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Dear Haynes,

Recently, I have been reading your paper "An algebraic analogue of a conjecture of GW Whitehead" which appeared in PAMS recently.

I am a little ~~puzzled by~~ concerned about a statement in the proof of Lemma 2.1. The sentence starts from 10th line of page : The leading index in each term of the admissible expansion of this element is less than $\frac{1}{2}(j + |I'|)$. (Recall that $j + |I'|$ is the degree of the element concerned.)

I presume admissible expansion here means expressing $Sq^j Sq^I$ as linear combination of Sq^J with $|J| = j + |I'|$ & J is admissible with length $\geq n$. If so, then the leading index in $J = (j_1, \cdots)$ ought to be $\geq \frac{1}{2}|J| = \frac{1}{2}(j + |I'|)$. (Proof: $j_1 \geq 2j_2$, $j_2 \geq 2j_3$, $\cdots$. Thus $|J| = j_1 + \cdots \leq j_1 + \frac{1}{2}j_1 + \frac{1}{4}j_1 + \cdots < 2j_1$.) If I understand this correctly, then it seems there is a gap in the proof of Lemma 2.1.(b)$_n$.

However, I can prove (b)$_n$ by induction over $|m|$ where $Sq^m \in F^n$. The proof is contained in enclosed.

I'll be glad of your comments.

Yours sincerely

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P.S. Thank you for your letter of 10 Dec 1982. I'll read the articles you mentioned.

Proof of Lemma 2.1 $(b)_n$

We proceed by induction over $|m|$.

Each m can be expressed as ΣSq^I with I admissible with length $\geq n$. Write $I = (i_1, i_2, \cdots)$. Then $i_n \geq 1$, $i_{n-1} \geq 2$, $\cdots$. Thus $|I| \geq 2^{n-1} + \cdots + 2 + 1$. It is therefore clear that $Sq^{2^{n-1}} \cdots Sq^2 Sq^1$ is the only non-zero element in F^n with minimum degree $(= 2^n - 1)$. That $Sq^k(Sq^{2^{n-1}} \cdots Sq^2 Sq^1) \in F^{n+1}$ whenever $k \geq 2^n - 1$ can be verified directly with the help of Adem relations & the minimality of $|Sq^{2^{n-1}} \cdots Sq^2 Sq^1|$. This provides the basis for induction.

(*) {Let $m \in F^n$ be s.t. $|m| > 2^n - 1$. Assume $(b)_n$ is true for all $m' \in F^n$ with $|m'| < |m|$. We only need to show that if I is admissible with $|I| = |m|$ and $k \geq |I| = |m|$, then
$$Sq^k Sq^I \in F^{n+1}.$$

Let's write $I = (i_1, I')$. Suppose $k < 2i_1$. The Adem relation gives
$$Sq^k Sq^I = \sum_{0 \leq 2j \leq k}^{*} Sq^{k+i_1-j} Sq^j Sq^{I'}.$$
The indexes in each non-zero term in the sum satisfies
$$i_1 - 1 - j \geq k - 2j$$
$$\Rightarrow \quad j \geq k - i_1 = |I'|.$$
Therefore by $(b)_{n-1}$, $Sq^j Sq^{I'} \in F^n$. Observe that
$$|Sq^j Sq^{I'}| = j + |I'| < |I|.$$
Thus $k + i_1 - j \geq k \geq |I| \geq j + |I'|$.

We can now apply induction hypothesis (*) to conclude that $Sq^{k+i_1-j} Sq^j Sq^{I'} \in F^{n+1}$. This ~~gives~~ proves $(b)_n$.

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