Appendix A
Category Theory Language

This Appendix provides a swift summary of some of the basic notions of category theory used in this book. Many of the terms are defined in Chapters 1 and 2, but we repeat them here for the convenience of the reader.

A.1 Categories

Definition A.1.1 A *category* $\mathcal{C}$ consists of the following: a class obj$(\mathcal{C})$ of *objects*, a set $\operatorname{Hom}_{\mathcal{C}}(A, B)$ of *morphisms* for every ordered pair (A, B) of objects, an *identity morphism* $\mathrm{id}_A \in \operatorname{Hom}_{\mathcal{C}}(A, A)$ for each object A, and a *composition function* $\operatorname{Hom}_{\mathcal{C}}(A, B) \times \operatorname{Hom}_{\mathcal{C}}(B, C) \to \operatorname{Hom}_{\mathcal{C}}(A, C)$ for every ordered triple (A, B, C) of objects. We write $f: A \to B$ to indicate that f is a morphism in $\operatorname{Hom}_{\mathcal{C}}(A, B)$, and we write gf or $g \circ f$ for the composition of $f: A \to B$ with $g: B \to C$. The above data is subject to two axioms:

Associativity Axiom: $(hg)f = h(gf)$ for $f: A \to B$, $g: B \to C$, $h: C \to D$
Unit Axiom: $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$ for $f: A \to B$.

Paradigm A.1.2 The fundamental category to keep in mind is the category **Sets** of sets. The objects are sets and the morphisms are (set) functions, that is, the elements of $\operatorname{Hom}_{\mathbf{Sets}}(A, B)$ are the functions from A to B. Composition of morphisms is just composition of functions, and id_A is the function $\mathrm{id}_A(a) = a$ for all $a \in A$. Note that the objects of **Sets** do not form a set (or else we would encounter Russell's paradox of a set belonging to itself!); this explains the pedantic insistence that obj$(\mathcal{C})$ be a class and not a set. Nevertheless, we shall often use the notation $C \in \mathcal{C}$ to indicate that C is an object of $\mathcal{C}$.

Examples A.1.3 Another fundamental category is the category **Ab** of abelian groups. The objects are abelian groups, and the morphisms are group homomorphisms. Composition is just ordinary composition of homomorphisms.

The categories **Groups** of groups (and group maps) and **Rings** of rings (and ring maps) are defined similarly.

If R is a ring, R–**mod** is the category of left R-modules. Here the objects are left R-modules, the morphisms are R-module homomorphisms, and composition has its usual meaning. The category **mod**–R of right R-modules is defined similarly, and it is the same as R–**mod** when R is a commutative ring.

A *discrete* category is one in which every morphism is an identity morphism. Every set (or class!) may be regarded as a discrete category, since composition is forced by discreteness.

Small categories A.1.4 A category $\mathcal{C}$ is *small* if obj($\mathcal{C}$) is a set (not just a class). **Sets**, **Ab** and R–**mod** are not small, but a poset or a group may be thought of as a small category as follows.

A *partially ordered set*, or *poset*, is a set P with a reflexive, transitive antisymmetric relation $\leq$. We regard a poset as a small category as follows. Given $p, q \in P$ the set $\mathrm{Hom}_P(p, q)$ is the empty set unless $p \leq q$, in which case there is exactly one morphism from p to q (denoted $p \leq q$ of course). Composition is given by transitivity and the reflexive axiom ($p \leq p$) yields identity morphisms.

A category with exactly one object $*$ is the same thing as a *monoid*, that is, a set M (which will be $\mathrm{Hom}(*, *)$) equipped with an associative law of composition and an identity element. In this way we may consider a group as a category with one object.

The word "category" is due to Eilenberg and MacLane (1947) but was taken from Aristotle and Kant. It is chiefly used as an organizing principle for familiar notions. It is also useful to have other words to describe familiar types of morphisms that we encounter in many different categories; here are a few.

A morphism $f: B \to C$ is called an *isomorphism* in $\mathcal{C}$ if there is a morphism $g: C \to B$ such that $gf = \mathrm{id}_B$ and $fg = \mathrm{id}_C$. The usual proof shows that if g exists it is unique, and we often write $g = f^{-1}$. An isomorphism in **Sets** is a set bijection; an isomorphism in the category **Top** of topological spaces and continuous maps is a homeomorphism; an isomorphism in the category of smooth manifolds and smooth maps is called a diffeomorphism. In most algebraic categories, isomorphism has its usual meaning. In a group (considered as a category), every morphism is an isomorphism.

A.1.5 A morphism $f: B \to C$ is called *monic* in $\mathcal{C}$ if for any two distinct morphisms $e_1, e_2: A \to B$ we have $fe_1 \neq fe_2$; in other words, we can cancel f on the left. In **Sets**, **Ab**, R–**mod**, ..., in which objects have an underlying set ("concrete" categories; see A.2.3), the monic morphisms are precisely the morphisms that are set injections (monomorphisms) in the usual sense. If $B \to C$ is monic, we will sometimes say that B is a *subobject* of C. (Technically a subobject is an equivalence class of monics, two monics being equivalent if they factor through each other.)

A morphism $f: B \to C$ is called *epi* in $\mathcal{C}$ if for any two distinct morphisms $g_1, g_2: C \to D$ we have $g_1 f \neq g_2 f$; in other words, we can cancel f on the right. In **Sets**, **Ab**, and R–**mod** the epi morphisms are precisely the onto maps (epimorphisms). In other concrete categories such as **Rings** or **Top** this fails; the morphisms whose underlying set map is onto are epi, but there are other epis.

Exercise A.1.1 Show that $\mathbb{Z} \subset \mathbb{Q}$ is epi in **Rings**. Show that $\mathbb{Q} \subset \mathbb{R}$ is epi in the category of Hausdorff topological spaces.

A.1.6 An *initial object* (if it exists) in $\mathcal{C}$ is an object I such that for every C in $\mathcal{C}$ there is exactly one morphism from I to C. A *terminal object* in $\mathcal{C}$ (if it exists) is an object T such that for every C in $\mathcal{C}$ there is exactly one morphism from C to T. All initial objects must be isomorphic, and all terminal objects must be isomorphic. For example, in **Sets** the empty set ϕ is the initial object and any 1-point set is a terminal object. An object that is both initial and terminal is called a *zero object*. There is no zero object in **Sets**, but 0 is a zero object in **Ab** and in R–**mod**.

Suppose that $\mathcal{C}$ has a zero object 0. Then there is a distinguished element in each set $\text{Hom}_{\mathcal{C}}(B, C)$, namely the composite $B \to 0 \to C$; by abuse we shall write 0 for this map. A *kernel* of a morphism $f: B \to C$ is a morphism $i: A \to B$ such that $fi = 0$ and that satisfies the following universal property: Every morphism $e: A' \to B$ in $\mathcal{C}$ such that $fe = 0$ factors through A as $e = ie'$ for a unique $e': A' \to A$. Every kernel is monic, and any two kernels of f are isomorphic in an evident sense; we often identify a kernel of f with the corresponding subobject of B. Similarly, a *cokernel* of $f: B \to C$ is a morphism $p: C \to D$ such that $pf = 0$ and that satisfies the following universal property: Every morphism $g: C \to D'$ such that $gf = 0$ factors through D as $g = g'p$ for a unique $g': D \to D'$. Every cokernel is an epi, and any two cokernels are isomorphic. In **Ab** and R–**mod**, kernel and cokernel have their usual meanings.

Exercise A.1.2 In **Groups**, show that monics are just injective set maps, and kernels are monics whose image is a normal subgroup.

Opposite Category A.1.7 Every category $\mathcal{C}$ has an *opposite category* $\mathcal{C}^{\text{op}}$. The objects of $\mathcal{C}^{\text{op}}$ are the same as the objects in $\mathcal{C}$, but the morphisms (and composition) are reversed, so that there is a 1–1 correspondence $f \mapsto f^{\text{op}}$ between morphisms $f: B \to C$ in $\mathcal{C}$ and morphisms $f^{\text{op}}: C \to B$ in $\mathcal{C}^{\text{op}}$. If f is monic, then f^{op} is epi; if f is epi, then f^{op} is monic. Similarly, taking opposites interchanges kernels and cokernels, as well as initial and terminal objects. Because of this duality, $\mathcal{C}^{\text{op}}$ is also called the *dual category* of $\mathcal{C}$.

Example A.1.8 If R is a ring (a category with one object), R^{op} is the ring with the same underlying set, but in which multiplication is reversed. The category (R^{op})–**mod** of left R^{op}-modules is isomorphic to the category **mod**–R of right R-modules. However, $(R$–**mod**$)^{\text{op}}$ cannot be S–**mod** for any ring S (see A.4.7).

Exercise A.1.3 (Pontrjagin duality) Show that the category $\mathcal{C}$ of finite abelian groups is isomorphic to its opposite category $\mathcal{C}^{\text{op}}$, but that this fails for the category $\mathcal{T}$ of torsion abelian groups. We will see in exercise 6.11.4 that $\mathcal{T}^{\text{op}}$ is the category of profinite abelian groups.

Products and Coproducts A.1.9 If $\{C_i : i \in I\}$ is a set of objects of $\mathcal{C}$, a *product* $\prod_{i \in I} C_i$ (if it exists) is an object of $\mathcal{C}$, together with maps $\pi_j: \prod C_i \to C_j$ $(j \in I)$ such that for every $A \in \mathcal{C}$, and every family of morphisms $\alpha_i: A \to C_i$ $(i \in I)$, there is a unique morphism $\alpha: A \to \prod C_i$ in $\mathcal{C}$ such that $\pi_i \alpha = \alpha_i$ for all $i \in I$. *Warning:* Any object of $\mathcal{C}$ isomorphic to a product is also a product, so $\prod C_i$ is not a well-defined object of $\mathcal{C}$. Of course, if $\prod C_i$ exists, then it is unique up to isomorphism. If $I = \{1, 2\}$, then we write $C_1 \times C_2$ for $\prod_{i \in I} C_i$. Many concrete categories (**Sets**, **Groups**, **Rings**, R–**mod**, ... A.2.3) have arbitrary products, but others (e.g., **Fields**) have no products at all.

Dually, a *coproduct* $\coprod_{i \in I} C_i$ of a set of objects in $\mathcal{C}$ (if it exists) is an object of $\mathcal{C}$, together with maps $\iota_j: C_j \to \coprod C_i$ $(j \in I)$ such that for every family of morphisms $\alpha_i: C_i \to A$ there is a unique morphism $\alpha: \coprod C_i \to A$ such that $\alpha \iota_j = \alpha_j$ for all $j \in I$. That is, a coproduct in $\mathcal{C}$ is a product in $\mathcal{C}^{\text{op}}$. If $I = \{1, 2\}$, then we write $C_1 \amalg C_2$ for $\coprod_{i \in I} C_i$. In **Sets**, the coproduct is disjoint union; in **Groups**, the coproduct is the free product; in R–**mod**, the coproduct is direct sum.

Exercise A.1.4 Show that $\text{Hom}_{\mathcal{C}}(A, \prod C_i) \cong \prod_{i \in I} \text{Hom}_{\mathcal{C}}(A, C_i)$ and that $\text{Hom}_{\mathcal{C}}(\coprod C_i, A) \cong \prod_{i \in I} \text{Hom}_{\mathcal{C}}(C_i, A)$.

Exercise A.1.5 Let $\{\alpha_i: A_i \to C_i\}$ be a family of maps in $\mathcal{C}$. Show that

1. If $\prod A_i$ and $\prod C_i$ exist, there is a unique map $\alpha: \prod A_i \to \prod C_i$ such that $\pi_i \alpha = \alpha_i \pi_i$ for all i. If every α_i is monic, so is α.

2. If $\coprod A_i$ and $\coprod C_i$ exist, there is a unique map $\alpha: \coprod A_i \to \coprod C_i$ such that $\iota_i \alpha_i = \alpha \iota_i$ for all i. If every α_i is an epi, so is α.

A.2 Functors

By a *functor* $F: \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ we mean a rule that associates an object $F(C)$ (or FC or even F_C) of $\mathcal{D}$ to every object C of $\mathcal{C}$, and a morphism $F(f): F(C_1) \to F(C_2)$ in $\mathcal{D}$ to every morphism $f: C_1 \to C_2$ in $\mathcal{C}$. We require F to preserve identity morphisms ($F(\mathrm{id}_C) = \mathrm{id}_{FC}$) and composition ($F(gf) = F(g)F(f)$). Note that F induces set maps

$$\mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \to \mathrm{Hom}_{\mathcal{D}}(FC_1, FC_2)$$

for every C_1, C_2 in $\mathcal{C}$. If $G: \mathcal{D} \to \mathcal{E}$ is another functor, the composite $GF: \mathcal{C} \to \mathcal{E}$ is defined in the obvious way: $(GF)(C) = G(F(C))$ and $(GF)(f) = G(F(f))$.

The identity functor $\mathrm{id}_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$ is the rule fixing all objects and morphisms, that is, $\mathrm{id}_{\mathcal{C}}(C) = C$, $\mathrm{id}_{\mathcal{C}}(f) = f$. Clearly, for a functor $F: \mathcal{C} \to \mathcal{D}$ we have $F \circ \mathrm{id}_{\mathcal{C}} = F = \mathrm{id}_{\mathcal{D}} \circ F$. Except for set-theoretic difficulties, we could form a category **CAT** whose objects are categories and whose morphisms are functors. Instead, we form **Cat**, whose objects are small categories; $\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ is the set (!) of all functors from $\mathcal{C}$ to $\mathcal{D}$, the identity of $\mathcal{C}$ is $\mathrm{id}_{\mathcal{C}}$, and composition is composition of functors.

Hom and Tensor Product A.2.1 Let R be a ring and M a right R-module. For every left R-module N the tensor product $M \otimes_R N$ is an abelian group and $M \otimes_R -$ is a functor from R–**mod** to **Ab**. For every right R-module N, $\mathrm{Hom}_R(M, N)$ is an abelian group and $\mathrm{Hom}_R(M, -)$ is a functor from **mod**–R to **Ab**. These two functors are discussed in Chapter 3.

Forgetful Functors A.2.2 A functor that does nothing more than forget some of the structure of a category is commonly called a *forgetful functor*, and written with a U (for "underlying"). For example, there is a forgetful functor from R–**mod** to **Ab** (forget the R-module structure), one from **Ab** to **Sets** (forget the group structure), and their composite from R–**mod** to **Sets**.

Faithful Functors A.2.3 A functor $F: \mathcal{C} \to \mathcal{D}$ is called *faithful* if the set maps $\mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(FC, FC')$ are all injections. That is, if f_1 and f_2 are distinct maps from C to C' in $\mathcal{C}$, then $F(f_1) \neq F(f_2)$. Forgetful functors are usually faithful functors, and a category $\mathcal{C}$ with a faithful functor $U: \mathcal{C} \to$ **Sets**

is called a *concrete category*. In a concrete category, morphisms are completely determined by their effect on the underlying sets. R–**mod** and **Ab** are examples of concrete categories.

A *subcategory* $\mathcal{B}$ of a category $\mathcal{C}$ is a collection of some of the objects and some of the morphisms, such that the morphisms of $\mathcal{B}$ are closed under composition and include id_B for every object B in $\mathcal{B}$. A subcategory is a category in its own right, and there is an (obvious) *inclusion functor*, which is faithful by definition.

A subcategory $\mathcal{B}$ in which $\mathrm{Hom}_{\mathcal{B}}(B, B') = \mathrm{Hom}_{\mathcal{C}}(B, B')$ for every B, B' in $\mathcal{B}$ is called a *full subcategory*. We often refer to it as "the full subcategory on the objects" $\mathrm{obj}(\mathcal{B})$, since this information completely determines $\mathcal{B}$.

A functor $F: \mathcal{C} \to \mathcal{D}$ is *full* if the maps $\mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(FC, FC')$ are all surjections. That is, every $g: F(C) \to F(C')$ in $\mathcal{D}$ is of the form $g = F(f)$ for some $f: C \to C'$. A functor that is both full and faithful is called *fully faithful*. For example, the inclusion of a full subcategory is fully faithful. The Yoneda embedding (see A.3.4) is fully faithful. Another example of a fully faithful functor is "reflection" onto a skeletal subcategory, which we now describe.

Skeletal Subcategories A.2.4 By a *skeletal subcategory* $\mathcal{S}$ of a category $\mathcal{C}$ we mean a full subcategory such that every object of $\mathcal{C}$ is isomorphic to exactly one object of $\mathcal{S}$. For example, the full subcategory of **Sets** on the cardinal numbers $0 = \phi$, $1 = \{\phi\}$, . . . is skeletal. The category of finitely generated R-modules is not a small category, but it has a small skeletal subcategory.

If we can select an object FC in $\mathcal{S}$ and an isomorphism $\theta_C: C \cong FC$ for each C in $\mathcal{C}$, then F extends to a "reflection" functor as follows: if $f: B \to C$, then $F(f) = \theta_C f \theta_B^{-1}$. Such a reflection functor is fully faithful. We will discuss reflections and reflective subcategories more in A.6.3 below. The set-theoretic issues involved here are discussed in [MacCW, I.6].

Contravariant Functors A.2.5 The functors we have been discussing are sometimes called *covariant* functors to distinguish them from *contravariant* functors. A *contravariant functor* $F: \mathcal{C} \to \mathcal{D}$ is by definition just a covariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$. That is, it associates an object $F(C)$ of $\mathcal{D}$ to every object C of $\mathcal{C}$, and a morphism $F(f): F(C_2) \to F(C_1)$ in $\mathcal{D}$ to every $f: C_1 \to C_2$ in $\mathcal{C}$. Moreover, $F(\mathrm{id}_C) = \mathrm{id}_{FC}$ and F reverses composition: $F(gf) = F(f)F(g)$.

The most important example in this book will be the contravariant functor $\mathrm{Hom}_R(-, N)$ from **mod**–R to **Ab** associated with a right R-module N. Its derived functors $\mathrm{Ext}^*_R(-, N)$ are also contravariant (see 2.5.2). Another example is a *presheaf* on a topological space X; this is by definition a contravariant functor from the poset of open subspaces of X to the category **Ab**.

A.3 Natural Transformations

Suppose that F and G are two functors from $\mathcal{C}$ to $\mathcal{D}$. A *natural transformation* $\eta: F \Rightarrow G$ is a rule that associates a morphism $\eta_C: F(C) \to G(C)$ in $\mathcal{D}$ to every object C of $\mathcal{C}$ in such a way that for every morphism $f: C \to C'$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{array}{ccc} F(C) & \xrightarrow{Ff} & F(C') \\ \eta\downarrow & & \downarrow\eta \\ G(C) & \xrightarrow{Gf} & G(C'). \end{array}$$

This gives a precise meaning to the informal usage, "the map $\eta_C: F(C) \to G(C)$ is *natural* in C." If each η_C is an isomorphism, we say that η is a *natural isomorphism* and write $\eta: F \cong G$.

Examples A.3.1

1. Let $T(A)$ denote the torsion subgroup of an abelian group A. Then T is a functor from **Ab** to itself, and the inclusion $T(A) \subseteq A$ is a natural transformation $T \Rightarrow \mathrm{id}_{\mathbf{Ab}}$.
2. Let $h: M \to M'$ be an R-module homomorphism of right modules. For every left module N there is a natural map $h \otimes N: M \otimes_R N \to M' \otimes_R N$, forming a natural transformation $M \otimes_R \Rightarrow M' \otimes_R$. For every right module N there is a natural map $\eta_N: \mathrm{Hom}_R(M', N) \to \mathrm{Hom}_R(M, N)$ given by $\eta_N(f) = fh$, forming natural transformation $\mathrm{Hom}_R(M', -) \Rightarrow \mathrm{Hom}_R(M, -)$. These natural transformations give rise to maps of Tor and Ext groups; see Chapter 3.
3. In Chapter 2, the definitions of δ-functor and universal δ-functor will revolve around natural transformations.

Equivalence A.3.2 We call a functor $F: \mathcal{C} \to \mathcal{D}$ an *equivalence of categories* if there is a functor $G: \mathcal{D} \to \mathcal{C}$ and there are natural isomorphisms $\mathrm{id}_{\mathcal{C}} \cong GF$, $\mathrm{id}_{\mathcal{D}} \cong FG$. For example, the inclusion of a skeletal subcategory is an equivalence (modulo set-theoretic difficulties, which we ignore). The category of based vector spaces (objects = vector spaces with a fixed basis, morphisms =

matrices) is equivalent to the usual category of vector spaces by the forgetful functor. Equivalence of categories is the useful version of "isomorphism" most often encountered in practice. As a case in point, the category of based vector spaces is not isomorphic to the category of vector spaces, in which the basis choices are not explicitly given.

Functor Categories A.3.3 Given a category I and a category $\mathcal{A}$, the functors $F: I \to \mathcal{A}$ form the objects of the *functor category* $\mathcal{A}^I$. The morphisms in $\mathcal{A}^I$ from F to G are the natural transformations $\eta: F \Rightarrow G$, the composition $\zeta\eta$ of η with $\zeta: G \Rightarrow H$ is given by $(\zeta\eta)_i = \zeta_i\eta_i$, and the identity morphism of F is given by $(id_F)_i = id_{F(i)}$. (*Exercise:* show that $\mathcal{A}^I$ is a category when I is a small category.) We list several examples of funtor categories in Chapter 1, section 7 in connection with abelian categories; if $\mathcal{A}$ is an abelian category, then so is $\mathcal{A}^I$ (exercise A.4.3). Here is one example: If G is a group, the $\mathbf{Ab}^G$ is the category of G-modules discussed in Chapter 6.

Example A.3.4 The *Yoneda embedding* is the functor $h: I \to \mathbf{Sets}^{I^{\mathrm{op}}}$ given by letting h_i be the functor $h_i(j) = \mathrm{Hom}_I(j, i)$. This is a fully faithful functor. If I is an **Ab**-category (see A.4.1 below), the Yoneda embedding is sometimes thought of as a functor from I to $\mathbf{Ab}^{I^{\mathrm{op}}}$ (which is an abelian category). In particular, the Yoneda embedding allows us to think of any **Ab**-category (or any additive category) as a full subcategory of an abelian category. We discuss this more in Chapter 1, section 6.

A.4 Abelian Categories

The notion of abelian category extracts the crucial properties of abelian groups out of **Ab**, and gives homological algebra much of its power. We refer the reader to [MacCW] or Chapter 1, section 3 of this book for more details.

A.4.1 A category $\mathcal{A}$ is called an **Ab**-*category* if every hom-set $\mathrm{Hom}_{\mathcal{A}}(C, D)$ in $\mathcal{A}$ is given the structure of an abelian group in such a way that composition distributes over addition. For example, given a diagram in $\mathcal{A}$ of the form

$$A \xrightarrow{f} B \underset{g}{\overset{g'}{\rightrightarrows}} C \xrightarrow{h} D$$

we have $h(g + g')f = hgf + hg'f$ in $\mathrm{Hom}(A, D)$. Taking $A = B = C = D$, we see that each $\mathrm{Hom}(A, A)$ is an associative ring. Therefore, an **Ab**-category with one object is the same thing as a ring. At the other extreme, R–**mod** is an

Ab-category for every ring R, because the sum of R-module homomorphisms is an R-module homomorphism.

We call $\mathcal{A}$ an *additive category* if it is an **Ab**-category with a zero object 0 and a product $A \times B$ for every pair A, B of objects of $\mathcal{A}$. This structure is enough to make finite products the same as finite coproducts, and it is traditional to write $A \oplus B$ for $A \times B$. Again, R–**mod** is an additive category, but so is the smaller category on objects $\{0, R, R^2, R^3, \dots\}$ with $\mathrm{Hom}(R^n, R^m) =$ all $m \times n$ matrices in R.

Definition A.4.2 An *abelian category* is an additive category $\mathcal{A}$ such that:

1. (AB1) Every map in $\mathcal{A}$ has a kernel and cokernel,
2. (AB2) Every monic in $\mathcal{A}$ is the kernel of its cokernel, and
3. Every epi in $\mathcal{A}$ is the cokernel of its kernel.

Thus monic = kernel and epi = cokernel in an abelian category. Again, R–**mod** is an abelian category (kernel and cokernel have the usual meanings).

Exercise A.4.1 Let $\mathcal{A}$ be an **Ab**-category and $f: B \to C$ a morphism. Show that:

1. f is monic $\Leftrightarrow$ for every nonzero $e: A \to B$, $\quad fe \neq 0$;
2. f is an epi $\Leftrightarrow$ for every nonzero $g: C \to D$, $\quad gf \neq 0$.

Exercise A.4.2 Show that $\mathcal{A}^{\mathrm{op}}$ is an abelian category if $\mathcal{A}$ is an abelian category.

Exercise A.4.3 Given a category I and an abelian cateory $\mathcal{A}$, show that the functor category $\mathcal{A}^I$ is also an abelian category and that the kernel of $\eta: B \to C$ is the functor A, $\; A(i) = \ker(\eta_i)$.

In an abelian category every map $f: B \to C$ factors as

$$B \xrightarrow{e} \mathrm{im}(f) \xrightarrow{m} C$$

with $m = \ker(\mathrm{coker}\, f)$ monic and e epi. Indeed, m is obviously monic; we leave the proof that e is epi as an exercise. The subobject $\mathrm{im}(f)$ of C is called the *image* of f, because in "concrete" abelian categories like R–**mod** (A.2.3) the image is $\mathrm{im}(f) = \{f(b): b \in B\}$ as a subset of C.

A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of maps in an abelian category is called *exact* (at B) if $\ker(g) = \mathrm{im}(f)$. This implies in particular that the composite

$gf: A \to C$ is zero. Homological algebra might be thought of as the study of the circumstances when sequences are exact in an abelian category.

A.4.3 The following axioms for an abelian category $\mathcal{A}$ were introduced by Grothendieck in [Tohoku]. Axioms (AB1) and (AB2) were described above. The next four are discussed in Chapter 1, section 3; Chapter 2, sections 3 and 6; and in Chapter 3, section 5.

(AB3) For every set $\{A_i\}$ of objects of $\mathcal{A}$, the coproduct $\coprod A_i$ exists in $\mathcal{A}$. The coproduct is often called the *direct sum* and is often written as $\oplus A_i$. Rather than say that $\mathcal{A}$ satisfies (AB3), we often say that $\mathcal{A}$ is *cocomplete* (see A.5.1).

(AB3*) For every set $\{A_i\}$ of objects of $\mathcal{A}$, the product $\prod A_i$ exists in $\mathcal{A}$. Rather than say that $\mathcal{A}$ satisfies (AB3*), we usually say that $\mathcal{A}$ is *complete* (see A.5.1 below).

Example A.4.4 **Ab** and R–**mod** satisfy both (AB3) and (AB3*), but the abelian category of finite abelian groups satisfies neither and the abelian category of torsion abelian groups satisfies (AB3) but not (AB3*). For purposes of homological algebra, it is often enough to assume that $\prod A_i$ and $\coprod A_i$ exist for countable sets of objects $\{A_i\}$; for example, this suffices to construct the total complexes of a double complex in 1.2.6 or the functor $\varprojlim^1$ of Chapter 3, section 5.

Exercise A.4.4 (Union and intersection) Let $\{A_i\}$ be a family of subobjects of an object A. Show that if $\mathcal{A}$ is cocomplete, then there is a smallest subobject $\sum A_i$ of A containing all of the A_i. Show that if $\mathcal{A}$ is complete, then there is a largest subobject $\cap A_i$ of A contained in all the A_i.

(AB4) $\mathcal{A}$ is cocomplete, and the direct sum of monics is a monic.

(AB4*) $\mathcal{A}$ is complete, and the product of epis is an epi.

Example A.4.5 **Ab** and R–**mod** satisfy both (AB4) and (AB4*). The abelian category Sheaves(X) of sheaves of abelian groups on a fixed topological space X (described in Chapter 1, section 7) is a complete abelian category that does not satisfy (AB4*).

Exercise A.4.5

1. Let $\mathcal{A}$ be a complete abelian category. Show that $\mathcal{A}$ satisfies (AB4*) if and only if products of exact sequences are exact sequences, that is,

for every family $\{A_i \to B_i \to C_i\}$ of exact sequences in $\mathcal{A}$ the product sequence

$$\prod A_i \longrightarrow \prod B_i \longrightarrow \prod C_i$$

is also an exact sequence in $\mathcal{A}$.

2. By considering $\mathcal{A}^{\text{op}}$, show that a cocomplete abelian category satisfies (AB4) if and only if direct sums of exact sequences are exact sequences.

A.4.6 For the last two axioms, we assume familiarity with filtered colimits and inverse limits (see A.5.3 below). These axioms are discussed in Chapter 2, section 6 and Chapter 3, section 5.

(AB5) $\mathcal{A}$ is cocomplete, and filtered colimits of exact sequences are exact. Equivalently, if $\{A_i\}$ is a lattice of subobjects of an object A, and B is any subobject of A, then

$$\sum(A_i \cap B) = B \cap (\sum A_i).$$

(AB5*) $\mathcal{A}$ is complete, and filtered inverse limits of exact sequences are exact. Equivalently, if $\{A_i\}$ is a lattice of subobjects of A and B is any subobject of A, then

$$\cap(A_i + B) = B + (\cap A_i).$$

Examples A.4.7

1. We show in 2.6.15 that **Ab** and R–**mod** satisfy (AB5). However, they do not satisfy (AB5*), and this gives rise to the obstruction $\varprojlim^1 A_i$ discussed in Chapter 2, section 7. Hence $(R\text{–}\mathbf{mod})^{\text{op}}$ cannot be S–**mod** for any ring S.
2. Sheaves(X) satisfies (AB5) but not (AB5*); see A.4.5.

Exercise A.4.6 Show that (AB5) implies (AB4), and (AB5*) implies (AB4*).

Exercise A.4.7 Show that if $\mathcal{A} \neq 0$, then $\mathcal{A}$ cannot satisfy both axiom (AB5) and axiom (AB5*). *Hint:* Consider $\oplus A_i \to \prod A_i$.

A.5 Limits and Colimits (see Chapter 2, section 6)

A.5.1 The *limit* of a functor $F: I \to \mathcal{A}$ (if it exists) is an object L of $\mathcal{A}$, together with maps $\pi_i: L \to F_i$ $(I \in I)$ in $\mathcal{A}$ which are "compatible" in the sense that for every $\alpha: j \to i$ in I the map π_i factors as $F_\alpha \pi_j: L \to F_j \to F_i$, and that satisfies a universal property: for every $A \in \mathcal{A}$ and every system of "compatible" maps $f_i: A \to F_i$ there is a unique $\lambda: A \to L$ so that $f_i = \pi_i \lambda$. This universal property guarantees that any two limits of F are isomorphic. We write $\lim_{i \in I} F_i$ for such a limit. For example, if I is a discrete category, then $\lim_{i \in I} F_i = \prod_{i \in I} F_i$, so products are a special kind of limit.

A category $\mathcal{A}$ is called *complete* if $\lim F_i$ exists for all functors $F: I \to \mathcal{A}$ in which the indexing category I is small. Many familiar categories like **Sets**, **Ab**, R–**mod** are complete. Completeness of an abelian category agrees with the notion (AB3*) introduced in A.4.3 by the following exercise, and will be crucial in our discussion of $\varprojlim^1$ in Chapter 3, section 5.

Exercise A.5.1 Show that an abelian category is complete iff it satisfies (AB3*).

Dually, the *colimit* of $F: I \to \mathcal{A}$ (if it exists) is an object $C = \operatorname{colim}_{i \in I} F_i$ of $\mathcal{A}$, together with maps $\iota_i: F_i \to C$ in $\mathcal{A}$ that are "compatible" in the sense that for every $\alpha: j \to i$ in I the map ι_j factors as $\iota_i F_\alpha: F_i \to F_i \to C$, and that satisfies a universal property: for every $A \in \mathcal{A}$ and every system of "compatible" maps $f_i: F_i \to A$ there is a unique $\gamma: C \to A$ so that $f_i = \gamma \iota_i$. Again, the universal property guarantees that the colimit is unique up to isomorphism, and coproducts are a special kind of colimit. Since $F: I \to \mathcal{A}$ is the same as a functor $F^{\mathrm{op}}: I^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$, it is also clear that a colimit in $\mathcal{A}$ is the same thing as a limit in $\mathcal{A}^{\mathrm{op}}$.

A category $\mathcal{A}$ is called *cocomplete* if $\operatorname{colim} F_i$ exists for all functors $F: I \to \mathcal{A}$ in which the indexing category I is small. Many familiar categories like **Sets**, **Ab**, R–**mod** are also cocomplete. Cocompleteness plays a less visible role in homological algebra, but we shall discuss it and axiom (AB3) briefly in Chapter 2, section 6.

Exercise A.5.2 Show that an abelian category is cocomplete iff it satisfies axiom (AB3).

As a Natural Transformation A.5.2 There is a diagonal functor $\Delta: \mathcal{A} \to \mathcal{A}^I$ that sends $A \in \mathcal{A}$ to the constant functor: $(\Delta A)_i = A$ for all $i \in I$. The compatibility of the maps $\pi_j: \lim(F_i) \to F_j$ is nothing more than the assertion that π is a natural transformation from $\Delta(\lim F_i)$ to F. Similarly, the compatibility of the maps $\iota_j: F_j \to \operatorname{colim} F_i$ is nothing more than the assertion that ι is a natural transformation from F to $\Delta(\operatorname{colim} F_i)$. We will see that lim and colim are adjoint functors to Δ in exercise A.6.1.

Filtered Categories and Direct Limits A.5.3 A poset I is called *filtered*, or *directed*, if every two elements $i, j \in I$ have an upper bound $k \in I$ ($i \leq k$ and $j \leq k$). More generally, a small category I is called *filtered* if

1. For every $i, j \in I$ there is a $k \in I$ and arrows $i \to k$, $j \to k$ in I.
2. For every two arrows $u, v: i \to j$ there is an arrow $w: j \to k$ such that $wu = wv$ in $\operatorname{Hom}(i, k)$.

This extra generality is to include the following example. Let M be an abelian monoid and write I for the "translation" category whose objects are the elements of M, with $\operatorname{Hom}_I(i, j) = \{m \in M: mi = j\}$. I is a filtered category, because the upper bound in (1) is $k = ij = ji$, and in axiom (2) we can take $w = i \in \operatorname{Hom}_I(j, ij)$.

A *filtered colimit* in a category $\mathcal{A}$ is just the colimit of a functor $A: I \to \mathcal{A}$ in which I is a filtered category. We shall give such a colimit the special symbol $\underrightarrow{\operatorname{colim}}(A_i)$, although (filtered) colimits over directed posets are often called *direct limits* and are often written $\varinjlim A_i$. We shall see in Chapter 1, section 6 that filtered colimits in R–**mod** (and other cocomplete abelian categories) are well behaved; for example, they are exact and commute with Tor. This provides an easy proof (3.2.2) that $S^{-1}R$ is a flat R-module, using the translation category of the monoid S.

Example A.5.4 Let I be the (directed) poset of nonnegative integers. A functor $A: I \to \mathcal{A}$ is just a sequence $A_0 \to A_1 \to A_2 \to \cdots$ of objects in $\mathcal{A}$, and the direct limit $\lim_{i\to\infty} A_i$ is our filtered colimit $\underrightarrow{\operatorname{colim}} A_i$. A contravariant functor from I to $\mathcal{A}$ is just a tower $\cdots \to A_2 \to A_1 \to A_0$, and the "inverse limit" is the filtered limit $\varprojlim A_i$ we discuss in Chapter 3, section 5.

A.6 Adjoint Functors (see sections 2.3 and 2.6)

A.6.1 A pair of functors $L: \mathcal{A} \to \mathcal{B}$ and $R: \mathcal{B} \to \mathcal{A}$ are called *adjoint* if there is a set bijection for all A in $\mathcal{A}$ and B in $\mathcal{B}$:

$$\tau = \tau_{AB}: \operatorname{Hom}_{\mathcal{B}}(L(A), B) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{A}}(A, R(B)),$$

which is "natural" in A and B in the sense that for all $f: A \to A'$ in $\mathcal{A}$ and $g: B \to B'$ in $\mathcal{B}$ the following diagram commutes.

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{B}}(L(A'), B) & \xrightarrow{Lf^*} & \mathrm{Hom}_{\mathcal{B}}(L(A), B) & \xrightarrow{g_*} & \mathrm{Hom}_{\mathcal{B}}(L(A), B') \\ \downarrow \tau & & \downarrow \tau & & \downarrow \tau \\ \mathrm{Hom}_{\mathcal{A}}(A', R(B)) & \xrightarrow{f^*} & \mathrm{Hom}_{\mathcal{A}}(A, R(B)) & \xrightarrow{Rg_*} & \mathrm{Hom}_{\mathcal{A}}(A, R(B')) \end{array}$$

That is, τ is a natural isomorphism between the functors $\mathrm{Hom}_{\mathcal{B}}(L, -)$ and $\mathrm{Hom}_{\mathcal{A}}(-, R)$ from $\mathcal{A}^{\mathrm{op}} \times \mathcal{B}$ to **Sets**. We say that L is the *left adjoint* of R, and R is the *right adjoint* of L. We also say that (L, R) is an *adjoint pair*.

Here is a familiar example of a pair of adjoint functors. Let k be a field and L: **Sets** $\to$ (k-vector spaces) the functor sending a set X to the vector space with basis X. ($L(X)$ is the set of formal linear combinations of elements of X). This is left adjoint to the forgetful functor U, because $\mathrm{Hom}_k(L(X), V)$ is the same as $\mathrm{Hom}_{\mathbf{Sets}}(X, U(V))$.

We will see many other examples of adjoint functors in Chapter 2, section 6. The most important for Chapter 3 is the following adjunction between Hom and tensor product. Let R be a ring and B a left R-module. For every abelian group C $\mathrm{Hom}_{\mathbf{Ab}}(B, C)$ is a right R-module: $(fr)(b) = f(rb)$. The resulting functor $\mathrm{Hom}_{\mathbf{Ab}}(B, -)\colon \mathbf{Ab} \to \mathbf{mod}\text{–}R$ has $L(A) = A \otimes_R B$ as its left adjoint. (See 2.3.8 and 2.6.2.)

Exercise A.6.1 Fix categories I and $\mathcal{A}$. When every functor $F\colon I \to \mathcal{A}$ has a limit, show that $\lim\colon \mathcal{A}^I \to \mathcal{A}$ is a functor. Show that the universal property of $\lim F_i$ is nothing more than the assertion that lim is right adjoint to Δ. Dually, show that the universal property of $\mathrm{colim}\, F_i$ is nothing more than the assertion that $\mathrm{colim}\colon \mathcal{A}^I \to \mathcal{A}$ is left adjoint to Δ.

Theorem A.6.2 *An adjoint pair $(L, R)\colon \mathcal{A} \to \mathcal{B}$ determines*

1. *A natural transformation $\eta\colon \mathrm{id}_{\mathcal{A}} \Rightarrow RL$ (called the* unit *of the adjunction), such that the right adjoint of $f\colon L(A) \to B$ is $R(f) \circ \eta_A\colon A \to R(B)$.*
2. *A natural transformation $\varepsilon\colon LR \Rightarrow \mathrm{id}_{\mathcal{B}}$ (called the* counit *of the adjunction), such that the left adjoint of $g\colon A \to R(B)$ is $\varepsilon_B \circ L(g)\colon L(A) \to B$.*

Moreover, both of the following composites are the identity:

$$(*) \quad L(A) \xrightarrow{L(\eta)} LRL(A) \xrightarrow{\varepsilon L} L(A) \quad \text{and} \quad R(B) \xrightarrow{\eta R} RLR(B) \xrightarrow{R(\epsilon)} R(B).$$

Proof The map $\eta_A\colon A \to RL(A)$ is the element of $\mathrm{Hom}(A, RL(A))$ corresponding to $\mathrm{id}_{LA} \in \mathrm{Hom}(L(A), L(A))$. The map $\varepsilon_B\colon LR(B) \to B$ is the element of $\mathrm{Hom}(LR(B), B)$ corresponding to $\mathrm{id}_{RB} \in \mathrm{Hom}(R(B), R(B))$. The rest of the assertions are elementary manipulations using the naturality of τ and are left to the reader as an exercise. The lazy reader may find a proof in [MacCW, IV.1]. $\diamond$

Exercise A.6.2 Suppose given functors $L: \mathcal{A} \to \mathcal{B}$, $R: \mathcal{B} \to \mathcal{A}$ and natural transformations $\eta: \mathrm{id}_{\mathcal{A}} \Rightarrow RL$, $\varepsilon: LR \Rightarrow \mathrm{id}_{\mathcal{B}}$ such that the composites (*) are the identity. Show that (L, R) is an adjoint pair of functions.

Exercise A.6.3 Show that $\varepsilon \circ (LR\varepsilon) = \varepsilon \circ (\varepsilon LR)$ and that $(RL\eta) \circ \eta = (\eta RL) \circ \eta$. That is, show that the following diagrams commute:

$$
\begin{array}{ccc}
LR(LR(B)) & \xrightarrow{LR\varepsilon} & LR(B) \\
\downarrow{\scriptstyle \varepsilon_{LR(B)}} & & \downarrow{\scriptstyle \varepsilon_B} \\
LR(B) & \xrightarrow{\cdot\varepsilon} & B
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\eta} & RL(A) \\
\downarrow{\scriptstyle \eta_A} & & \downarrow{\scriptstyle \eta_{RLA}} \\
RL(A) & \xrightarrow{RL\eta} & RL(RL(A))
\end{array}
$$

Reflective Subcategories A.6.3 A subcategory $\mathcal{B}$ of $\mathcal{A}$ is called a *reflective subcategory* if the inclusion functor $\iota: \mathcal{B} \subseteq \mathcal{A}$ has a left adjoint $L: \mathcal{A} \to \mathcal{B}$; L is often called the *reflection* of $\mathcal{A}$ onto $\mathcal{B}$. If $\mathcal{B}$ is a full subcategory, then by the above exercise $B \cong R(B)$ for all B in $\mathcal{B}$. The "reflection" onto a skeletal subcategory is a reflection in this sense.

Here are two examples of reflective subcategories. **Ab** is reflective in **Groups**; the reflection is the quotient $L(G) = G/[G, G]$ by the commutator subgroup. In 2.6.5 we will see that for every topological space X the category of sheaves on X is a reflective subcategory of the category of presheaves on X; in this case the reflection functor is called "sheafification."