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# Recent Work on the Homotopy Theory of Classifying Spaces of Finite Groups

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A fair proportion of the community of homotopy theorists has been spending a lot of time in the last few years thinking about various aspects of the homotopy theory of classifying spaces of finite groups. I will try to tell you two or three theorems that have emerged.

To begin, let me recall for you a slight extension of an idea of Dennis Sullivan's. Let $G$ be a finite group, and suppose $X$ is a nice space on which $G$ acts nicely: say, $X$ is a $G$ - cw complex. Then the fixed point set $X^G$ is a subcomplex, and Sullivan proposed to recover it (up to weak homotopy type) from homotopy theory. For this, write

$$X^G = Map^G(*,X).$$

Now general ideas indicate that $*$ is not a nice G-space to map out of. It is natural to blow it up to a <u>free</u> G-space $EG$ such that the equivariant map $EG \to *$ is a homotopy equivalence. There are of course many constructions of such a space; any one will do. It is the total space of the universal principal G-bundle $EG \to BG$; $BG$ is the orbit space $EG/G$. In any case, we have a natural map

$$i : X^G \to Map^G(EG,X).$$

The target is "homotopy theoretic"; for instance, if $f : X \to Y$ is a G-map and a homotopy equivalence, then $Map^G(EG,f)$ is a homotopy equivalence. The map $f^G$ is not, of course; think of $EG \to *$ as a choice of $f$.

Sullivan's idea was that under the proper circumstances, $i$ should be close to a homotopy equivalence. Obviously it can't be in general: if $X = EG$, then $X^G = \emptyset$, while the right hand side contains at least the identity map. To avoid such examples, some restriction must be imposed on $X$. The most convenient one, though perhaps not the most natural, is to require that $X$ be finite-dimensional.

If $X$ is free, so that $X^G = \emptyset$, the projection $X \to X/G$ is a principal G-bundle, and so is classified by a map $X/G \to BG$. Suppose we had G-map $EG \to X$. It induces a map $BG \to X/G$, and the composite $BG \to X/G \to BG$ must be homotopic to the identity, since it pulls the bundle $EG \to BG$ back to itself. But $X/G$ is of finite category, while $BG$ is not, so we have a contradiction. We conclude that $Map^G(EG,X) = \emptyset$ also, so $i$ is a homotopy-equivalence in this case.

The first theorem I'd like to put before you represents the other extreme:

Theorem A. The map $i$ is a weak equivalence if $G$ is finite, $X$ is finite-dimensional, and the G-action on $X$ is trivial.

Of course, in this case

$$Map^G(EG,X) \cong Map(BG,X).$$

If $X$ is connected, the theorem admits a further reformulation, as follows. Pick a point $* \in BG$, and look at the evaluation fibration:

$$\begin{array}{ccc} & & Map_*(BG,X) \\ & & \downarrow \\ X & \xrightarrow{\ i\ } & Map(BG,X) \\ & \searrow^{=} & \downarrow p \\ & & X \end{array}$$

Since $pi = 1$, Theorem A is equivalent to

Theorem A'. Let $G$ be a finite group and $X$ a connected finite-dimensional cw-complex. Then $Map_*(BG,X)$ has the weak homotopy type of a point.

I shall say just a word about the proof, before showing you an extension and an application. One shows using K-theory that any map $BG \to X$ lifts to the universal cover of $X$, and so one can assume that $X$ is simply connected. A reduction, due in its present form to Mike Hopkins, allows one then to assume $G = Z_p$ ; this is the basic case. Unstable Adams spectral sequence methods of Bousfield and Kan show that vanishing of a suitable Ext is sufficient. This Ext involves both the cup-product structure and the action of the Steenrod algebra A. One shows that certain homological constructions on commutative coalgebras preserve boundedness, and then that certain other homological constructions on A-modules yield 0, given bounded input. The latter step involves an observation due for $p = 2$ to Gunnar Carlsson, and is closely related to the following result.

Theorem. $\overline{H}^*(BZ_p)$ is injective in the category of unstable left A-modules of finite type.

The extension of Theorem A' I have in mind is due to Alex Zabrodsky.

Theorem. Let $W$ be a connected space with $\pi_i W$ finite for all $i$, and 0 for all large $i$. Let $X$ be a connected finite dimensional cw complex. Then $Map_*(W,X)$ is weakly contractible.

Let me show you the idea of the proof. Suppose $G$ is a topological group--for instance, $K(\pi,n)$, $\pi$ Abelian--and let $E$ be a G-space. One then has the Milnor filtration of $EG \times_G E$ :

$$(*)\qquad \begin{array}{ccccccccc} E & \to & F_1 & \to & F_2 & \to & \cdots & \to & EG \times_G E \\ & & \downarrow & & \downarrow & & & & \\ & & \Sigma G \wedge E^+ & & \Sigma^2 G^{(2)} \wedge E^+ & & & & \end{array}$$

The vertical maps are the cofibers. Assume that $Map_*(G,X) \simeq *$. Then, using adjointness,

$$Map_*(K \wedge G,X) \cong Map_*(K,Map_*(G,X)) \cong * \ .$$

Mapping (*) into X, we obtain a tower of fibrations, in which each projection map is a weak equivalence. Thus

$$\mathrm{Map}_*(E,X) \overset{\simeq}{\leftarrow} \mathrm{Map}_*(EG \times_G E,X) .$$

As a case I take $E = *$ ; this shows that $\mathrm{Map}_*(BG,X) \simeq *$. Thus by induction $\mathrm{Map}_*(K(\pi,n),X) \simeq *$ for any finite $\pi$ and finite-dimensional $X$.

Next take $E$ principal, with orbit space $B$. Then $EG \times_G E \overset{\simeq}{\rightarrow} B$, so we have

$$\mathrm{Map}_*(E,X) \overset{\simeq}{\leftarrow} \mathrm{Map}_*(B,X).$$

This is the inductive step to go to a finite Postnikov system as source.

By supplementing this result with other techniques, Zabrodsky has removed the finiteness assumption on $\pi_i(W)$.

<u>Theorem</u>. Let $W$ be a connected space with $\pi_1(W)$ finite, $\pi_i(W)$ finitely generated for each $i$, and $\pi_i(W) = 0$ for all large $i$. Let $X$ be a connected finite dimensional CW complex. Then every map from $W$ to $\Omega^i X$ is phantom; and $\pi_i(\mathrm{Map}_*(W,X),f) = \prod_n \mathrm{Ext}^1_{\mathbf{Z}}(H_n(W;Q), \pi_{n+i+1}(X))$ for any $f : W \to X$.

The application is due to Chuck McGibbon and Joe Neisendorfer.

<u>Theorem</u>. Let $X$ be a nilpotent space such that $H_*(X; \mathbf{Z}_{(p)})$ is bounded and of finite type over $\mathbf{Z}_{(p)}$. Then either $X \simeq_{(p)} BN$ for a finitely generated torsion-free nilpotent group $N$, or else the p-torsion in $\pi_* X$ is unbounded.

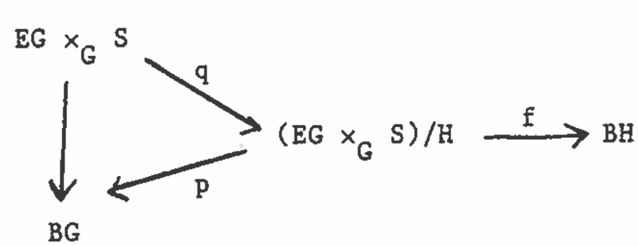
You will, of course, recognize that this is an extension of a thoeorem of Serre's, and resolves affirmatively a conjecture of his. The proof is quite easy.

Now I turn to stable homotopy theory. Traditionally one thinks of stabilizing as a device for simplifying problems to the point where they may be addressed, if not solved. In the present situation, however, it seems that one merely trades one set of problems for another; and in fact the Sullivan conjecture seems quite unrelated to any stable analogue.

Given a space $Y$, let $QY = \varinjlim \Omega^n \Sigma^n Y$. $QY$ is an n-fold loop space for any $n$, so $[X,QY]$ is an Abelian group for any $X$. Write $\{X,Y\}$ for this; it is the group of stable maps from $X$ to $Y$. I propose to describe $\{BG^+,BH^+\}$ to you explicitly, for any finite groups $G$ and $H$.

Of course, a map of spaces $X \to Y$ gives rise to a stable map $X^+ \to Y^+$; but there is another construction of stable maps as well, whose existence I recall for you. Given a finite cover $E \to B$, there is a stable "transfer" map $B^+ \to E^+$. If $B$ is a finite complex, it may be obtained via the Pontjagin-Thom construction applied to an embedding over $B$ of $E$ into $B \times \mathbb{R}^n$. If the finite cover has the form $BH \to BG$, for $H < G$, then this map induces the usual group-theoretic transfer in cohomology.

I will show how to produce a large collection of stable maps $BG^+ \to BH^+$ using a combination of these two constructions. The initial data consists in a finite set $S$, on which $G$ acts from the left and $H$ from the right, such that (1) the two actions are compatible, and (2) the H-action is free. Then we have a diagram



in which f classifies the principal H-bundle Q. Now the transfer associated to the #(S/H)-fold cover p, composed with $f^+$, induces $\alpha(S) \in \{BG^+,BH^+\}$. The association $\alpha$ is clearly additive, and so yields a map

$$\alpha : A(G,H) \to \{BG^+,BH^+\}$$

from the Grothendieck group of isomorphism classes of such (G,H)-sets S.

As a case of particular interest, take $H = 1$. Then $A(G,1)$ is the well-known Burnside ring $A(G)$ (the product comes from putting the diagonal G-action on products of G-sets), and the right hand side is the zero-dimensional unreduced stable cohomotopy $\pi^0(BG)$. On the other hand, we can regard $A(G,H)$ as morphisms from G to H in a category, which we call the "Burnside category." Composition is given by

$$({}_GS_H \ ; \ {}_HT_K) \mapsto ({}_GS \times_H T_K) \ .$$

The map $\alpha$ then gives a functor from the Burnside category to the stable category. The Burnside category is the natural source for any "representation-theoretic" functor.

We regard $A(G,H)$ as a discrete approximation to $\{BG^+,BH^+\}$. Since $BG^+$ is infinite-dimensional, however, we anticipate that the latter group will be quite large. Indeed, it is complete with respect to the topology defined by kernels of restrictions to skelata of $BG^+$. So we expect to have to complete $A(G,H)$ somehow to improve the chances of $\alpha$ being an isomorphism. There is a ring-homomorphism $\varepsilon : A(G) \to \mathbf{Z}$ given by sending S to its cardinality. Let $IA(G) = \ker \varepsilon$. $A(G)$ acts on $A(G,H)$ by

$$({}_GS, \ {}_GT_H) \mapsto ({}_GS \times T_H) \ ,$$

$G$ acting diagonally. We check that $\alpha$ is continuous, using the IA(G)-topology on the left and the skeleton topology on the right. There results a map

$$\hat{\alpha}: \hat{A}(G,H) \to \{BG^+, BH^+\}.$$

<u>Theorem B</u>. (Carlsson) $\hat{\alpha}$ is an isomorphism. Moreover, for any spectrum $K$, $\{K \wedge BG^+, BH^+\}$ can be expressed in terms of the groups $\{K, BW^+\}$ for certain associated groups $W$. In particular, for all $q > 0$,

$$\{BG^+, \Sigma^q BH^+\} = 0 .$$

When $H = 1$ this reads: $\hat{A}(G) \xrightarrow{\cong} \pi^0(BG)$, $\pi^q(BG) = 0$ for $q > 0$, and $\pi^{-q}(BG)$ for $q > 0$ may be expressed in terms of $\pi_*$ of certain associated classifying spaces--namely, Weyl groups of subgroups of $G$. This is of course reminiscent of M. F. Atiyah's theorem about the K-theory of a classifying space, and, indeed, Atiyah's theorem was doubtless an inspiration to Segal, who is said to have proposed this case as a conjecture.

Notice right away that this is <u>not</u> a stabilization of the Sullivan conjecture. For instance,

$$\varinjlim [\Sigma^n BG^+, S^n] = \varinjlim Z = Z$$

while

$$\{BG^+, S^0\} = \hat{A}(G)$$

is generally much larger. The point is that the maps involved in $\{BG^+, S^0\}$ cannot be defined as maps from all of $\Sigma^n BG^+$ for any finite $n$ ; this is an illustration of the wisdom of Frank Adams' dictum, "cells now, maps later."

The proof of this theorem has occupied many of us for some time, and I'll say a word or two about it, leaving aside many contributions not directly in the line of its current formulation. However, I can't go without mentioning that the initial step was taken by W. H. Lin. In 1978 he proved the case $G = \mathbf{Z}/2$, $H = 1$, using the Adams spectral sequence. While his algebraic methods, even as later improved by others, are not now used, there is no doubt that his success caught everyone by surprise and started the ball rolling. I should also say that some useful early work was done by E. Laitinen.

Rather direct preliminary work (due e.g., to Lewis, May, and McClure) reduces us to the case in which $G$ is a p-group and $H = 1$. On the other hand, Adams, Gunawardena, and I proved the case $G = (\mathbf{Z}/p)^n$. Along the way, we were forced to allow $H$ to be various elementary Abelian p-groups; indeed, the exigencies of that proof are what led me to formulate the version of the "Segal conjecture" outlined above. Our method was to express the Segal conjecture as the assertion that a certain map is a homotopy equivalence, check that it induces an isomorphism at $E_2$ of the Adams spectral sequence, and prove a convergence theorem. The algebra involved used essentially ideas of W. Singer, and has since been clarified in part by Priddy and Wilkerson. I quote one result from this work. Let $V = (\mathbf{Z}/p)^n$, regard $V$ as $H^1(V^*)$, and consider $\beta V \subset H^2(V^*)$. Inverting the product of the nonzero elements of $\beta V$, we obtain a localization $H^*(V^*)_{\beta V}$ ; it is an A-algebra in a unique way making the localization map A-linear. The automorphism group $GL(V)$ acts on $H^*(V^*)_{\beta V}$ by A-linear maps, and so it acts also on the module of indecomposables $\mathbf{F}_p \otimes_A H^*(V^*)_{\beta V}$. It is possible and useful to identify this GL(V)-module, which turns out to be concentrated in degree -n. As it happens, it is projective over $\mathbb{F}_p GL(V)$. This fact puts a lower bound on its size, as follows. Let $U$ be a p-Sylow subgroup of $GL(V)$: it is a group of "unipotent matrices," upper triangular with 1 's down the diagonal, when written out in a suitable basis. Since $\mathbb{F}_p U$ is local, any projective is free. It turns out there is an essentially unique irreducible $\mathbb{F}_p GL(V)$-module which as an $\mathbb{F}_p U$-module is free of rank 1 ; it was discovered by R. Steinberg, and I shall denote it by $St(V)$. Steinberg in fact wrote down an idempotent $\varepsilon$ in $\mathbb{F}_p GL(V)$ which splits $St(V)$ off the group algebra.

Now I can tell you the theorem.

__Theorem__. (i) As $\mathbb{F}_p GL(V)$-modules,

$$\mathbb{F}_p \otimes_A H^*(V^*)_{\beta V} \cong \Sigma^{-n} St(V) .$$

(ii) The resulting quotient map $H^*(V^*)_{\beta V} \to \Sigma^{-n} St(V)$ induces a $Tor_*^A(\mathbb{F}_p,-)$-equivalence on G-fixed point submodules for any subgroup $G$ of $GL(V)$.

The remaining, and hardest, task was accomplished last year by Gunnar Carlsson. He worked with a formulation of the Segal conjecture as a statement about equivariant stable homotopy theory, and indeed his result represents the premier theorem in this new field. Without entering into definitions, the conjecture, for a p-group $G$, is equivalent to the claim that $EG \to *$ induces an isomorphism in p-completed equivariant stable cohomotopy. This is analogous to Segal's reformulation of Atiyah's theorem. Carlsson discovered an ingenious induction, using a blow-up of the singular locus of certain G-spaces, showing that the conjecture was valid provided it was true for elementary Abelian groups. Actually, his context reduces the conjecture in the Abelian group case to a check that a certain boundary map is an isomorphism, and that check can be carried out using somewhat less than the end result of Adams, Gunawardena, and Miller; May and Priddy have a paper on this point. Finally, I mention that substantial clarification of Carlsson's proof have been proposed recently by Frank Adams. I quote one interesting theorem which occurs as part of an inductive cycle in Frank's approach. If $V$ is a G-representation, then the one-point compactification $S^V$ is a pointed G-space, with base-point at $\infty$. Since $0$ is also fixed, we have a G-map $S^0 \to S^V$. As spaces, this map is of course null-homotopic; but if $V^G = 0$, it is non-null, and, indeed, of fundamental importance, as a G-map; it is a kind of Euler class. For any G-cw complex $X$ we may form the sequence of G-maps

$$X \to S^V \wedge X \to S^{2V} \wedge X \to \dots$$

__Theorem.__ $\lim\limits_{\leftarrow n} \hat{\pi}^*_G(S^{nV} \wedge X) = 0$ where $G$ is a p-group and $\hat{\pi}^*_G$ denotes p-complete G-equivariant stable cohomotopy.

As a final taste of the work being done on classifying spaces, I should mention the work of S. Mitchell and S. Priddy on stable splittings of classifying spaces. Let $Sp^n X = X^n/\Sigma_n$ ; this construction stabilizes easily to a functor on spectra. For example, one may form $\Sigma^{-n}(Sp^{p^n} S^0/Sp^{p^{n-1}} S^0)_{(p)} = L(n)$.

__Theorem C.__ (Mitchell-Priddy) $\Sigma^\infty B(\mathbf{Z}/p)^n$ contains a wedge of $p^{\binom{n}{2}}$ copies of $L(n)$ as a retract.

These summands are obtained as follows. The group algebra $R = \hat{\mathbf{Z}}_p GL(V)$ acts naturally on $\Sigma^\infty BV$, $V = (\mathbf{Z}/p)^n$. Any idempotent $\varepsilon$ in $R$ yields a splitting of $\Sigma^\infty BV$ ; one factor is the mapping telescope of

$$\Sigma^\infty BV \xrightarrow{\varepsilon} \Sigma^\infty BV \xrightarrow{\varepsilon} \dots$$

If we take for $\epsilon$ the idempotent of Steinberg mentioned earlier, then this mapping telescope is $L(n) \vee L(n-1)$. I point out that this splitting of the telescope must (by the Segal conjecture) correspond to a further decomposition of $\epsilon$ in the "monoid algebra" $\mathbb{F}_p \mathrm{End}(V)$. Such a splitting seems to be beyond the ken of the algebraists.

Mitchell and Priddy also obtain splittings of certain other classifying spaces, such as dihedral groups, with transfer techniques. Some early work on this was done by Celia Whitten in her thesis under Jim Milgram.

This fact turned out to be just what N. J. Kuhn needed to complete a proof of the "Whitehead Conjecture" for $p = 2$. The Dold-Thom theorem implies that $Sp^{\infty}S^0$ is the integral Eilenberg-MacLane spectrum $H$. Form the diagram of cofibration sequences

$$\begin{array}{ccccccc} H & \longrightarrow & H/S^0 & \longrightarrow & H/Sp^2S^0 & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ S^0 & & Sp^2S^0/S^0 & & Sp^4S^0/Sp^2S^0 & & \end{array}$$

and then write down the sequence of boundary maps:

$$* \longleftarrow H \longleftarrow S^0 \longleftarrow L(1) \longleftarrow L(2) \longleftarrow \cdots$$

<u>Theorem D</u>. (N. J. Kuhn) This sequence is exact in homotopy localized at 2.