# ON JONES'S KAHN–PRIDDY THEOREM

Haynes Miller[1]
Massachusetts Institute of Technology
Cambridge, MA 02139/USA

For Hirosi Toda on his sixtieth birthday

In this note we record a simple proof of a beautiful result of J.D.S. Jones, stating that the Mahowald root invariant of a stable homotopy class $\alpha$ has dimension at least twice that of $\alpha$. This result is a natural strengthening of the Kahn–Priddy theorem. Our contribution is simply to provide a postcard–length proof of the key diagram (3.1) (which occurs on p. 481 of [2]), but we take the opportunity to restate the notions of root invariant and quadratic construction, and the connection with the Kahn–Priddy theorem. We also deal with odd primes, after J. P. May [1]. We end with a proof of Mahowald's theorem that $\alpha_i \in R(p^i)$, and the observation that Jones's theorem allows one to translate a 20–year old result of Toda's into the assertion that up to a unit, $\beta_1 \in R(\alpha_1)$.

## §1. Mahowald's root invariant.

We begin with a discussion of the "root invariant" introduced by Mahowald in 1967 [5]. A reference for the constructions in this section is [1]. We begin with $p = 2$. Let $\lambda$ denote the canonical line bundle over a real projective space, and let $P_{-n}^{t-1}$ be the Thom spectrum of $-n\lambda|\mathbb{R}P^{t+n-1}$. This is the suspension spectrum of $\mathbb{R}P^{t-1}/\mathbb{R}P^{-n-1}$ if $n < 0$; it is $\mathbb{R}P_{+}^{t-1}$ if $n = 0$; and if $n > 0$ one embeds $n\lambda|\mathbb{R}P^{t+n-1}$ in a trivial vector bundle and considers the complementary subbundle. Atiyah duality shows that the 0–dual of $P_{-n}^{t-1}$ is

$$DP_{-n}^{t-1} \simeq \Sigma P_{-t}^{n-1} . \tag{1.1}$$

There are natural inclusions $P_{-n}^{t-1} \to P_{-n}^{t}$ and collapses $P_{-n}^{t-1} \to P_{-n+1}^{t-1}$, which will be represented by unnamed arrows. By forming direct limits we get spectra $P_{-n}$ with collapse maps $P_{-n} \to P_{-n+1}$.

The pinch map $\pi : P_0^{t-1} = \mathbb{R}P_{+}^{t-1} \to S^0$ dualizes to a map $S^{-1} \to P_{-t}^{-1}$. Composing with the inclusion we get maps $\iota : S^{-1} \to P_{-t}$ which are compatible under the collapse maps and yield a diagram

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$$\begin{array}{ccc} & & \downarrow \\ & \overset{\iota}{\nearrow} & P_{-t-1} \\ & & \downarrow \\ S^{-1} & \xrightarrow{\iota} & P_{-t} \\ & & \downarrow \\ & \underset{\iota}{\searrow} & P_{-t+1} \\ & & \downarrow \end{array} \tag{1.2}$$

Now let $p \neq 2$ and set $q = 2(p-1)$. Then Adams' stable homology approximation to $B\Sigma_p$ (see [1], p. 146) extends to give spectra $P_{-n}$ with analogous properties. Here $n \equiv 0$ or $1 \bmod q$; $P_{-n}$ has one cell in dimension i for each $i \equiv 0$ or $-1 \bmod q$ with $i \geq -n$; $P_{q-1} \simeq \Sigma^\infty B\Sigma_p$; (1.1) holds with $t \equiv 0$ or $1 \bmod q$; and there is an evident analogue of (1.2). See [1], pp. 146 and 42.
For $1 \leq i \leq q-1$, let $P_{-qk-i} = P_{-qk-1}$.

The *Mahowald filtration* on the p–completed stable homotopy group $\pi_t$ is

$$M^s \pi_t = \ker(\iota_* : \pi_{t-1}(S^{-1}) \longrightarrow \pi_{t-1}(P_{-s})). \tag{1.3}$$

Clearly $M^0 \pi_t = \pi_t$ for all t. There is a cofiber sequence

$$S^{-1} \xrightarrow{\iota} P_{-1} \longrightarrow P_0 \xrightarrow{\tau} S^0$$

in which $\tau$ is the "transfer" map. The Kahn–Priddy theorem may thus be stated as

$$M^1 \pi_t = \pi_t \qquad \text{for all } t \geq 1. \tag{1.4}$$

The theorem of Jones asserts

$$M^s \pi_t = \pi_t \qquad \text{for } s \leq t, \text{ for } p = 2. \tag{1.5}$$

We will show also that for any prime p,

$$M^s\pi_t = \pi_t \text{ for } s \leq qk-\epsilon, \text{ if } t = 2k-\epsilon \tag{1.6}$$

Lin's theorem [4] (Gunawardena's when $p \neq 2$) says that the map $\iota : S^{-1} \longrightarrow \operatorname{holim} P_{-t}$ is p–adic completion. This implies

$$\bigcap_s M^s \pi_* = 0 \,. \tag{1.7}$$

If $\alpha \in M^s \pi_t - M^{s+1} \pi_t$, the *root invariant* of $\alpha$ is defined as

$$R(\alpha) = \left\{\beta : \begin{array}{ccccc} & & S^{-s-1} & & \\ & \overset{\beta}{\nearrow} & & \overset{j}{\searrow} & \\ S^{t-1} & & & & P_{-s-1} \\ & \underset{\alpha}{\searrow} & & \underset{\iota}{\nearrow} & \\ & & S^{-1} & & \end{array}\right\} \subseteq \pi_{s+t} \qquad (1.8)$$

where j is the inclusion of the bottom cell. Jones's result is thus that $|R(\alpha)| \geq 2|\alpha|$ . when p = 2, and we find that for any prime p, $|R(\alpha)| \geq 2(pk-\epsilon)$ if $|\alpha| = 2k-\epsilon$.

One may of course define an invariant "$R_s(\alpha)$" by means of (1.8) for any s. But if $\alpha \in M^{s+1}\pi_t$, then $R_s(\alpha) = \{\beta : j \circ \beta = *\}$, and if $\alpha \notin M^s \pi_t$, then $R_s(\alpha) = \emptyset$ : in either case it is independent of $\alpha$ and of little interest.

The root invariant of $\alpha$ may be interpreted in terms of the homotopy spectral sequence associated to the tower (1.2): it is the set of representatives of $\alpha$ in $E^1_{-s,s+t} = \pi_{s+t}$. Lin's theorem shows that this spectral sequence converges to $\pi_*((S^{-1})^\wedge_p)$; so

$$E^\infty_{u,q} = 0 \quad \text{for } q < -(u+1). \qquad (1.9)$$

From (1.4), we have

$$E^\infty_{u,*} = 0 \ \text{ for } u \geq 0 , \qquad (1.10)$$

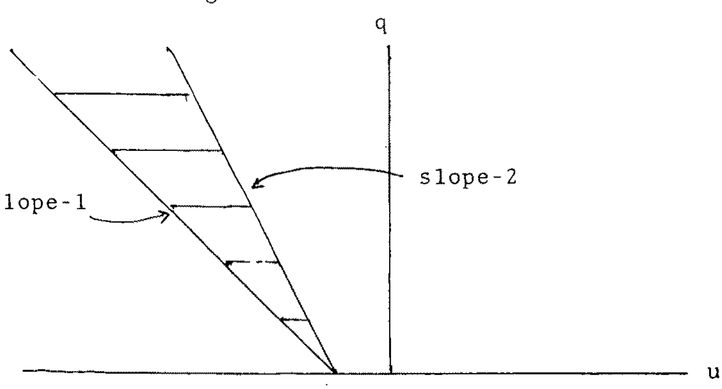
and the Kahn–Priddy theorem asserts

$$E^\infty_{-1,q} = 0 \quad \text{for } q \geq 1. \qquad (1.11)$$

For p = 2, Jones's theorem is

$$E^\infty_{u,q} = 0 \quad \text{for } q > -2(u+1). \qquad (1.12)$$

Thus $E^\infty$ is concentrated in a wedge:



More precise information about the position of $E^\infty$ is tied up with the conjecture that the root invariant converts "$v_n$–periodic" families to "$v_{n+1}$–periodic" families. This leads one to expect that $E^\infty$ should be concentrated near rays with slopes $-(2-2^{-n})$. Similar analytic geometry holds at odd primes.

## §2. The Cup–i construction.

Jones proves his theorem by relating the root invariant to the cup–i construction. We recall this construction from [2].

The p–adic construction

$$D_pX = \frac{E\sigma \times_\sigma X^{(p)}}{E\sigma \times_\sigma *}, \quad \sigma = \Sigma_p,$$

of a pointed space X extends naturally to spectra; see [3], [1]. It is easy to see that $D_pS^{2n} = \Sigma^{2n}P_{qn}$ and $D_pS^{2n-1} = \Sigma^{2n-1}P_{qn-1}$ for $n > 0$, and this holds for $n \le 0$ as well. There is a natural map

$$\varphi_K : K \wedge D_pX \longrightarrow D_p(K \wedge X)$$

for K a pointed space and X a spectrum. If X is a space this given by

$$k;\, e, x_1,\dots x_p \longmapsto e, (k,x_1),\dots(k,x_n).$$

If $K = S^1$ and $X = S^n$, this is the "collapse map," $\Sigma^{n+1}P_n \longrightarrow \Sigma^{n+1}P_{n+1}$ when $p = 2$, and analogously for $p \neq 2$.

For $p = 2$ and $\alpha \in \pi_t$ define $Q(\alpha) : P_t \to S^{-t}$ as the desuspension of the composite

$$\Sigma^tP_t = D_2(S^t) \xrightarrow{D_2(\alpha)} D_2(S^0) = P_0 \xrightarrow{\pi} S^0 . \tag{2.1}$$

The *quadratic filtration* of $\pi_t$ is given by

$$F^n\pi_t = \{\alpha : Q(\alpha) \,|\, P_t^{t+n-1} \simeq *\} . \tag{2.2}$$

Thus

$$F^0\pi_t = \pi_t \tag{2.3}$$

and

$$F^1\pi_t = \{\alpha : \alpha^2 = 0\} . \tag{2.4}$$

We will see that

$$\bigcap_n F^n\pi_t = 0 \ . \tag{2.5}$$

If $\alpha \in F^n\pi_t - F^{n+1}\pi_t$, we define the *cup–n construction* to be

$$C(\alpha) = \left\{ \beta : \begin{array}{ccccc} & & S^{t+n} & & \\ & \nearrow & & \overset{\beta}{\searrow} & \\ P_{-t}^{t+n} & & & & S^{-t} \\ & \searrow & & \nearrow_{Q(\alpha)} & \\ & & P_t & & \end{array} \right\} \subseteq \pi_{2t+n} \ . \tag{2.6}$$

Thus for example if $\alpha \notin F^1\pi_t$ then $C(\alpha) = \{\alpha^2\}$ .

It is useful to extend this definition by considering the composite

$$P_{-s} \xrightarrow{\ c\ } P_t \xrightarrow{\ Q(\alpha)\ } S^{-t}$$

for $s \geq -t$, where $c$ is the collapse map. We then have a *modified quadratic filtration*

$$F^n_s\pi_t = \{\alpha : (Q(\alpha)\circ c)\,|\,P_{-s}^{t+n-1} \simeq *\} \tag{2.7}$$

Then

$$F^n\pi_t = F^n_{-t}\pi_t \subseteq F^n_{-t+1}\pi_t \subseteq \cdots, \tag{2.8}$$

and $F^n_s\pi_t$ is independent of $s$ for $s \geq t+1$. We call this last the *stable quadratic filtration* and write $F^n_\infty\pi_t$ for it.

For $s \geq -t$ and $\alpha \in F^n_s\pi_t - F^{n+1}_s\pi_t$ we may define the *modified cup–n construction*

$$C_s(\alpha) = \left\{ \beta : \begin{array}{ccccc} & & S^{t+n} & & \\ & \nearrow & & \overset{\beta}{\searrow} & \\ P_{-s}^{t+n} & & & & S^{-t} \\ & \searrow & & \nearrow_{Q(\alpha)} & \\ & & P_t & & \end{array} \right\} \subseteq \pi_{2t+n} \tag{2.9}$$

For $s \geq t+1$, $C_s(\alpha)$ is independent of $s$; it is the *stable cup–n construction* $C_\infty(\alpha)$.

For $p \neq 2$ and $\alpha \in \pi_t$, define $Q(\alpha)$ analogously as the composite

$$\Sigma^{2k-\epsilon}P_{qk-\epsilon} = D_p(S^t) \xrightarrow{\ D_p(\alpha)\ } D_p(S^o) \xrightarrow{\ \pi\ } S^o$$

if $t = 2k-\epsilon$ with $\epsilon \in \{0,1\}$. If $t = 2k$ then $Q(\alpha)|S^{pt} = \alpha^p$. If $t = 2k-1$, $Q(\alpha)|S^{2pk-2}$ is the "restricted p–fold Massey product" $<\alpha>^p$, depending not just on the product in the sphere–spectrum but on the commuting homotopies as well. For example, in [7] Toda proves $<\alpha_1>^p = \beta_1$ up to a unit in $\mathbb{F}_p$. The proof uses little more than the non–existence of elements of mod p Hopf invariant one corresponding to $P^p$.

The rest of the definitions go just as for $p = 2$; we have the p–adic filtration and the stable p–adic filtration, the cup–construction and the stable cup–construction. For instance, if $|\alpha|$ is odd and $<\alpha>^p \neq 0$, then $C(\alpha) = \{<\alpha>^p\}$; otherwise $\alpha \in F^1\pi_t$.

## §3. The proof.

Jones's key observation is the

**Lemma 3.1.** Let $\alpha \in \pi_t$, with $t = 2k-\epsilon$, let $2\ell+\delta \geq t+1$, and let $r = q\ell+\delta$. Then

$$\begin{array}{ccc} P_{-r} & \xrightarrow{\ \pi\ } & S^0 \\ \downarrow & & \downarrow \alpha \\ P_{qk-\epsilon} & \xrightarrow{\ Q(\alpha)\ } & S^{-t} \end{array}$$

commutes.

This lemma relates the Mahowald filtration to the stable quadratic filtration, and the root invariant to the stable cup–construction. To see this we note that for any $s \equiv 0$ or $-1 \bmod q$ and any $\beta \in \pi_{s+t}$,

$$\begin{array}{ccc} S^{t-1} & \xrightarrow{\ \alpha\ } & S^{-1} \\ \downarrow \beta & & \downarrow \iota \\ S^{-s-1} & \xrightarrow{\ j\ } & P_{-s-1} \end{array}$$

commutes iff

$$\begin{array}{ccc} S^{t-1} & \xrightarrow{\ \alpha\ } & S^{-1} \\ \downarrow \beta & & \downarrow \iota \\ S^{-s-1} & \xrightarrow{\ j\ } & P^{r-1}_{-s-1} \end{array}$$

does. By (1.1), together with the fact that the dual of $\alpha$ is $\alpha$, this commutes iff

$$\begin{array}{ccc} P^s_{-r} & \xrightarrow{\ \pi\ } & S^0 \\ \downarrow & & \downarrow \alpha \\ S^s & \xrightarrow{\ \beta\ } & S^{-t} \end{array}$$

does. But by (3.1), this commutes iff

$$\begin{array}{ccc} P^s_{-r} & \longrightarrow & P_{qk-\epsilon} \\ \downarrow & & \downarrow Q(\alpha) \\ S^s & \xrightarrow{\ \beta\ } & S^{-t} \end{array}$$

does. Thus (taking $\beta = 0$) $S^{t-1} \xrightarrow{\alpha} S^{-1} \xrightarrow{\iota} P_{-s-1}$ is null iff $P^s_{-r} \longrightarrow P_{qk-\epsilon} \xrightarrow{Q(\alpha)} S^{-t}$ is null. The latter is surely null if $s < qk-\epsilon$; and this is Jones's theorem (1.5) and its extension to odd primes, (1.6). In fact this shows (as Jones notes) that

$$M^s \pi_t = F_\infty^{s-t}\, \pi_t\,, \tag{3.2}$$

and that the root invariant coincides with the stable cup construction:

$$R(\alpha) = C_\infty(\alpha). \tag{3.3}$$

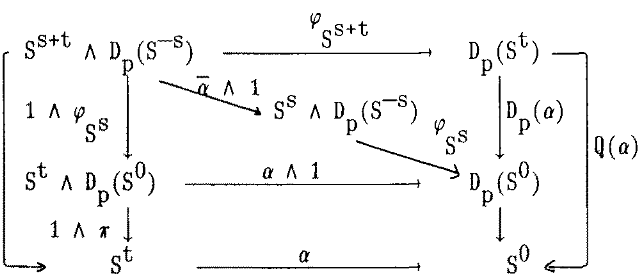
Since $M^s\pi_t = F_\infty^{s-t}\,\pi_t \supseteq F^{s-t}\,\pi_t$, Lin's theorem leads via (1.7) to (2.5), which may be restated as:

(3.4) The p–adic construction (with $t = 2k-\epsilon$)

$$Q : \pi^{-t} \longrightarrow \pi^{-t}\, P_{qk-\epsilon}$$

is injective.

The proof of (3.1) is quite simple. Let $s = 2\ell+\delta$. Since $s \geq t+1$, $\alpha \in \pi_t$ is represented by a map $\bar{\alpha} : S^{s+t} \longrightarrow S^s$. The following diagram then commutes by naturality in K of $\varphi_K$, and this proves (3.1).




## §4. Two examples.

Let $\alpha_0 = \iota$, and for $s > 0$ let $\alpha_s \in \pi_*$ be a minimal dimensional nonzero p–torsion element of Adams filtration s. These are represented in Adams' $E_2$ along Adams' edge. The element $\alpha_s$ is well–defined up to a unit in $\mathbb{F}_p$.

For $s > 0$, the dimension of $\alpha_s$ is $q(s)-1$, where if $p > 2$, $q(s) = qs$; if $q = 2$, $q(s+4) = q(s) + 8$ and

| s | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| q(s) | 0 | 2 | 3 | 4 |

Theorem (Mahowald). $\alpha_s \in R(p^s\iota)$.

This says that the diagram

$$\begin{array}{ccc} S^{-1} & \xrightarrow{\quad p^s \quad} & S^{-1} \\ {\scriptstyle \alpha_s}\big\downarrow & & \big\downarrow{\scriptstyle \iota} \\ S^{-q(s)} & \xrightarrow{\quad j \quad} & P_{-q(s)} \end{array}$$

commutes, and that the common composite is essential.

Since $E_2^{s,s-1}(S^{-q(s)}) = \mathbb{Z}/p$ maps isomorphically to $E_2^{s,s-1}(P_{-q(s)})$, and $E_2^{r,r-1}(P_{-q(s)}) = 0$ for $r > s$, it will suffice to show that $0 \neq p^s\,\iota \in \pi_{-1}P_{-q(s)}$.

Take $p = 2$. When $s \equiv 1$ or $2 \bmod 4$, this may be checked by projecting on into $bo \wedge P_{-q(s)}$, which is not hard to compute [6]. In the other cases, one may detect the class in J–theory. Using James periodicity to translate Mahowald's analysis in [6] of $J_*(\mathbb{R}P^n)$ to stunted projective spaces, one finds

$$J_{-1}(P_{-q(s)}) = \mathbb{Z}/2^{s+1}$$

generated by the Hurewicz image of $\iota$.

When $p > 2$, a similar but simpler analysis shows that $J_{-1}(P_{-qs}) = \mathbb{Z}/p^{s+1}$, and the result again follows.

In our second example, we will use Toda's computation of $C(\alpha_1)$; as recalled above, it is a unit multiple of $\beta_1$. Thus by Lemma 3.1

$$\begin{array}{ccccc} & & S^{q-1} & & \\ & \nearrow^{\pi} & & \searrow^{\alpha_1} & \\ \Sigma^{q-1}P_{-s}^{(p-1)q-1} & \longrightarrow & \Sigma^{q-1}P_{(p-1)q-1} & \xrightarrow{\;Q(\alpha_1)\;} & S^0 \\ & \searrow & \big\uparrow & \nearrow_{u\beta_1} & \\ & & S^{pq-2} & & \end{array}$$

commutes. If the horizontal composite is nonzero for large $s$, then $u\beta_1 \in R(\alpha_1)$. The dual of the bottom composite is

$$S^{q-2} \xrightarrow{\;\beta_1\;} S^{-(p-1)q} \longrightarrow P_{-(p-1)q}$$

(after dropping the unit, suspending, and including into the infinite projective space). To see this is essential one checks that it is nonzero at Adams' $E_2$. Since $E_2^{0,q-1}(P_{-(p-1)q}) = 0$, this element survives.

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