BECK MODULES AND ALTERNATIVE ALGEBRAS

NISHANT DHANKHAR, HAYNES MILLER, AND ALI TAHBOUB

Abstract. We set out the general theory of "Beck modules" in a variety of algebras and describe them as modules over suitable "universal enveloping" unital associative algebras. We pay particular attention to the somewhat nonstandard case of "alternative algebras," defined by a restricted associative law, and determine the Poincaré polynomial of the universal enveloping algebra in the homogenous case.

1. Introduction

The notion of a "module" occupies an important place in the study of general algebraic systems. Most of these diverse notions are united under the theory of "Beck modules." Given an object A in any category $\mathbf{C}$, one may consider the "slice category" $\mathbf{C}/A$ of objects in $\mathbf{C}$ equipped with a map to A. A Beck module for A is then an abelian group object in $\mathbf{C}/A$. If $\mathbf{C}$ is the category of commutative rings, for example, a Beck module for A is simply an A-module, while if $\mathbf{C}$ is the category of associative algebras, a Beck module for A is an A-bimodule. Many other examples occur in the literature.

This definition occurs in the thesis [5] of Jonathan Beck written under the direction of Samuel Eilenberg. Eilenberg himself had discussed such objects in [8], at least in the linear context, as the kernel of a "square zero extension." These kernels were understood to constitute "representations" of the algebra, and this structure was made explicit in various cases.

We review below the context of a "variety" $\mathbf{V}$ of algebras over a commutative ring K. In this case, for every $\mathbf{V}$-algebra A the category $\mathbf{Mod}_A$ of Beck A-modules is an abelian category with a single projective generator. As a result, the category $\mathbf{Mod}_A$ is equivalent to the category of right modules over a canonical unital associative K-algebra $U_{\mathbf{V}}(A)$, the "universal enveloping algebra" for A.

This raises the question of identifying the structure of $U_{\mathbf{V}}(A)$ for various varieties $\mathbf{V}$ and $\mathbf{V}$-algebras A. We review some of the standard examples, and then focus on a somewhat less standard one, the variety of "alternative algebras" over K. This example has been considered before, but even over a field basic features of the universal enveloping algebra for an alternative algebra, such as its dimension, have remained obscure. In 1954, Nathan Jacobson [12] wrote "The introduction of the universal associative algebras

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for the birepresentations [his term for Beck modules] enables one to split the representation problem into two parts: (1) determination of the structure of $U(A)$, (2) representation theory for the associative algebra $U(A)$. In practice, however, it seems to be difficult to treat (1) as a separate problem. Only in some special cases is it feasible to attack this directly." Richard Schafer [15] observed in 1966 that if $\mathbf{V}$ is the variety of alternative K-algebras, with K a field, and $\dim_K A = n$, then $\dim_K U(A) \leq 4^n$. But the precise dimension, even the case of "homogeneous" alternative algebras – those with trivial product – over a field, has eluded analysis.

In that case, the universal enveloping algebra admits a natural grading, by word length, or, as we call it, by weight. Write K^n for the free K-module on n generators, regarded as a homogeneous alternative K-algebra. The principal new result in this paper is the description of an explicit basis for the universal enveloping algebra of K^n when K is a field. A surprise is that characteristic 3 diverges from all other characteristics.

Theorem 1.1. *Let K be a field. If the field K is of characteristic different from 3,*

$$\dim U(K^n)_k = \begin{cases} \binom{2n}{k} & \text{if } k \leq 2 \\ 2\binom{n}{k} & \text{if } k \geq 3\,. \end{cases}$$

The same is true if the characteristic is 3 except when $k = 3$, in which case the dimension is $4\binom{n}{3}$.

In particular,

$$U(K^n)_k = 0 \quad \text{for} \quad k > n\,.$$

A base-change property of the universal enveloping algebra construction leads to the following corollary.

Corollary 1.2. *The weight k component of the graded ring $U(\mathbb{Z}^n)_*$ is free of rank $\binom{2n}{k}$ if $k \leq 2$ and $2\binom{n}{k}$ if $k > 3$, while in weight 3 it is the sum of $4\binom{n}{3}$ cyclic groups of which half are free and half are finite 3-groups.*

These calculations show that at least in the homogeneous case, the growth rate of the universal enveloping algebra is indeed exponential, but much slower than the upper bound observed by Schafer.

Our tool is the theory of Gröbner bases for noncommutative graded algebras. We employ the computer algebra system Sage to determine a Gröbner basis for the ideal of relations defining $U(K^n)$, with $K = \mathbb{Q}$, for $n \leq 5$. The structure of this basis for these small values of n turns out to imply that a basis with the same structure exists for all n. The set of normal monomials with respect to this basis (which projects to a basis for $U(K^n)$) is then easy to determine. An analysis of the Sage output indicates that the same is true for any characteristic different from 3, and gives the modified answer when the characteristic is 3. To our knowledge, this is the first example of the use of a computer algebra system to prove an infinite family of results.

After a review of the theory of varieties of algebras in §2, and a reminder of some particular features of alternative algebras in §3, we describe in §4 the theory of Beck modules in this generality and describe how they appear in our various examples. In §5 we discuss the universal enveloping algebra and some of its features, and in §6 we specialize to the case of alternative algebras. Finally, in §7, we review some of the essential features of the theory of Gröbner bases.

This work is intended as a first step in the study of the Quillen homology and cohomology of algbraic systems such as alternative algebras.

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2. Varieties of algebras

We will work with algebras defined by a product operation, though much of this work can be carried out in much greater generality. Following the lead of Bourbaki [6, §7.1], we make the following definition.

Definition 2.1. *A* magma *is a set X with a binary operation $X \times X \to X$ (written as juxtaposition). A* unital magma *is a magma X equipped with an element e such that $ex = x = xe$ for all $x \in X$.*

We will also restrict our attention to linear examples, and work over a commutative ring K. So a *magmatic K-algebra* (or just *K-algebra*) is a K-module A equipped with a K-bilinear product $A \otimes A \to A$ (written as juxtaposition). It becomes *unital* if it is equipped with an element 1 such that $1a = a = a1$ for all $a \in A$.

Magmatic K-algebras constitute the objects in a category $\mathbf{Mag}_K$. The forgetful functor to sets has a left adjoint $Mag_K : \mathbf{Set} \to \mathbf{Mag}_K$, so we can define the free magma on a set. The free magma $Mag(S)$ generated by a set S is the set of bracketed strings of elements of S; see [6, §7.1]. The free magmatic K-algebra on a set S is the free K-module generated by $Mag(S)$: $Mag_K(S) = KMag(S)$.

We can adjoin axioms using the following device [2]. An *equation* is an element of the free magmatic K-algebra on a finite set. Given a magmatic K-algebra A, we will say that an equation $\omega \in Mag_K(S)$ is *satisfied by A* if for any set map $S \to A$ the induced map $Mag_K(S) \to A$ sends ω to 0. A set of equations defines a *variety* of K-algebras, namely the subcategory of $\mathbf{Mag}_K$ cut out by (that is, satisfying) these equations. An object of a variety of K-algebras $\mathbf{V}$ is a "$\mathbf{V}$-algebra."

A variety of K-algebras is an "algebraic category" [9]. It is complete and cocomplete. Any subalgebra of a $\mathbf{V}$-algebra is again a $\mathbf{V}$-algebra. The forgetful functor $u : \mathbf{V} \to \mathbf{Mod}_K$ has a left adjoint

$$F : \mathbf{Mod}_K \to \mathbf{V}\,.$$

Examples 2.2. Here are four standard examples, beyond $\mathbf{Mag}_K$ itself.

- $\mathbf{Ass}_K$, the variety of associative algebras, is defined by the equation
$$(xy)z - x(yz) \in Mag_K\{x, y, z\}\,.$$
- Adding the further equation
$$xy - yx \in Mag_K\{x, y\}$$
gives us the variety of commutative K-algebras, $\mathbf{Com}_K$.
- A Lie algebra (in $\mathbf{Lie}_K$) is a K-algebra satisfying the equations
$$xx \in Mag_K\{x\}\,, \quad (xy)z + (yz)x + (zx)y \in Mag_K\{x, y, z\}\,.$$
- An alternative algebra is a magmatic K-algebra satisfying the equations
$$(xx)y - x(xy)\,,\, (xy)y - x(yy) \,\in\, Mag_K\{x, y\}\,.$$
These are the objects in the variety $\mathbf{Alt}_K$.

Note that we do not assume a unit element in any of these examples. There are many other axioms in use today – Vinberg algebras, Novikov algebras, Leibniz algebras, divided power algebras ....

There is a reversal involution $\overline{(-)} : \mathbf{Mag} \to \mathbf{Mag}$. It comes with a natural bijection of underlying sets $X \to \overline{X}$ that we will also denote with an overline, and $\overline{x}\,\overline{y} = \overline{yx}$. It extends to an involution on $\mathbf{Mag}_K$. To any variety $\mathbf{V}$ of K-algebras we can associate an "opposite" variety $\overline{\mathbf{V}}$, with defining equations given by reversing the defining equations of $\mathbf{V}$. By sending a $\mathbf{V}$-algebra to the same K-module with opposite multiplication, you get a natural equivalence of categories

$$\mathbf{V} \to \overline{\mathbf{V}}\,, \quad A \mapsto \overline{A}\,.$$

A variety $\mathbf{V}$ is *symmetric* $\overline{\mathbf{V}} = \mathbf{V}$. All the examples above are symmetric, but, for example, the variety of "left alternative K-algebras," satisfying $(xx)y - x(xy)$ but perhaps not $(xy)y - x(yy)$, is not symmetric; its opposite is the variety of right alternative K-algebras. If $\mathbf{V}$ is a symmetric variety, the isomorphism $\mathbf{V} \to \overline{\mathbf{V}}$ becomes an involution on $\mathbf{V}$, sending an algebra to the same K-module with the opposite multiplication.

We end with a discussion of base-change. Let $\alpha : K \to L$ be homomorphism of commutative rings. An equation in $\mathbf{Mag}_K(S)$ induces an equation in $\mathbf{Mag}_L(S)$. So a variety of K-algebras, say $\mathbf{V}_K$, induces a variety of L-algebras, which we'll denote by $\mathbf{V}_L$. This is a transitive operation. Moreover, the functor $L \otimes_K - : \mathbf{Mod}_K \to \mathbf{Mod}_L$ lifts to a functor $L \otimes_K - : \mathbf{V}_K \to \mathbf{V}_L$.

3. Alternative algebras

The example of alternative algebras is less familiar than the others and we spend a moment introducing it. Schafer's book [15] provides a good reference.

Any associative K-algebra is alternative, and Emil Artin proved that any alternative K-algebra with two generators is associative [7]. The alternative identities imply that the further "flexible" equation

$$(xy)x - x(yx)$$

is satisfied. The algebra of octonions [3] is a well-known example of a nonassociative alternative algebra.

The *associator* in a magmatic K-algebra is the trilinear form

$$(x, y, z) = (xy)z - x(yz)$$

In an alternative algebra the associator is an alternating form: transpositions reverse the sign. This suggests adding a further basic example, one defined by weight 3 equations:

- An almost alternative K-algebra is a magmatic K-algebra for which the associator is an alternating form; that is to say, satisfying the equations

$$(xy)z - x(yz) + (xz)y - x(zy)\,, \quad (xy)z - x(yz) + (yx)z - y(xz)\,.$$

Write $\mathbf{AAlt}_K$ for this variety. If 2 is invertible in K these axioms are equivalant to the alternative axioms. In some respects this "almost alternative" condition is better behaved than the alternative condition itself; it is operadic, for example. If K is an $\mathbb{F}_2$-algebra, multiplication table

	a	b	c
a	a	b	0
b	0	0	0
c	c	0	b

defines an almost alternative K-algebra that is not alternative.

A monoid X in **Set** defines an associative algebra in $\mathbf{Mod}_K$ by forming the free K-module on X. The equations for alternative algebras make sense in **Set**, so one can talk about "alternative sets." An alternative product on X determines a magmatic K-algebra structure on KX, but it is not necessarily alternative. For example the multiplication table

	a	b	c
a	a	a	c
b	a	b	b
c	c	b	c

is commutative and alternative, and hence even flexible, but the K-module that it generates is not alternative. We thank Hadeel AbuTabeekh for this example.

4. Beck modules

Let $\mathbf{V}$ be a variety of K-algebras and A a $\mathbf{V}$-algebra. The "slice category" $\mathbf{V}/A$ has as objects morphisms in $\mathbf{V}$ with target A, and as morphisms maps compatible with the projections to A. This slice category again has good properties; in particular it is complete and cocomplete. We can thus speak of abelian group objects in $\mathbf{V}/A$.

An abelian group structure on a $\mathbf{V}$-algebra over A, $p : B \downarrow A$, begins with a unit: a map from the terminal object of $\mathbf{V}/A$, that is, a section $\eta : A \uparrow B$ of p. This unit defines an "axis inclusion" $i : B \coprod B \to B \times_A B$ in $\mathbf{V}/A$. A magma structure on B is an extension of the "fold map" $\nabla : B \coprod B \to B$ over the product. In these algebraic situations, the map i is an epimorphism, so such an extension is unique if it exists: Being a unital magma in $\mathbf{V}/A$ is a *property* of a pointed object, not further structure on it. Furthermore, the unique unital magma structure with given unit, when it exists, is an abelian group structure. We call an object of this type an *abelian object*.

Definition 4.1. [5] *Let A be a $\mathbf{V}$-algebra. A* Beck A-module *is an abelian object in the slice category $\mathbf{V}/A$:*

$$\mathbf{Mod}_A = \mathrm{Ab}(\mathbf{V}/A)\,.$$

Proposition 4.2. [1, Theorem 3.16] *and* [4, Chapter 2, Theorem 2.4] $\mathbf{Mod}_A$ *is an abelian category.*

In our K-linear situation, write M for the kernel of $p : B \downarrow A$. Suppose $B \downarrow A$ has the structure of a unital magma in $\mathbf{V}/A$. This consists of two pieces of structure: the "unit" is a map from the terminal object in $\mathbf{Mod}_K/A$, that is, a section of $p : B \downarrow A$, and the "addition" is a map $\alpha : B \times_A B \to B$ over A. Since

$$B \times_A B = (A \oplus M) \times_A (A \oplus M) = A \oplus M \oplus M$$

the structure map has the form $\alpha : A \oplus M \oplus M \to A \oplus M$. Using linearity and unitality it's easy to see that the abelian group structure is actually determined by the addition in M:

$$\alpha(a, x, y) = (a, x + y)\,.$$

The K-algebra structure on $A \oplus M$ is described by left and right "actions"

$$A \otimes M \to M\,, \quad M \otimes A \to M$$

both of which we denote by juxtaposition. Together they determine the multiplication on $A \oplus M$ by

$$(a, x)(b, y) = (ab, ay + xb)\,.$$

Absent further axioms, these action maps satisfy no properties. This describes the category of magmatic Beck A-modules. If we are working with a general variety of K-algebras $\mathbf{V}$, the axioms of $\mathbf{V}$ will determine further properties of these two actions. Here's how this works out in our examples. In describing them it will be useful to adjoin a unit element to an object A in $\mathbf{Ass}_K$; so let's write

$$A_+ = K \oplus A$$

with product given by $(p, a)(q, b) = (pq, pb + qa)$.

- $\mathbf{V} = \mathbf{Ass}_K$: Applying the associativity equation to the triple (a, x), (b, y), (c, z) in $A \oplus M$ and setting all but one of x, y, z to 0 gives

  $$(xb)c = x(bc)\,, \quad (ay)c = a(yc)\,, \quad (ab)z = a(bz)\,.$$

  In other words, $\mathbf{Mod}_A$ is the usual category of bimodules over A_+ (for which K acts the same way on both sides).
- $\mathbf{V} = \mathbf{Com}_K$: The actions satisfy these axioms together with

  $$ax = xa$$

  In other words, $\mathbf{Mod}_A$ is the usual category of A_+-modules.
- $\mathbf{V} = \mathbf{Lie}_K$: $(a, x)(a, x) = 0$ gives $ax + xa = 0$, while the Jacobi identity $((a, x)(b, y))(c, z) + ((b, y)(c, z))(a, x) + ((c, z)(a, x))(b, y) = 0$ gives

  $$(xb)c + (bc)x + (cx)b = 0 \quad (ay)c + (yc)a + (ca)y = 0$$
  $$(ab)z + (bz)a + (za)b = 0\,.$$

  These three equations all say the same thing, which we may re-express using $ax = -xa$ as

  $$x(ab) = (xa)b - (xb)a\,.$$
- $\mathbf{V} = \mathbf{Alt}_K$: The equation $((a, x)(a, x))(b, y) = (a, x)((a, x)(b, y)$ gives

  $$(ax)b + (xa)b = a(xb) + x(ab)\,, \quad (aa)y = a(ay)$$

  while the other alternative axiom gives

  $$(xb)b = x(bb)\,, \quad (ab)y + (ay)b = a(by) + a(yb)\,.$$

  Rearranging gives

  $$(ab)x - a(bx) = a(xb) - (ax)b = (xa)b - x(ab)$$
  $$(aa)x = a(ax)\,, \quad (xb)b = x(bb)\,.$$
- $\mathbf{V} = \mathbf{AAlt}_K$: The the associator (u, v, w) makes sense whenever two of $\{u, v, w\}$ are in the K-algebra A and the third is in a Beck module for it. The linearization of the almost alternating condition is then simply that

  $$(a, x, b)$$

  is alternating for $a, b \in A$ and x in a Beck module M. These are the "alternative modules" of [15, p. 66], where the connection between such objects and extensions of the alternative algebra A is made explicit.

Let $\mathbf{V}$ be a variety of K-algebras and A a $\mathbf{V}$-algebra. The forgetful functor $\mathbf{Mod}_A \to \mathbf{Mod}_K$ sending M to its underlying K-module has a left adjoint

$$F_A : \mathbf{Mod}_K \to \mathbf{Mod}_A .$$

In **Ass** this is given by

$$F_A(V) = A \otimes V \otimes A .$$

In **Com** it is given by

$$F_A(V) = V \otimes A .$$

The free Beck module construction in the other cases is most conveniently described in terms of the universal enveloping algebra; see §5.

Finally, we take note of a canonical object in $\mathbf{Mod}_A$, the "regular representation" of A:

Lemma 4.3. *In any variety of K-algebras, A itself is a Beck A-module, using its multiplication for both actions.*

Proof. Given a magmatic algebra A, define the "dual number" magmatic algebra to be $A \oplus Ae$ with product given by $(a+xe)(b+ye) = ab+(ay+xb)e$. This is an object in $\mathrm{Ab}(\mathbf{Mag}_K/A)$. The map $a \to (a,0)$ is a section, and the resulting actions of A on the kernel establish A as a Beck module over itself. So the result holds for magmatic Beck modules.

Let S be a finite set and ω an element of $KMag(S)$. We claim that if ω is satisfied by A then it is also satisfied by $A \oplus Ae$. Applying this to all the equations defining the variety $\mathbf{V}$, we reach our conclusion.

Given a map $S = \{s_1, \ldots, s_n\} \to A$ to the underlying set of A sending s_i to a_i, denote by $\omega(a_1, \ldots, a_n)$ the image of ω under the extension of this map to a K-algebra map. Then

$$\omega(a_1 + x_1 e, \ldots, a_n + x_n e) = \omega(a_1, \ldots, a_n) + \sum_i \omega(a_1, \ldots, x_i, \ldots, a_n)e$$

and all the terms on the right vanish. □

5. The universal enveloping algebra

In many cases, such as $\mathbf{Lie}_K$ and $\mathbf{Alt}_K$, it is useful to use the Freyd embedding theorem [10, p. 106] to express the category $\mathbf{Mod}_A$ as the category of modules over an appropriate associative K-algebra with unit, $U_{\mathbf{V}}(A)$, which we term the *universal enveloping algebra*. In the operadic context, this construction was described for example by Ginzburg and Kapranov [11, §1.6].

So let $\mathbf{V}$ be a variety of K-algebras and $A \in \mathbf{V}$. The forgetful functor $u : \mathbf{Mod}_A \to \mathbf{Mod}_K$ has a left adjoint $F_A : \mathbf{Mod}_K \to \mathbf{Mod}_A$. Since

$$\mathbf{Mod}_A(F_A(K), M) = \mathbf{Mod}_K(K, uM) = uM ,$$

the functor $\mathbf{Mod}_A(F_A K, -)$ is faithful and $F_A K$ is a projective object: it serves as the projective generator for the category $\mathbf{Mod}_A$ required by Freyd's theorem.

The associative K-algebra $U_{\mathbf{V}}(A)$ is defined as the endomorphism K-algebra of the free Beck A-module on one generator:

$$U_{\mathbf{V}}(A) = \mathrm{End}_A(F_A(K))\,.$$

By the universal property of $F_A(K)$, the underlying K-module of $U_{\mathbf{V}}(A)$ is the same as that of $F_A(K)$. The identity endomorphism e is the image in $F_A(K)$ of $1 \in K$ under the natural map $K \to uF_A(K)$; it is the unit element in this associative K-algebra.

Now the K-module underlying a Beck module M has the structure of a right $U_{\mathbf{V}}(A)$-module: Represent $x \in M$ by $\hat{x} : F_A(K) \to M$ and let $f \in U_{\mathbf{V}}(A)$, i.e. $f : F_A(K) \to F_A(K)$. Then xf is the element represented by $\hat{x} \circ f$.

This construction is of course natural in the algebra A. Any variety of K-algebras has a terminal object, the trivial K-module 0, and the defining property of the universal enveloping algebra implies that $U_{\mathbf{V}}(0) = K$. So a universal enveloping algebra always has a canonical augmentation

$$\epsilon : U_{\mathbf{V}}(A) \to K\,.$$

There is a K-linear map $A \oplus A \to U_{\mathbf{V}}(A)$ given by

$$(a, b) \mapsto ae + eb\,.$$

Let's write $l_a \in U_{\mathbf{V}}(A)$ for the element corresponding to ae and $r_b \in U_{\mathbf{V}}(A)$ for the element corresponding to eb.

This map extends to a surjection of unital associative K-algebras

$$\mathrm{Tens}_K(A \oplus A) \to U_{\mathbf{V}}(A)\,.$$

If $\mathbf{V} = \mathbf{Mag}_K$, this map is an isomorphism. In general, the kernel will depend on what axioms one has in $\mathbf{V}$. For any finite set S and $\omega \in \mathbf{Mag}_K S$, the process of writing down the corresponding relations in $U_{\mathbf{V}}(A)$ for $\mathbf{V}$ a variety of K-algebras satisfying ω is a form of noncommutative partial differentiation. Here are our standard examples again.

- $\mathbf{V} = \mathbf{Ass}$: The identities read

  $$r_a r_b = r_{ab}\,, \quad l_a r_b = r_b l_a\,, \quad l_{ab} = l_b l_a\,.$$

  The quotient of $\mathrm{Tens}_K(A \oplus A)$ by this relation is the usual "extended algebra":

  $$U_{\mathbf{Ass}}(A) = A^e = A_+^{op} \otimes A_+$$

  In this representation, $r_a = 1 \otimes a$ and $l_a = a \otimes 1$.
- $\mathbf{V} = \mathbf{Com}$: Now $U_{\mathbf{Com}}(A) = A_+$, and both l and r are the natural embedding of A into the direct sum $K \oplus A$.
- $\mathbf{V} = \mathbf{Lie}$: Now $r = -l$, and the second equation in our description of Beck modules in $\mathbf{Lie}$ gives

  $$r_{ab} = r_a r_b - r_b r_a$$

  This equation defines the usual Lie universal enveloping algebra.

- $\mathbf{V} = \mathbf{Alt}$: Here are the relations among the maps l and r (cf. [14, equation (6) on p. 2]):

$$l_a^2 = l_{aa}\,, \quad r_b^2 = r_{bb}$$
$$l_{ab} - l_b l_a = r_b l_a - l_a r_b = r_a r_b - r_{ab}$$

This use of the term "universal enveloping algebra" differs from the classical Lie perspective. In that case, one has a "forgetful" functor $\mathbf{Ass}_K \to \mathbf{Lie}_K$ that sends an associative K-algebra to the Lie algebra structure on the K-module A given by $[a, b] = ab - ba$; and U is the left adjoint of this functor. In our generality, there is no "underlying" associative algebra; the universal enveloping algebra has a different defining property. But it turns out to produce the same result in the case of $\mathbf{Lie}_K$.

The reversal endomorphism of $\mathbf{Mag}$ induces natural isomorphisms

$$U_{\overline{\mathbf{V}}}(\overline{A}) = U_{\mathbf{V}}(A)^{op}$$

that swaps the K-module maps r, l from the K-module A.

Formation of the universal enveloping algebra enjoys a strong base-change property.

Proposition 5.1. *Let $\alpha : K \to L$ be a homomorphism of commutative rings, $\mathbf{V}_K$ a variety of K-algebras, and $A \in \mathbf{V}_K$. There is a natural map $\alpha : U_{\mathbf{V}_K}(A) \to U_{\mathbf{V}_L}(L \otimes_K A)$ of unital K-algebras that extends to an isomorphism*

$$e_{\mathbf{V}} : L \otimes_K U_{\mathbf{V}_K}(A) \to U_{\mathbf{V}_L}(L \otimes_K A)$$

of unital L-algebras.

Proof. To begin with, extension of scalars gives us a map

$$e : \mathrm{Tens}_K(A \oplus A) \to \mathrm{Tens}_L((L \otimes_K A) \oplus (L \otimes_K A))\,.$$

Let $\omega \in \mathbf{Mag}_K(S)$ be an equation satisfied by objects of $\mathbf{V}_K$. Each map $a : S \to A$ determines an element $\omega'(a)$ of $\mathrm{Tens}_K(A \oplus A)$ that vanishes in $U_{\mathbf{V}_K}(A)$, by the process exemplified above. The image of ω in $\mathbf{Mag}_L(S)$ together with the composite $S \to A \to L \otimes_K A$ determines an element of $\mathrm{Tens}_L((L \otimes_K A) \oplus (L \otimes_K A))$ that vanishes in $U_{\mathbf{V}_L}(L \otimes_K A)$. This element is none other than $e(\omega'(a))$. This shows that the map on tensor algebras descends to a map on universal enveloping algebra quotients.

Since $L \otimes_K (A^{\otimes_K n}) \cong (L \otimes_K A)^{\otimes_L n}$, the map $e_{\mathbf{V}}$ is an isomorphism in the absence of equations. The ideals match up since elements $e(\omega'(a))$ generate the ideal of relations in $U_{\mathbf{V}_L}(L \otimes_K A)$. □

We have three closely related varieties of K-algebras, related by forgetful right adjoints

$$\mathbf{Com}_K \xrightarrow{u} \mathbf{Ass}_K \xrightarrow{u} \mathbf{Alt}_K\,.$$

The left adjoints are given by forming the maximal associative quotient of an alternative algebra, and the maximal commutative quotient of an associative algebra. These functors induce right adjoints

$$\mathbf{Com}_K/A \to \mathbf{Ass}_K/uA$$

for $A \in \mathbf{Com}_K$ and

$$\mathbf{Ass}_K/A \to \mathbf{Alt}_K/uA$$

for $A \in \mathbf{Ass}_K$. As right adjoints, they preserve products, and hence induce functors

$$\mathrm{Ab}(\mathbf{Com}_K/A) \to \mathrm{Ab}(\mathbf{Ass}_K/uA)\,, \quad \mathrm{Ab}(\mathbf{Ass}_K/A) \to \mathrm{Ab}(\mathbf{Alt}_K/uA)\,.$$

These functors can be described by means of ring homomorphisms between the corresponding universal enveloping algebras: There is a K-algebra surjection natural in $A \in \mathbf{Ass}_K$

$$U_{\mathbf{Alt}}(uA) \to U_{\mathbf{Ass}}(A)\,, \quad l_a \mapsto a \otimes 1\,, \quad r_b \mapsto 1 \otimes b\,,$$

and a K-algebra surjection natural in $A \in \mathbf{Com}_K$

$$U_{\mathbf{Ass}}(uA) \to U_{\mathbf{Com}}(A)\,, \quad a \otimes b \mapsto ab\,.$$

6. Universal enveloping algebras for $\mathbf{Alt}_K$

Let K^n denote the free K-module of rank n regarded as an alternative K-algebra with trivial product. Write l_i and r_i, $1 \le i \le n$, for the images in $U(K^n)$ of the standard basis elements under $l, r : K^n \to U(K^n)$. This associative K-algebra is graded by weight. Clearly $U(K^0) = K$ and $U(K^1)$ has basis $\{1, l_1, r_1, l_1 r_1\}$; in fact $l_1^2 = r_1^2 = l_1 r_1 - r_1 l_1 = 0$.

Theorem 6.1. *Let K be a field of characteristic different from 3. Then*

$$\dim U(K^n)_k = \begin{cases} \binom{2n}{k} & k \le 2 \\ 2\binom{n}{k} & k \ge 3\,. \end{cases}$$

In particular, $U(K^n)_k = 0$ for $k > n$ as long as $n > 1$. The growth of $\dim_K U(K^n)$ with n is exponential;

$$\dim_K U(K^n) = 2 \cdot 2^n + (n^2 - 1)\,.$$

Proof. In fact we can make a more precise statement, specifying for each k a set of monomials in $S = \{l_1, r_1, \ldots, l_n, r_n\}$ forming a basis for $U(K^n)_k$. Order S as shown; order monomials first by weight and within a given weight left-lexicographically. The "leading monomial" in a polynomial will be the least term.

In weight 0, we have only 1. In weight 1 we have S. In weight 2 the basis consists of the elements xy where $x, y \in S$ with $x > y$. In weight $k \ge 3$, the basis elements are

$$r_{i(1)} r_{i(2)} \cdots r_{i(k-1)} r_{i(k)} \quad \text{and} \quad r_{i(1)} r_{i(2)} \cdots r_{i(k-1)} l_{i(k)}$$

where i ranges over all strictly decreasing sequences of length k.

This follows from the theory of Gröbner bases, which we review briefly in the next section, assisted by a computation with Sage. We claim that for any n the following list constitutes a reduced Gröbner basis for $I = \ker(\mathrm{Tens}_K\{l_1, r_1, \ldots, l_n, r_n\} \to U(K^n))$. We put leading monomials first.

- weight 2, length 1: l_il_i, r_ir_i for all i
- weight 2, length 2: $l_ir_i - r_il_i$ for all i
- weight 2, length 2: $r_jr_i + l_il_j$ and $l_jl_i + r_ir_j$ for $i > j$
- weight 2, length 3: $r_jl_i - l_ir_j - r_ir_j$ and $l_jr_i - l_ir_j - r_il_j$ for $i > j$
- weight 3, length 1: $r_ir_jl_j$, $l_ir_jl_j$, $r_il_ir_j$, and $r_il_il_j$ for $i > j$
- weight 3, length 2: $r_il_jl_k - r_ir_jr_k$, $l_ir_jr_k - r_ir_jl_k$, $l_il_jr_k - r_ir_jr_k$, and $l_il_jl_k - r_ir_jl_k$ for $i > j > k$
- weight 3, length 3: $r_il_jr_k + r_ir_jl_k + r_ir_jr_k$ and $l_ir_jl_k + r_ir_jl_k + r_ir_jr_k$ for $i > j > k$

For $n \leq 5$, this is verified over $\mathbb{Q}$ by a Sage computation. This computation actually suffices to prove the general case in characteristic 0. To see this, notice that in this range the only new relations beyond the defining relations are of weight 3. Thus polynomials arising from overlap differences involving 4 or more generators do not result in any additions to the basis. Stated this way, the same calculation applies for all larger values of n.

Now one must verify that none of the given monomials is divisible by the leading monomial of any Gröbner basis element. The weight 2 Gröbner polynomials enforce the condition that the indices in a normal basis element are weakly increasing, and strictly increasing except that r_il_i is allowed. This gives the basis in weights less than 3. The six weight three patterns exclude l's in normal monomials of weight greater than or equal to 3, except possibly at the end.

This argument suffices to verify the theorem if the characteristic is zero. For K of finite characteristic one must certify that the given basis serves as a Gröbner basis over K as well. For this, one first executes the overlap procedure for pairs of weight 2 polynomial relations. The result differs from the list given by Sage. But one checks that each of the overlap polynomials is in fact an integral linear combination of the polynomials given by Sage, and conversely each of the Sage polynomials is a linear combination of the overlap polynomials. This latter step precisely requires 3 to be invertible, so it succeeds over any field of characteristic not 3.

Here is some more detail. The generators of the defining ideal I are given by the set R_0 consisting in

$$r_ir_i\,,\quad l_ir_i - r_il_i\,,\quad l_il_i\,,\quad \text{for } 1 \leq i \leq n$$
$$r_jr_i + l_il_j\,,\quad l_jl_i + r_ir_j\,,\quad \text{for } 1 \leq j < i \leq n$$
$$r_jl_i - l_ir_j - r_ir_j\,,\quad l_jr_i - l_ir_j - r_il_j\,,\quad \text{for } 1 \leq j < i \leq n\,.$$

The overlap differences come in three types, depending on whether one, two, or three indices are involved. The overlaps with just one index are already divisible by $LM(R_0)$ since r_ir_i and l_il_i are. For fixed $j < i$, half of the sixteen cases are not needed for the same reason. Each of the remaining eight overlap polynomials reduces using $LM(R_0)$ to an elment of the span of

$$r_ir_jl_j\,,\quad r_il_ir_j\,,\quad r_il_il_j\,,\quad l_ir_jl_j\,,$$

and together they span this subspace, so we add these terms to the basis.

For each sequence $i > j > k$ between 1 and n we can consider overlaps of elements of R_0 whose leading terms have indices given by $(k, j), (j, i)$ There are eight overlap patterns, indexed by the sequence of letters in the leading entries. Each overlap difference can be reduced using R_0 to a linear combination of monomials with subscripts (i, j, k). They are given by the following table of coefficients.

	rrr	rrl	rlr	rll	lrr	lrl	llr	lll
rr, rr	1	1	1	−1	1	0	1	−1
rr, rl	0	0	1	1	0	1	1	2
rl, lr	1	0	1	0	2	1	1	0
rl, ll	1	0	1	0	1	1	1	1
lr, rr	1	0	1	−1	0	−1	0	0
lr, rl	0	1	1	2	0	1	0	1
ll, lr	1	1	1	1	1	1	0	0
ll, ll	−1	1	0	1	−1	1	1	1

Sage suggests that the rational vector space spanned by these eight vectors should have basis given by the six vectors with coefficients given by the table

rrr	rrl	rlr	rll	lrr	lrl	llr	lll
0	-1	0	0	0	0	0	1
-1	0	0	0	0	0	1	0
1	1	0	0	0	1	0	0
0	-1	0	0	1	0	0	0
-1	0	0	1	0	0	0	0
1	1	1	0	0	0	0	0

It is easily checked that in fact all eight vectors are integral linear combinations of these six. Now the row rank of the square matrix above is 6 in characteristic not 3, but only 5 in characteristic 3, so the second six vectors can be expressed as linear combinations of the first eight except in characteristic 3, where they cannot be so expressed. This accounts for the divergence of characteristic 3.

But in characteristic not 3, one now adjoins these six vectors for each $i > j > k$ to obtain a larger basis R_1. Then one checks that all new ovelap differences reduce to 0 using R_1, so we have obtained the Gröbner basis as described. □

Theorem 6.2. *If the characteristic of K is 3,*

$$\dim U(K^n)_k = \begin{cases} \binom{2n}{k} & k \leq 2 \\ 4\binom{n}{3} & k = 3 \\ 2\binom{n}{k} & k \geq 4\,. \end{cases}$$

Proof. The proof is the same, with the following change. The normal monomials of weight not 3 are as before, but in weight 3 we have twice as many:

$$r_ir_jr_k,\ r_ir_jl_k,\ r_il_jr_k,\ r_il_jl_k\,,\quad i>j>k\,.$$

This follows from the fact that a reduced Gröbner basis is given by the following set.

- weight 2, length 1: l_il_i , r_ir_i for all i
- weight 2, length 2: $l_ir_i - r_il_i$ for all i
- weight 2, length 2: $r_jr_i + l_il_j$ and $l_jl_i + r_ir_j$ for $i > j$
- weight 2, length 3: $r_jl_i - l_ir_j - r_ir_j$ and $l_jr_i - l_ir_j - r_il_j$ for $i > j$
- weight 3, length 1: $r_ir_jl_j$, $l_ir_jl_j$, $r_il_ir_j$, and $r_il_il_j$ for $i > j$
- weight 3, length 4: $l_ir_jr_k - r_il_jl_k - r_il_jr_k + r_ir_jl_k$ and $l_ir_jl_k + r_il_jl_k - r_il_jr_k - r_ir_jr_k$ for $i > j > k$
- weight 3, length 5: $l_il_jr_k + r_il_jl_k + r_il_jr_k + r_ir_jl_k - r_ir_jr_k$ and $l_il_jl_k + r_il_jl_k - r_il_jr_k + r_ir_jl_k + r_ir_jr_k$ for $i > j > k$
- weight 4, length 2: $r_ir_jl_kl_m - r_ir_jr_kr_m$ for $i > j > k > m$
- weight 4, length 3: $r_ir_jl_kr_m + r_ir_jr_kl_m + r_ir_jr_kr_m$ for $i > j > k > m$

Sage verifies this table for $n \leq 7$, and the general result follows as before. □

Thus if K is a field of characteristic 3,

$$\dim U(K^n) = 2\cdot 2^n + \frac{n^3+2n-3}{3}\,.$$

If A is an alternative algebra over $\mathbb{Z}$,

$$U(K\otimes A) = K\otimes U(A)$$

for any commutative ring K by Proposition 5.1. Since each weight component of $U(\mathbb{Z}^n)$ is a finitely generated abelian group, we discover that $U(\mathbb{Z}^n)_k$ is torsion-free (of known rank) unless $k = 3$, and that $U(\mathbb{Z}^n)_3$ is a sum of $2\binom{n}{3}$ infinite cyclic groups and the same number of nontrivial finite cyclic 3-groups.

7. Gröbner bases

We elaborate briefly on the Gröbner process.

Let S be a set equipped with a well-founded partial order: a partial order in which every strictly decreasing sequence is finite. The free monoid B generated by S inherits a partial order – first by weight, and left-lexicographically within a given weight – that is again well-founded.

Let K be a field. The free K-module generated by B is the tensor algebra on S, $T = KS$.

Any nonzero element in T has a “leading monomial,” the least monomial occurring with nonzero coefficient in its expression as a linear combination of elements of B. Write

$$LM : T^* \to B$$

for this function, where we write $I^* = I - \{0\}$ for any ideal I in T. The partial order on B pulls back to a well-founded weak ordering on T^*.

For $u, v \in B$, say $u|v$ if there are monomials s, t such that $v = sut$. Divisiblity is transitive.

Definition 7.1. *Let I be an ideal in T. A subset G of I^* is a* Gröbner basis *for I if for all $r \in I^*$ there is $g \in G$ such that $LM(g)|LM(r)$.*

A Gröbner basis G for I yields an efficient algorithm for deciding whether $z \in T^*$ lies in I^*. We may assume z is monic; that is, the coefficient of $LM(z)$ is 1. If $LM(z)$ is not divisble by $LM(g)$ for any $g \in G$, then $z \notin I^*$. If instead

$$LM(z) = sLM(g)t$$

for some $g \in G$ and $s, t \in B$, then form

$$z' = z - sgt\,.$$

This is in I if and only if z is. If $z' = 0$, we have established that $z \in I$. If not, at least we can say that the leading monomials cancel, so z' is strictly less than z in the well order on T^*. Divide z' by the coefficient of its leading monomial, and repeat this process, which terminates because the ordering is well-founded. The original element z is in I if and only if the element you wind up with is 0, in which case you have written z as an explicit linear combination of terms divisible by elements of G. This shows that G generates I as an ideal.

Given a subset R of I that generates I as an ideal, we attempt to enlarge it to a Gröbner basis for I. We may assume that each element of R is monic. Say that $r, s \in R$ *overlap* if there are $a, b, c \in B$ such that $LM(r) = ab$ and $LM(s) = bc$. For each overlapping pair r, s, form the "overlap difference"

$$rc - as\,.$$

This difference again lies in I, but exhibits a new leading monomial, one that was hidden in the original generating set R.

Now use the reduction process with respect to R to simplify each of the overlap differences.

Then adjoin all the nonzero reduced overlap differences, made monic, to the set R to get a new generating set R'. One may want to precede this step by doing some linear algebra to find a simpler basis for the space spanned by these polyomials.

Proposition 7.2 ([13], Prop. 5.2,p. 95)**.** *If R is finite, this process terminates, and the result is a Gröbner basis for I.*

A Gröbner basis is minimal (no subset is a Gröbner basis for the same ideal) if and only if it is reduced (no divisibility relations among its leading monomials) [13, p. 67]. One may always refine a Gröbner basis to a reduced one.

A monomial u is *normal* mod I if it is not divisible by any element of $LM(I^*)$. If G is a Gröbner basis for I, it suffices to check non-divisibility by the leading monomials of elements of G: Suppose that $r \in I^*$ is such that $LM(r)|u$, and let $g \in G$ be such that $LM(g)|LM(r)$: then $LM(g)|u$.

Proposition 7.3 ([13], Prop. 3.3, p. 70)**.** *The set of monomials that are normal mod I projects to a vector space basis for T/I.*

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