DIVIDED POWERS AND KÄHLER DIFFERENTIALS

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Abstract. Divided power algebras form an important variety of non-binary universal algebras. We identify the universal enveloping algebra and Kähler differentials associated to a divided power algebra over a general commutative ring, simplifying and generalizing work of Roby and Dokas.

1. Introduction

Divided power algebras were introduced by Henri Cartan [3] to describe the homology of Eilenberg Mac Lane spaces. They have subsequently been intensively studied [9] and have played important roles in other parts of mathematics, such as algebraic geometry, where they form the basis of the construction of crystalline cohomology [2]. They constitute an important example of a variety of algebras that are "non-binary," in the sense that their structure is not entirely encoded in a binary product.

In [8], Dan Quillen described a uniform process for defining a cohomology theory in any one of a wide class of algebraic structures. An important role in that construction is played by the category of "Beck modules" (see [1], for example) for an algebra A in a specified variety $\mathbf{V}$. This is the category, recognized long ago by Sammy Eilenberg [7] as providing the appropriate meaning of a "representation" in a general context, consists of the abelian objects in the slice category $\mathbf{V}/A$. In linear cases it can be identified with the category of modules over a unital associative algebra $U(A)$, the "universal enveloping algebra" of A. Beck modules form the coefficients in the Quillen cohomology theory defined on $\mathbf{V}$, and in fact Quillen homology is an appropriately defined derived functor of the abelianization functor, evaluated on the terminal object $1_A : A \downarrow A$ in $\mathbf{V}/A$. If $\mathbf{V}$ is the variety of commutative rings, $\mathrm{Ab}_A(1_A)$ is the A-module of Kähler differentials, and this suggests defining $\Omega_A^{\mathbf{V}} = \mathrm{Ab}_A(1_A)$ in a general variety of algebras.

This construction has been considered in detail by Ionnnis Dokas [6, 5] in case one is working with divided power algebras over a field. The goal of the present paper is to show how this works out over a general ring. Dokas proves the interesting result that if A is a DP algebra over a field then the module of divided power Kähler differentials is simply a DP Beck module structure on the usual commutative algebra module of Kähler differentials.

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We show that the same result holds in general, and along the way we simplify some of his arguments.

The classical module of Kähler differentials of divided power algebras was also the subject of a study by Norbert Roby [10]. He described the Kähler differentials of a free divided power algebra. We show how his result follows easily from the identification $\Omega_A^{DP} = \Omega_A^{CA}$.

In the first section of this short paper we gather some reminders about divided power algebras. We find it convenient to work with *non-unital* commutative algebras. We then study the structure of abelian divided power algebras – those whose product is trivial. These coincide with Beck modules over the zero divided power algebra. In §4 we identify the universal enveloping algebra of a general divided power algebra. Next we define the module of Kähler differentials, and state and prove the main theorem. Finally, in §6, we give some consequences of this theorem.

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2. Divided power algebras and their universal enveloping algebras

We recall the definition of divided power algebras and some of their basic properties. We will always work over a fixed commutative ring R, and an un-decorated "$\otimes$" will indicate the tensor product over R. By an "algebra" we mean a commutative non-unital R-algebra.

Definition 2.1. *A DP* structure *on the algebra A is a family of maps $\gamma_i : A \to A, i > 0$ such that for all $a, b \in A$ and $r \in R$,*

$$\gamma_1(a) = a$$
$$\gamma_n(a+b) = \gamma_n(a) + \sum_{i+j=n} \gamma_i(a)\gamma_j(b) + \gamma_n(b)$$
$$\gamma_n(ra) = r^n \gamma_n(a)$$
$$\gamma_n(ab) = a^n \gamma_n(b) \quad \text{and} \quad \gamma_n(rb) = r^n \gamma_n(b)$$
$$\gamma_m(a)\gamma_n(a) = \frac{(m+n)!}{m!n!}\gamma_{m+n}(a)$$
$$\gamma_m(\gamma_n(a)) = \frac{(mn)!}{m!(n!)^m}\gamma_{mn}(a)\,.$$

A morphism of DP algebras is an algebra map commuting with these "divided power" operations.

Write $\mathbf{DPAlg}_R$ for the category of DP algebras.

Remark 2.2. The commutative ring R might be "large," including the nonunital algebra A as an ideal. Then one may define $\gamma_0 = 1$ and the axioms become slightly more compact; this is the expression one finds in [2] and elsewhere.

But R may also be "small." In fact any nonunital R-algebra A is an ideal in the augmented unital R-algebra

$$A_+ = A \oplus R$$

with product $(a, r)(b, s) = (ab, rb + sa)$. This is the context discussed by Roby [9] and Dokas [5, 6]. (Our use of the subscripted $+$ is the opposite of theirs, but agrees with the usage in [4].)

The following well-known facts will be useful.

Proposition 2.3 ([9])**.** *The forgetful functor* $\mathbf{DPAlg}_R \to \mathbf{Mod}_R$ *has a left adjoint, sending the* R*-module* V *to the "free DP algebra"* $\Gamma_R(V)$ *generated by* V*.*

A ring homorphism $f : R \to S$ induces an adjoint pair

$$f_* : \mathbf{DPAlg}_R \rightleftarrows \mathbf{DPAlg}_S : f^*$$

in which f^*N is N regarded as an R-module, and $f_*M = S \otimes_R M$. This adjunction induces an isomorphism, natural in $V \in \mathbf{Mod}_R$,

$$S \otimes_R \Gamma_R(V) \to \Gamma_S(S \otimes_R V)\,.$$

For example, for any abelian group V

$$\mathbb{Q} \otimes \Gamma_{\mathbb{Z}}(V) \cong \Gamma_{\mathbb{Q}}(\mathbb{Q} \otimes V)\,.$$

Over the rationals, any commutative algebra admits unique a DP structure, and the free DP algebra functor is just the symmetric algebra functor.

The coproduct of two algebras, A and B, is the algebra

$$A \coprod B = (A \otimes R) \oplus (A \otimes B) \oplus (R \otimes B)$$

with the evident product. The canonical inclusions take $a \in A$ to $a \otimes 1$ and $b \in B$ to $1 \otimes b$.

Proposition 2.4 ([9])**.** *Let* A *and* B *be DP algebras. Then there is a unique DP structure on* $A \coprod B$ *such that the two inclusions are DP maps, and it serves as the coproduct in* $\mathbf{DPAlg}_R$*.*

The product of two DP algebras is a DP structure on their product as algebras, which has as its underlying R-module the product R-module. The terminal DP algebra is the unique structure on the trivial R-module 0. Any algebra has a unique "point," i.e. a unique map from the terminal algebra.

3. Abelian DP-algebras

We are interested in abelian group objects in the category $\mathbf{DPAlg}_R$. A unital product on a R-algebra A is an algebra map $\mu : A \times A \to A$ such that $\mu(a, 0) = a$ and $\mu(0, b) = b$. From this we find that

$$ab = \mu(a, 0)\mu(0, b) = \mu(0, 0) = 0\,.$$

Also, μ is a *additive* map, so

$$\mu(a,b) = \mu((a,0)+(0,b)) = \mu(a,0)+\mu(0,b) = a+b$$

so the product is none other than the sum in the R-module A, which is R-linear by distributivity. This product is thus automatically an abelian group structure, so the abelian objects in $\mathbf{CAlg}_R$ are the algebras with trivial product. Since the product in $\mathbf{DPAlg}_R$ is the same, an abelian DP algebra also has trivial algebra structure. This imposes strong restrictions on the divided powers.

Proposition 3.1. *A DP algebra is abelian as a DP algebra if and only if it is abelian as a commutative algebra. In such a DP algebra, $\gamma_n = 0$ unless n is a power of a prime. For any prime number p, γ_p is additive, $p\gamma_p = 0$, $\gamma_{p^e} = \gamma_p^e$, and, for all $a \in A$ and $r \in R$, $\gamma_p(ra) = r^p\gamma_p(a)$.*

Proof. We begin by verifying these properties. Suppose A is a DP algebra that is abelian as a commutative algebra. The deviation from additivity of γ_n is a sum of products, so vanishes if products vanish. If $i+j=n$ with $i,j \geq 1$, the product rule for divided powers shows that $(i,j)\gamma_n = 0$. Since

$$\gcd\{(i,j) : i,j \geq 1, i+j=n\} = \begin{cases} 1 & \text{if } n \text{ is not a prime power} \\ p & \text{if } n = p^e \end{cases}$$

we see that $\gamma_n = 0$ if n is not a prime power and $p\gamma_{p^e} = 0$. Finally, following [3, 7-09], we claim that

$$\gamma_{kp} \equiv \gamma_k \gamma_p \mod p\,.$$

That is to say,

$$\frac{(kp)!}{k!(p!)^k} \equiv 1 \mod p\,.$$

To see this, write the numerator as

$$\prod_{i=1}^{p} \big(i(p+i)(2p+i)\cdots((k-1)p+i)\big)\,.$$

The denominator is $((p-1)!)^k$ times the $i = p$ factor in the numerator. For $i < p$, the ith factor in the numerator is congruent mod p to i^k, so the product of those terms is congruent mod p to $((p-1)!)^k$.

It follows by induction that $\gamma_{p^e} \equiv \gamma_p^e \mod p$.

Now observe that if A is DP algebra with trivial product then the addition map, which is the only candidate for an abelian group product, is indeed a DP algebra map. This follows from linearity of the divided powers on such an algebra. □

Define the unital associative algebra

$$U(0) = R[\phi_p : p \text{ prime}]/(p\phi_p)\,,$$

with the twisted product defined by $\phi_p r = r^p \phi_p$. We have proved:

Lemma 3.2. *The category of abelian DP algebras is equivalent to the category of left $U(0)$-modules*

The ϕ's record the divided powers. We have shifted notation for clarity of later constructions. If p and q are distinct primes, then both p and q kill $\phi_p\phi_q$, which therefore vanishes.

A DP ideal in a DP algebra A is an ideal in A that is closed under the divided powers. It is easy to see that if I is a DP ideal in A then the divided powers descend to give a divided power structure on A/I, their non-additivity notwithstanding.

The ideal A^2 is closed under the divided powers, so A/A^2 is an example of an abelian divided power algebra; indeed, it is the abelianization of A.

4. Beck modules

Fix a DP algebra A and consider the slice category $\mathbf{DPAlg}_R/A$. An object of $\mathbf{DPAlg}_R/A$ is a DP algebra B equipped with a DP map $\pi : B \downarrow A$. A morphism from $\pi' : B' \downarrow A$ to $\pi : B \downarrow A$ is a DP algebra map $f : B' \to B$ such that $\pi f = \pi'$.

Definition 4.1. *A* DP A-module *is an abelian object in* $\mathbf{DPAlg}_R/A$.

These are the "Beck modules" in the theory of DP algebras. Write $\mathbf{Mod}_A$ for the category of DP A-modules.

Proposition 4.2. $\mathbf{Mod}_A$ *is an abelian category, equivalent to the category of left modules over the associative unital algebra $U(A)$ given by*

$$U(A) = A_+ \otimes_R U(0)$$

as R-module, with product determined by

$$(a \otimes 1)(b \otimes 1) = ab \otimes 1\,, \quad (1 \otimes u)(1 \otimes v) = 1 \otimes uv$$
$$(a \otimes 1)(1 \otimes u) = a \otimes u$$
$$(1 \otimes \phi_p)(a \otimes 1) = 0$$

with $a, b \in A$ and $u, v \in U(0)$.

Proof. Given an abelian object $\pi : E \downarrow A$ over A in $\mathbf{DPAlg}_R$, let $M = \ker \pi$. This kernel is a sub DP algebra of E. The abelian structure of $E \downarrow A$ restricts to an abelian structure on $M \downarrow 0$, so the product on M is trivial and the divided powers are given by an action of $U(0)$ on M.

The unital algebra A_+ acts on A, and M is a submodule. It remains to describe how the action of A_+ interacts with the divided power structure. But for $n > 1$, in $A \oplus M$,

$$(0, \gamma_n(ax)) = \gamma_n(0, ax) = \gamma_n((a, 0)(0, x)) = \gamma_n(a, 0)(0, x)^n = 0$$

so $\gamma_n(ax) = 0$.

Conversely, given a $U(A)$-module M, the R-module $A \oplus M$ becomes an abelian object over A with

$$(a,x)(b,y) = (ab, ay+bx)\,, \quad \gamma_n(a,x) = \Big(\gamma_n a, \phi_n x + \sum_{i+j=n} \gamma_i(a)\phi_j(x)\Big)\,.$$

These constructions are inverse to each other. □

It's interesting to observe that the DP structure of A plays no role in this description of the category of DP A-modules.

For example, fix a prime p and suppose that R is a $\mathbb{Z}_{(p)}$-algebra. Then $U(A)$ is too, so $\phi_q = 0$ for primes q other than p, and

$$U(A) = A_+ \widetilde{\otimes} R[\phi_p]/(p\phi_p)$$

with product twisted by $\phi_p r = r^p \phi_p$ for $r \in R$ and $\phi_p a = 0$ for $a \in A$. Compare with [6].

Remark 4.3. Divided power algebras often occur in a graded setting; this is how Cartan encountered them in [3], for example. If the Koszul sign rule applies, one may divide by the ideal generated by the elements of odd degree to obtain a graded algebra that is commutative in the sense that $ab = ba$.

In this graded context, it is appropriate to grade $U(0)$ on the commutative monoid $\mathbb{Z}^\times_{>0}$ of positive integers under multiplication; so for example $|1| = |\phi_1| = 1$. This monoid acts additively on $\mathbb{Z}$, and the action of the divided powers is compatible with that action. The algebra $U(A)$ is naturally $\mathbb{Z}$-graded; an element $a \otimes \phi_n \in A_+ \otimes U(0)$ has degree $n|a|$. Then the category of abelian objects in the category of graded DP algebras over A is equivalent to the category of graded left $U(A)$-modules.

5. Derivations and Kähler differentials

The category $\mathbf{DPAlg}_R$ has its own proper theory of derivations and differentials.

Definition 5.1. *Let A be a DP algebra and M a DP A-module. A DP* derivation *is an R-linear map $s : A \to M$ such that $a \mapsto (a, s(a))$ is a DP algebra section of* $\mathrm{pr}_1 : A \oplus M \downarrow A$.

Lemma 5.2. *An R-linear map $s : A \to M$ is a DP derivation if and only if*

$$s(ab) = as(b) + bs(a)$$
$$s(\gamma_n(a)) = \phi_n(sa) + \sum_{i+j=n} \gamma_i(a)\phi_j(sa)\,.$$

Proof. The first equality follows from the map $a \mapsto (a, sa)$ being an algebra map:

$$(ab, s(ab)) = (a, s(a))(b, s(b)) = (ab, as(b) + bs(a))\,.$$

The second follows from commutation with divided powers:

$$\begin{aligned}(\gamma_n a, \phi_n(sa)) &= \gamma_n(a, sa) = \gamma_n((a,0) + (0, sa)) \\ &= (\gamma_n a, 0) + \sum_{i+j=n} (\gamma_i a, 0)(0, \phi_j(sa)) + (0, \phi_n(sa)) \\ &= (\gamma_n a, 0) + \sum_{i+j=n} (0, (\gamma_i a)(\phi_j(sa))) + (0, \phi_n(sa))\,.\end{aligned}$$

Equating second entries gives the result.

Conversely, these conditions imply that $a \mapsto (a, s(a))$ is a DP algebra map splitting the projection. □

Remark 5.3. We note some implications of these equations. First of all,

$$\phi_p(s(ab)) = 0$$

since $s(ab) = as(b) + bs(a)$ and ϕ_p kills elements of the form ax. Next,

$$\phi_p(s(\gamma_q a)) = \phi_p\phi_q(sa) + \sum_{i+j=q} \phi_p(\gamma_i(a)\phi_j(sa))) = \begin{cases} \phi_p^2(sa) & \text{if } p = q \\ 0 & \text{otherwise} \end{cases}$$

since ϕ_p is additive, and, again, kills elements of the form ax.

There is obviously a universal DP derivation out of A which we write

$$d : A \to \Omega^{DP}_{A/R}\,.$$

One construction is as follows. Any R-module map $s : A \to M$ extends to a $U(A)$-module map $U(A) \otimes A \to M$. As a left $U(A)$-module,

$$\Omega^{DP}_{A/R} = (U(A) \otimes A)/S$$

where S is the sub $U(A)$-module generated by the elements

$$a \otimes b - 1 \otimes ab + b \otimes a$$

$$1 \otimes \gamma_n a - \phi_n \otimes a + \sum_{i+j=n} (\gamma_i(a)\phi_j \otimes a\,.$$

In this model, the universal derivation is given by $da = [1 \otimes a]$.

This expression obscures the simplicity of $\Omega^{DP}_{A/R}$ in general.

Theorem 5.4. *Let A be a DP algebra. There is a unique DP A-module structure on $\Omega^{CA}_{A/R}$ such that $d : A \to \Omega^{CA}_{A/R}$ is a DP derivation, and it serves as the universal DP derivation.*

The case in which R is a field of characteristic p is proven in [6].

Proof. We follow [6] in a proof of the main theorem. To begin with, we employ one of the standard constructions of the Kähler differentials to construct a DP A-module structure on $\Omega^{CA}_{A/R}$.

The "fold" map $\nabla : A\coprod A \to A$ is characterized as a DP algebra map by the equations $\nabla \circ \mathrm{in}_1 = 1_A = \nabla \circ \mathrm{in}_2$. The only algebra map satisfying them sends each of the the factors in $A\coprod A = (A\otimes R)\oplus(A\otimes A)\oplus(R\otimes A)$ to A by the product, so the product map is a DP algebra map.

The kernel I of this map is thus a DP subalgebra of $A\coprod A$. One construction of $\Omega^{CA}_{A/R}$ is as the quotient I/I^2, which thus has a natural structure of an abelian DP algebra.

$\Omega^{CA}_{A/R}$ is of course also a module over A_+. We have to see that this A_+-module structure coheres with the divided powers; that is, that $\gamma_p((a\otimes 1)\omega)\in I^2$ for any $\omega\in I$. By the product formula this is $(a^p\otimes 1)\gamma_p\omega$. But $a^p = p!\gamma_p(a)$ and $p\gamma_p\omega\in I^2$ since $p\phi_p = 0$ in I/I^2.

So $\Omega^{CA}_{A/R}$ is a left $U(A)$-module.

Next, the universal derivation $d : A\to\Omega^{CA}_{A/R}$ is in fact a DP derivation. We refer to [6] for this; we observe that the argument for Proposition 2.8 in that paper does not require one to work over a field of characteristic p, but rather is a mod p result and so is applicable since $p\gamma_p = 0$.

Finally, suppose that $s : A\to M$ is any DP derivation. Since it is in particular a derivation in $\mathbf{CAlg}_R$, there is a unique A_+-module map $f : \Omega^{CA}_{A/R}\to M$ such that $s = f\circ d$. We claim that f is in fact a map of DP A-modules.

The DP algebra over A determined by M, $A\oplus M$, admits two DP algebra maps from A: the unit map and the map corresponding to the derivation s. Together they induce a DP algebra map $A\coprod A\to A\oplus M$. We claim that its restriction to I factors through the inclusion $M\hookrightarrow A\oplus M$. To see this let $\sum a_i\otimes b_i\in I$, so that $\sum a_ib_i = 0$ in A. Its image in $A\oplus M$ is

$$\sum(a_i,0)(b_i,s(b_i)) = \sum(a_ib_i, a_is(b_i)) = (0,\sum a_is(b_i))\,.$$

Since M is an abelian algebra, products vanish in it, so the map $I\to M$ factors through the quotient $I/I^2 = \Omega^{CA}_{A/R}$. This construction of $f : \Omega^{CA}_{A/R}\to M$ makes it clear that it is indeed a DP module homomorphism. $\square$

6. Examples

There is one case in which the module of Kähler differentials is easy to compute for formal reasons.

Lemma 6.1 (cf. [6, Theorem 2.9])**.** *Let V be an R-module and let $A = \Gamma_R(V)$ be the free DP algebra generated by the R-module V. Then there is a unique DP derivation $d : A\to U(A)\otimes V$ such that $dv = 1\otimes v$ for $v\in V$, and it is the universal DP derivation;*

$$\Omega^{DP}_{A/R} = U(A)\otimes V\,.$$

Proof. We simply verify that $(U(A)\otimes V, d)$ satisfies the universal property. Let M be any $U(A)$-module and consider the corresponding abelian object over A. Let $i : V \to A$ denote the inclusion of generators. Then:

$$\mathrm{Der}_R^{DP}(A, M) = \left\{ \begin{array}{ccc} & & A \oplus M \\ & \nearrow & \downarrow \\ A & \xrightarrow{1} & A \end{array} \right\}_{\mathbf{DPAlg}_R} = \left\{ \begin{array}{ccc} & & A \oplus M \\ & \nearrow & \downarrow \\ V & \xrightarrow{i} & A \end{array} \right\}_{\mathbf{Mod}_R}$$

$$= \mathrm{Hom}_R(V, M) = \mathrm{Hom}_{U(A)}(U(A) \otimes V, M)\,.$$

We leave the check that the universal differential is as stated to the reader.
□

Theorem 5.4 has the following consequence.

Corollary 6.2 ([10])**.** *The module of Kähler differentials of the free DP algebra $A = \Gamma_R(V)$ when regarded as merely a commutative algebra is*

$$\Omega^{CA}_{A/R} = U(A) \otimes V$$

as an A_+-module.

Example 6.3. For example suppose that V is free of rank 1 over R, with generator x. Then $A = \Gamma_R(R)$ is the free R-module on $\{\gamma_n(x) : n > 0\}$. Its universal enveloping algebra has the form

$$A_+ \oplus \bigoplus_p (A_+/p)[\phi_p(dx)]$$

as an A_+-module. To lighten notation, abbreviate $\gamma_n(x)$ to γ_n and $\phi_p(dx)$ as ϕ_p.

On the other hand, $\Omega^{CA}_{A/R}$ is generated as an A_+-module by the elements $d\gamma_n(x)$ (which we will abbreviate to $d\gamma_n$). This expression does not reveal its structure as an A_+-module. Since $d : A \to \Omega_{A/R}$ is a DP derivation, we have the relations

$$d\gamma_n = \phi_n + \gamma_1\phi_{n-1} + \cdots + \gamma_{n-1}\phi_1\,.$$

The $d\gamma_n$s are determined by the ϕ_j's by means of an invertible lower triangular band matrix. These equations have a unique solution, namely

$$\phi_n = d\gamma_n - \gamma_1 d\gamma_{n-1} + \cdots + (-1)^{n-1}\gamma_{n-1} d\gamma_1\,.$$

To see this, substitute these values for the ϕ_j's into the right hand side of the equation. It is the sum of the following sums:

$$\begin{array}{l}
d\gamma_n - \gamma_1 d\gamma_{n-1} + \gamma_2 d\gamma_{n-2} - \cdots + (-1)^{n-2}\gamma_{n-2} d\gamma_2 + (-1)^{n-1}\gamma_{n-1} d\gamma_1 \\
\qquad \gamma_1 d\gamma_{n-1} - \gamma_1\gamma_1 d\gamma_{n-2} + \cdots + (-1)^{n-3}\gamma_1\gamma_{n-3} d\gamma_2 + (-1)^{n-2}\gamma_1\gamma_{n-2} d\gamma_1 \\
\qquad\qquad \gamma_2 d\gamma_{n-2} - \cdots + (-1)^{n-4}\gamma_2\gamma_{n-4} d\gamma_2 + (-1)^{n-3}\gamma_2\gamma_{n-3} d\gamma_1 \\
\qquad\qquad\qquad\qquad \cdots\cdots \\
\qquad\qquad\qquad\qquad \gamma_{n-2} d\gamma_2 - \gamma_{n-2}\gamma_1 d\gamma_1 \\
\qquad\qquad\qquad\qquad\qquad \gamma_{n-1} d\gamma_1
\end{array}$$

For $k>0$, the kth column is of the form

$$\begin{aligned}(-1)^k\left(\gamma_k+\sum_{i=1}^{k-1}(-1)^i\gamma_i\gamma_{k-i}+(-1)^k\gamma_k\right)d\gamma_{n-k}\\ =(-1)^k\left(\sum_{i=0}^{k}(-1)^i\binom{k}{i}\right)\gamma_k d\gamma_{n-k}\,,\end{aligned}$$

since $\gamma_i\gamma_{k-i}=\binom{k}{i}\gamma_k$. But the alternating sum of binomial coefficients vanishes, leaving only $d\gamma_n$ as claimed.

This provides an explict description of $\Omega_{A/R}$ as an R-module:

$$\Omega_{A/R}=A_+\langle d\gamma_1\rangle\oplus\bigoplus_p\bigoplus_{e\geq 1}(A_+)/p\langle\phi_{p^e}\rangle\,.$$

Remark 6.4. This example actually provides a general expression for $\phi_n(a)$ in terms of $\{d\gamma_i(a):i\leq n\}$, for any DP algebra A and any $a\in A$, since there is a unique DP algebra map $\Gamma\langle x\rangle\to A$ sending x to a. By naturality of the operators γ_n and ϕ_n,

$$\phi_n(da)=d\gamma_n(a)+\sum_{i+j=n}(-1)^i\gamma_i(a)d\gamma_j(a)$$

in the $U(A)$-module $\Omega_{A/R}$. These elements generate $\Omega_{A/R}$ as an A_+-module. Their dependence on a is additive and "Frobenius linear" in the sense that $\phi_n(r\,da)=r^n\phi_n(da)$ for $r\in R$. And they satisfy the other properties of the ϕ_n's: $\phi_n(da)=0$ unless n is a prime power and $p\phi_{p^e}(da)=0$ for any prime number p and any $e\geq 1$. Moreover, the equations in 5.3 show that $\phi_n(a)$ depends only on the class of a in the "module of DP indecomposables" of A: the maximal quotient QA of A in which products and divided powers γ_n for $n>1$ vanish. So these elements are determined by their values on a choice of lifts of R-module generators of QA.

As observed by Roby, Theorem 5.4 and Lemma 6.1 together determine the indecomposables in a free divided power algebra:

Corollary 6.5 ([10])**.** *The indecomposable quotient of the free DP algebra $A=\Gamma_R(V)$ is*

$$A/A^2=U(0)\otimes V\,.$$

Proof. First, for any algebra A, the augmentation $\epsilon:A_+\to R$ puts an A_+-module structure such that $AV=0$ on any R-module V. A derivation $s:A\to V$ satisfies $s(ab)=as(b)+bs(a)=0$, and so factors uniquely through an R-module map $A/A^2\to V$. Conversely, for R-module map $A/A^2\to V$ the composite $A\to A/A^2\to V$ is a derivation. This implies

$$R\otimes_{A_+}\Omega_{A/R}=A/A^2\,.$$

For $A=\Gamma_R(V)$, we can now calculate

$$A/A^2=R\otimes_{A_+}\Omega_{A/R}=R\otimes_{A_+}(U(A)\otimes V)=U(0)\otimes V$$

where the last equality follows from $U(A) = A_+ \otimes U(0)$. □

Remark 6.6. Note that the DP module $\Omega^{DP}_{A/R}$ is the abelianization of the identity map $A \downarrow A$ as an object of $\mathbf{DPAlg}/A$. This special case of the abelianization functor

$$\mathrm{Ab}_A : \mathbf{DPAlg}/A \to \mathrm{Ab}(\mathbf{DPAlg}/A) = \mathbf{Mod}_{U(A)}$$

– the left adjoint of the inclusion – in fact determines the whole functor: Given $B \downarrow A$,

$$\mathrm{Ab}_A(B) = U(A) \otimes_{U(B)} \Omega^{DP}_{B/R} .$$

This is easily seen [5] using the fact that the pullback of $A \oplus M \downarrow A$ along $B \to A$ is $B \oplus M|_B \downarrow B$.

Remark 6.7. The canonical expression for $\Omega^{CA}_{A/R}$ is

$$\Omega^{CA}_{A/R} = A_+ \otimes A/(a \otimes b - 1 \otimes ab + b \otimes a)$$

and the natural map $\Omega^{CA}_{A/R} \to \Omega^{DP}_{A/R}$ includes the unit in $U(0)$ and collapses the relation involving the divided powers. So the theorem shows that the effect of those relations is precisely to kill the augmentation ideal in $U(0)$. The relations clearly have that effect; what is in question is whether they do any more.

References

[1] M. Barr, *Acyclic models*, Centre de Recherches Mathématiques Monograph Series 17, 2002.

[2] P. Berthelot and A. Ogus, *Notes on Crystalline Cohomology*, Annals of Mathematics Studies, 1978.

[3] H. Cartan, *Séminaire Henri Cartan, 7e année : 1954/1955 : Algèbres d'Eilenberg-Mac Lane*, École Normale Supérieure, 1955.

[4] N. Dhankhar, H. Miller, A. Tahboub, and V. Yin, Beck modules and alternative algebras, Communications in Algebra, to appear.

[5] I. Dokas, Triple cohomology and divided powers algebras in prime characteristic, Journal of the Australian Mathematical Society 87 (2009) 161–173.

[6] I. Dokas, On Kähler differentials of divided power algebras, Journal of Homotopy and Related Structures 18 (2023) 153–176.

[7] S. Eilenberg, Extensions of general algebras, Annales de la Société Polonaise de Mathématique 21 (1948) 125–134.

[8] D. Quillen, On the (co-)homology of commutative rings, Proceedings of Symposia on Pure Mathematics XVII (1970) 65–87, American Mathematical Society.

[9] N. Roby, Les algèbres à puissances divisées, Bulletin des Sciences Mathématiques 89 (1965) 75–91.

[10] N. Roby, Sur l'algèbre des puissances divisés dun module et le module de ses différentielles, Annales scientifiques de l'École Normale Supérieur 83:2 (1966) 75–89.

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