1 The fixed point formula

The fixed point theorem is occurs in two ways in the story of elliptic cohomology. First, Witten came to consider elliptic genera by imagining the result of applying it to the circle action of the free loop space of a manifold. Second, the present proofs of the rigidity theorem use the fixed point formula in an essential way (as was also foreseen by Witten).

A convenient reference for this material is the paper [?] of Atiyah and Bott.

Let $\mathbb{T}$ denote the circle group $\mathbb{R}/2\pi\mathbb{Z}$ and let X be a finite $\mathbb{T}$-CW complex. Let E be a complex-oriented spectrum, so that $E^*(E\mathbb{T} \times_{\mathbb{T}} X)$ is a module for $E^*(B\mathbb{T}) = E^*[[x]]$. Let $i : X^{\mathbb{T}} \to X$ be the inclusion of the fixed point subcomplex.

Lemma 1.1 *The map $i^* : E^*(E\mathbb{T} \times_{\mathbb{T}} X) \to E^*(E\mathbb{T} \times_{\mathbb{T}} X^{\mathbb{T}})$ becomes an isomorphism after inverting x and tensoring with $\mathbb{Q}$. In fact, it suffices to invert the primes dividing the orders of the finite isotropy groups of the action.*

Proof. Consider first the case $X = \mathbb{T}/K$, where K is the subgroup of $\mathbb{T}$ of order n. Then $E\mathbb{T} \times_{\mathbb{T}} X = BK$. Since $X^{\mathbb{T}} = \emptyset$, we must show that $E^*(BK)$ localizes to zero. The trick to understanding $E^*(BK)$ is to think of BK as the total space of the sphere-bundle of the n-fold tensor power of the tautologous bundle λ over $B\mathbb{T}$. As a general notation, write B_ξ for the Thom space of the bundle ξ over B. The map $\sigma : B \to B_\xi$ induced by the inclusion of the zero-section is up to homotopy the cofiber of the projection map from the sphere-bundle of ξ, and by definition it pulls the Thom class back to the Euler class of ξ. With $\xi = \lambda^n$, this is the n-series $[n](x)$. σ induces in E-cohomology a module-homomorphism over $E^*(B\mathbb{T})$, so to show that it becomes an isomorphism after localization it suffices to show that $[n](x)$ becomes a unit. But $[n](x) \equiv nx$ modulo x^2, so in the localization it is a unit since

$$\frac{[n](x)}{nx} = 1 + \cdots$$

is invertible.

The result now follows by induction over a $\mathbb{T}$-CW structure on X. //

Now assume X is a compact smooth $\mathbb{T}$-manifold. General theorems imply that X admits the structure of a $\mathbb{T}$-CW complex. Let C be a component of the fixed point set. It is a smooth submanifold. Let ν_C denote its normal bundle. The $\mathbb{T}$-action on X induces an action of $\mathbb{T}$ without fixed-points on each fiber of ν_C. Such an action decomposes naturally into eigenspaces, where on the nth factor the eigenvalues of $t \in \mathbb{T}$ are $e^{\pm int}$ (with equal multiplicity). For each $n > 0$, these eigenspaces assemble into a subbundle $\epsilon_C(n)$, and ν_C is the direct sum of these eigenbundles.

We endow the eigenbundle $\epsilon_C(n)$ with a complex structure by letting multiplication by i coincide with the action by $\pi/2n \in \mathbb{R}/2\pi\mathbb{Z}$. $\epsilon_C(n)$ thus becomes $\mathbb{T}$-equivariant complex bundle over C.

ν_C and its subbundles $\epsilon_C(n)$ determine vector-bundles $\tilde{\nu}_C = E\mathbb{T} \times_{\mathbb{T}} \nu_C$ and $\tilde{\epsilon}_C(n) = E\mathbb{T} \times_{\mathbb{T}} \epsilon_C(n)$ over $E\mathbb{T} \times_{\mathbb{T}} C = B\mathbb{T} \times C$. By construction,

$$\tilde{\epsilon}_C(n) = \lambda^n \otimes \epsilon_C(n)\,.$$

the external tensor product. I claim that the Euler class $\tilde{e}_C$ of $\tilde{\nu}_C$ is invertible in $E^*(B\mathbb{T} \times C)[x^{-1}] \otimes \mathbb{Q}$.

Pick $c \in C$ and consider the restriction map $j^* : E^*(B\mathbb{T} \times C) \to E^*(B\mathbb{T})$. Its kernel is nilpotent. One way to see this is to consider the Atiyah-Hirzebruch-Serre spectral sequence

$$H^*(C; E^*(B\mathbb{T})) \Rightarrow E^*(B\mathbb{T} \times C).$$

C is a finite complex, so the spectral sequence has finite width and an element of $\ker\, j$ is nilpotent by multiplicativity of the spectral sequence.

The kernel of any localization of j^* is thus nilpotent as well, so it suffices to show that $\tilde{e}_C$ restricts to a unit in $E^*(B\mathbb{T})[x^{-1}] \otimes \mathbb{Q}$. But the bundle $\tilde{\nu}_C$ resticts to a sum of powers of the canonical bundle over $B\mathbb{T}$, so its Euler class is invertible in the localization just as before.

Let $i_C : B\mathbb{T} \times C \to E\mathbb{T} \times_{\mathbb{T}} X$ be the inclusion. For $a \in E^*(B\mathbb{T} \times C)$, we have

$$i_C^* i_{C*} a = \tilde{e}_C \cup a.$$

In particular, $\tilde{e}_C = i_C^* i_{C*} 1$. Since i_C^* localizes to an isomorphism, $i_{C*}1$ is invertible in $E^*(E\mathbb{T} \times_{\mathbb{T}} X)[x^{-1}] \otimes \mathbb{Q}$. In the localization of $E^*(B\mathbb{T} \times C)$,

$$a = \frac{i_C^* i_{C*} a}{\tilde{e}_C} = i_C^* \frac{i_{C*} a}{i_{C*} 1} = i_C^* i_{C*}\left(\frac{a}{\tilde{e}_C}\right),$$

where for the last step we used the fact that i_{C*} is a module-homomorphism over $E^*(E\mathbb{T} \times_T X)$. Thus for $b \in E^*(E\mathbb{T} \times_{\mathbb{T}} X)$,

$$i^* b = \sum_C i_C^* i_{C*}\left(\frac{i_C^* b}{\tilde{e}_C}\right),$$

and so in $E^*(E\mathbb{T} \times_{\mathbb{T}} X)[x^{-1}] \otimes \mathbb{Q}$ we have

$$b = \sum_C i_{C*} \left(\frac{i_C^* b}{\tilde{e}_C} \right).$$

This is the basic formula. It is usually employed, however, to compute the localization of the Gysin homomorphism associated to the projection map

$$\pi : E\mathbb{T} \times_{\mathbb{T}} X \to B\mathbb{T}$$

under the assumption that π is orientable in E-cohomology. Since Gysin maps compose, we have

$$\pi_* b = \sum_C \pi_{C*} \left(\frac{i_C^* b}{\tilde{e}_C} \right)$$

where $\pi_C = \pi \circ i_C : B\mathbb{T} \times C \to B\mathbb{T}$.

For example, suppose E is complex K-theory, with the usual complex orientation given by

$$e_K(L) = 1 - L. \tag{1}$$

(Here we have omitted mention of the Bott class.) Let's then assume that X admits an invariant weakly-almost complex structure, so that each C has a WAC as well (complementary to the complex structure on its normal bundle in the restriction of the complex structure of X.) We wish to compute the equivariant Todd genus of X.

Our approach to this formula will use the formalism of operations on vector bundles. It follows from (1) and the splitting principal that the K-theory Euler class of a complex vector bundle is

$$e_K(\xi) = \Lambda_{-1}(\xi).$$

If μ is a line bundle, then

$$\Lambda^k(\mu \otimes \xi) = \mu^k \otimes \Lambda^k(\xi),$$

so

$$e_K(\lambda^n \otimes \xi) = \sum_k (-1)^k \lambda^{kn} \Lambda^k(\xi) = \Lambda_{-\lambda^n}(\xi).$$

Thus

$$e_K(\tilde{\nu}_C) = \bigotimes_{n>0} \Lambda_{-\lambda^n}(\epsilon_C(n))$$

and so

$$\pi_*^K(1) = \sum_C {\pi_C}_*^K \left(\bigotimes_{n>0} S_{\lambda^n} \epsilon_C(n) \right).$$

We can express this result cohomologically (without loss of information, since the fixed point theorem is a rational result anyway) using the Riemann-Roch formula associated to the multiplicative transformation from K-theory to rational cohomology given by the Chern character. The corresponding exponential characteristic class ρ is characterized by the property that for a line bundle λ it takes on the value

$$\rho(\lambda) = \frac{ch\ e_K(\lambda)}{e_H(\lambda)}.$$

The standard convention sets $e_H(\lambda) = -c_1(\lambda)$, and we let x denote this class in the universal case. We will also use the notation

$$q = ch\ \lambda = e^{-x}.$$

Then the multiplicative class ρ is the *inverse* of the Todd class, which is characterized by

$$\mathrm{Td}(\lambda) = \frac{e_H(\lambda)}{ch\ e_K(\lambda)} = \frac{x}{1 - e^{-x}}.$$

Thus the Riemann Roch theorem gives

$$ch\ {\pi_C}_*^K(a) = {\pi_C}_*^H(\mathrm{Td}(\tau_C) \cup ch\ a).$$

where $\mathrm{Td}(\tau_C) \in H^*(C;\mathbb{Q})$ acts on $H^*(B\mathbb{T} \times C;\mathbb{Q})$ through the projection map $p : B\mathbb{T} \times C \to C$. ($p^*\tau_C = E\mathbb{T} \times_{\mathbb{T}} \tau_C$ is the bundle of tangents along the fiber of π.)

Using the splitting principle we may formally write $\epsilon_C(n)$ as a sum of line bundles:

$$\epsilon_C(n) = \bigoplus_i \lambda_{n,i}, \qquad x_{n,i} = -c_1(\lambda_{n,i}).$$

Then

$$ch\ \Lambda_{-\lambda^n}(\epsilon_C(n)) = ch\ \bigotimes_i \Lambda_{-\lambda^n} \lambda_{n,i} = \prod_i ch\ (1 - \lambda^n \lambda_{n,i}) = \prod_i (1 - q^n e^{-x_{n,i}})$$

and

$$ch\ \pi_*^K(1) = \sum_C \pi_{C*}^H \left(\mathrm{Td}(\tau_C) \cup \prod_{n>0} \prod_i \frac{1}{(1 - q^n e^{-x_{n,i}})} \right).$$

Now consider the case of KO. The natural orientation here is the Atiyah-Bott-Shapiro orientation of spin-bundles. It is useful to notice that a stable spin structure on an oriented vector bundle—i.e., a lifting of the classifying map $X \to BSO$ through the covering map $BSpin \to BSO$—automatically destabilizes to a spin-reduction of the oriented frame bundle, since

$$\begin{array}{ccc} BSpin(n) & \to & BSpin \\ \downarrow & & \downarrow \\ BSO(n) & \to & BSO \end{array}$$

is a homotopy-pullback square.

Let M be an oriented smooth closed Riemannian n-manifold, with oriented orthonormal frame bundle $P \to M$. Suppose $\mathbb{T}$ acts smoothly on M. The action lifts canonically to an action on P, by differentiation. Suppose $Q \to P$ is a spin-structure on M. The action is compatible with the spin-structure, or "even," if the action on P lifts to an action on Q. Since $Q \to P$ is a double cover, the induced action of the double cover $\hat{\mathbb{T}}$ is always even.

If we have an even action on M, there is a Gysin homomorphism

$$\pi_*^{KO} : KO^*(E\mathbb{T} \times_{\mathbb{T}} M) \to KO^*(B\mathbb{T}).$$

To apply the fixed point theorem as we have described it, we must have a complex-oriented theory. KO is not complex oriented, but it becomes orientable after inverting 2. On the other hand, the bundle of tangents along the fiber of π has a spin-structure as well as a complex structure. The structure group for such bundles is the pull-back in the diagram

$$\begin{array}{ccc} \hat{U}(n) & \xrightarrow{\hat{r}} & Spin(2n) \\ \downarrow & & \downarrow \\ U(n) & \xrightarrow{r} & SO(2n) \end{array}.$$

Since $c_1(\xi)$ reduces mod 2 to $w_2(r\xi)$, the group $\hat{U}(n)$ is equally well the pull-back in the diagram

$$\begin{array}{ccc} \hat{U}(n) & \longrightarrow & \hat{\mathbb{T}} \\ \downarrow & & \downarrow \\ U(n) & \xrightarrow{\det} & \mathbb{T} \end{array}$$

where $\hat{\mathbb{T}} \to \mathbb{T}$ is the double cover. In short, a spin structure on the real bundle underlying a complex vector bundle is simply a square root of its determinant line bundle.

If I have such a bundle ξ over X, it has an Atiyah-Bott-Shapiro Euler class (namely the difference between the two half-spin bundles) $e_{KO}(\xi) \in KO(X)$ that complexifies to the class

$$e_{\hat{A}}(\xi) = \frac{\Lambda_{-1}(\xi)}{\sqrt{\det\ \xi}} \in KU(X).$$

It is a multiplicative class on bundles with structure groups $\hat{U}(n)$ which on line bundles λ (which are their own determinant bundles) takes value

$$e_{\hat{A}}(\lambda) = \frac{1-\lambda}{\sqrt{\lambda}}.$$

It serves as an Euler class if we invert 2. If μ is a line bundle, We have

$$e_{\hat{A}}(\mu \otimes \xi) = \frac{\mu^{-d/2}\Lambda_{-\mu}(\xi)}{\sqrt{\det\ \xi}}$$

where d is the fiber-dimension of ξ.

Thus in the situation of the fixed point theorem,

$$e_{\hat{A}}(\lambda^n \epsilon_C(n))^{-1} = \lambda^{nd_n/2}\sqrt{\det\ \epsilon_C(n)}S_{\lambda^n}(\epsilon_C(n)).$$

where $d_n = \dim\ \epsilon_C(n)$. Thus, with $m = \sum nd_n$ (which is even if the action is even),

$$\pi_*^{\hat{A}}(1) = \sum_C \pi_{C*}^{\hat{A}}\left(\lambda^{m/2}\sqrt{\det\ \nu_C}\bigotimes_{n>0} S_{\lambda^n}(\epsilon_C(n))\right)$$

To reduce to ordinary cohomology, note that

$$ch\ e_{\hat{A}}(\lambda) = \frac{1-e^{-x}}{e^{-x/2}} = 2\sinh(x/2).$$

The corresponding multiplicative characteristic class is the $\hat{A}$-genus, characterized by

$$\hat{A}(\lambda) = \frac{e_H(\lambda)}{ch\ e_{\hat{A}}(\lambda)} = \frac{x/2}{\sinh(x/2)},$$

so the Riemann-Roch formula asserts that

$$ch\ \pi_*^{\hat{A}}(a) = \pi_{C*}^{H}(\hat{A}(\tau_C) \cup ch\ a),$$

Thus, with the usual conventions,

$$ch\ \pi_*^{\hat{A}}(1) = \sum_C \pi_{C*}^{H} \left(\hat{A}(\tau_C) \cup q^{m/2} e^{-x/2} \prod_{n>0} \prod_i \frac{1}{(1 - q^n e^{-x_{n,i}})} \right)$$

where $x = -c_1(\nu_C)$.

Finally, the L-genus arises from the $K[\frac{1}{2}]$-theoretic class

$$e_L(\lambda) = \frac{1-\lambda}{1+\lambda}.$$

(One has to introduce 1/2 here in order to form

$$(1+\lambda)^{-1} = (2 - (1-\lambda))^{-1} = \frac{1}{2} \sum_{n \geq 0} \left(\frac{1-\lambda}{2} \right)^n$$

in $K[\frac{1}{2}](\mathbb{C}P^\infty)$.) In terms of exterior and symmetric algebras, this is

$$e_L(\lambda) = \Lambda_{-1}(\lambda) \otimes S_{-1}(\lambda).$$

This formula extends by multiplicativity to a general complex vector bundle ξ twisted by a line bundle λ:

$$e_L(\lambda\xi) = \Lambda_{-\lambda}(\xi) \otimes S_{-\lambda}(\xi).$$

In the situation of the fixed point theorem, we have

$$e_L(\lambda^n \epsilon_C(n))^{-1} = S_{\lambda^n}(\epsilon_C(n)) \otimes \Lambda_{\lambda^n}(\epsilon_C(n)),$$

so

$$\pi_*^L(1) = \sum_C \pi_{C*}^{L} \left(\bigotimes_{n>0} (S_{\lambda^n}(\epsilon_C(n)) \otimes \Lambda_{\lambda^n}(\epsilon_C(n))) \right). \qquad (2)$$

To reduce to ordinary cohomology, note that

$$ch\ e_L(\lambda) = \frac{1-q}{1+q} = \frac{1-e^{-x}}{1+e^{-x}} = \frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}} = \tanh(x/2).$$

If $\xi = \bigoplus_i L_i$ is a splitting into line bundles then

$$\Lambda_{-s}(\xi) = \bigotimes_i (1 - sL_i), \qquad S_{-s}\xi = \bigotimes_i (1 + sL_i)^{-1},$$

and so if $x_i = -c_1(L_i)$, we find

$$ch\ e_L(\lambda^n \otimes \xi) = \prod_i \frac{1 - q^n e^{-x_i}}{1 + q^n e^{-x_i}}.$$

The corresponding exponential characteristic class is the L-genus, characterized by

$$L(\lambda) = \frac{e_H(\lambda)}{ch\ e_L(\lambda)} = \frac{x}{\tanh(x/2)},$$

so the Riemann-Roch formula asserts that

$$ch\ \pi_{C*}^{L}(a) = \pi_{C*}^{H}(L(\tau_C) \cup ch\ a).$$

Thus, with the usual conventions,

$$ch\ \pi_*^L(1) = \sum_C \pi_{C*}^{H} \left(L(TM) \cup \prod_{n>0} \prod_i \frac{1 + q^n e^{-x_{n,i}}}{1 - q^n e^{-x_{n,i}}} \right).$$

If ξ is a complex vector bundle and $\sqrt{\det \xi}$ is given, the K-theoretic *spinor class* is given by

$$\Delta(\xi) = \frac{e_L(\xi)}{e_{\hat{A}}(\xi)} = \frac{(\det\ \xi)^{d/2}}{\Lambda_1(\xi)}.$$

Thus

$$\Delta(\lambda) = \frac{\sqrt{\lambda}}{1 + \lambda}.$$

This implies that the quotient multiplicative characteristic class $L(\xi)/\hat{A}(\xi)$ is characterized by

$$\frac{L(\lambda)}{\hat{A}(\lambda)} = \frac{1 + \lambda}{\sqrt{\lambda}} = \lambda^{-1/2} + \lambda^{1/2}.$$

This class is associated to the spinor representation: if τ_C has a spin structure, and $\Delta(\tau_C)$ denotes the corresponding spinor bundle, then

$$ch\ \Delta(\tau_C) = \prod_i (e^{-x_i/2} + e^{x_i/2}),$$

where the x_i are the formal roots of the tangent bundle. The equivariant signature may thus be expressed in terms of the $\hat{A}$-genus (from (2)):

$$\pi_*^L(1) = \sum_C \pi_{C*}^{\hat{A}} \left(\Delta(\tau_C) \otimes \bigotimes_{n>0} (S_{\lambda^n}(\epsilon_C(n)) \otimes \Lambda_{\lambda^n}(\epsilon_C(n))) \right).$$