dots|x_{(q)} \otimes t_q]x_{(q+1)} \otimes y. \tag{1·3}$$

Here $\Sigma x_{(1)} \otimes \dots \otimes x_{(q+1)} \in S^{\otimes q} \otimes M$ is the q-fold diagonal of $x \in M$, and the sign is the usual one required to pass the t_i's across the $x_{(j)}$'s. This formula is obtained by considering the Alexander–Whitney map for the cosimplicial *DG* R-modules of which $\Omega(S; M)$ and $\Omega(T; N)$ are the normalizations. It is clearly natural and associative.

Now suppose G is a *DG* Hopf algebra; then for G-comodules M and N we have an internal pairing given by the composite

$$\Omega(G; M) \otimes \Omega(G; N) \to \Omega(G \otimes G; M \otimes N) \to \Omega(G; M \otimes^\Delta N).$$

For example, if G is the dual Steenrod algebra and $M = H_*(X)$, $N = H_*(Y)$, then this is the smash-product pairing at E_1 of the Adams spectral sequence. This formula is presumably well-known, but I have not located it in the literature.

In particular, if B is an algebra over G then $\Omega(G; B)$ is a *DG* algebra, and if N is a

B-module over G then $\Omega(G;N)$ is an $\Omega(G;B)$-module. In Proposition 1·2, we take $B = \Omega S$ and $N = \Omega(S;M)$.

The twisting morphism $\theta\colon (S \tilde{\otimes} G) \to \Omega(G;\Omega S)$ is determined by $\theta(s \otimes 1) = -[\;][s]$, $\theta(1 \otimes g) = -[g][\;]$, and $\theta | JS \otimes JG = 0$. The check that θ is in fact a twisting morphism is an amusing exercise which we leave to the reader.

A contracting homotopy for $(S \tilde{\otimes} G) \otimes_\theta \Omega(G;\Omega(S;M))$ is given by c such that

$$c((s \otimes g)[h_1| \dots |h_p][t_1| \dots |t_q]x)$$
$$= \begin{cases} (s \otimes \bar{g}h_1)[h_2| \dots |h_p][t_1| \dots |t_q]x & \text{if } p > 0 \\ \Sigma \pm (\bar{s}t_1 \otimes \bar{g}t_{2'} \dots t_{q'}x_{'})[\;][t_{2''}| \dots |t_{q''}]x_{''} & \text{if } p = 0 \text{ and } q > 0 \\ 0 & \text{if } p = q = 0. \end{cases} \tag{1·4}$$

Here $\bar{g} = \eta\epsilon g \in G$ and similarly for $\bar{s}$; and the sign is the usual one required to achieve the indicated permutation.

To avoid distracting flatness assumptions we assume in the following Corollary that k is a field.

COROLLARY 1·5. *Let G be a DG k-Hopf algebra with commutative multiplication, let S be a supplemented DG coalgebra over G, and let M be an S-comodule over G. Write*

$$A = S \tilde{\otimes} G,$$

and assume that S, G and M are zero in negative dimensions. Then

(*a*) *$\bar{\theta}\colon \Omega A \to \Omega(G;\Omega S)$ is a homology isomorphism of algebras and*

$$\Omega(A;M) \to \Omega(G;\Omega(S;M))$$

is a homology isomorphism of modules over $\bar{\theta}$;

(*b*) $\mathrm{Cotor}_A(k,k) \cong \mathrm{Cotor}_G(k,\Omega S)$ *as algebras and*

$$\mathrm{Cotor}_A(k,M) \cong \mathrm{Cotor}_G(k,\Omega(S;M))$$

as modules; and

(*c*) *there is a spectral sequence with*

$$E_2 = \mathrm{Cotor}_{H(G)}(k, \mathrm{Cotor}_S(k,M)) \Rightarrow \mathrm{Cotor}_A(k,M).$$

Proof. (*a*) follows from J. C. Moore's spectral comparison theorem ((5), theorem B), and (*a*) implies (*b*). The spectral sequence is the Eilenberg–Moore spectral sequence ((7), theorem 9·2), where we have rewritten the limit term $\mathrm{Cotor}_G(k,\Omega(S;M))$ using (*b*).

2. *The localization theorem.* To describe the principal result of this section we need some notation.

Let E be the exterior Hopf algebra over the field k with a single generator e of dimension n. Then $\Omega E = k[q]$, a polynomial algebra with a single generator $q = [e]$ of dimension $n-1$, with trivial differential. If X is a DG E-comodule then, neglecting the differential, $\Omega(E;X) = \Omega E \otimes X$. Let $K = k[q, q^{-1}]$, and define a differential K-module

$$tX = q^{-1}\Omega(E;X) = K \otimes_{\Omega E} \Omega(E;X).$$

Since localization is exact,

$$q^{-1}\,\mathrm{Cotor}_E(k,X) = H(tX). \tag{2·1}$$

THEOREM 2·2. *Let E be an exterior algebra over k on one generator, and let q, K and t be as above. Let S be a supplemented DG coalgebra over E and let M be an S-comodule over E. Write $A = S \tilde{\otimes} E$. Assume that E, S and M are zero in negative dimensions. Then there is a natural isomorphism*

$$q^{-1}\,\mathrm{Cotor}_A(k, M) \cong \mathrm{Cotor}_{tS}(K, tM).$$

Proof. We compute:

$$\begin{aligned} q^{-1}\,\mathrm{Cotor}_A(k, M) &= q^{-1}\,\mathrm{Cotor}_E(k, \Omega(S; M)) \\ &= H(t\Omega(S; M)) = H(\Omega(tS; tM)) \\ &= \mathrm{Cotor}_{tS}(K, tM). \end{aligned}$$

The first equality comes from Corollary 1·5, and the second is (2·1). The third equality holds because $t\Omega(S; M) = \Omega(tS; tM)$. Here we have extended somewhat the usual range of applicability of the cobar construction by allowing the ground-ring to be *graded*; tS is a K-coalgebra. The last equality holds by definition. ∎

One can now attempt to compute this localization by means of the usual algebraic Eilenberg–Moore spectral sequence:

$$E_2 = \mathrm{Cotor}^{p,q}_{H(tS)}(K, H(tM)) \underset{p}{\Rightarrow} \mathrm{Cotor}^{q-p}_{tS}(K, tM). \tag{2·3}$$

This is obtained by filtering the cobar construction $\Omega(tS; tM)$ by homological degree. It is a cohomological spectral sequence lying in the right half-plane. It converges in the sense that a map $(S', M') \to (S, M)$ inducing an isomorphism at E_2 induces an isomorphism $\mathrm{Cotor}_{tS'}(K, tM') \to \mathrm{Cotor}_{tS}(K, tM)$.

3. *Examples from the Steenrod algebra.* Let k be the field with p elements and let P be the Hopf algebra $k[\xi_1, \xi_2, \ldots]$, $|\xi_n| = 2(p^n - 1)$, with diagonal

$$\Delta\xi_n = \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \xi_i,$$

where $\xi_0 = 1$. Let $E(0)$ be the exterior Hopf algebra generated by

$$\tau_0, \tau_1, \ldots, |\tau_n| = 2p^n - 1.$$

Give $E(0)$ the structure of a Hopf algebra over P by requiring

$$\psi\tau_n = \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \tau_i.$$

Write $A(0)$ for the semi-tensor product Hopf algebra. If $p > 2$ then $A(0)$ is simply the dual A of the Steenrod algebra. For comments on the situation when $p = 2$ see Remark 4·9.

More generally, let $E(n) = E[\tau_n, \tau_{n+1}, \ldots]$; it is a quotient Hopf algebra over P of $E(0)$. Let $A(n) = E(n) \tilde{\otimes} P$; it is a quotient Hopf algebra of $A(0)$. By dualizing (6), XVI, § 5 $(2)_2$, we have a spectral sequence

$$E_2 = \mathrm{Cotor}_{A(0)}(\mathrm{Cotor}_{A(n)}(k, A(0)), N) \Rightarrow \mathrm{Cotor}_{A(n)}(k, N)$$

for any $A(0)$-comodule N. By the theorem of Milnor and Moore (13), $A(0)$ is an injective $A(n)$-comodule, so the spectral sequence collapses and

$$\mathrm{Cotor}_{A(0)}(k \,\square_{A(n)} A(0), N) = \mathrm{Cotor}_{A(n)}(k, N).$$

It is easy to see that the left-hand side is isomorphic to

$$\mathrm{Cotor}_{A(0)}(k, (A(0) \,\square_{A(n)} k) \otimes^{\Delta} N),$$

and that $A(0) \,\square_{A(n)} k = E[\tau_0, \ldots, \tau_{n-1}]$.

Suppose that $p > 2$ and that $V(n-1)$ is a spectrum such that

$$H_* V(n-1) = E[\tau_0, \ldots, \tau_{n-1}]$$

as an A-comodule. Then we have seen that $\mathrm{Cotor}_{A(n)}(k, H_* X)$ is the E_2-term of the Adams spectral sequence for $\pi_*(V(n-1) \wedge X)$. In any case $\mathrm{Cotor}_{A(n)}(k, k)$ is of interest algebraically; in section 5 we will see that it is central to an understanding of the cohomology of the Steenrod algebra. For the present, notice that $\tau_n \in A(n)$ is primitive and so determines an element $q_n \in \mathrm{Cotor}^{1, 2p^n-1}_{A(n)}(k, k)$ represented by $[\tau_n] \in \Omega A(n)$. Our goal is now to understand $q_n^{-1} \mathrm{Cotor}_{A(n)}(k, N)$.

The Hopf algebra $A(n)$ may be obtained as a semi-tensor product

$$A(n) = A(n+1) \,\tilde{\otimes}\, E[e_n]$$

where $E[e_n]$ coacts by

$$\left.\begin{aligned} \psi\bar{\tau}_{n+i} &= 1 \otimes \bar{\tau}_{n+i} + e_n \otimes \bar{\xi}_i^{p^n}, \\ \psi\bar{\xi}_i &= 1 \otimes \bar{\xi}_i \end{aligned}\right\} \tag{3·1}$$

($i \geqslant 0$); here $\bar{x}$ is the Hopf conjugate of $x \in A(n+1)$. Thus by Theorem 2·2,

$$q_n^{-1} \mathrm{Cotor}_{A(n)}(k, N) = \mathrm{Cotor}_{tA(n+1)}(K(n), tN) \tag{3·2}$$

where $K(n) = k[q_n, q_n^{-1}]$. The spectral sequence (2·3) involves $H(tA(n+1))$, which we compute by means of (3·1) to be $K(n) \otimes P(n)$, where

$$P(n) = P/(\bar{\xi}_1^{p^n}, \bar{\xi}_2^{p^n}, \ldots) = P/(\xi_1^{p^n}, \xi_2^{p^n}, \ldots).$$

Thus (2·3) reads

$$\mathrm{Cotor}_{P(n)}(k, H(tN)) \Rightarrow q_n^{-1} \mathrm{Cotor}_{A(n)}(k, N) \tag{3·3}$$

Suppose for example that $n = 0$. Then the spectral sequence collapses at $E_2 = H(tN)$; so for $p > 2$ the q_0-localization of the Adams E_2-term for X is just the same as the Bockstein E_2-term for the homology of X.

Remark 3·4. In fact, the two spectral sequences coincide. To see this note that the homology Bockstein spectral sequence for X may be regarded as the Adams spectral sequence for $H \wedge X$ based on the ring-spectrum $V(0)$; here H is the integral Eilenberg–MacLane spectrum. It is easy to construct a map of Adams resolutions from the usual Adams resolution for X into the $V(0)$-Adams resolution for $H \wedge X$. This induces a map of spectral sequences agreeing at E_2 with the map shown above to localize to an isomorphism. The result follows.

Notice that the natural projection $tA(n+1) \to K(n) \otimes P(n)$ is a homology isomorphism and a map of differential $K(n)$-coalgebras. Therefore, whenever this map can be covered by a homology isomorphism $tN \to H(tN)$, the spectral sequence (3·3) collapses. This is the case for example when $N = k$ (so $tN = K(n)$). Thus:

COROLLARY 3·5. *The natural Hopf algebra map* $A(n) \to E[\bar{\tau}_n] \otimes P(n)$ *induces an isomorphism in* Cotor *after localizing at* q_n:

$$q_n^{-1}\,\mathrm{Cotor}_{A(n)}(k,k) \xrightarrow{\cong} K \otimes \mathrm{Cotor}_{P(n)}(k,k).$$

The first interesting case here is $n = 1$. Since $P(1)$ is biprimitive,

$$\mathrm{Cotor}_{P(1)}(k,k) = E[h_{i,0}: i \geqslant 1] \otimes P[b_{i,0}: i \geqslant 1]$$

with
$$h_{i,0} = \{[\bar{\xi}_i]\} \in \mathrm{Cotor}_{P(1)}^{1,2(p^i-1)}(k,k)$$

$$b_{i,0} = \left\{\sum_{j=1}^{p-1} \frac{\binom{p}{j}}{p}\,[\bar{\xi}_i^j|\bar{\xi}_i^{p-j}]\right\} \in \mathrm{Cotor}_{P(1)}^{2,2p(p^i-1)}(k,k).$$

Thus we have

COROLLARY 3·6. *For* $p > 2$, *the Adams* E_2*-term for a Moore space localizes at* q_1 *to*

$$P[q_1, q_1^{-1}] \otimes E[h_{i,0}: i \geqslant 1] \otimes P[b_{i,0}: i \geqslant 1].$$

Notice that $P(2)$ is the odd-primary analogue of the Hopf algebra whose cohomology is the E_2-term for MSp. Its cohomology thus may be expected to be quite complicated.

4. *Bigradings and vanishing lines.* It is convenient to study $A(n)$ by means of the semi-tensor product decomposition $A(n) = E(n) \tilde{\otimes} P$. By Corollary 1·5 there is an algebra map $\Omega A(n) \to \Omega(P; \Omega E(n))$ which is a homotopy equivalence.

Let $Q(0) = P[q_0, q_1, \ldots]$ be the algebra over P with coaction determined by

$$\psi q_n = \Sigma \xi_{n-i}^{p^i} \otimes q_i$$

and with zero differential. Then $I_n = (q_0, \ldots, q_{n-1})$ is an invariant ideal (i.e. an ideal and a sub P-comodule) so $Q(n) = Q(0)/I_n$ is again an algebra over P. There is a twisting morphism $\theta: E(n) \to Q(n)$ sending τ_j to q_j. It is a map of P-comodules and the associated total space $E(n) \otimes_\theta Q(n)$ is acyclic.

It follows from these two remarks that the composite

$$\Omega A(n) \to \Omega(P; \Omega E(n)) \to \Omega(P; Q(n))$$

is an algebra map and a homotopy equivalence, and that, for any bounded below $A(n)$-comodule M, the composite $\Omega(A(n); M) \to \Omega(P; \Omega(E(n); M)) \to \Omega(P; Q(n) \otimes_\theta M)$ is a module map and a homotopy equivalence. In particular,

$$\mathrm{Cotor}_{A(n)}(k,k) \cong \mathrm{Cotor}_P(k, Q(n)) \tag{4·1}$$

as algebras and $\mathrm{Cotor}_{A(n)}(k, M) \cong \mathrm{Cotor}_P(k, Q(n) \otimes_\theta M)$ as modules.

The equality (4·1) follows also from the fact that the extension spectral sequence for

$$k \to P \to A(n) \to E(n) \to k \tag{4·2}$$

collapses at $E_2 = \mathrm{Cotor}_P(k, Q(n))$ since τ_n can be endowed with an extra 'Cartan degree' so that the E_2-term lies along a diagonal. This phenomenon was I believe first noticed by Novikov ((15), theorem 12·1). The above proof has the advantage of providing a more explicit isomorphism and a more satisfactory explanation.

Remark 4·3. One can now continue by projecting $\Omega(P; Q(n) \otimes_\theta M)$ to the Λ-algebra for P with $Q(n) \otimes_\theta M$ coefficients (4). With $M = k$, this gives the smallest known associative algebra resolution for $A(n)$. It has been studied by S. Rosen and M. C. Tangora.

In these terms, the localization theorem reads

COROLLARY 4·4. *The natural map $Q(n) \to k[q_n]$ is an algebra map over the Hopf algebra map $P \to P(n)$ and for any bounded below $A(n)$-module M the induced map in* Cotor *localizes to an isomorphism*

$$q_n^{-1}\,\mathrm{Cotor}_{A(n)}(k, M) \xrightarrow{\cong} \mathrm{Cotor}_{P(n)}(k, tM). \tag{4·5}$$

In particular $$q_n^{-1}\,\mathrm{Cotor}_{A(n)}(k, k) \xrightarrow{\cong} K(n) \otimes \mathrm{Cotor}_{P(n)}(k, k). \tag{4.6}$$

Furthermore, (4·6) *is an algebra map and* (4·5) *respects the module structures.*

We need to fix indexing notation. $Q(n)$ is bigraded, with $|q_i| = (1, 2(p^i - 1))$; the first index is the 'Cartan degree' and is respected by the P-coaction. Then

$$\mathrm{Cotor}_P(k, Q(n))$$

is trigraded, say by (s, t, u) with $s =$ homological degree, $t =$ Cartan degree, $u =$ complementary degree. Thus

$$\mathrm{Cotor}^{a,b}_{A(n)}(k, k) = \bigoplus_{\substack{s+t=a\\ u+t=b}} \mathrm{Cotor}^{s,t,u}_P(k, Q(n)).$$

For $s \geq 0$ let $U(2s) = pqs$ and $U(2s+1) = pqs + q$, $q = 2(p-1)$, and write

$$U(-1) = \infty.$$

Then by (2), $\mathrm{Cotor}^{s,u}_P(k, M) = 0$ for $u < U(s) + m$ if M is $(m-1)$-connected. In particular, $\mathrm{Cotor}^{s,t,u}_P(k, Q(n+1)) = 0$ for $u < U(s) + 2(p^{n+1} - 1)t$. By studying the long exact sequence associated to the short exact sequence of P-comodules

$$0 \longrightarrow Q(n) \xrightarrow{q_n} Q(n) \longrightarrow Q(n+1) \longrightarrow 0,$$

we see that $q_n | \mathrm{Cotor}^{s,t,u}_P(k, Q(n))$ is monic if $u < U(s-1) + 2(p^{n+1} - 1)(t+1) - 2(p^n - 1)$ and epic if $u < U(s) + 2(p^{n+1} - 1)(t+1) - 2(p^n - 1)$. From the definition of localization it then follows that $\mathrm{Cotor}^{s,t,u}_P(k, Q(n)) \to q_n^{-1}\mathrm{Cotor}^{s,t,u}_P(k, Q(n))$ is monic or epic under the same conditions. Since $\mathrm{Cotor}^{s,t,u}_P(k, Q(n)) = 0$ generally only for

$$u < U(s) + 2(p^n - 1)t,$$

the localizations described in section 3 become 'visible' in a wedge widening with t.

When $n > 1$, the $U(s)$ vanishing line (i.e. powers of $b_{1,0}$) mask the q_n-periodicity in $\mathrm{Cotor}_{A(n)}(k, k)$. However, when $n \leq 1$ we have by similar methods

PROPOSITION 4·7. (*a*) *Let $p > 2$ and let M be an $A(1)$-comodule which is zero in negative dimensions. Then*

$$\mathrm{Cotor}_{A(1)}(k, M) \to q_1^{-1}\,\mathrm{Cotor}_{A(1)}(k, M)$$

in bidegree (s,t) is epic for $t < U(s+1)-2p+1$ and an isomorphism for

$$t < U(s)-2p+1.$$

(*b*) *Let M be an $A(0)$-comodule which is zero in negative dimensions. Then*

$$\text{Cotor}_{A(0)}(k,M) \to q_0^{-1}\,\text{Cotor}_{A(0)}(k,M)$$

is epic for $t < (2p-1)(s+1)-1$ and an isomorphism for $t < (2p-1)s-1$.

Part (*b*) together with Remark 3·4 implies for $p > 2$ the folk-theorem that for $t < (2p-1)s-2$, the Adams spectral sequence coincides with the homology Bockstein spectral sequence. In future work we shall apply techniques of Novikov (15) to show that in the q_1-localization of the Adams spectral sequence for a Moore space at an odd prime,

$$d_2 h_{i,0} = \pm q_1 b_{i-1,0} \tag{4·8}$$

for all $i > 1$. Thus $E_3 = P[q_1] \otimes E[h_{1,0}]$ in bidegree (s,t) such that $t < U(s)-2p+1$, and as these classes survive, this is E_∞ as well.

Remark 4·9. If $p = 2$ then the dual Steenrod algebra A sits in a *non-split* extension of Hopf algebras

$$k \to P \to A \to E(0) \to k.$$

The corresponding spectral sequence has

$$E_2 = \text{Cotor}_P(k, Q(0)) \simeq \text{Cotor}_{A(0)}(k,k),$$

but does not collapse. Indeed, it is closely related (15) to the mod 2 Novikov spectral sequence for the sphere.

5. *A chromatic spectral sequence.* The localization theorem of section 2 plays a role in a program to elucidate the structure of the cohomology of the Steenrod algebra analogous to the role of (11) in the study of the Novikov E_2-term initiated in (12). We outline this program in this section. A comparison of this with (12) clarifies some differences between the Adams and Novikov E_2-terms.

Let $Q = Q(0)$ and let $I_n = (q_0, \ldots, q_{n-1}) \subset Q$. Q is an algebra over P and I_n is an invariant ideal, i.e. an ideal and a sub P-comodule. Just as in (11), we prove

LEMMA 5·1. *If M is a Q-module over P such that $I_n^k M = 0$ for some k, then multiplication by $q_n^{p^s}$ on M is a P-comodule map for all sufficiently large s.*

LEMMA 5·2. *If M is a Q-module over P such that for all $x \in M$ there exists $k \geq 0$ for which $I_n^k x = 0$, then $q_n^{-1}M$ supports a unique P-comodule structure such that $M \to q_n^{-1}M$ is a map of Q-modules over P.*

Now let $R_n^0 = Q/I_n$, and suppose R_n^s has been defined and is such that for all $x \in R_n^s$, $I_{n+s}^k x = 0$ for some $k \geq 0$. Then $Q_n^s = q_{n+s}^{-1} R_n^s$ is a Q-module over P by Lemma 5·2, and R_n^{s+1} is defined by the exactness of

$$0 \to R_n^s \to Q_n^s \to R_n^{s+1} \to 0. \tag{5·3}$$

Then we have a long exact sequence

$$0 \to Q/I_n \to Q_n^0 \to Q_n^1 \to \ldots. \tag{5·4}$$

Applying $\mathrm{Cotor}_P(k, \text{-})$, we obtain a spectral sequence converging to

$$\mathrm{Cotor}_P(k, Q/I_n) = \mathrm{Cotor}_{A(n)}(k, k),$$

with

$$E_1^{s,t}(n) = \mathrm{Cotor}_P^t(k, Q_n^s).$$

This is the 'chromatic spectral sequence'.

Continuing in analogy with (12), consider the short exact sequence

$$0 \to Q_{n+1}^{s-1} \to Q_n^s \xrightarrow{q_n} Q_n^s \to 0$$

and its associated 'Bockstein' exact couple linking $\mathrm{Cotor}_P(k, Q_n^s)$ to $\mathrm{Cotor}_P(k, Q_{n+1}^{s-1})$. By a succession of these exact couples, $E_1^s(n)$ is related to

$$\mathrm{Cotor}_P(k, Q_{n+s}^0) = \mathrm{Cotor}_P(k, q_{n+s}^{-1} Q/I_{n+s}) = q_{n+s}^{-1} \mathrm{Cotor}_{A(n+s)}(k, k).$$

We finish by pointing out some differences between the Adams and Novikov E_2-terms which are exposed by a comparison of their respective chromatic spectral sequences.

To begin with, the most striking feature of the BP situation is Morava's theorem (14): $q_n^{-1}\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}^{s,t}(\mathrm{BP}_*, \mathrm{BP}_*/I_n)$ is a Poincaré duality algebra of formal dimension n^2 (so $= 0$ for $s > n^2$) if $(p-1)$ does not divide n, and is finitely generated over a polynomial algebra on one generator if $(p-1)$ divides n. This remarkable finiteness property is completely lacking in the present case. Indeed, for $n > 0$, $q_n^{-1}\,\mathrm{Cotor}_P(k, Q(n))$ is not finitely generated as an algebra, and has infinite Krull dimension. It enjoys a kind of stability property, since $P(n) \leftarrow P(n+1)$; so the groups $\mathrm{Cotor}_{P(n)}(k, k)$, which provide the backbone of the q_n-periodic subquotient of $\mathrm{Cotor}_{A(0)}(k, k)$, converge to the cohomology of the algebra of reduced powers as n becomes large.

Furthermore, this extra complexity in $\mathrm{Cotor}_P(k, Q_s^0)$ makes the analysis of the Bockstein exact couples described above much more difficult. For example, it is almost trivial to obtain $E_1^{1,*}(0)$ in the BP case ((12), § 4), where as in the present case it is tantamount to computing $\mathrm{Cotor}_{A(0)}^{s,t}(k, k)$ for $t > U(s)$ and appears to be quite formidable.

Yet the Adams E_2-term, which is the object of these computations, is less closely related (at least for $p \neq 2$) to the real object of interest, stable homotopy, than is the Novikov E_2-term. Many of the algebraic complexities are thus due to the inadequate image of homotopy theory present in the Steenrod algebra.

REFERENCES

(1) ADAMS, J. F. A finiteness theorem in homological algebra. *Proc. Cambridge Philos. Soc.* **57** (1961), 31–36.

(2) ADAMS, J. F. A periodicity theorem in homological algebra. *Proc. Cambridge Philos. Soc.* **62** (1966), 365–377.

(3) ANDERSON, D. W. and DAVIS, D. W. A vanishing theorem in homological algebra. *Comment Math. Helv.* **48** (1973), 318–327.

(4) BOUSFIELD, A. K. and CURTIS, E. B. A spectral sequence for the homotopy of nice spaces. *Trans. Amer. Math. Soc.* **151** (1970), 457–479.

(5) CARTAN, H. *DGA-modules (Suite): Notion de construction*, exp. 3, Seminaire H. Cartan, 1954/1955, Paris.

(6) CARTAN, H. and EILENBERG, S. *Homological algebra* (Princeton University Press, 1956).

(7) EILENBERG, S. and MOORE, J. C. Homology and fibrations. I. Coalgebras, cotensor product and its derived functors. *Comment Math. Helv.* **40** (1966), 199–236.

(8) HUSEMOLLER, D., MOORE, J. C. and STASHEFF, J. Differential homological algebra and homogeneous spaces. *J. Pure and Appl. Alg.* **5** (1974), 113–185.

(9) MAHOWALD, M. The order of the image of the J-homomorphism. *Bull. Amer. Math. Soc.* **76** (1970), 1310–1313.

(10) MASSEY, W. S. and PETERSON, F. P. The cohomology structure of certain fibre spaces: I. *Topology* **4** (1965), 47–65.

(11) MILLER, H. R. and RAVENEL, D. C. Morava stabilizer algebras and the localization of Novikov's E_2-term. *Duke J. Math.* **44** (1977), 433–447.

(12) MILLER, H. R., RAVENEL, D. C. and WILSON, W. S. Periodic phenomena in the Adams–Novikov spectral sequence. *Ann. of Math.* **106** (1977), 469–516.

(13) MILNOR, J. and MOORE, J. C. On the structure of Hopf algebras. *Ann. of Math.* **81** (1965), 211–264.

(14) MORAVA, J. Extensions of cobordism comodules. (To appear.)

(15) NOVIKOV, S. P. The methods of algebraic topology from the viewpoint of cobordism theory. *Math. USSR Izvestija* **1** (1967), 827–913.

(16) STEENROD, N. E. and EPSTEIN, D. B. A. *Cohomology operations* (Princeton University Press, 1962).