18.781 Problem Set 5: Due Wednesday, March 22. *Second edition*

1. (a) Find all the reduced quadratic irrationals of discriminant 328; this is the set $R(328)$. (There are ten.) Record also the coefficients a, b, c for each one.

(b) Determine the action of the continued fraction operator $\phi(\alpha) = \frac{1}{\alpha - [\alpha]}$ on $R(328)$.

2. (This is a continuation of PS4#1.) **(a)** Let a and c be relatively prime, with $c > 0$. Show that there is exactly one solution in integers x, y with $0 \leq y < c$ to each of the equations

$$ay - cx = 1; \quad ay - cx = -1. \tag{1}$$

Explain how to find them, using the continued fraction expansion of a/c. Find them for example in case $a = 61, c = 23$.

(b) Given a proper continued fraction $\langle q_0, q_1, \ldots \rangle$, let

$$W_n = \begin{pmatrix} a_n & a_{n-1} \\ b_n & b_{n-1} \end{pmatrix}.$$

(I prefer this slight variant of a matrix discussed in class because it leads to more uniform expressions.) Explain that for all n

$$W_n = W_{n-1} \begin{pmatrix} q_n & 1 \\ 1 & 0 \end{pmatrix}$$

and that we could start this with $W_{-1} = I$. Thus

$$W_n = \begin{pmatrix} q_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} q_n & 1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

Use this matrix expression to solve (**??**) in case $a = 61, c = 23$, again.

(c) Show that a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ is a product of the form (**??**) (with $q_i \in \mathbb{Z}, q_i > 0$ for $i > 0$) if and only if either

(i) $c \geq 2$ and $0 \leq d < c$, or

(ii) $c = 1$ and $2d - 1 = \det A$.

(This alternative reflects the fact that $b_n > b_{n-1}$ except possibly for $n = 1$.) Show that then the sequence of q_n's is unique. (Hint: a, c, and $\det A$ determine A, using **(a)**.)

3. We saw that the unique reduced quadratic irrational α of discriminant $d = 4m$ with $a = 1$ in its primitive polynomial is $q + \sqrt{m}$ with $q = [\sqrt{m}]$, and that this element has a continued fraction expansion of the form $\langle \overline{q_0, \ldots, q_{k-1}} \rangle$, where q_0 is even and the sequence $q_1, \ldots, q_{k-1}$ is palindromic. Observe that in the primitive polynomial α, b is even; and that $-b = q_0$.

Now let $d = 4m + 1$ be an *odd* discriminant. Then b is odd as well. Express α, the unique reduced quadratic irrational of discriminant d with $a = 1$ in its primitive polynomial, in terms of $\frac{1+\sqrt{d}}{2}$, and show that the exact same statement holds: $\alpha = \langle \overline{q_0, \ldots, q_{k-1}} \rangle$ with $q_0 = -b$ and the sequence $q_1, \ldots, q_{k-1}$ is palindromic.

Had I inherited a fortune I would probably not have fallen prey to mathematics.—Lagrange, *fide* E. T. Bell.