HARRISON HOMOLOGY AND THE QUILLEN COHOMOLOGY OF COMMUTATIVE MONOIDS

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Abstract. The cohomology theory for commutative monoids developed by P. A. Grillet is a case of a graded form of Harrison homology.

In his book *Homotopical Algebra* [30], Daniel Quillen described a homotopy theory of simplicial objects in any of a wide class of universal algebras, and corresponding theories of homology and cohomology. Quillen homology is defined as derived functors of an abelianization functor, and in many cases can be computed using a cotriple resolution [4]. Coefficients for these theories are "Beck modules," that is, abelian objects in a slice category. The case of commutative rings was studied at the same time by Michel André [1].

An example of such an algebraic theory, one of long standing and increasing importance, is provided by the category **ComMon** of commutative monoids. The prime exponent of the study of commutative monoids has for years been Pierre Grillet [15, 16, 17, 18, 21, 22] (but see also [8] for example). Among other things, Grillet provided the beginning of a small cochain complex, based on multilinear maps subject to certain symmetry conditions, whose cohomology he showed to be isomorphic in low dimensions to the Quillen cohomology $H^*_{CM}(X;M)$ of the commutative monoid X with coefficients in a Beck module over X; and in [20] a corresponding resolution in Beck modules was developed. This was surprising, since Quillen cohomology is defined by means of a simplicial resolution and does not generally admit such an efficient computation.

It is easy [26, 15] to see that $\mathbf{Mod}_X$ is equivalent to the category of covariant functors from the "Leech category" L_X to the category **Ab** of abelian groups. The Leech category has object set X; $L_X(x,y) = \{z : y = zx\}$; and composition is given by multiplication in the commutative monoid.

In this brief note, we observe that Grillet's construction is in fact subsumed by the theory of Harrison cohomology of commutative rings, once this theory has been extended to the graded context. As pointed out by Bourbaki [5, Ch. 2 §11], one can speak of rings graded by a commutative monoid: an X-graded object in a category $\mathbf{C}$ is an assignment of an object $C_x \in \mathbf{C}$ for each $x \in X$. An X-graded ring is a ring object in this category; that is, an X-graded abelian group R together with maps $R_x \otimes R_y \to R_{x+y}$ and an

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element $1 \in R_0$ (writing the commutative monoid additively), satisfying the evident unital and associativity conditions.

The first observation, simple enough as to need no proof, is that there is a natural X-graded commutative ring $\widetilde{\mathbb{Z}}X$ in which, for each $x \in X$, $(\widetilde{\mathbb{Z}}X)_x$ is the free abelian group generated by an element we will write 1_x, with the evident unit and multiplication. This is the "X-graded monoid algebra" of X.

The next observation, equally simple, is that the category of Beck X-modules is equivalent to the category of X-graded left modules over $\widetilde{\mathbb{Z}}X$.

These two observations bring into play the entire highly developed homological theory of commutative rings. Our main result along these lines 5.2 is that

$$HQ^*_{CM}(X;M) = HQ^*_{CA}(\widetilde{\mathbb{Z}}X;M)$$

where the right hand term denotes the well-studied André-Quillen cohomology [31, 1, 33], extended to the graded context. This generalizes an observation of Kurdiani and Pirashvili [25], who considered the case of Beck modules pulled back from the trivial monoid, in which case one arrives at the André-Quillen cohomology of $\mathbb{Z}X$ as an ungraded commutative ring.

André-Quillen cohomology is of course hard to compute, but there are well known approximations to it. One such approximation is given by Harrison cohomology [23, 12, 2, 34]. This theory is most neatly expressed by restricting to Hochschild cochains that annihilate shuffle decomposables; or, equivalently, to cochains that satisfy appropriate "partition" symmetry conditions. This characterization was apparently suggested by Mac Lane, and adopted in [23], but Harrison's original invariance property involved a different characterization of the same symmetry conditions, using "monotone" permutations. The equivalence of these two definitions can be found as Corollary 4.2 in [12].

Exactly the same monotone symmetry conditions occur in the partial complex described by Grillet (modified, in dimensions 4 and 6 in a manner suggested by Barr [34]); Grillet's partial complex is essentially the beginning of the Harrison complex for the graded monoid algebra.

This approximation definitely breaks down at some point: The André-Quillen cohomology of a polynomial algebra vanishes in positive dimensions, but Michael Barr showed [2] that the Harrison cohomology of the polynomial algebra over a field of characteristic p is nonzero in dimension $2p$, and in her thesis Sarah Whitehouse showed [34] that in characteristic 2 Barr's variant is nonzero in dimension 5, so the conjecture made in [21] fails at least in dimension 5.

Grillet's identification of his cohomology with the Quillen cohomology of commutative monoids goes beyond what seems to be known about Harrison cohomology in general, and suggests a variety of questions about André-Quillen homology.

We note as well that the observation that Beck modules over a commutative monoid are just graded modules over its graded monoid algebra suggests that the description of Quillen homology for commutative monoids carried out in [25] is in fact a special case of a graded extension of Pirashvili's earlier work [29].

We begin in §1 with a recollection of Quillen homology, along with the cotriple resolution that may be used to compute it. §2 sets out some elementary but sometimes surprising facts about X-gradings, and in §3 we explain how the grading behaves in homological algebra. Change of grading is explained in §4. The next section is the core of the work, proving the identifications of Quillen homology for commutative monoids and certain X-graded commutative rings. We then turn to interprenting the work of Grillet. This requires developing the Hochschild complex with its shuffle product, and the various indecomposable quotients occuring in the definitions of Harrison and Barr homology. Finally, in §10, we review the motivating work of Pierre Grillet and relate it to Harrison and Barr homology.

We are grateful to Pierre Grillet for forwarding us an early copy of a paper in which a similar story is worked out, in response to a letter from us outlining the results presented here. He uses somewhat different language – his "multi" objects are our graded objects – but he did not make the connection with Harrison homology that we establish here.

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1. Quillen homology

A cotriple G on a category $\mathbf{C}$ determines a functor $G_\bullet$ to the category of simplicial objects over $\mathbf{C}$, augmented to the identity functor: the "cotriple resolution" [4]. This allows one to define G-derived functors for any functor $E : \mathbf{C} \to \mathbf{A}$ to an abelian category:

$$L_n E(C) = H_n(\mathrm{ch}\, E(G_\bullet C))$$

where ch denote formation of the chain complex associated to a simplicial object in an abelian category. If $\mathbf{C}$ is the category of F-algebras, for a triple F on **Set**, then we may take for G the adjoint cotriple on $\mathbf{C}$. Quillen's definition [31] of homology involves the category of abelian objects in the slice category over $A \in \mathbf{C}$: the category $\mathbf{Mod}_A$ of "Beck modules." Very often $\mathbf{Mod}_A$ is an abelian category (see [11] for example) and the forgetful functor (which is typically faithful) $\mathbf{Mod}_A \to \mathbf{C}/A$ has a left adjoint, "abelianization" $\mathrm{Ab}_A : \mathbf{C}/A \to \mathbf{Mod}_A$. In this case we can form the Quillen homology

$$HQ_n(A) = L_n \mathrm{Ab}_A(A)$$

– a sequence of Beck A-modules. If $\mathbf{C}$ is the category of algebras for a triple on **Set** and G is the adjoint triple, the these adjoint functors may be computed as G-derived functors:

$$HQ_n(A) = H_n(\mathrm{ch}\,\mathrm{Ab}_A(G_\bullet A))\,.$$

When $\mathbf{C}$ is the category $\mathbf{ComAlg}_K$ of commutative K-algebras, the category of Beck A-modules is equivalent to the category of left A-modules: An abelian object over A, $p : B \downarrow A$, first of all has a section, the "zero-section," which provides an identification of K-modules $B \cong A \oplus M$ where M is the kernel of p as an A-module. The abelian structure forces the multiplication on $A \oplus M$ to be given by $(a, m)(b, n) = (ab, an + bm)$. Under this identification, a section of $A \oplus M \downarrow A$ in $\mathbf{ComAlg}_K$ is given by $a \mapsto (a, da)$ where $d \in \mathrm{Der}_K(A, M)$. The abelianization of $\mathrm{id} : A \downarrow A$ is the A-module such that $\mathrm{Hom}_A(\mathrm{Ab}_A(A), M) = \mathrm{Der}_K(A, M)$; that is, $\mathrm{Ab}_A(A)$ is the usual module $\Omega_{A/K}$ of Kähler differentials. More generally, for $B \downarrow A$ in $\mathbf{ComAlg}_K/A$

$$\mathrm{Ab}_A(B) = A \otimes_B \Omega_{B/K}\,.$$

So in that case, we have the "cotangent complex"

$$\mathbf{L}_{A/K} = A \otimes_{G_\bullet A} \Omega_{G_\bullet A/K}$$

and the André-Quillen homology is its homotopy:

$$HQ_n(A) = H_n(\mathrm{ch}\,\mathbf{L}_{A/K})\,.$$

We can also afflict this construction with coefficients, and equally well consider a cohomology version: If M is an A-module,

$$HQ_n(A; M) = H_*((\mathrm{ch}\,\mathbf{L}_{A/K}) \otimes_A M)\,,$$
$$HQ^n(A; M) = H^*(\mathrm{Hom}_A(\mathrm{ch}\,\mathbf{L}_{A/K}, M))\,.$$

When necessary we will indicate the ground ring K as well, writing $HQ^n(A/K; M)$, for example.

2. Gradings

Let X be a commutative monoid, which we will write additively. Following Bourbaki [5] we say that an X-graded object $C_\bullet$ in a category $\mathbf{C}$ is a choice of object C_x of $\mathbf{C}$ for each $x \in X$. Write $\mathbf{C}^X$ for the category of X-graded objects in $\mathbf{C}$. A morphism $C_\bullet \to C'_\bullet$ is a morphism $C_x \to C'_x$ for each $x \in X$. A functor $F : \mathbf{C} \to \mathbf{D}$ induces $F^X : \mathbf{C}^X \to \mathbf{D}^X$, and an adjunction between E and F induces a canonical adjunction between E^X and F^X.

A symmetric monoidal category $(\mathbf{C}, \mathbb{1}, \otimes, c)$ is *distributive* if as a category it has finite coproducts for which the canonical maps

$$(C \otimes D)\coprod(C' \otimes D) \to (C\coprod C') \otimes D$$
$$(C \otimes D)\coprod(C \otimes D') \to C \otimes (D\coprod D')$$

are isomorphisms. (It follows that $\mathbb{0} \otimes D = \mathbb{0}$ and $C \otimes \mathbb{0} = \mathbb{0}$, where $\mathbb{0}$ is the initial object.) Assuming that $\mathbf{C}$ has coproducts of large enough sets of objects, there is then a canonical distributive symmetric monoidal structure on $\mathbf{C}^X$, in which

$$\mathbb{1}_x = \begin{cases} \mathbb{1} & \text{for } x = 0 \\ \mathbb{0} & \text{for } x \neq 0 \end{cases}$$
$$(C_\bullet \otimes D_\bullet)_z = \coprod_{x+y=z} C_x \otimes D_y\,.$$

The symmetry $c : (C_\bullet \otimes D_\bullet)_z \to (D_\bullet \otimes C_\bullet)_z$ is such that for all x, y with $x + y = z$,

$$c_{C_\bullet,D_\bullet} \circ \mathrm{in}_{x,y} = \mathrm{in}_{y,x} \circ c_{C_x,D_y}$$

where $c_{C,D} : C \otimes D \to D \otimes C$ is the symmetry in $\mathbf{C}$. For example, a commutative monoid in the symmetric monoidal category $\mathbf{Set}^X$ consists of a set T_x for each $x \in X$ together with an element $1 \in T_0$ and maps $\mu : T_x \times T_y \to T_{x+y}$ satisfying evident conditions. This is to be distinguished from an X-graded commutative monoid, an object of $\mathbf{ComMon}^X$!

Let K be a commutative ring. The category of X-graded K-modules $\mathbf{Mod}_K^X$ admits a symmetric monoidal structure given by the "graded tensor product," with

$$(A_\bullet \otimes_K B_\bullet)_z = \bigoplus_{x+y=z} A_x \otimes_K B_y$$

and unit given by the X-graded K-module with K in degree 0 and 0 in all other degrees. An X-graded K-algebra is a monoid for this tensor product. Once again, beware of this use of language; this is not an X-graded object in $\mathbf{ComAlg}_K$. Write $\mathbf{ComAlg}(\mathbf{Mod}_K^X)$ for this category.

A left module for the X-graded K-algebra $A_\bullet$ is an action of this monoid; write $\mathbf{Mod}_{A_\bullet}$ for this category; then

$$\mathbf{Mod}_{A_\bullet} = \mathrm{Ab}(\mathbf{ComAlg}(\mathbf{Mod}_K^X)/A_\bullet)$$

The relationship with the Leech category (section 5 below) suggests that rather than defining a right $A_\bullet$-module as a right action, we should say this:

Definition 2.1. *A right $A_\bullet$-module is an X-graded K-module $M^\bullet$ together with homomorphisms*

$$\varphi_{x,y} : M^{x+y} \otimes A_y \to M^x$$

such that $\varphi_{x,0}(m \otimes 1) = m$ *and*

$$\begin{array}{ccc} M^{x+y+z} \otimes A_z \otimes A_y & \xrightarrow{1\otimes\mu_{z,y}} & M^{x+y+z} \otimes A_{z+y=y+z} \\ \Big\downarrow{\scriptstyle \varphi_{x+y,z}\otimes 1} & & \Big\downarrow{\scriptstyle \varphi_{x,y+z}} \\ M^{x+y} \otimes A_y & \xrightarrow{\quad\varphi_{x,y}\quad} & M^x \end{array}$$

commutes.

Write $\mathbf{RMod}_{A_\bullet}$ for the category of right $A_\bullet$-modules.

If X has inverses, so is in fact an abelian group, this is equivalent to a right $A_\bullet$-module in the usual sense, using "lower indexing" $M_x = M^{-x}$.

Let $N_\bullet$ be a left $A_\bullet$-module and $M^\bullet$ a right $A_\bullet$ module. Their tensor product over $A_\bullet$, $N^\bullet \otimes_{A_\bullet} M_\bullet$, is the K-module defined as the coequalizer of the two maps

$$f, g : P = \bigoplus_{x,y} M^{x+y} \otimes A_y \otimes N_x \rightrightarrows \bigoplus_z M^z \otimes N_z$$

Each of these maps is defined by giving the composite with an inclusion $\mathrm{in}_{x,y} : M^{x+y} \otimes A_y \otimes N_x \to P$:

$$\begin{aligned} f \circ \mathrm{in}_{x,y} &= \mathrm{in}_{x+y} \circ (1 \otimes \varphi_{x,y})\,, \\ g \circ \mathrm{in}_{x,y} &= \mathrm{in}_x \circ (\varphi_{x,y} \otimes 1)\,. \end{aligned}$$

The default symmetry in $\mathbf{Mod}_K^X$ is of course given by $a \otimes b \mapsto b \otimes a$, but there are variants. A *sign rule* on a commutative monoid X is a function

$$\sigma : X \times X \to \{\pm 1\}$$

such that

$$\sigma(x+y, z) = \sigma(x,z)\sigma(y,z)\,, \quad \sigma(x, y+z) = \sigma(x,y)\sigma(x,z)\,,$$
$$\sigma(x,y) = \sigma(y,x)$$

The "topologist's sign rule," with $X = \mathbb{Z}$, is $\sigma(x,y) = (-1)^{xy}$. A sign rule σ determines a new symmetric monoidal structure with symmetry c_σ on $\mathbf{Mod}_X$ given by

$$\mathrm{c}_\sigma : A_x \otimes B_y \to B_y \otimes A_x\,, \quad a \otimes b \mapsto \sigma(x,y) b \otimes a\,.$$

If we are interested in taking values in the category of modules over a given commutative ring K, we can consider sign rules as above but taking values in $K^\times$ rather than $\mathbb{Z}^\times = \{\pm 1\}$.

Given a sign rule σ one can speak of σ-commutative X-graded rings. Much of what follows could be carried out with a general sign rule, but for simplicity we will restrict attention to the default rule.

3. Graded homological algebra

From now on we will write just A rather than $A_\bullet$ and so on. Let A be an X-graded K-algebra. For each $x \in X$, there is a left A-module $P^x(A)$ together with $\iota \in P^x_x(A)$ such that for any left A-module N the map

$$\mathrm{Hom}_A(P^x(A), N) \to N_x\,, \quad f \mapsto f(\iota)$$

is an isomorphism. Explicitly,

$$P^x_y(A) = \bigoplus_{x+z=y} A_z$$

and ι is the image of $1 \in A_0$ under $\mathrm{in}_0 : A_0 \to P^x_x(A)$. Given $n \in N_x$, the corresponding map $\widehat{n} : P^x(A) \to N$ is defined by

$$\widehat{n} \circ \mathrm{in}_z(a) = an\,, \quad a \in A_z\,.$$

The A-module $P^x(A)$ is projective since $N \mapsto N_x$ is an exact functor.

For any A-module N let

$$p : \bigoplus_x P^x(A) \otimes N_x \to N$$

be the map defined by

$$p_y \circ (\mathrm{in}_z \otimes 1)(a \otimes n) = an$$

for any z such that $x + z = y$, $a \in A_z$, $n \in N_x$. This map is surjective, and so any projective A-module is a retract of a direct sum of $P^x(A)$'s.

Each $y \in X$ also determines a projective right A-module $P_y(A)$, given by the same formula. Then

$$P_y(A) \otimes_A N = N_y\,.$$

This shows that $P_y(A)$ is "flat," since $N \mapsto N_y$ is an exact functor.

The account of Quillen homology given above goes through in the graded context without essential change. The category $\mathrm{Ab}(\mathbf{ComAlg}(\mathbf{Mod}^X_K)/A)$ can be identified with $\mathbf{Mod}_A$. A section of $\mathrm{pr}_1 : A \oplus N \downarrow A$ is a (graded) derivation $d \in \mathrm{Der}_K(A, N)$; this is a graded K-module map such that $d(ab) = a\,db + b\,da$ as usual. The functor $N \mapsto \mathrm{Der}_K(A, N)$ is co-represented by the (graded) Kähler differentials: $\Omega_{A/K} \in \mathbf{Mod}_A$. Expressed in terms of generators and relations, this A-module is the cokernel of the map

$$d : \bigoplus_{x,y} P^{x+y}(A) \to \bigoplus_z P^z(A)$$

determined by

$$d \circ \mathrm{in}_{x,y} = \mathrm{in}_x \circ y^* - \mathrm{in}_{x+y} + \mathrm{in}_y \circ x^*\,.$$

This is $\mathrm{Ab}_A A$. To describe $\mathrm{Ab}_A B$, for $p : B \to A$ in $\mathbf{ComAlg}(\mathbf{Mod}^X_K)$, notice that for each $x \in X$ the right A-module $P_x(A)$ can be regarded as a right B-module through the map p. Then

$$(\mathrm{Ab}_A B)_x = P_x(A) \otimes_B \Omega_{B/K}\,.$$

The left A-module structure on $\mathrm{Ab}_A B$ arises from the A-bimodule structure of $P(A)$.

Given an X-graded set $T_\bullet$, one can construct the free commutative monoid $\mathbb{N}T_\bullet$ in $\mathbf{Set}^X$ as

$$(\mathbb{N}T_\bullet)_x = \coprod_{\Sigma Y = x} \prod_{y \in Y} T_y$$

the disjoint union taken over finite subsets Y of X with sum x.

The construction of the free commutative monoid generated by an X-graded set is useful in building the free commutative X-graded K-algebra generated by $T_\bullet \in \mathbf{Set}^X$: It is in degree x just the free K-module generated by $(\mathbb{N}T_\bullet)_x$. This provides us with an adjoint pair

$$F_X : \mathbf{Set}^X \rightleftarrows \mathbf{ComAlg}(\mathbf{Mod}_K^X) : u_X$$

giving a cotriple G_X on $\mathbf{ComAlg}(\mathbf{Mod}_K^X)$. Following [4], this in turn leads to a natural simplicial object $G_{X\bullet}A$ augmented to A, with $G_{Xn}A = G^{n+1}A$, which can be used to derive functors on $(\mathbf{ComAlg}(\mathbf{Mod}_K^X)/A$.

The Quillen homology of $A \in \mathbf{ComAlg}(\mathbf{Mod}_K^X)$ is defined as the derived functors of $\mathrm{Ab}_A : (\mathbf{ComAlg}(\mathbf{Mod}_K^X)/A) \to \mathbf{Mod}_A$. Thus for each n the Quillen homology $HQ_n(A)$ is itself an A-module. Any right A-module M results in the sequence of K-modules

$$HQ_n(A; M) = H_n(\mathrm{ch}\,(M \otimes_A \mathrm{Ab}_A G_{X\bullet}A))\,.$$

For any $x \in X$, we can recover $HQ_n(A)_x$ by using the right A-module $P_x(A)$ for coefficients:

$$HQ_n(A)_x = HQ_n(A; P_x(A))\,.$$

For $N \in \mathbf{Mod}_A$ we can define the André-Quillen cohomology:

$$HQ^n(A; N) = H^n(\mathrm{Hom}_A(\mathrm{ch}\, G_{X\bullet}A, N))\,.$$

4. Change of grading monoid

Let $\alpha : X \to Y$ be a map of commutative monoids. A Y-graded object C in $\mathbf{C}$ determines an X-graded object $\alpha^* C$ by

$$(\alpha^* C)_x = C_{\alpha(x)}\,.$$

If $\mathbf{C}$ has coproducts, the functor $\alpha^* : \mathbf{C}^Y \to \mathbf{C}^X$ has a left adjoint given by

$$(\alpha_* C)_y = \coprod_{\alpha(x) = y} C_x\,.$$

Let K be a commutative ring. The functor $\alpha_* : \mathbf{Mod}_K^X \to \mathbf{Mod}_K^Y$ is symmetric monoidal –

$$\alpha_*(M \otimes N) = \alpha_* M \otimes \alpha_* N$$

– so α_* sends X-graded commutative K-algebras to Y-graded commutative K-algebras. Let $A \in \mathbf{ComAlg}(\mathbf{Mod}_K^X)$. Then for the same reason α induces adjoint pairs

$$\alpha_* : \mathbf{Mod}_A \rightleftarrows \mathbf{Mod}_{\alpha_* A} : \alpha^*$$

and

$$\alpha_* : \mathbf{RMod}_A \rightleftarrows \mathbf{RMod}_{\alpha_* A} : \alpha^*$$

Let N be an $\alpha_* A$-module. It is straightforward to construct a natural isomorphism

$$\mathrm{Der}_K(A, \alpha^* N) = \mathrm{Der}_K(\alpha_* A, N)$$

and so a natural isomorphism of $\alpha_* A$-modules

$$\Omega_{\alpha_* A/K} = \alpha_* \Omega_{A/K} \,.$$

The adjoint pairs (F_X, u_X) are compatible under change of grading monoid: A monoid homomorphism $\alpha : X \to Y$ determines the following squares.

$$\begin{array}{ccc} \mathbf{Set}^X & \xleftarrow{u_X} & \mathbf{ComAlg}(\mathbf{Mod}_K^X) \\ \uparrow{\scriptstyle \alpha^*} & & \uparrow{\scriptstyle \alpha^*} \\ \mathbf{Set}^Y & \xleftarrow{u_Y} & \mathbf{ComAlg}(\mathbf{Mod}_K^Y) \end{array} \qquad \begin{array}{ccc} \mathbf{Set}^X & \xrightarrow{F_X} & \mathbf{ComAlg}(\mathbf{Mod}_K^X) \\ \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} \\ \mathbf{Set}^Y & \xrightarrow{F_Y} & \mathbf{ComAlg}(\mathbf{Mod}_K^Y) \end{array}$$

The diagram of right adjoints clearly commutes, so the diagram of left adjoints does too:

$$\alpha_* F_X(T) = F_Y(\alpha_* T) \in \mathbf{ComAlg}(\mathbf{Mod}_K^Y) \,.$$

Passing to the cotriples,

$$\alpha_* G_X(A) = \alpha_* F_X u_X(A) = F_Y \alpha_* u_X(A) = F_Y u_Y(\alpha_* A) = G_Y(\alpha_* A)$$

and so to simplicial resolutions:

$$\alpha_* G_{X\bullet}(A) = G_{Y\bullet}(\alpha_* A) \,.$$

Assembling all this, we find:

Proposition 4.1. *Let $\alpha : X \to Y$ be a monoid homomorphism. There is an isomorphism natural in $A \in \mathbf{ComAlg}(\mathbf{Mod}_K^X)$*

$$\alpha_* HQ_*(A) = HQ_*(\alpha_* A)$$

as well as isomorphisms

$$HQ_*(A; \alpha^* M) = HQ_*(\alpha_* A; M)$$

natural in $M \in \mathbf{RMod}_{\alpha_* A}$ and

$$HQ^*(A; \alpha^* N) = HQ^*(\alpha_* A; N)$$

natural in $N \in \mathbf{Mod}_{\alpha_* A}$.

An important example is provided by taking Y to be the one-element monoid, e, and $\alpha : X \to e$ the unique map. Then $\alpha_* A$ is the "degrading" of A, an ungraded commutative K-algebra, and N is a module for it; $\alpha^* N$ is the "constant" X-graded K-module with $(\alpha^* N)_x = N$ for all $x \in X$, and A acting among them in the obvious way; and $HQ_*(\alpha_* A)$, $HQ_*(\alpha_* A; M)$, and $HQ^*(\alpha_* A; N)$ are the usual André-Quillen groups.

5. Commutative monoids

We now regard commutative monoids as the category of homological interest, rather than as a source of gradings.

Let $X \in \mathbf{ComMon}$. It is easy [26, 15] to identify the category of Beck modules over X in terms of the *Leech category*, L_X, with object set X and $L_X(x, z) = \{y \in X : x + y = z\}$ with unit and compositon determined by the commutative monoid X. Write $y_* : x \to (x + y)$.

A map $\alpha : Y \to X$ of commutative monoids induces a functor

$$\alpha^* : \mathbf{Mod}_X \to \mathbf{Mod}_Y$$

which, under the equivalence with functors from the Leech categories, may be regarded as induced by post-composition with the functor $\alpha : L_Y \to L_X$. The left adjoint α_* is then induced by left Kan extension along α.

A section of an abelian object over X is a "derivation," and under the identification of abelian objects over X with functors on L_X a derivation with values in $M : L_X \to \mathbf{Ab}$ is an assignment of an element $s(x) \in M_x$ for each $x \in X$ such that

$$s(x + y) = x_* s(y) + y_* s(x)$$

There is a universal example, the Beck module of "Kähler differentials" Ω_X^{CM}, which provides a distinguished object of $\mathbf{Mod}_X$. For any $\alpha : Y \to X$, $\mathrm{Ab}_X Y = \alpha_* \Omega_Y^{CM}$.

A commutative monoid X defines a canonical commutative X-graded K-algebra $\widetilde{K}X$ in which $(\widetilde{K}X)_x = K$ for each $x \in X$, with generator 1_x, $1 = 1_0 \in (\widetilde{K}X)_0$, and

$$\mu_{x,y}(a_x \otimes b_y) = (ab)_{x+y} = ab1_{x+y} \,, \quad a, b \in K \,.$$

This object of $\mathbf{ComAlg}_K^X$ co-represents the functor sending an object A to the set of sections a of the grading function $A \to X$ such that $a_x a_y = a_{x+y}$.

For a Beck K-module M over X we may define a left module over $\widetilde{K}X$, which we will again denote by M. The K-module in degree x is M_x and the multiplication by $\widetilde{K}X$ is given by the action of elements of X, regarded as morphisms in L_X. The following lemma is simply a change in perspective.

Lemma 5.1. *The category of covariant functors $L_X \to \mathbf{Mod}_K$ is canonically isomorphic to the category of $\widetilde{K}X$-modules. The category of contravariant functors $L_X \to \mathbf{Mod}_K$ is canonically isomorphic to the category of right $\widetilde{K}X$-modules.*

Of course the usual monoid-algebra is obtained from $\widetilde{K}X$ as

$$KX = \alpha_* \widetilde{K}X\,, \quad \alpha : X \to e\,.$$

These isomorphisms are compatible with attendant structure: the functors induced by a map $\alpha : Y \to X$ of commutative monoids agree with the functors induced by $\widetilde{\mathbb{Z}}\alpha : \widetilde{\mathbb{Z}}Y \to \widetilde{\mathbb{Z}}X$. Under this identification the Kähler differential objects match up:

$$\Omega_X^{CM} = \Omega_{\widetilde{\mathbb{Z}}X}^{CA}$$

and more generally $\alpha_* \widetilde{\mathbb{Z}}Y \in \mathbf{ComAlg}(\mathbf{Mod}_K^X)/\widetilde{\mathbb{Z}}X$ and

$$\mathrm{Ab}_X(Y) = \mathrm{Ab}_{\widetilde{\mathbb{Z}}X}(\alpha_* \widetilde{\mathbb{Z}}Y)$$

in $\mathbf{Mod}_X = \mathbf{Mod}_{\widetilde{\mathbb{Z}}X}$. As a further example, given a contravariant functor $M : L_X \to \mathbf{Mod}_K$ and a covariant functor $N : L_X \to \mathbf{Mod}_K$,

$$M \otimes_{\widetilde{\mathbb{Z}}X} N = M \otimes_{L_X} N$$

where the right hand side is the usual tensor product over the category L_X, as considered by Kurdiani and Pirashvili in [25].

The left adjoint Sym_K^X of the forgetful functor $\mathbf{ComAlg}(\mathbf{Mod}_K^X) \to \mathbf{Set}^X$ can be described as follows. Let $\mathbb{N}$ denote the free commutative monoid cotriple on $\mathbf{ComMon}$, let $Y \in \mathbf{Set}^X$, and write $\alpha : \mathbb{N}Y \to Y \to X$ for the structure map. Then

$$\mathrm{Sym}_K^X(Y) = \alpha_* \widetilde{K}\mathbb{N}Y\,.$$

To see this, observe that both sides co-represent the functor sending $A \in \mathbf{ComAlg}(\mathbf{Mod}_K^Y)$ to the set of maps $Y \to A$ over X.

We need a category of graded algebras in which the grading commutative monoid can vary. So let $\mathbf{ComAlg}_K^*$ be the category whose objects are pairs (X, A) where X is a commutative monoid and A is a commutative X-graded K-algebra. A morphism $(X, A) \to (Y, B)$ consists of a monoid homomorphism $\alpha : X \to Y$ together with a morphism $f : \alpha_* A \to B$ in $\mathbf{ComAlg}(\mathbf{Mod}_K^Y)$ (or equivalently a morphism $\hat{f} : A \to \alpha^* B$ in $\mathbf{ComAlg}(\mathbf{Mod}_K^X)$). Given a second morphism $(\beta, g) : (Y, B) \to (Z, C)$, the composite is defined as $(\beta \circ \alpha, g \circ \beta_* f)$, or, equivalently as $(\beta \circ \alpha, \alpha^* \hat{g} \circ \hat{f})$). Then the construction $\widetilde{K}$ provides a functor

$$\widetilde{K} : \mathbf{ComMon} \to \mathbf{ComAlg}_K^*\,.$$

Theorem 5.2. *Let X be a commutative monoid. There are natural isomorphisms*

$$HQ_n^{CM}(X) = HQ_n^{CA}(\widetilde{\mathbb{Z}}X)$$

where we have identified Beck modules over X with modules over $\widetilde{\mathbb{Z}}X$. For any right Beck module M over X,

$$HQ_n^{CM}(X; M) = HQ_n^{CA}(\widetilde{\mathbb{Z}}X; M)$$

and for any left Beck module N

$$HQ^n_{CM}(X;N) = HQ^n_{CA}(\widetilde{\mathbb{Z}}X;N)$$

Proof. This proof relies on the "fundamental theorem of homological algebra," that is, the possibility of computing derived functors from any "resolution." In this simplicial context the appropriate version of a resolution is a cofibrant replacement in a Quillen model category structure. The fact that any cofibrant replacement can be used is contained, for example, in [11, Proposition 3.9].

So we begin by recalling (from [24] for example) the notion of a model category, with its factorization and lifting axioms, and from [30, Ch II §4] the existence of a model category structure on simplicial objects on categories like **ComMon** and $\mathbf{ComAlg}(\mathbf{Mod}_K^X)$. Quillen requires the existence of a set of small projective generators. The free commutative monoid $\mathbb{N}$ serves this purpose in **ComMon**, and the X-graded commutative K-algebras freely generated by an element of degree x for each $x \in X$ play this role in $\mathbf{ComAlg}(\mathbf{Mod}_K^X)$.

Let $\alpha : Y_\bullet \to X$ be a cofibrant replacement of $X \in \mathbf{ComMon}$ regarded as a constant simplical object. We could take the cotriple resolution $\mathbb{N}_\bullet X$ for example. This is a weak equivalence in $s\mathbf{ComMon}$, which is to say a weak equivalence of simplicial sets. The entire simplicial set $Y_\bullet$ splits into a disjoint union of the pre-images of the points of X under α, and each of these components is a contractible simplicial set.

Now apply the functor $\widetilde{K}$ to $Y_\bullet$, to get a simplicial object in $\mathbf{ComAlg}_K^*$ with an augmentation $\widetilde{K}Y_\bullet \to \widetilde{K}X$; that is, a map

$$\alpha_*\widetilde{K}Y_\bullet \to \widetilde{K}X$$

in $\mathbf{ComAlg}(\mathbf{Mod}_K^X)$. The object $\alpha_*\widetilde{K}Y_\bullet$ of $s\mathbf{Mod}_K^X$ splits as a direct sum of the free K-module functor applied to the components of $Y_\bullet$ above the points of X. Since $Y_\bullet \to X$ is an equivalence, each of these components is contractible. But for any pointed simplicial set $Z_\bullet$,

$$\pi_*(KZ_\bullet) = H_*(Z_\bullet, *; K)\,.$$

Since the homology of a contractible simplicial set is just K in degree 0, the map $\alpha_*\widetilde{K}Y_\bullet \to \widetilde{K}X$ is a weak equivalence.

The second observation is that $\alpha_*\widetilde{K}\mathbb{N}_\bullet X$ is cofibrant in the model category of simplicial objects in $\mathbf{ComAlg}(\mathbf{Mod}_K^X)$. Indeed, it is almost free in the sense of [27]. A simplicial object $Z_\bullet \in s\mathbf{ComMon}$ is "almost free" if there are subsets $G_s \subseteq Z_s$, for each s, that are respected by all face and degeneracy maps except for d_0, and such that Z_s is freely generated by G_s. The cotriple resolution $\mathbb{N}_\bullet X$ is easily seen to be almost free. If we then apply the functor $\alpha_*\widetilde{K}$ to it, the same generating sets, now regarded as in $\mathbf{Set}^X$, again generate as objects in $\mathbf{ComAlg}(\mathbf{Mod}_X^K)$.

The simplicial object $\alpha_*\widetilde{K}\mathbb{N}_\bullet X \to \widetilde{K}X$ thus joins $\mathrm{Sym}^X_{K\bullet}(\widetilde{K}X) \to \widetilde{K}X$ as a cofibrant replacement, and so (see [11, Proposition 3.9]) can be used to compute Quillen homology. Thus (finally taking $K = \mathbb{Z}$)

$$\mathrm{Ab}_X \mathbb{N}_\bullet X = \mathrm{Ab}_{\widetilde{\mathbb{Z}}X}(\alpha_* \widetilde{\mathbb{Z}}\mathbb{N}_\bullet X) \simeq \mathrm{Ab}_{\widetilde{\mathbb{Z}}X}(\mathrm{Sym}^X_{\mathbb{Z}_\bullet}(\widetilde{\mathbb{Z}}X))$$

as simplicial objects in $\mathbf{Mod}_X = \mathbf{Mod}_{\widetilde{\mathbb{Z}}X}$. Applying the functors $\pi_*(-) = H_*(\mathrm{ch}\,(-))$, $\pi_*(M \otimes_{\widetilde{\mathbb{Z}}X} -)$, and $H_*(\mathrm{Hom}_{\widetilde{\mathbb{Z}}X}(-, N))$ gives us the results. □

Corollary 5.3 ([25]). *Let X be a commutative monoid and let $\alpha : X \to e$ be the unique monoid map to the trivial monoid. There are isomorphisms*

$$\alpha_* H_n^{CM}(X; \alpha^* M) = H_n^{CA}(\mathbb{Z}X; M)$$

natural in the right $\mathbb{Z}X$-module M, and

$$\alpha_* H^n_{CM}(X; \alpha^* N) = H^n_{CA}(\mathbb{Z}X; N)$$

natural in the left $\mathbb{Z}X$-module N.

6. The Hochschild complex

This section reviews well known material (e.g. [33]) in order to establish notation. We take the opportunity to point out the intrinsic simplicity and symmetry of the Hochschild complex, hidden by standard treatments.

Let A be an associative K-algebra. There is a canonical simplicial A-bimodule $C_\bullet(A)$ over K with

$$C_n(A) = A^{\otimes(n+2)}$$

for $n \geq 0$, augmented to A, and

$$d_i = 1^{\otimes i} \otimes \mu \otimes 1^{\otimes(n-i)} : A^{\otimes(n+2)} \to A^{\otimes(n+1)}\,,\ 0 \leq i \leq n$$

$$s_i = 1^{\otimes(i+1)} \otimes \eta \otimes 1^{\otimes(n-i)} : A^{\otimes(n+1)} \to A^{\otimes(n+2)}\,,\ 0 \leq i \leq n-1$$

where $\mu : A \otimes A \to A$ is the multiplication and $\eta : K \to A$ includes the unit.

There are also maps

$$s_{-1} = \eta \otimes 1^{\otimes(n+1)}\,, \quad s_n = 1^{\otimes(n+1)} \otimes \eta : A^{\otimes(n+1)} \to A^{\otimes(n+2)}\,.$$

The first is a right A-module map and the second is a left A-module map, and they provide contracting homotopies of the simplicial object regarded as either a right or a left A-module. In fact it is just the simplicial bar resolution of A as a left or right A-module. Thus the chain complex associated to $C_\bullet(A)$, $\mathrm{ch}\, C_\bullet(A)$, is a relative projective resolution of A as an A-bimodule, the *Hochschild resolution*. If A is projective as a K-module, it's an absolute projective resolution.

Let Q_A be the functor from A-bimodules to K-modules given by

$$Q_A(M) = M/(ax - xa : a \in A, x \in M)\,,$$

where the denominator indicates the sub-K-module generated by these elements. More generally, given an A-bimodule N, define the functor Q_N^A, or Q_N if A is understood, from A-bimodules to K-modules by

$$Q_N(M) = M \otimes N/(ax \otimes y - x \otimes ya, xa \otimes y - x \otimes ay)\,.$$

We recover Q_A by regarding A as a bimodule over itself using left and right multiplication. A bimodule is the same thing as a module over $A^e = A \otimes A^{op}$, and under this equivalence

$$Q_N(M) = M \otimes_{A^e} N$$

In general this is just a K-module, but if A is commutative then $Q_A(M)$ is naturally an A-module, since A is then an (A, A^e)-bimodule.

Apply this functor to $C_\bullet(A)$ to get a simplicial object in K-modules, $Q_N C_\bullet(A)$, equipped with an augmentation to $Q_N A$. This is the *Hochschild complex* with coefficients in N, and by definition

$$Hoch_n(A; N) = H_n(\mathrm{ch}\, Q_N C_\bullet(A)) .$$

To understand this better, notice the isomorphism

$$Q_N(A \otimes V \otimes A) \to N \otimes V$$

for a K-module V, given by factoring

$$x \otimes a \otimes v \otimes b \mapsto bxa \otimes v$$

through $Q_N(A \otimes V \otimes A)$. The inverse sends $x \otimes v$ to $[x \otimes 1 \otimes v \otimes 1]$.

This isomorphism breaks symmetry. But using it we may write the augmented simplicial K-module $Q_N C_\bullet(A)$ as

$$Q_N(A) \leftarrow N \Leftarrow N \otimes A \cdots .$$

The operators in $C_\bullet(A)$ descend to operators in $Q_N C_\bullet(A)$ which under these isomorphisms are given by

$$d_i(x \otimes a_1 \otimes \cdots \otimes a_n) = \begin{cases} xa_1 \otimes a_1 \otimes \cdots \otimes a_n & i = 0 \\ x \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & 0 < i < n \\ a_n x \otimes a_1 \otimes \cdots \otimes a_{n-1} & i = n \end{cases}$$

$$s_i(x \otimes a_1 \otimes \cdots \otimes a_n) = x \otimes a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n .$$

This simplicial structure can be put in attractive form: if $\phi : [m] \to [n]$ is a simplicial operator, then (with $a_0 = x$)

$$\phi^*(a_0 \otimes a_1 \otimes \cdots \otimes a_n) =$$
$$(a_{\phi(m)+1} \cdots a_n a_0 a_1 \cdots a_{\phi(0)}) \otimes (a_{\phi(0)+1} \cdots a_{\phi(1)}) \otimes \cdots \otimes (a_{\phi(m-1)+1} \cdots a_{\phi(m)}) .$$

If A and B are two K-algebras and M and N bimodules for them, there is a natural isomorphism

$$Q_A(M) \otimes Q_B(N) \to Q_{A \otimes B}(M \otimes N)$$

under the identity map on $M \otimes N$. The fact that it is an isomorphism follows from the identity

$$(a \otimes b)(m \otimes n) - (m \otimes n)(a \otimes b) = am \otimes (bn - nb) + (am - ma) \otimes nb$$

We get natural isomorphisms of simplicial objects

$$C_\bullet(A) \otimes C_\bullet(B) \to C_\bullet(A \otimes B)$$
$$Q_A C_\bullet(A) \otimes Q_B C_\bullet(B) \to Q_{A\otimes B} C_\bullet(A \otimes B)$$

If A is commutative we may take $A = B$ and compose with the K-algebra map $\mu : A \otimes A \to A$ to obtain a simplicial commutative A-algebra structures on $Q_A C_\bullet(A)$, and $Q_N C_\bullet(A)$ becomes a module over $Q_A C_\bullet(A)$.

Passing to associated chain complexes, the Eilenberg-Zilber or shuffle map ([9, p. 64] or [28, p. 39]) results in the structure of a commutative (in the signed sense) differential graded A-algebra structure on $\operatorname{ch} Q_A C_\bullet(A)$ and hence a graded commutative A-algebra structure on its homology $Hoch_\bullet(A)$.

Dually, denote by $R_N(M)$ the K-module of A-bimodule maps from M to N. The Hochschild cochain complex with coefficients in N is then $\operatorname{ch} R_N C_\bullet(A)$, and its homology is the Hochschild cohomology $Hoch_\bullet(A; N)$. It is well known and easy to verify that

$$Hoch_1(A) = \Omega_{A/K} = HQ_0^{Com_K}(A)$$

and

$$Hoch^1(A; M) = \operatorname{Der}_K(A; M) = HQ^0_{Com_K}(A; M)\,.$$

This entire discussion goes through without change in the presence of a grading by a commutative monoid.

7. Hochschild and Quillen

Proposition 7.1. *There are maps*

$$Hoch_n(A) \to HQ_{n-1}(A)$$

of A-modules natural on the category of commutative K-algebras A such that $\operatorname{Tor}_q^K(A, A) = 0$ *for $q > 0$, and maps*

$$Hoch_n(A; M) \to HQ_{n-1}(A; M)$$
$$Hoch^n(A; M) \leftarrow HQ^{n-1}(A; M)$$

of K-modules natural in A and the A-module M.

Proof. This is best seen as arising from comparing different varieties of K-algebras. The category $\mathbf{Mod}_A^{Alg}$ of Beck modules over A in the category $\mathbf{Alg}_K$ of associative K-algebras are precisely A-bimodules over K. The abelianization of A as an algebra over itself is [31, §4] the A-bimodule

$$\Omega^{Alg}_{A/K} = \ker(\mu : A \otimes A \to A)\,,$$

and the functor $f_* : \mathbf{Mod}_A^{Alg} \to \mathbf{Mod}_B^{Alg}$ left adjoint to the pullback functor induced by $f : A \to B$ is given by

$$M \mapsto A^e \otimes_{B^e} M\,.$$

so

$$\mathrm{Ab}_B A = f_* \Omega^{Alg}_{B/K} = A^e \otimes_{B^e} \Omega^{Alg}_{B/K}\,.$$

It is well known, and follows from general principles, that if A is in fact a commutative K-algebra,

$$\Omega^{CA}_{A/K} = Q_A \Omega^{Alg}_{A/K} = A \otimes_{A^e} \Omega^{Alg}_{A/K} .$$

The Quillen homology of $A \in \mathbf{Alg}_K$ is obtained by choosing a cofibrant replacement $Y_\bullet \to A$ in the Quillen model structure on the category $s\mathbf{Alg}_K$ of simplicial associative K-algebras and forming the homotopy of the "associative cotangent complex"

$$\mathbf{L}^{Alg}_{A/K} = A^e \otimes_{Y^e_\bullet} \Omega^{Alg}_{Y_\bullet/K}$$

If A is commutative, this can be compared with the commutative K-algebra cotangent complex

$$\mathbf{L}^{ComAlg}_{A/K} = A^e \otimes_{X_\bullet} \Omega^{ComAlg}_{X_\bullet/K}$$

constructed from a cofibrant replacement $X_\bullet \to A$ in $\mathbf{ComAlg}_K$. A map in $s\mathbf{ComAlg}_K$ is an acyclic fibration if and only if it is an acyclic fibration in $s\mathbf{Alg}_K$: in both cases it is simply the a surjective weak equivalence of simplicial K-modules. So the lifting axiom in the model category structure on $s\mathbf{Alg}_K$ provides the dotted arrow in

$$\begin{array}{ccc} K & \longrightarrow & X_\bullet \\ \downarrow & & \downarrow \\ Y_\bullet & \longrightarrow & A \end{array}$$

From this we obtain a map

$$\mathbf{L}^{Alg}_{A/K} = A^e \otimes_{Y^e_\bullet} \Omega^{Alg}_{Y_\bullet} \to A^e \otimes_{X^e_\bullet} \Omega^{Alg}_{X_\bullet/K}$$

Now the commutative diagram

$$\begin{array}{ccc} X^e_\bullet & \longrightarrow & X_\bullet \\ \downarrow & & \downarrow \\ A^e & \longrightarrow & A \end{array}$$

provides the continuation

$$\cdots \to A \otimes_{X^e_\bullet} \Omega^{Alg}_{X_\bullet/K} = A \otimes_{X_\bullet} (X_\bullet \otimes_{X^e_\bullet} \Omega^{Alg}_{X_\bullet/K}) = A \otimes_{X_\bullet} \Omega^{CA}_{X_\bullet/K} = \mathbf{L}^{CA}_{A/K} .$$

Passing to homotopy gives a map natural in the commutative K-algebra A

$$HQ^{Alg}_\bullet(A) \to HQ^{CA}_\bullet(A) .$$

We can also first tensor with an A-module M, or map into one, and obtain maps

$$\mathbf{L}^{Alg}_{A/K} \otimes_A M \to \mathbf{L}^{CA}_{A/K} \otimes_A M$$

$$\mathrm{Hom}_A(\mathbf{L}^{Alg}_{A/K}, M) \leftarrow \mathrm{Hom}_A(\mathbf{L}^{CA}_{A/K}, M)$$

inducing

$$HQ_n^{Alg}(A;M) \to HQ_n^{CA}(A;M)$$
$$HQ^n_{Alg}(A;M) \leftarrow HQ^n_{CA}(A;M)$$

On the other hand, Quillen proved [31] that there are isomorphisms

$$Hoch_n(A) \cong HQ^{Ass}_{n-1}(A)$$
$$Hoch_n(A;M) \cong HQ^{Ass}_{n-1}(A;M)$$
$$Hoch^n(A;M) \cong HQ^{n-1}_{Ass}(A;M)$$

natural in K-modules such that $\mathrm{Tor}^A_q(K,K) = 0$ for $q > 0$. □

This entire discussion goes through without change in the presence of a grading.

8. Harrison homology

Let A be an X-graded commutative K-algebra. Then $Q_A A = A$; the Hochschild complex $\mathrm{ch}\, Q_A C_\bullet(A)$ is augmented to A. Let $I_\bullet(A)$ denote the kernel of this augmentation; this is the ideal of positive-dimensional elements in the commutative differential graded A-algebra $\mathrm{ch}\, Q_A C_\bullet(A)$. The *Harrison complex* [23] is the differential graded module of indecomposables in $\mathrm{ch}\, Q_A C_\bullet(A)$, $I_\bullet(A)/I_\bullet(A)^2$. The *Harrison homology* of A is the homology of this DG A-module:

$$Harr_n(A) = H_n(I_\bullet(A)/I_\bullet(A)^2)\,.$$

We can equip it with coefficients in an A-module M:

$$Harr_n(A;M) = H_n((I_\bullet(A)/I_\bullet(A)^2) \otimes_A M)\,.$$

The Harrison cohomology with coefficients in an A-module M is

$$Harr^n(A;M) = H^n(\mathrm{Hom}_A(I_\bullet(A)/I_\bullet(A)^2, M))sAs\,.$$

Clearly

$$Harr_0(A) = 0 \quad \text{and} \quad Harr_1(A) = Hoch_1(A)$$

The shuffle product defines a commutative graded K-algebra structure on the $\mathbb{N}$-graded K-module $\overline{C}_\bullet(A)$ with

$$\overline{C}_n(A) = A^{\otimes n}\,.$$

As graded A-modules,

$$\mathrm{ch}\, C_\bullet(A) \otimes_{A^e} A = A \otimes \overline{C}_\bullet(A)\,.$$

Only the differential depends on the algebra structure, and it is not the A-linear extension of a differential on $\overline{C}_\bullet(A)$.

The Harrison cochains can be re-expressed in terms of the graded K-algebra $\overline{C}_\bullet(A)$. Let $\overline{I}_\bullet(A)$ be its augmentation ideal; then

$$I_\bullet = A \otimes \overline{I}_\bullet\,, \quad I^2_\bullet = A \otimes \overline{I}^2_\bullet\,,$$

and so

$$I_\bullet(A)/I_\bullet(A)^2 = A \otimes (\overline{I}_\bullet(A)/\overline{I}_\bullet(A)^2)\,.$$

A Harrison n-cochain (for $n > 0$) with coefficients in M is thus a K-linear map

$$s : A^{\otimes n} \to M$$

that annihilates decomposables. This may be phrased as a symmetry condition on the cochain: given i, j, both positive and summing to n, let $\Sigma(i, j)$ be the set (i, j)-shuffles; that is, the set of elements of Σ_n that preserve the order of $\{1, \ldots, i\}$ and of $\{i+1, \ldots, n\}$. The symmetry condition (i, j) on a Hochschild cochain $s : A^{\otimes n} \to M$ is

$$\sum_{\sigma\in\Sigma(i,j)} \mathrm{sgn}(\sigma) s\circ\sigma = 0\,.$$

Since the shuffle product is commutative, we may assume that $i \leq j$; there are $\lfloor n/2\rfloor$ independent conditions.

An alternative symmetry condition (apparently the one originally conceived of by Harrison; the shuffle description is said to be due to Mac Lane) is described in [12]. Think of an element of Σ_n as an ordering of $\{1, 2, \ldots, n\}$. Let $1 \leq k \leq n$. An element $\sigma \in \Sigma_n$ is a *k-monotone* permutation if the lead element is k, the numbers $1, 2, \ldots, k$ occur in decreasing order, and the numbers $k + 1, \ldots, n$ occur in increasing order. There are $\binom{n-1}{k-1}$ of them. For example the only n-monotone permutation in Σ_n corresponds to the sequence $n, n-1, \cdots, 2, 1$, and the 4-monotone permutations in Σ_6 are

432156, 432516, 432561, 435216, 435261, 435621, 453216, 453261, 453621, 456321

Let $M_k(n)$ be the set of k-monotone permutations in Σ_n. Let $dr(\sigma)$ be the sum of the positions occupied by $1, 2, \ldots, k-1$ in the permutation $\sigma \in M_k(n)$.

Lemma 8.1 ([12], Theorem 4.1)**.** *A map $s : A^{\otimes n} \to M$ is a Harrison cochain if and only if*

$$s = \sum_{\sigma\in M_k(n)} (-1)^{dr(\sigma)} s\circ\sigma\,, \quad 2 \leq k \leq n\,.$$

So for example a Hochschild 4-cochain s is a Harrison cochain if and only if

$$\begin{aligned} s(a_1, a_2, a_3, a_4) &= s(a_2, a_1, a_3, a_4) - s(a_2, a_3, a_1, a_4) + s(a_2, a_3, a_4, a_1) \\ &= -s(a_3, a_2, a_1, a_4) + s(a_3, a_2, a_4, a_1) - s(a_3, a_4, a_2, a_1) \\ &= -s(a_4, a_3, a_2, a_1) \end{aligned}$$

The 4-monotone and 2-monotone symmetries combine to give

$$s(a_4, a_3, a_2, a_1) = -s(a_2, a_1, a_3, a_4) + s(a_2, a_3, a_1, a_4) - s(a_2, a_3, a_4, a_1)$$

which is the same as the 3-monotone condition (after rearranging the labels); so we can dispense with either one of the first two conditions in this list. The same argument shows that one need only assume the n-monotone condition together with one condition from each pair $\{2, n-1\}, \{3, n-2\}, \ldots$: so $\lfloor n/2 \rfloor$ conditions suffice. This matches with the number of independent shuffle conditions.

9. Barr homology

Michael Barr suggested possible variations on Harrison's symmetry conditions, in an attempt to come closer to Quillen homology. As explained by Sarah Whitehouse [34], these variations still fail, though they may give a better approximation. Gerstenhaber and Schack [12, Remark, p. 232] (see also [34]) suggest that one of Barr's ideas was to divide the Hochschild complex not just by shuffle decomposables but by the full divided power structure as well. While a divided power structure on the even homotopy groups of a simplicial commutative algebra was implicit in the works of Eilenberg and Mac Lane [9] and Henri Cartan [7, Exp. 8], its construction on the level of the Hochschild complex was at best a folk result at the time of Barr's question, and even at when Gerstenhaber and Schack were writing. It seems to have first been set out, in the associated chain complex of a simplicial commutative algebra, by Siegfried Brüderle and Ernst Kunz [6] in 1994; see also [32] and [13].

From these sources one obtains the following formula for the divided power structure on $\overline{C}_{even}(A)$, where A is a commutative K-algebra. Let $S_k(kn)$ be the set of shuffles associated to the partition of $\{1, 2, \ldots, kn\}$ into k intervals. Let $S'_k(kn)$ be the subset of these such that the leading terms of the k sequences occur in order. Then, for n even,

$$\gamma_k[a_1|\cdots|a_n] = \sum_{\sigma \in S'_k(kn)} \operatorname{sgn}(\sigma)[a_1|\cdots|a_n|a_1|\cdots|a_n|\cdots|a_1|\cdots|a_n] \circ \sigma$$

where the sequence $a_1|\cdots|a_n$ is repeated k times. For example

$$\gamma_2[a|b] = [a|b|a|b]\,, \quad \gamma_3[a|b] = [a|b|a|b|a|b]\,,$$
$$\gamma_2[a|b|c|d] = [a|b|c|d|a|b|c|d] - [a|b|c|a|d|b|c|d] + [a|b|c|a|b|d|c|d]$$
$$+[a|b|a|c|d|b|c|d] - [a|b|a|c|b|d|c|d] + 2[a|b|a|b|c|d|c|d]\,.$$

We can put at least one restriction on the tensors occurring in the expression for the divided powers. To express it, notice that there is a universal Hochschild n-chain, $[a_1|\cdots|a_n] \in K[a_1, \ldots, a_n]^{\otimes n}$.

Lemma 9.1. *No decomposable tensor with entries chosen from $\{a_1, \ldots, a_n\}$ occurring with nonzero coefficient in $\gamma_k[a_1|\cdots|a_n]$ has consecutive occurrences of any a_i.*

Proof. We show how such terms cancel in pairs in the expression for the divided power, by defining a free involution on the set of terms with neighboring repeated letters with the property that the elements of each orbit occur with opposite signs. The involution will leave unchanged all the letters up to and including the left-most neighboring repeated pair.

If the repeated pair is $a_1|a_1$, swap the positions of the remaining letters in the two blocks initiated by these letters. We get an identical word, but since n is even this is an odd number of transpositions, so the terms cancel.

If the repeated pair is $a_i|a_i$ for $i > 1$, just swap those two entries. This is allowed since the leading term of both blocks precedes both entries in the repeated pair. □

Since every term in the expression for $\gamma_k[a_1|\cdots|a_n]$ has leading entry a_1, we obtain:

Corollary 9.2. $\gamma_k[a|b] = [a|b|a|b|\cdots|a|b]$.

In general the expression for γ_k seems very complicated. For example, a computer calculation shows that $\gamma_3[a|b|c|d]$ has 53 terms, with coefficients ranging from -4 to 6.

Write $Barr_*(A)$ for the homology of the resulting DG A-module. Since $k!\gamma_k(\omega)$ is decomposable, we have an exact sequence

$$0 \to Harr_{2k+1}(A) \to Barr_{2k+1}(A) \to T_{2k} \to Harr_{2k}(A) \to Barr_{2k}(A) \to 0$$

where $k!T_{2k} = 0$. Thus

$$Harr_{2k+1}(A) \to Barr_{2k+1}(A) \text{ is injective with cokernel killed by } k!\,,$$
$$Harr_{2k}(A) \to Barr_{2k}(A) \text{ is surjective with kernel killed by } k!\,.$$

We can also form the "Barr cohomology" with coefficients in an A-module. Its cochains consist of Hochschild cochains satisfying the Harrison symmetry conditions with the additional conditions

$$s(\gamma_k(\omega)) = 0\,, \quad k > 0\,,$$

for $|\omega|$ even; for example $s(a,b,a,b) = 0$ in dimension 4; $s(a,b,a,b,a,b) = 0$ in dimension 6; and in dimension 8 there are two additional symmetries,

$$s(a,b,a,b,a,b,a,b) = 0$$

guaranteeing annihilation of $\gamma_4[a|b]$, and

$$s(a,b,c,d,a,b,c,d) - s(a,b,c,a,d,b,c,d) + s(a,b,c,a,b,d,c,d)$$
$$+s(a,b,a,c,d,b,c,d) - s(a,b,a,c,b,d,c,d) + 2s(a,b,a,b,c,d,c,d) = 0$$

to annihilate $\gamma_2[a|b|c|d]$.

Remark. We would conjecture that the natural map of A-modules $Hoch_n(A) \to HQ_{n-1}(A)$ factors as

$$Hoch_n(A) \to Harr_n(A) \to Barr_n(A) \to HQ_{n-1}(A)\,.$$

10. Grillet's work

In a series of papers, Pierre Grillet associates to a commutative monoid X and a Beck module M over it the beginning of a cochain complex and proves or conjectures that it computes the low-dimensional components of the Quillen cohomology $HQ^*(X;M)$. We observe that the symmetry conditions he imposes are precisely the monontone conditions, with two variations which correspond to Barr's variation on the Harrison complex. We will not attempt a complete survey of Grillet's work on this subject, but merely note the occurrence of symmetry conditions that we now see as Harrison or Barr symmetry conditions on Hochschild cochains, and where they are proved to yield cohomology groups isomorphic to Quillen's.

To begin with, for any X graded K-algebra and A-module M,

$$Harr^1(A;M) \cong Hoch^1(A;M) \cong HQ^0_{Com_K}(A;M) \cong Der_K(A,M)$$

so

$$Harr^1(\widetilde{\mathbb{Z}}X;M) \cong HQ^0(X;M)\,.$$

In 1974 [15] Grillet used 2-cocyles s with the symmetry

$$s(a,b) = s(b,a)$$

to classify extensions of commutative monoids. This is of course the 2-monotone symmetry. Twenty years later, in [16], he returned to this by invoking Quillen cohomology as an intermediary, thus showing that

$$Harr^2(\widetilde{\mathbb{Z}}X;M) \cong HQ^1(X;M)\,.$$

(Grillet chooses to index Quillen homology following the Hochschild convention, so he would write $HQ^2(X;M)$.)

This paper was supplemented by [17], which confirmed this result by direct computation and extended it to dimension 3 using the symmetry conditions

$$s(a,b,c) + s(c,b,a) = 0$$
$$s(a,b,c) + s(b,c,a) + s(c,a,b) = 0\,.$$

Taken together these are equivalent to the k-monotone symmetries for $k=2$ and 3. A remarkable calculation then verifies that

$$Harr^3(\widetilde{\mathbb{Z}}X;M) \cong HQ^2(X;M)$$

This work was consolidated and summarized in his book [18]

After another twenty years, Grillet returned again to this project, in [19], extending his calculation to dimension 4 using cochains satisfying the symmetry conditions simplified in [20] to

$$s(a,b,c,d) - s(b,a,c,d) + s(b,c,a,d) - s(b,d,d,a) = 0$$
$$s(a,b,c,d) + s(d,c,b,a) = 0$$
$$s(a,b,b,a) = 0$$

The reader will recognize the first two as the 2-monotone and 4-monotone symmetries. The first equation implies $s(a,b,a,b) = s(b,a,a,b)$, so the third condition is equivalent to the Barr variant $s(a,b,a,b) = 0$. And here again, Grillet obtains the surprising result

$$Barr^4(\widetilde{\mathbb{Z}}X; M) \cong HQ^3(X; M)\,.$$

This work was quickly followed by [21], in which Grillet proposes symmetric conditions on Hochschild cocycles extending into dimensions 5 and 6. In dimension 5 he proposes

$$s(a,b,c,d,e) - s(b,a,c,d,e) + s(b,c,a,d,e) - s(b,c,d,a,e) + s(b,c,d,e,a) = 0$$
$$s(a,b,c,d,e) + s(e,d,c,b,a) = 0\,.$$

We recognize these as the 2-monotone and 5-monotone conditions, which suffice to determine Harrison cohomology. In dimension 6 his proposed symmetries are precisely the k-monotone conditions for $k = 2, 3$, and 6, augmented by the Barr variant $s(a,b,a,b,a,b) = 0$.

The identifications with Quillen cohomology are left as conjectures. We can now see that at least the 5-dimensional case was too optimistic. Write α for the unique map of commutative monoids $\mathbb{N} \to e$. Then

$$Barr^\bullet(\widetilde{\mathbb{Z}}\mathbb{N}; \alpha^*\mathbb{F}_p) = Barr^\bullet(\mathbb{Z}[x]; \mathbb{F}_p) = Barr^\bullet(\mathbb{F}_p[x]; \mathbb{F}_p)\,.$$

To see the last equality,write $A = \mathbb{Z}[x]$ and notice that a Hochschild cochain $C_n(A) \to \mathbb{F}_p$ factors through the divided power indecomposables of $A^{\otimes n} \to A^{\otimes n}/p = (A/p)^{\otimes n}$. The augmentation ideals of the second and third terms here coincide, and the map on divided power decomposables is surjective. This is enough to conclude that the map on quotients by those decomposables is an isomorphism.

In her thesis, Sarah Whitehouse [34] showed that $Barr^5(\mathbb{F}_2[x]; \mathbb{F}_2) \neq 0$. This conflicts with the value of the corresponding André-Quillen cohomology: $HQ^4(\mathbb{Z}[x]/\mathbb{Z}; \mathbb{F}_2) = 0$ since $\mathbb{Z}[x]$ is a free commutative $\mathbb{Z}$-algebra.

Grillet's calculations appear to make extensive use of the special features of $\widetilde{\mathbb{Z}}X$ as an X-graded commutative ring, so it may well be that Harrison cohomology fails to coincide with Quillen cohomology in the dimensions he studies for more general graded commutative K-algebras.

On the other hand, the evident graded extension of Barr's theorem shows this:

Theorem 10.1. *For any commutative monoid X*

$$HQ_{*-1}(X) \otimes \mathbb{Q} \cong Harr_*(\widetilde{\mathbb{Z}}X) \otimes \mathbb{Q}$$

as Beck modules. For any Beck X-module N,

$$HQ^{*-1}(X; N) \otimes \mathbb{Q} \cong Harr^*(\widetilde{\mathbb{Z}}X; N) \otimes \mathbb{Q}\,.$$

*For any right $\widetilde{\mathbb{Z}}X$-module M – any contravariant functor $L_X \to$ **Ab** –*

$$HQ_{*-1}(X; M) \otimes \mathbb{Q} \cong Harr_*(\widetilde{\mathbb{Z}}X; M) \otimes \mathbb{Q}\,.$$

Barr [2] and Fleury [10] assume that their commutative ring is a $\mathbb{Q}$-algebra, but since $\mathbb{Q}$ is flat over $\mathbb{Z}$ we may push it inside the computation of homology. In the case of Quillen homology, this is "flat descent" ([31, Theorem 5.3], [33, 8.8.4]).

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