HARRISON HOMOLOGY AND THE QUILLEN COHOMOLOGY OF COMMUTATIVE MONOIDS

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Abstract. We observe that Beck modules for a commutative monoid are exactly modules over a graded commutative ring associated to the monoid. Under this identification, the Quillen cohomology of commutative monoids is a special case of André-Quillen cohomology for graded commutative rings, generalizing a result of Kurdiani and Pirashvili. To verify this we develop the necessary grading formalism. The partial cochain complex developed by Pierre Grillet for computing Quillen cohomology appears as the start of a modification of the Harrison cochain complex suggested by Michael Barr.

In his book *Homotopical Algebra* [35], Daniel Quillen described a homotopy theory of simplicial objects in any of a wide class of universal algebras, and corresponding theories of homology and cohomology. Quillen homology is defined as derived functors of an abelianization functor, and in many cases can be computed using a cotriple resolution [4]. Coefficients for these theories are "Beck modules," that is, abelian objects in a slice category. The case of commutative rings was studied at the same time by Michel André [1].

An example of such an algebraic theory, one of long standing and increasing importance, is provided by the category **ComMon** of commutative monoids. The prime exponent of the study of commutative monoids has for years been Pierre Grillet [18, 19, 20, 21, 24, 25] (but see also [10] and [29] for example). Among other things, Grillet provided the beginning of a small cochain complex, based on multilinear maps subject to certain symmetry conditions, whose cohomology he showed to be isomorphic in low dimensions to the Quillen cohomology $H^*_{CM}(X;M)$ of the commutative monoid X with coefficients in a Beck module M for X; and in [23] a corresponding resolution in Beck modules was developed. These results are surprising, since Quillen cohomology is defined by means of a simplicial resolution and does not generally admit such an efficient computation.

It is well-known ([30, 18] and [39, p. 29]) and easy to see that the category of (left) Beck modules for X, $\mathbf{LMod}_X$, is equivalent to the category of covariant functors from the "Leech category" L_X to the category $\mathbf{Ab}$ of abelian groups. The Leech category has object set X; $L_X(x,y) = \{z : y =$

2010 *Mathematics Subject Classification.* 20M14, 13D03.
Key words and phrases. commutative monoid, Harrison homology, Quillen cohomology.

$z + x\}$ (writing the monoid structure additively); and composition is given by addition in the commutative monoid.

In this note, we observe that Grillet's construction is in fact subsumed by the theory of Harrison cohomology of commutative rings, slightly augmented as suggested by Michael Barr, once this theory has been extended to the graded context. As pointed out by Bourbaki [7, Ch. 2 §11], one can speak of rings graded by a commutative monoid: an X-graded object in a category $\mathbf{C}$ is an assignment of an object $C_x \in \mathbf{C}$ for each $x \in X$. If $\mathbf{C}$ is the category $\mathbf{Mod}_K$ of modules over some commutative ring K, there is a natural symmetric monoidal structure on the category $\mathbf{Mod}_K^X$ of X-graded K-modules, and we may define X-graded K-algebras, and modules over them, accordingly.

The first observation, simple enough as to need no proof, is that there is a natural X-graded commutative K-algebra $\widetilde{K}X$ in which, for each $x \in X$, $(\widetilde{K}X)_x$ is the free K-module generated by an element we will write 1_x, with the evident unit and multiplication. This is the "X-graded monoid K-algebra" of X.

The next observation, equally simple, is that the category of Beck X-modules is equivalent to the category of (X-graded) left modules over $\widetilde{\mathbb{Z}}X$.

These two observations bring into play the entire highly developed homological theory of commutative rings. Our first main result 5.2 is that

$$HQ^*_{CM}(X; M) = HQ^*_{CA}(\widetilde{\mathbb{Z}}X; M)$$

where the right hand term denotes the well-studied André-Quillen cohomology [36, 1, 38], extended to the graded context. This generalizes an observation of Kurdiani and Pirashvili [29], who considered the case of Beck modules pulled back from the trivial monoid, in which case one arrives at the André-Quillen cohomology of $\mathbb{Z}X$ as an ungraded commutative ring.

André-Quillen cohomology is of course hard to compute, but there are well known approximations to it. One such approximation is given by Harrison cohomology [26, 15, 2, 40] $Harr^*(A; M)$. This theory is most neatly expressed by restricting to Hochschild cochains that annihilate shuffle decomposables; or, equivalently, to cochains that satisfy appropriate "partition" symmetry conditions. This characterization was apparently suggested by Mac Lane, and was adopted in [26], but Harrison's original invariance property involved a different characterization of the same symmetry conditions, using "monotone" permutations. The equivalence of these two definitions can be found as Corollary 4.2 in [15].

This approximation definitely breaks down in finite characteristic: The André-Quillen cohomology of a polynomial algebra vanishes in positive dimensions, but Michael Barr showed [2] that the Harrison cohomology of the polynomial algebra over a field of characteristic p is nonzero in dimension $2p$. Barr himself proposed a variant of the Harrison construction, restricting Hochschild cochains that not only by vanish on shuffle decomposables but also on divided powers. This overcomes the obstacle in dimension $2p$, but in her thesis [40] Sarah Whitehouse proved that this variant also fails to give André-Quillen cohomology, by showing that Barr cohomology in dimension 5 does not vanish on $\mathbb{F}_2[x]$.

Our second observation is that exactly the same monotone symmetry and divided-power annihilation conditions occur in the partial complex described by Grillet; Grillet's partial complex is precisely the beginning of the "Barr complex" for the graded monoid algebra. In later work [24], Grillet proposed that this complex correctly computes Quillen cohomology in higher dimensions as well, but Whitehouse's counterexample shows that this conjecture fails at least in dimension 5.

Grillet's identification of his cohomology with the Quillen cohomology of commutative monoids goes well beyond what seems to be known about the relationship between Harrison cohomology and André-Quillen cohomology in general, and suggests a variety of questions.

We note that the observation that Beck modules over a commutative monoid are just graded modules over its graded monoid algebra suggests that the description of Quillen homology for commutative monoids carried out in [29] is in fact a special case of a graded extension of Pirashvili's earlier work [34].

We begin in §1 with a recollection of Quillen homology, along with the cotriple resolution that may be used to compute it. §2 sets out some elementary but sometimes surprising facts about X-gradings, and in §3 we explain how the grading behaves in homological algebra. Change of grading is explained in §4. The next section is the core of the work, proving the identifications of Quillen homology for commutative monoids and certain X-graded commutative rings. We then turn to interprenting the work of Grillet. This requires developing the Hochschild complex with its shuffle product and divided power structure (and we provide some new general information about the latter), and the various indecomposable quotients occuring in the definitions of Harrison and Barr homology. In §9, we review the motivating work of Pierre Grillet and relate it to Harrison and Barr cohomology.

Acknowledgements. We are grateful to Pierre Grillet for forwarding us, in response to a letter from us outlining the results presented here, an early copy of a paper in which a similar story is worked out. He uses somewhat different language – his "multi" objects are our graded objects – but he did not make the connection with Harrison homology that we establish here. We also acknowledge the encouragement of the referee to be more explicit about basic homotopical algebra, and for giving us the opportunity to make other improvements in the paper.

This work was carried out under the auspices of a program, supported by MIT's Jameel World Education Laboratory, designed to foster collaborative research projects involving students from MIT and Palestinian universities. We acknowledge with thanks the contributions made by early particpants in this program – Mohammad Damaj and Ali Tahboub of Birzeit University and Hadeel AbuTabeekh of An-Najah National University – as well as the support of Palestinian faculty – Reema Sbeih and Mohammad Saleh at Birzeit and Khalid Adarbeh and Muath Karaki at NNU. We thank Professor Victor Kac for pointing out to us the relevance of [15]. The first author acknowledges support by the MIT UROP office.

1. Quillen homology

In [36], Daniel Quillen proposed a uniform definition of the "homology" and "cohomology" of objects in a very general class of categories. The marquis example was that of commutative algebras, but his definition applies much more generally and subsumes many of the ad hoc definitions that were already in use at the time and have been considered subsequently.

Quillen proposed that the construction of "homology" should be a special case of a general procedure for deriving a functor. One of the motivations for his development of the theory of "model categories" [35, 28, 27, 17] was precisely to provide a context for defining derived functors of non-additive functors. This theory "internalizes" homological algebra, in the sense that objects playing the role of projective resolutions (called "cofibrant objects") and maps playing the role of quasi-isomorphisms (called "weak equivalences") exist in the category (rather than in some auxiliary category such as a category of chain complexes). One of the axioms asserts that an object admits a weak equivalence from a cofibrant object; this "cofibrant replacement" plays the role of a projective resolution.

In [35, II§4], Quillen establishes the existence of a model structure on the category of simplicial objects over any one of a very general class of categories with suitable mild properties. We refer to Quillen's book or [14] for the definitions.

Theorem 1.1. [35, II§4, Theorem 4] *Let* $\mathbf{C}$ *be any cocomplete category admitting a set* $\mathcal{P}$ *of small projective generators (a "quasi-algebraic category" in the language of* [14]*). Then there is a model structure on the category of simplicial objects over* $\mathbf{C}$ *in which the weak equivalences are the morphisms* f *such that* $\mathbf{C}(P, f)$ *is a weak equivalence of simplicial sets for all* $P \in \mathcal{P}$*.*

All normally occurring categories of universal algebras satisfy these axioms.

This model structure on $s\mathbf{C}$ allows us to define derived functors for any functor $E : \mathbf{C} \to \mathbf{A}$, where $\mathbf{A}$ is an abelian category: For any X in $\mathbf{C}$, let $P_\bullet \to X$ be a cofibrant replacement and define

$$L_n E(X) = \pi_n(EP_\bullet)\,.$$

See [14] for an elaboration of the naturality of this construction. The fact that any cofibrant replacement can be used is contained, for example, in [14, Proposition 3.9].

Explicit cofibrant replacements can often be constructed as a "cotriple resolution" [4]. An adjoint pair

$$\mathbf{C} \rightleftarrows \mathbf{D}$$

defines a triple F on $\mathbf{C}$ and a cotriple G on $\mathbf{D}$; see [12, 5]. For example the free commutative monoid functor $\mathbb{N}$ is left adjoint to the forgetful functor –

$$\mathbb{N} : \mathbf{Set} \rightleftarrows \mathbf{ComMon} : u$$

– and this adjoint pair defines triple $u\mathbb{N}$ on $\mathbf{Set}$ and a cotriple $\mathbb{N}u$ on $\mathbf{ComMon}$. The natural numbers (also denoted by $\mathbb{N}$) is a small projective generator for $\mathbf{ComMon}$, and $u(S) = \mathbf{ComMon}(\mathbb{N}, S)$.

A cotriple G on $\mathbf{C}$ determines a functor $G_\bullet$ to the category $s\mathbf{C}$ of simplicial objects over $\mathbf{C}$, augmented to the identity functor: this is the "cotriple resolution": see [4] or [38, Chapter 8]. To relate it to the model category structure we need a further restriction: A category $\mathbf{C}$ is *algebraic* (in Frankland's sense) if it is quasi-algebraic and Barr-exact ([3, p. 35],[14, p. 91]).

Proposition 1.2. [36, p. 69] *Let* $\mathbf{C}$ *be an algebraic category and* $\mathcal{P}$ *a set of small projective generators. When the cotriple* G *is associated to the adjoint pair* $\mathbf{C} \rightleftarrows \mathbf{Set}^{\mathcal{P}}$*, the cotriple resolution* $G_\bullet A \to A$ *serves as a cofibrant replacement of* A *(regarded as a constant simplicial object) in the Quillen model structure on* $s\mathbf{C}$*.*

This allows one to calculate the derived functors for any functor $E : \mathbf{C} \to \mathbf{A}$ to an abelian category:

$$L_n E(C) = \pi_n(E(G_\bullet C)) = H_n(\operatorname{ch} E(G_\bullet C))$$

where ch denote formation of the chain complex associated to a simplicial object in an abelian category; see [38, Theorem 8.4.1], for example.

Quillen's definition [36] of homology and cohomology of an object in a category $\mathbf{C}$ involves the notion of Beck modules.

Definition 1.3. [5, Definition 5] *A* Beck module *over an object* A *in* $\mathbf{C}$ *is an abelian object in the slice category* $\mathbf{C}/A$*.*

With the evident morphisms, Beck A-modules form a category $\mathbf{LMod}_A$. The terminal object in $\mathbf{LMod}_A$ is the identity map $\mathrm{id}_A : A \downarrow A$ with its unique abelian structure. If $\mathbf{C}$ is quasi-algebraic, so is $\mathrm{Ab}(\mathbf{C}/A)$ [14, 3.40]. Under mild additional conditions $\mathrm{Ab}(\mathbf{C}/A)$ is an abelian category:

Proposition 1.4. [36, p. 69] [1], [3, Chapter 2, Theorem 2.4] *Let* $\mathcal{C}$ *be an algebraic category and* $A \in \mathcal{C}$*. Then both* $\mathcal{C}/A$ *and* $\mathrm{Ab}(\mathcal{C}/A)$ *are algebraic;* $\mathrm{Ab}(\mathcal{C}/A)$ *is abelian; and the forgetful functor* $\mathrm{Ab}(\mathcal{C}/A) \to \mathcal{C}/A$ *has a left adjoint* $\mathrm{Ab}_A : \mathcal{C}/A \to \mathrm{Ab}(\mathcal{C}/A)$*.*

We are now in position to recall the following definition.

[1]But Quillen inadvertantly omits the exactness condition.

Definition 1.5. [35] *Let* $\mathbf{C}$ *be an algebraic category. The* Quillen homology *of an object* A *in* $\mathbf{C}$ *is the sequence of Beck* A*-modules given by*

$$HQ_n(A) = L_n\mathrm{Ab}_A(\mathrm{id}_A) = \pi_n(\mathrm{Ab}_A(P_\bullet))$$

where $P_\bullet \to A$ *is a cofibrant replacement regarded as an object in* $s\mathbf{C}/A$*. The* Quillen cohomology *of* A *with coefficients in a Beck* A*-module is the sequence of abelian groups*

$$HQ^n(A; M) = H^n(\mathrm{Hom}_A(\mathrm{ch}\, P_\bullet, M)) .$$

In terms of the cotriple resolution,

$$HQ_n(A) = \pi_n(\mathrm{Ab}_A(G_\bullet A))$$
$$HQ^n(A; M) = H^n(\mathrm{Hom}_A(\mathrm{Ab}_A(G_\bullet A), M)) .$$

For example [36, §4] when $\mathbf{C}$ is the category $\mathbf{ComAlg}_K$ of commutative K-algebras, the category of Beck A-modules is equivalent to the category of left A-modules: An abelian object over A, $p : B \downarrow A$, first of all has a section, the "zero-section," which provides an identification of K-modules $B \cong A \oplus M$ where M is the kernel of p as an A-module. The abelian structure forces the multiplication on $A\oplus M$ to be given by $(a, m)(b, n) = (ab, an+bm)$. Under this identification, a section of $A \oplus M \downarrow A$ in $\mathbf{ComAlg}_K$ is given by $a \mapsto (a, da)$ where $d \in \mathrm{Der}_K(A, M)$. The abelianization of $\mathrm{id} : A \downarrow A$ is the A-module such that $\mathrm{Hom}_A(\mathrm{Ab}_A(A), M) = \mathrm{Der}_K(A, M)$; that is, $\mathrm{Ab}_A(A)$ is the usual module $\Omega_{A/K}$ of Kähler differentials. More generally, for $B \downarrow A$ in $\mathbf{ComAlg}_K/A$

$$\mathrm{Ab}_A(B) = A \otimes_B \Omega_{B/K} .$$

So in that case we have the "cotangent complex"

$$\mathbf{L}_{A/K} = A \otimes_{G_\bullet A} \Omega_{G_\bullet A/K} .$$

The André-Quillen homology is its homotopy –

$$HQ_n(A) = \pi_n(\mathbf{L}_{A/K})$$

– and the André-Quillen cohomology is

$$HQ^n(A; M) = H^n(\mathrm{Hom}_A(\mathrm{ch}\, \mathbf{L}_{A/K}, M)) .$$

In this case we can also define homology with coefficients in an A-module M:

$$HQ_n(A; M) = \pi_n(\mathbf{L}_{A/K} \otimes_A M) .$$

2. Gradings

Let X be a commutative monoid, which we will write additively. Following Bourbaki [7, Ch. 2 §11], we say that an X-graded object $C_\bullet$ in a category $\mathbf{C}$ is a choice of object C_x of $\mathbf{C}$ for each $x \in X$. Write $\mathbf{C}^X$ for the category of X-graded objects in $\mathbf{C}$. A morphism $C_\bullet \to C'_\bullet$ is a morphism $C_x \to C'_x$ for each $x \in X$. A functor $F : \mathbf{C} \to \mathbf{D}$ induces $F^X : \mathbf{C}^X \to \mathbf{D}^X$, and an adjunction between E and F induces a canonical adjunction between E^X and F^X.

Structure on $\mathbf{C}$ often induces structure on $\mathbf{C}^X$. For example suppose that $(\mathbf{C}, \mathbf{1}, \otimes, c)$ is a closed symmetric monoidal category [6, §6.1]. Assuming that $\mathbf{C}$ has coproducts of large enough sets of objects, there is then a canonical symmetric monoidal structure on $\mathbf{C}^X$, also closed if $\mathbf{C}$ has products of large enough sets of objects, in which

$$\mathbf{1}_x = \begin{cases} \mathbf{1} & \text{for } x = 0 \\ \mathbf{0} & \text{for } x \neq 0 \end{cases}$$

$$(C_\bullet \otimes D_\bullet)_z = \coprod_{x+y=z} C_x \otimes D_y \,.$$

The symmetry $c : (C_\bullet \otimes D_\bullet)_z \to (D_\bullet \otimes C_\bullet)_z$ is such that for all x, y with $x + y = z$,

$$c \circ \mathrm{in}_{x,y} = \mathrm{in}_{y,x} \circ c_{C_x, D_y}$$

where $c_{C,D} : C \otimes D \to D \otimes C$ is the symmetry in $\mathbf{C}$. For example, a commutative monoid in the symmetric monoidal category $\mathbf{Set}^X$ consists of a set T_x for each $x \in X$ together with an element $1 \in T_0$ and maps $\mu : T_x \times T_y \to T_{x+y}$ satisfying evident conditions. This is to be distinguished from an X-graded commutative monoid, an object of $\mathbf{ComMon}^X$!

Let K be a commutative ring. The category of X-graded K-modules $\mathbf{Mod}_K^X$ admits a symmetric monoidal structure given by the "graded tensor product," with

$$(A_\bullet \otimes_K B_\bullet)_z = \bigoplus_{x+y=z} A_x \otimes_K B_y$$

and unit given by the X-graded K-module with K in degree 0 and 0 in all other degrees. The symmetry sends $x \otimes y$ to $y \otimes x$. An X-graded K-algebra is a monoid for this tensor product. Once again, beware of this use of language; this is not an X-graded object in $\mathbf{ComAlg}_K$. Write $\mathbf{ComAlg}(\mathbf{Mod}_K^X)$ for this category. A Beck module for the commutative X-graded K-algebra $A_\bullet$ is an action of this monoid.

The relationship with the Leech category (section 5 below) suggests that rather than defining a right $A_\bullet$-module as a right action, we should say this:

Definition 2.1. *A right $A_\bullet$-module is an X-graded K-module $M^\bullet$ together with homomorphisms*

$$\varphi_{x,y} : M^{x+y} \otimes A_y \to M^x$$

such that $\varphi_{x,0}(m \otimes 1) = m$ and

$$\begin{array}{ccc} M^{x+y+z} \otimes A_z \otimes A_y & \xrightarrow{1 \otimes \mu_{z,y}} & M^{x+y+z} \otimes A_{z+y=y+z} \\ \downarrow {\scriptstyle \varphi_{x+y,z} \otimes 1} & & \downarrow {\scriptstyle \varphi_{x,y+z}} \\ M^{x+y} \otimes A_y & \xrightarrow{\quad \varphi_{x,y} \quad} & M^x \end{array}$$

commutes.

Write $\mathbf{RMod}_{A_\bullet}$ for the category of right $A_\bullet$-modules. If X has inverses, so is in fact an abelian group, the category $\mathbf{RMod}_{A_\bullet}$ is equivalent to the category of a right $A_\bullet$-modules in the usual sense, using "lower indexing" $M_x = M^{-x}$.

Let $N_\bullet$ be a left $A_\bullet$-module and $M^\bullet$ a right $A_\bullet$ module. Their *tensor product* over $A_\bullet$, $M^\bullet \otimes_{A_\bullet} N_\bullet$, is the K-module defined as the coequalizer of the two maps

$$f, g : P = \bigoplus_{x,y} M^{x+y} \otimes A_y \otimes N_x \rightrightarrows \bigoplus_z M^z \otimes N_z \,.$$

Each of these maps is defined by giving the composite with an inclusion $\mathrm{in}_{x,y} : M^{x+y} \otimes A_y \otimes N_x \to P$:

$$f \circ \mathrm{in}_{x,y} = \mathrm{in}_{x+y} \circ (1 \otimes \varphi_{y,x}) \,,$$
$$g \circ \mathrm{in}_{x,y} = \mathrm{in}_{x} \circ (\varphi_{x,y} \otimes 1) \,.$$

3. Graded homological algebra

From now on we will write just A rather than $A_\bullet$ and so on. Let A be an X-graded K-algebra. For each $x \in X$, there is a left A-module P^x together with $\iota \in P^x_x$ such that for any left A-module N the map

$$\mathrm{Hom}_A(P^x, N) \to N_x \,, \quad f \mapsto f(\iota)$$

is an isomorphism. Explicitly,

$$P^x_y = \bigoplus_{x+z=y} A_z \tag{1}$$

and ι is the image of $1 \in A_0$ under $\mathrm{in}_0 : A_0 \to P^x_x$. Given $n \in N_x$, the corresponding map $\widehat{n} : P^x \to N$ is defined by

$$\widehat{n} \circ \mathrm{in}_z(a) = an \,, \quad a \in A_z \,.$$

Lemma 3.1. *The set $\{P^x : x \in X\}$ is a generating set of small projective A-modules.*

Proof. The A-module P^x is projective since $N \mapsto N_x$ is an exact functor. An object is small if the functor it co-represents preserves filtered colimits; this is clear since colimits in $\mathbf{LMod}_A$ are computed component-wise. For any A-module N, the map

$$\bigoplus_{x\in X} \bigoplus_{n \in N_x} P^x \to N \,,$$

given by $\widehat{n}$ on the component indexed by (x, n), is an epimorphism; this shows that $\{P^x : x \in X\}$ is a generating set. □

It follows that any projective A-module is a retract of a direct sum of P^x's.

The account of Quillen homology given above goes through in the graded context without essential change. For an X-graded K-algebra A we have identified the category $\mathbf{LMod}_A$ with the category of left actions of A, with N corresponding to the abelian object in $\mathbf{ComAlg}_K/A$ given by $\mathrm{pr}_1 : A \oplus N \downarrow A$ with unit section $a \mapsto (a, 0)$ and product given by $(a, x)(b, y) = (ab, ay + xb)$. A section of this object of $\mathbf{ComAlg}_K/A$ is a (degree-preserving) derivation, that is, an X-graded K-module map $d : A \to N$ such that $d(ab) = a\,db + b\,da$, as usual. The functor $N \mapsto \mathrm{Der}_K(A, N)$ is co-represented by the (graded) A-module of Kähler differentials: $\Omega_{A/K} \in \mathbf{LMod}_A$. Expressed in terms of generators and relations, this A-module is the cokernel of the map

$$d : \bigoplus_{x,y} P^{x+y} \to \bigoplus_{z} P^{z}$$

determined by

$$d \circ \mathrm{in}_{x,y} = \mathrm{in}_x \circ y^* - \mathrm{in}_{x+y} + \mathrm{in}_y \circ x^* \,.$$

This is $\mathrm{Ab}_A A$. To describe $\mathrm{Ab}_A B$, for $p : B \to A$ in $\mathbf{ComAlg}(\mathbf{Mod}_K^X)$, notice that for each $x \in X$ the right A-module P_x can be regarded as a right B-module through the map p. Then

$$(\mathrm{Ab}_A B)_x = P_x \otimes_B \Omega_{B/K} \,.$$

The left A-module structure on $\mathrm{Ab}_A B$ arises from the A-bimodule structure of P.

Given an X-graded set T, one can construct the free commutative monoid $\mathbb{N}T$ in $\mathbf{Set}^X$ as

$$(\mathbb{N}T)_x = \coprod_{\Sigma Y = x} \prod_{y \in Y} T_y$$

the disjoint union taken over finite subsets Y of X with sum x.

This construction is useful in building the free commutative X-graded K-algebra generated by $T \in \mathbf{Set}^X$: It is in degree x just the free K-module generated by $(\mathbb{N}T)_x$. This provides us with an adjoint pair

$$F_X : \mathbf{Set}^X \rightleftarrows \mathbf{ComAlg}(\mathbf{Mod}_K^X) : u_X$$

giving a cotriple Sym_K^X on $\mathbf{ComAlg}(\mathbf{Mod}_K^X)$. Following [4], this in turn leads to a natural simplicial object $\mathrm{Sym}_{K\bullet}^X A$ augmented to A, with $\mathrm{Sym}_{Kn}^X A = (\mathrm{Sym}_K^X)^{n+1} A$, which can be used to derive functors on $\mathbf{ComAlg}(\mathbf{Mod}_K^X)/A$.

The Quillen homology of $A \in \mathbf{ComAlg}(\mathbf{Mod}_K^X)$ is defined as the derived functors of $\mathrm{Ab}_A : \mathbf{ComAlg}(\mathbf{Mod}_K^X)/A \to \mathbf{LMod}_A$, evaluated at the object $\mathrm{id}_A : A \downarrow A$. Thus for each n the Quillen homology $HQ_n(A)$ is itself an A-module, and $HQ_0(A) = \Omega_{A/K}$. Any right A-module M results in the sequence of K-modules

$$HQ_n(A; M) = \pi_n(M \otimes_A \mathrm{Ab}_A G_{X\bullet} A) \,.$$

For any $x \in X$, we can recover $HQ_n(A)_x$ by using the right A-module P_x for coefficients:

$$HQ_n(A)_x = HQ_n(A; P_x) \,.$$

For $N \in \mathbf{LMod}_A$ we can define the André-Quillen cohomology:

$$HQ^n(A; N) = H^n(\mathrm{Hom}_A(\mathrm{ch}\, G_{X\bullet} A, N))\,.$$

4. Change of grading monoid

Let $\alpha : Y \to X$ be a map of commutative monoids. An X-graded object C in $\mathbf{C}$ determines a Y-graded object $\alpha^* C$ by

$$(\alpha^* C)_y = C_{\alpha(y)}\,.$$

If $\mathbf{C}$ has coproducts, the functor $\alpha^* : \mathbf{C}^X \to \mathbf{C}^Y$ has a left adjoint given by

$$(\alpha_* C)_x = \coprod_{\alpha(y)=x} C_y\,.$$

Let K be a commutative ring. The functor $\alpha_* : \mathbf{Mod}_K^Y \to \mathbf{Mod}_K^X$ is symmetric monoidal –

$$\alpha_*(M \otimes N) = \alpha_* M \otimes \alpha_* N$$

– so α_* sends Y-graded commutative K-algebras to X-graded commutative K-algebras. Let $A \in \mathbf{ComAlg}(\mathbf{Mod}_K^Y)$. Then for the same reason α induces adjoint pairs

$$\alpha_* : \mathbf{LMod}_A \rightleftarrows \mathbf{LMod}_{\alpha_* A} : \alpha^*$$

and

$$\alpha_* : \mathbf{RMod}_A \rightleftarrows \mathbf{RMod}_{\alpha_* A} : \alpha^*$$

Let N be an $\alpha_* A$-module. It is straightforward to construct a natural isomorphism

$$\mathrm{Der}_K(A, \alpha^* N) = \mathrm{Der}_K(\alpha_* A, N)$$

and so a natural isomorphism of $\alpha_* A$-modules

$$\Omega_{\alpha_* A/K} = \alpha_* \Omega_{A/K}\,.$$

The adjoint pairs (F_X, u_X) are compatible under change of grading monoid: A monoid homomorphism $\alpha : Y \to X$ determines the following squares.

$$\begin{array}{ccc} \mathbf{Set}^Y & \xleftarrow{u_X} & \mathbf{ComAlg}(\mathbf{Mod}_K^Y) \\ \uparrow{\scriptstyle \alpha^*} & & \uparrow{\scriptstyle \alpha^*} \\ \mathbf{Set}^X & \xleftarrow{u_X} & \mathbf{ComAlg}(\mathbf{Mod}_K^X) \end{array} \qquad \begin{array}{ccc} \mathbf{Set}^Y & \xrightarrow{F_Y} & \mathbf{ComAlg}(\mathbf{Mod}_K^Y) \\ \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} \\ \mathbf{Set}^X & \xrightarrow{F_X} & \mathbf{ComAlg}(\mathbf{Mod}_K^X) \end{array}$$

The diagram of right adjoints clearly commutes, so the diagram of left adjoints does too:

$$\alpha_* F_Y(T) = F_X(\alpha_* T) \in \mathbf{ComAlg}(\mathbf{Mod}_K^X)\,.$$

Passing to the cotriples,

$$\alpha_* \mathrm{Sym}_K^Y(A) = \alpha_* F_Y u_X(A) = F_X \alpha_* u_Y(A) = F_X u_X(\alpha_* A) = \mathrm{Sym}_K^X(\alpha_* A)$$

and so to simplicial resolutions:

$$\alpha_*\mathrm{Sym}^Y_{K\bullet}(A) = \mathrm{Sym}^X_{K\bullet}(\alpha_* A)\,.$$

Assembling all this, along with the fact that α_* is exact, we find:

Proposition 4.1. *Let $\alpha : Y \to X$ be a monoid homomorphism. There are isomorphisms of $\alpha_* A$-modules natural in $A \in \mathbf{ComAlg}(\mathbf{Mod}^Y_K)$*

$$\alpha_* HQ_n(A) = HQ_n(\alpha_* A)$$

as well as isomorphisms of K-modules

$$HQ_n(A; \alpha^* M) = HQ_n(\alpha_* A; M)$$

natural in $M \in \mathbf{RMod}_{\alpha_* A}$ and

$$HQ^n(A; \alpha^* N) = HQ^n(\alpha_* A; N)$$

natural in $N \in \mathbf{LMod}_{\alpha_* A}$.

An important example is provided by taking X to be the one-element monoid, e, and $\alpha : Y \to e$ the unique map. Then $\alpha_* A$ is the "degrading" of A, an ungraded commutative K-algebra, and N is a module for it; $\alpha^* N$ is the "constant" Y-graded K-module with $(\alpha^* N)_y = N$ for all $y \in Y$, and A acting among them in the obvious way; and $HQ_*(\alpha_* A)$, $HQ_*(\alpha_* A; M)$, and $HQ^*(\alpha_* A; N)$ are the usual André-Quillen groups.

5. Commutative monoids

We now regard commutative monoids as the category of homological interest, rather than as a source of gradings.

Let $X \in \mathbf{ComMon}$. It is easy [30, 18] to identify the category of Beck modules over X in terms of the *Leech category*, L_X, with object set X and $L_X(x, z) = \{y \in X : x + y = z\}$ with unit and composition determined by the commutative monoid X. Write $y_* : x \to (x + y)$ for a morphism in this category. The category of Beck modules over X is canonically equivalent to the category of functors from L_X to the category $\mathbf{Ab}$ of abelian groups.

A map $\alpha : Y \to X$ of commutative monoids induces a functor

$$\alpha^* : \mathbf{LMod}_X \to \mathbf{LMod}_Y$$

which, under the equivalence with functors from the Leech categories, may be regarded as induced by pre-composition with the functor $\alpha : L_Y \to L_X$. The left adjoint α_* is then induced by left Kan extension [31, Chapter X] along α.

A section of an abelian object over X is a "derivation," and under the identification of abelian objects over X with functors on L_X a derivation with values in $N : L_X \to \mathbf{Ab}$ is an assignment of an element $s(x) \in N_x$ for each $x \in X$ such that

$$s(x + y) = x_* s(y) + y_* s(x)$$

There is a universal example, the Beck module of "Kähler differentials" Ω_X^{CM}, which provides a distinguished object of $\mathbf{LMod}_X$. For any $\alpha : Y \to X$, $\mathrm{Ab}_X Y = \alpha_* \Omega_Y^{CM}$.

A commutative monoid X defines a canonical commutative X-graded K-algebra $\widetilde{K}X$ in which $(\widetilde{K}X)_x = K$ for each $x \in X$, with generator 1_x, $1 = 1_0 \in (\widetilde{K}X)_0$, and

$$\mu_{x,y}(a_x \otimes b_y) = (ab)_{x+y} = ab1_{x+y}\,, \quad a, b \in K\,.$$

This object of $\mathbf{ComAlg}_K^X$ co-represents the functor sending an object A to the set of sections a of the grading function $A \to X$ such that $a_x a_y = a_{x+y}$.

For a Beck K-module N over X we may define a left module over $\widetilde{K}X$, which we will again denote by N. The K-module in degree x is N_x and the multiplication by $\widetilde{K}X$ is given by the action of elements of X, regarded as morphisms in L_X. The following lemma is simply a change in perspective.

Lemma 5.1. *The category of covariant functors $L_X \to \mathbf{Mod}_K$ is canonically equivalent to the category of $\widetilde{K}X$-modules. The category of contravariant functors $L_X \to \mathbf{Mod}_K$ is canonically equivalent to the category of right $\widetilde{K}X$-modules.*

Of course the usual monoid-algebra is obtained from $\widetilde{K}X$ as

$$KX = \alpha_* \widetilde{K}X\,, \quad \alpha : X \to e\,.$$

These isomorphisms are compatible with attendant structure: the functors induced by a map $\alpha : Y \to X$ of commutative monoids agree with the functors induced by $\widetilde{\mathbb{Z}}\alpha : \widetilde{\mathbb{Z}}Y \to \widetilde{\mathbb{Z}}X$. Under this identification the Kähler differential objects match up:

$$\Omega_Y^{CM} = \Omega_{\widetilde{\mathbb{Z}}Y}^{CA}$$

and more generally $\alpha_* \widetilde{\mathbb{Z}}Y \in \mathbf{ComAlg}(\mathbf{Mod}_K^X)/\widetilde{\mathbb{Z}}X$ and

$$\mathrm{Ab}_X(Y) = \mathrm{Ab}_{\widetilde{\mathbb{Z}}X}(\alpha_* \widetilde{\mathbb{Z}}Y)$$

in $\mathbf{LMod}_X = \mathbf{LMod}_{\widetilde{\mathbb{Z}}X}$. As a further example, given a contravariant functor $M : L_X \to \mathbf{Mod}_K$ and a covariant functor $N : L_X \to \mathbf{Mod}_K$,

$$M \otimes_{\widetilde{\mathbb{Z}}X} N = M \otimes_{L_X} N$$

where the right hand side is the usual tensor product over the category L_X, as considered by Kurdiani and Pirashvili in [29].

In order to express the functoriality of $\widetilde{K}$, we need a category of graded algebras in which the grading commutative monoid can vary. So let $\mathbf{ComAlg}_K^*$ be the category whose objects are pairs (X, A) where X is a commutative monoid and A is a commutative X-graded K-algebra. A morphism $(X, A) \to (Y, B)$ consists of a monoid homomorphism $\alpha : X \to Y$ together with a morphism $f : \alpha_* A \to B$ in $\mathbf{ComAlg}(\mathbf{Mod}_K^Y)$ (or equivalently a morphism $\hat{f} : A \to \alpha^* B$ in $\mathbf{ComAlg}(\mathbf{Mod}_K^X)$). Given a second morphism

$(\beta, g) : (Y, B) \to (Z, C)$, the composite is defined as $(\beta \circ \alpha, g \circ \beta_* f)$, (or equivalently as $(\beta \circ \alpha, \alpha^* \hat{g} \circ \hat{f})$). Then the construction $\widetilde{K}$ provides a functor

$$\widetilde{K} : \mathbf{ComMon} \to \mathbf{ComAlg}_K^* .$$

Theorem 5.2. *There are isomorphisms natural in the commutative monoid X*

$$HQ_n^{CM}(X) = HQ_n^{CA}(\widetilde{\mathbb{Z}}X)$$

where we have identified Beck modules over X with modules over $\widetilde{\mathbb{Z}}X$. For any right Beck module M over X,

$$HQ_n^{CM}(X; M) = HQ_n^{CA}(\widetilde{\mathbb{Z}}X; M)$$

and for any left Beck module N

$$HQ^n_{CM}(X; N) = HQ^n_{CA}(\widetilde{\mathbb{Z}}X; N)$$

Proof. Let $\alpha : Y_\bullet \to X$ be a cofibrant replacement of $X \in \mathbf{ComMon}$ regarded as a constant simplical object. We could take the cotriple resolution $\mathbb{N}_\bullet X$ for example, and for definiteness we will do so. We claim that $\alpha_* \widetilde{K}\mathbb{N}_\bullet X \to \widetilde{K}X$ is a cofibrant replacement for $\widetilde{K}X$ in $s\mathbf{ComAlg}(\mathbf{Mod}_K^X)$.

The augmentation $\alpha : \mathbb{N}_\bullet X \to X$ is a weak equivalence in $s\mathbf{ComMon}$, which is to say a weak equivalence of simplicial sets. The entire simplicial set $\mathbb{N}_\bullet X$ splits into a disjoint union of the pre-images under α of the points of X, and each of these components is a weakly contractible simplicial set.

Now apply the functor $\widetilde{K}$ to $\mathbb{N}_\bullet X$, to get a simplicial object in $\mathbf{ComAlg}_K^*$ with an augmentation $\widetilde{K}\mathbb{N}_\bullet X \to \widetilde{K}X$; that is, a map

$$\alpha_* \widetilde{K}\mathbb{N}_\bullet X \to \widetilde{K}X$$

in $s\mathbf{ComAlg}(\mathbf{Mod}_K^X)$. The object $\alpha_* \widetilde{K}\mathbb{N}_\bullet X$ of $s\mathbf{Mod}_K^X$ splits as a direct sum of the free K-module functor applied to the components of $\mathbb{N}_\bullet X$ above the points of X. But a weak equivalence $Z_\bullet \to Y_\bullet$ of simplicial sets induces a weak equivalence $KZ_\bullet \to KY_\bullet$ [17, III, Prop. 2.16], so the map $\alpha_* \widetilde{K}\mathbb{N}_\bullet X \to \widetilde{K}X$ is a weak equivalence.

The second observation is that $\alpha_* \widetilde{K}\mathbb{N}_\bullet X$ is cofibrant in the model category of simplicial objects in $\mathbf{ComAlg}(\mathbf{Mod}_K^X)$. Indeed, it is almost free in the sense of [32], and hence, as observed there, is cofibrant. A simplicial object $Z_\bullet \in s\mathbf{ComMon}$ is "almost free" if there are subsets $G_s \subseteq Z_s$, for each s, that are respected by all face and degeneracy maps except for d_0, and such that Z_s is freely generated by G_s. A cotriple resolution, such as $\mathbb{N}_\bullet X$, is easily seen to be almost free. If we then apply the functor $\alpha_* \widetilde{K}$ to it, the same generating sets, now regarded as in $\mathbf{Set}^X$, again generate as objects in $\mathbf{ComAlg}(\mathbf{Mod}_K^X)$.

The map $\alpha_* \widetilde{K}\mathbb{N}_\bullet X \to \widetilde{K}X$ thus joins $\mathrm{Sym}_{K\bullet}^X(\widetilde{K}X) \to \widetilde{K}X$ as a cofibrant replacement, and so (e.g. [14, Proposition 3.9]) can be used to compute Quillen homology. Thus (finally taking $K = \mathbb{Z}$)

$$\mathrm{Ab}_X \mathbb{N}_\bullet X = \mathrm{Ab}_{\widetilde{\mathbb{Z}}X}(\alpha_* \widetilde{\mathbb{Z}}\mathbb{N}_\bullet X) \simeq \mathrm{Ab}_{\widetilde{\mathbb{Z}}X}(\mathrm{Sym}_{\mathbb{Z}\bullet}^X(\widetilde{\mathbb{Z}}X))$$

as simplicial objects in $\mathbf{LMod}_X = \mathbf{LMod}_{\widetilde{\mathbb{Z}}X}$. Applying the functors $\pi_*(-) = H_*(\mathrm{ch}\,(-))$, $\pi_*(M\otimes_{\widetilde{\mathbb{Z}}X} -) = H_*(M\otimes_{\widetilde{\mathbb{Z}}X}\mathrm{ch}\,(-))$, and $H^*(\mathrm{Hom}_{\widetilde{\mathbb{Z}}X}(\mathrm{ch}\,(-), N))$ gives us the results. □

Corollary 5.3 ([29]). *Let X be a commutative monoid and let $\alpha : X \to e$ be the unique monoid map to the trivial monoid. There are isomorphisms*

$$\alpha_* H_n^{CM}(X; \alpha^* M) = H_n^{CA}(\mathbb{Z}X; M)$$

natural in the right X-module M, and

$$\alpha_* H^n_{CM}(X; \alpha^* N) = H^n_{CA}(\mathbb{Z}X; N)$$

natural in the left X-module N.

6. The Hochschild complex

This section reviews well known material (e.g. [38, 41]) in order to establish notation. We take the opportunity to point out the intrinsic simplicity and symmetry of the Hochschild complex, hidden by standard treatments.

Let A be an associative K-algebra. There is a canonical simplicial A-bimodule $B_\bullet(A)$ over K with

$$B_n(A) = A^{\otimes(n+2)}$$

for $n \geq 0$, augmented to A, and

$$d_i = 1^{\otimes i} \otimes \mu \otimes 1^{\otimes(n-i)} : A^{\otimes(n+2)} \to A^{\otimes(n+1)}\,,\ 0 \leq i \leq n$$

$$s_i = 1^{\otimes(i+1)} \otimes \eta \otimes 1^{\otimes(n-i)} : A^{\otimes(n+1)} \to A^{\otimes(n+2)}\,,\ 0 \leq i \leq n-1$$

where $\mu : A\otimes A \to A$ is the multiplication and $\eta : K \to A$ includes the unit.

There are also maps

$$s_{-1} = \eta \otimes 1^{\otimes(n+1)}\,, \quad s_n = 1^{\otimes(n+1)} \otimes \eta : A^{\otimes(n+1)} \to A^{\otimes(n+2)}\,.$$

The first is a right A-module map and the second is a left A-module map, and they provide contracting homotopies of the simplicial object regarded as either a simplicial right A-module or a simplicial left A-module. In fact it is just the simplicial bar resolution of A as a left or right A-module. Thus the chain complex associated to $B_\bullet(A)$, $\mathrm{ch}\, B_\bullet(A)$, is a relative projective resolution of A as an A-bimodule, the *Hochschild resolution*. If A is projective as a K-module, it's an absolute projective resolution.

Let Q_A be the functor from A-bimodules to K-modules given by

$$Q_A(M) = M/(ax - xa : a \in A, x \in M)\,,$$

where the denominator indicates the sub-K-module generated by these elements. More generally, given an A-bimodule N, define the functor Q_N^A, or Q_N if A is understood, from A-bimodules to K-modules by

$$Q_N(M) = M \otimes N/(ax \otimes y - x \otimes ya, xa \otimes y - x \otimes ay)\,.$$

We recover Q_A by regarding A as a bimodule over itself using left and right multiplication. A bimodule is the same thing as a module over $A^e = A \otimes A^{op}$, and under this equivalence

$$Q_N(M) = M \otimes_{A^e} N$$

In general this is just a K-module, but if A is commutative then $Q_A(M)$ is naturally an A-module, since A is then an (A, A^e)-bimodule.

Apply this functor to $B_\bullet(A)$ to get a simplicial object in K-modules, $Q_N B_\bullet(A)$, equipped with an augmentation to $Q_N A$. This is the *Hochschild complex* with coefficients in N,

$$C_\bullet(A/K; N) = \operatorname{ch} Q_N B_\bullet(A)$$

and by definition

$$Hoch_n(A/K; N) = H_n(C_\bullet(A/K; N))\,.$$

When the ground ring K is understood we will drop it from the notation. When $N = A$ with its natural A-bimodule structure, we may drop it from the notation as well:

$$C_\bullet(A) = C_\bullet(A/K; A) \quad \text{and} \quad Hoch_\bullet(A) = Hoch_\bullet(A/K; A)\,.$$

To understand this better, notice the isomorphism

$$Q_N(A \otimes V \otimes A) \to N \otimes V$$

for a K-module V, given by factoring

$$x \otimes a \otimes v \otimes b \mapsto bxa \otimes v$$

through $Q_N(A \otimes V \otimes A)$. The inverse sends $x \otimes v$ to $[x \otimes 1 \otimes v \otimes 1]$.

This isomorphism breaks symmetry. But using it we may write the augmented simplicial K-module $Q_N B_\bullet(A)$ as

$$Q_N(A) \leftarrow N \Leftarrow N \otimes A \cdots\,,$$

so

$$C_n(A; N) = N \otimes A^{\otimes n}\,.$$

If A and Z are two K-algebras and M and N bimodules for them, there is a natural isomorphism

$$Q_A(M) \otimes Q_Z(N) \to Q_{A \otimes Z}(M \otimes N)$$

under the identity map on $M \otimes N$. The fact that it is an isomorphism follows from the identity

$$(a \otimes z)(m \otimes n) - (m \otimes n)(a \otimes z) = am \otimes (zn - nz) + (am - ma) \otimes nz$$

We get natural isomorphisms of simplicial objects

$$B_\bullet(A) \otimes B_\bullet(Z) \to B_\bullet(A \otimes Z)$$
$$Q_A B_\bullet(A) \otimes Q_Z B_\bullet(Z) \to Q_{A \otimes Z} B_\bullet(A \otimes Z)$$

If A is commutative we may take $A = Z$ and compose with the K-algebra map $\mu : A \otimes A \to A$ to obtain a simplicial commutative A-algebra structure on $Q_A B_\bullet(A)$, and $Q_N B_\bullet(A)$ becomes a module over $Q_A B_\bullet(A)$.

Passing to associated chain complexes, the Eilenberg-Zilber or shuffle map ([11, p. 64] or [33, p. 39]) results in the structure of a commutative (in the signed sense) differential graded A-algebra on $C_\bullet(A)$ and hence a graded commutative A-algebra structure on its homology $Hoch_\bullet(A)$.

Dually, denote by $R_N(M)$ the K-module of A-bimodule maps from M to N. The Hochschild cochain complex with coefficients in N is then $R_N B_\bullet(A)$, and its homology is the Hochschild cohomology $Hoch^\bullet(A; N)$.

It is well known and easy to verify that

$$Hoch_1(A) = \Omega_{A/K} = HQ_0^{CA}(A)$$

and

$$Hoch^1(A; M) = \mathrm{Der}_K(A; M) = HQ^0_{CA}(A; M)\,.$$

This entire discussion goes through without change in the presence of a grading by a commutative monoid.

7. Harrison homology

Let A be an X-graded commutative K-algebra. Then $Q_A A = A$; the Hochschild complex $C_\bullet(A)$ is augmented to A. Let $I_\bullet(A)$ denote the kernel of this augmentation; this is the ideal of positive-dimensional elements in the commutative differential graded A-algebra $C_\bullet(A)$. The *Harrison complex* [26] is the differential graded module of indecomposables in $C_\bullet(A)$, $I_\bullet(A)/I_\bullet(A)^2$. The *Harrison homology* of A is the homology of this chain complex of A-modules:

$$Harr_n(A) = H_n(I_\bullet(A)/I_\bullet(A)^2)\,.$$

We can equip it with coefficients in an A-module M:

$$Harr_n(A; M) = H_n((I_\bullet(A)/I_\bullet(A)^2) \otimes_A M)\,.$$

The Harrison cohomology with coefficients in an A-module M is

$$Harr^n(A; M) = H^n(\mathrm{Hom}_A(I_\bullet(A)/I_\bullet(A)^2, M))\,.$$

Clearly

$$Harr_0(A) = 0 \quad \text{and} \quad Harr_1(A) = Hoch_1(A)$$

The shuffle product defines a sign-commutative graded K-algebra structure on the $\mathbb{N}$-graded K-module $\overline{C}_\bullet(A)$ with

$$\overline{C}_n(A) = A^{\otimes n}\,.$$

As graded A-algebras

$$C_\bullet(A) = A \otimes \overline{C}_\bullet(A)\,.$$

Only the differential depends on the algebra structure, and it is not the A-linear extension of a differential on $\overline{C}_\bullet(A)$.

The Harrison cochains can be re-expressed in terms of the graded K-algebra $\overline{C}_\bullet(A)$. Let $\overline{I}_\bullet(A)$ be its augmentation ideal; then

$$I_\bullet = A \otimes \overline{I}_\bullet\,, \quad I_\bullet^2 = A \otimes \overline{I}_\bullet^2\,,$$

and so

$$I_\bullet(A)/I_\bullet(A)^2 = A \otimes (\overline{I}_\bullet(A)/\overline{I}_\bullet(A)^2)\,.$$

A Harrison n-cochain (for $n > 0$) with coefficients in M is thus a K-linear map

$$s : A^{\otimes n} \to M$$

that annihilates decomposables. This may be phrased as a symmetry condition on the cochain: given i, j, both positive and summing to n, let $\Sigma(i,j)$ be the set (i,j)-shuffles; that is, the set of elements of Σ_n that preserve the order of $\{1,\ldots,i\}$ and of $\{i+1,\ldots,n\}$. The symmetry condition (i,j) on a Hochschild cochain $s : A^{\otimes n} \to M$ is

$$\sum_{\sigma\in\Sigma(i,j)} \operatorname{sgn}(\sigma) s\circ\sigma = 0\,.$$

Since the shuffle product is commutative, we may assume that $i \le j$; there are $\lfloor n/2\rfloor$ independent conditions.

An alternative symmetry condition (apparently the one originally conceived of by Harrison; the shuffle description is said to be due to Mac Lane) is described in [15]. Think of an element of Σ_n as an ordering of $\{1,2,\ldots,n\}$. Let $1 \le k \le n$. An element $\sigma \in \Sigma_n$ is a *k-monotone* permutation if the lead element is k, the numbers $1,2,\ldots,k$ occur in decreasing order, and the numbers $k+1,\ldots,n$ occur in increasing order. There are $\binom{n-1}{k-1}$ of them. For example the only n-monotone permutation in Σ_n corresponds to the sequence $n, n-1, \cdots, 2, 1$, and the 4-monotone permutations in Σ_6 are

432156, 432516, 432561, 435216, 435261, 435621, 453216, 453261, 453621, 456321

Let $M_k(n)$ be the set of k-monotone permutations in Σ_n. Let $dr(\sigma)$ be the sum of the positions occupied by $1,2,\ldots,k-1$ in the permutation $\sigma \in M_k(n)$.

Lemma 7.1 ([15], Theorem 4.1)**.** *A map $s : A^{\otimes n} \to M$ is a Harrison cochain if and only if*

$$s = \sum_{\sigma\in M_k(n)} (-1)^{dr(\sigma)} s\circ\sigma\,, \quad 2 \le k \le n\,.$$

So for example a Hochschild 4-cochain s is a Harrison cochain if and only if

$$\begin{aligned} s(a_1,a_2,a_3,a_4) &= s(a_2,a_1,a_3,a_4) - s(a_2,a_3,a_1,a_4) + s(a_2,a_3,a_4,a_1)\\ &= -s(a_3,a_2,a_1,a_4) + s(a_3,a_2,a_4,a_1) - s(a_3,a_4,a_2,a_1)\\ &= -s(a_4,a_3,a_2,a_1) \end{aligned}$$

The 4-monotone and 2-monotone symmetries combine to give

$$s(a_4,a_3,a_2,a_1) = -s(a_2,a_1,a_3,a_4) + s(a_2,a_3,a_1,a_4) - s(a_2,a_3,a_4,a_1)$$

which is the same as the 3-monotone condition (after rearranging the labels); so we can dispense with either one of the first two conditions in this list. The same argument shows that one need only assume the n-monotone condition together with one condition from each pair $\{2,n-1\},\{3,n-2\},\ldots$: so $\lfloor n/2\rfloor$ conditions suffice. This matches with the number of independent shuffle conditions.

8. Divided powers and Barr homology

Michael Barr suggested possible variations on Harrison's symmetry conditions, in an attempt to come closer to Quillen homology. As explained by Sarah Whitehouse [40], these variations still fail, though they may give better approximations.

Gerstenhaber and Schack [15, Remark, p. 232] (see also [40]) suggest that one of Barr's ideas was to divide the Hochschild complex not just by shuffle decomposables but by the divided power structure as well. While a divided power structure on the even homotopy groups of a simplicial commutative algebra was implicit in the works of Eilenberg and Mac Lane [11] and Henri Cartan [9, Exp. 8], its construction on the level of the Hochschild complex was at best a folk result at the time of Barr's question, and even when Gerstenhaber and Schack were writing. It seems to have first been set out, in the associated chain complex of a simplicial commutative algebra $B_\bullet$, by Siegfried Brüderle and Ernst Kunz [8] in 1994; see also [37] and [16]. The result is a natural family of maps

$$\gamma_k : B_{2n} \to B_{2kn}$$

such that $k!\gamma_k x = x^k$.

From these sources one obtains the following formula for the divided power structure on $\overline{C}_{even}(A)$, where A is a commutative K-algebra. Let $S_k(kn)$ be the set of shuffles associated to the partition of $\{1,2,\ldots,kn\}$ into k intervals of length n. Let $S'_k(kn)$ be the subset of these such that the leading terms of the k sequences occur in order. Then, for n even,

$$\gamma_k[a_1|\cdots|a_n] = \sum_{\sigma\in S'_k(kn)} \operatorname{sgn}(\sigma)[a_1|\cdots|a_n|a_1|\cdots|a_n|\cdots|a_1|\cdots|a_n]\circ\sigma$$

where the sequence $a_1|\cdots|a_n$ is repeated k times. For example

$$\gamma_2[a|b] = [a|b|a|b]\,, \quad \gamma_3[a|b] = [a|b|a|b|a|b]\,,$$

$$\gamma_2[a|b|c|d] = [a|b|c|d|a|b|c|d] - [a|b|c|a|d|b|c|d] + [a|b|c|a|b|d|c|d]$$
$$+[a|b|a|c|d|b|c|d] - [a|b|a|c|b|d|c|d] + 2[a|b|a|b|c|d|c|d]\,.$$

We can put at least one restriction on the tensors occurring in the expression for the divided powers. To express it, notice that there is a universal Hochschild n-chain, $[a_1|\cdots|a_n]\in K[a_1,\ldots,a_n]^{\otimes n}$.

Lemma 8.1. *No decomposable tensor with entries chosen from* $\{a_1, \ldots, a_n\}$ *occurring with nonzero coefficient in* $\gamma_k[a_1|\cdots|a_n]$ *has consecutive occurrences of any* a_i.

Proof. We show how such terms cancel in pairs in the expression for the divided power, by defining a free involution on the set of terms with neighboring repeated letters with the property that the elements of each orbit occur with opposite signs. The involution will leave unchanged all the letters up to and including the left-most neighboring repeated pair.

If the repeated pair is $a_1|a_1$, swap the positions of the remaining letters in the two blocks initiated by these letters. We get an identical word, but since n is even this is an odd number of transpositions, so the terms cancel.

If the repeated pair is $a_i|a_i$ for $i > 1$, just swap those two entries. This is allowed since the leading term of both blocks precedes both entries in the repeated pair. □

Since every term in the expression for $\gamma_k[a_1|\cdots|a_n]$ has leading entry a_1, we obtain:

Corollary 8.2. $\gamma_k[a|b] = [a|b|a|b|\cdots|a|b]$.

In general the expression for γ_k seems very complicated. For example, a computer calculation shows that $\gamma_3[a|b|c|d]$ has 53 terms, with coefficients ranging from -4 to 6.

Write $Barr_*(A)$ for the homology of the chain complex of A-modules obtained from $C_\bullet(A)$ quotienting out by decomposables and the image of divided powers. Since $k!\gamma_k(\omega)$ is decomposable, we have an exact sequence

$$0 \to Harr_{2k+1}(A) \to Barr_{2k+1}(A) \to T_{2k} \to Harr_{2k}(A) \to Barr_{2k}(A) \to 0$$

where $k!T_{2k} = 0$. Thus

$$Harr_{2k+1}(A) \to Barr_{2k+1}(A) \text{ is injective with cokernel killed by } k!\,,$$
$$Harr_{2k}(A) \to Barr_{2k}(A) \text{ is surjective with kernel killed by } k!\,.$$

We can also form the "Barr cohomology" with coefficients in an A-module. Its cochains consist of Hochschild cochains s satisfying the Harrison symmetry conditions with the additional conditions

$$s(\gamma_k(\omega)) = 0\,, \quad k > 0\,,$$

for $|\omega|$ even; for example $s(a, b, a, b) = 0$ in dimension 4; $s(a, b, a, b, a, b) = 0$ in dimension 6; and in dimension 8 there are two additional symmetries,

$$s(a, b, a, b, a, b, a, b) = 0$$

guaranteeing annihilation of $\gamma_4[a|b]$, and

$$\begin{aligned} &s(a, b, c, d, a, b, c, d) - s(a, b, c, a, d, b, c, d) + s(a, b, c, a, b, d, c, d) \\ &+s(a, b, a, c, d, b, c, d) - s(a, b, a, c, b, d, c, d) + 2s(a, b, a, b, c, d, c, d) = 0 \end{aligned}$$

to annihilate $\gamma_2[a|b|c|d]$.

Remark 8.3. It is natural to hope that the natural map of A-modules $Hoch_n(A) \to HQ_{n-1}(A)$ factors as

$$Hoch_n(A) \to Harr_n(A) \to Barr_n(A) \to HQ_{n-1}(A)\,,$$

but this seems unlikely to us except in low dimensions.

9. Grillet's work

In a series of papers, Pierre Grillet associates to a commutative monoid X and a Beck module M over it the beginning of a cochain complex and proves or conjectures that it computes the low-dimensional components of the Quillen cohomology $HQ^*_{CM}(X;M)$. We observe that the symmetry conditions he imposes are precisely the monontone conditions, with two variations which correspond to Barr's variation on the Harrison complex. We will not attempt a complete survey of Grillet's work on this subject, but merely note the occurrence of symmetry conditions that we now see as Harrison or Barr symmetry conditions on Hochschild cochains, and where in Grillet's work they are proved to yield cohomology groups isomorphic to Quillen's.

To begin with, for any X graded K-algebra and A-module M,

$$Harr^1(A;M) \cong Hoch^1(A;M) \cong HQ^0_{CA}(A;M) \cong Der_K(A,M)$$

so

$$Harr^1(\widetilde{\mathbb{Z}}X;M) \cong HQ^0_{CM}(X;M)\,.$$

In 1974 [18] Grillet used 2-cocycles s with the symmetry

$$s(a,b) = s(b,a)$$

to classify extensions of commutative monoids. This is of course the 2-monotone symmetry. Twenty years later, in [19], he returned to this by invoking Quillen cohomology as an intermediary, thus showing that

$$Harr^2(\widetilde{\mathbb{Z}}X;M) \cong HQ^1_{CM}(X;M)\,.$$

(Grillet chooses to index Quillen homology following the Hochschild convention, so he would write $HQ^2_{CM}(X;M)$.)

This paper was supplemented by [20], which confirmed this result by direct computation and extended it to dimension 3 using the symmetry conditions

$$s(a,b,c) + s(c,b,a) = 0$$
$$s(a,b,c) + s(b,c,a) + s(c,a,b) = 0\,.$$

Taken together these are equivalent to the k-monotone symmetries for $k=2$ and 3. A remarkable calculation then verifies that

$$Harr^3(\widetilde{\mathbb{Z}}X;M) \cong HQ^2_{CM}(X;M)$$

This work was consolidated and summarized in his book [21].

After another twenty years, Grillet returned again to this project, in [22], extending his calculation to dimension 4 using cochains satisfying the symmetry conditions simplified in [23] to

$$\begin{gathered} s(a,b,c,d) - s(b,a,c,d) + s(b,c,a,d) - s(b,d,d,a) = 0\,, \\ s(a,b,c,d) + s(d,c,b,a) = 0\,, \\ s(a,b,b,a) = 0\,. \end{gathered}$$

The reader will recognize the first two as the 2-monotone and 4-monotone symmetries. The first equation implies $s(a,b,a,b) = s(b,a,a,b)$, so the third condition is equivalent to the Barr variant $s(a,b,a,b) = 0$. And here again, Grillet obtains the surprising result

$$Barr^4(\widetilde{\mathbb{Z}}X; M) \cong HQ^3_{CM}(X; M)\,.$$

This work was quickly followed by [24], in which Grillet proposes symmetric conditions on Hochschild cocycles extending into dimensions 5 and 6. In dimension 5 he proposes

$$\begin{gathered} s(a,b,c,d,e) - s(b,a,c,d,e) + s(b,c,a,d,e) - s(b,c,d,a,e) + s(b,c,d,e,a) = 0 \\ s(a,b,c,d,e) + s(e,d,c,b,a) = 0\,. \end{gathered}$$

We recognize these as the 2-monotone and 5-monotone conditions, which suffice to determine Harrison cohomology. In dimension 6 his proposed symmetries are precisely the k-monotone conditions for $k = 2, 3$, and 6, augmented by the Barr variant $s(a,b,a,b,a,b) = 0$.

In these higher dimensions the identifications with Quillen cohomology are left as conjectures. We can now see that at least the 5-dimensional case was too optimistic. Write α for the unique map of commutative monoids $\mathbb{N} \to e$. Then

$$Barr^\bullet(\widetilde{\mathbb{Z}}\mathbb{N}; \alpha^*\mathbb{F}_p) = Barr^\bullet(\mathbb{Z}[x]; \mathbb{F}_p)\,.$$

Let $c : \mathbb{F}_2[x]^{\otimes 5} \to \mathbb{F}_2$ be the non-bounding Barr cocycle described by Whitehouse [40]. Its composite with $\mathbb{Z}[x]^{\otimes 5} \to \mathbb{F}_2[x]^{\otimes 5}$ is again a cocycle and satisfies the same invariance properties, and if this composite were a coboundary then c would be too. So $Barr^5(\mathbb{Z}[x]; \mathbb{F}_2)$, which is the cohomology in degree 5 of Grillet's complex for the free commutative monoid $\mathbb{N}$ with coefficients in $\alpha^*\mathbb{F}_2$, is nontrivial.

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