The bar construction spectral sequence of an iterated loop space

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Here are some guesses about the structure of the mod 2 homology bar construction spectral sequence for an iterated loop space.

First suppose X is an infinite loop space. Ligaard and Madsen construct "vertical" DL operations

$$Q^i : E^2_{s,t} \to E^2_{s,t+i}\,, \quad i \geq t\,.$$

If $x \in E^2_{s,t}$ survives to $a \in H_{s+t}(BX)$, these operations are supposed to represent $Q^i a$. But $Q^i a = 0$ for $i < s+t$, so there must be canonical differentials killing $Q^i x$ for $i < s+t$.

I think that $Q^t x = 0$ already at E^2. Killing the rest is the job of the Cartan-Bousfield-Dwyer operations: We have natural elements

$$\delta_i x \in E^2_{s+i,2t}\,, \quad 2 \leq i \leq s$$

and

$$d^i(\delta_i x) = Q^{t+i-1}x\,, \quad 2 \leq i \leq s$$

with the usual caveats about indeterminacy.

When X is an $(n+1)$-fold loop space, the situation is more complicated because there aren't enough vertical operations. Here's what I think happens.

For a start, as John Ni showed, the Browder bracket, of degree n on $H_*(X)$, gives rise to a bracket

$$E^2_{s,t} \otimes E^2_{s',t'} \to E^2_{s+s'-1,t+t'+n}$$

corresponding to the degree $n-1$ bracket on $H_*(BX)$.

When $n = 1$, the E^2 term is commutative but $H_*(BX)$ is not, and the bracket represents the commutator bracket. There should also be a "restriction" map

$$\xi : E^2_{s,t} \to E^2_{2s-1,2t+1}$$

(possibly with some indeterminacy) that represents the square and satisfies

$$\xi(x+y) = \xi(x) + [x,y] + \xi(y)$$

if x and y are in the same bidegree. Moreover, there should be a vertical DL operator

$$Q^{t+1} : E^2_{s,t} \to E^2_{s,2t+1}$$

that coincides with ξ when $s = 1$. But for $x \in E^2_{s,t}$ with $s \geq 2$,

$$d^2(\delta_2 x) = Q^{t+1} x \, .$$

For larger n I would expect vertical operations

$$Q^{t+i} : E^2_{s,t} \to E^2_{s,t+i} \, , \quad 0 \leq i \leq n$$

The pattern of differentials and non-vertical operations will depend on the relation between n and s.

For $s > n$, I expect differentials

$$d^i(\delta_i x) = Q^{t+i-1} x \, , \quad 2 \leq i \leq n+1$$

and extra operations

$$Q^{s+t+i} : E^2_{s,t} \to E^2_{2s-n+i,2t+n} \, , \quad 0 \leq i < n$$

representing the DL operator of this name on elements in $H_{s+t}(BX)$.

For $s \leq n$ there will be the differentials as above but only for $2 \leq i \leq s$. So vertical operators $Q^{t+i}x$ remain for $s \leq i \leq n$, but we need more and they will be given by

$$Q^{s+t+i} : E^2_{s,t} \to E^2_{2s-n+i,2t+n} \, , \quad n - s < i < n$$

In both cases, the top operation $Q^{s+t+n-1} x \in E^2_{2s-1,2t+n}$ is the restriction (and so should be nonlinear).