9
Hochschild and Cyclic Homology

In this chapter we fix a commutative ring k and construct several homology theories based on chain complexes of k-modules. For legibility, we write $\otimes$ for $\otimes_k$ and $R^{\otimes n}$ for the n-fold tensor product $R \otimes \cdots \otimes R$.

9.1 Hochschild Homology and Cohomology of Algebras

9.1.1 Let R be a k-algebra and M an $R-R$ bimodule. We obtain a simplicial k-module $M \otimes R^{\otimes *}$ with $[n] \mapsto M \otimes R^{\otimes n}$ $(M \otimes R^{\otimes 0} = M)$ by declaring

$$\partial_i(m \otimes r_1 \otimes \cdots \otimes r_n) = \begin{cases} mr_1 \otimes r_2 \otimes \cdots \otimes r_n & \text{if } i = 0 \\ m \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n & \text{if } 0 < i < n \\ r_n m \otimes r_1 \otimes \cdots \otimes r_{n-1} & \text{if } i = n \end{cases}$$

$$\sigma_i(m \otimes r_1 \otimes \cdots \otimes r_n) = m \otimes \cdots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \cdots \otimes r_n,$$

where $m \in M$ and the r_i are elements of R. These formulas are k-multilinear, so the ∂_i and σ_i are well-defined homomorphisms, and the simplicial identities are readily verified. (Check this!) The *Hochschild homology* $H_*(R, M)$ of R with coefficients in M is defined to be the k-modules

$$H_n(R, M) = \pi_n(M \otimes R^{\otimes *}) = H_n C(M \otimes R^{\otimes *}).$$

Here $C(M \otimes R^{\otimes *})$ is the associated chain complex with $d = \sum (-1)^i \partial_i$:

$$0 \longleftarrow M \xleftarrow{\partial_0 - \partial_1} M \otimes R \xleftarrow{d} M \otimes R \otimes R \xleftarrow{d} \cdots .$$

For example, the image of $\partial_0 - \partial_1$ is the k-submodule $[M, R]$ of M that is generated by all terms $mr - rm$ $(m \in M, r \in R)$. Hence $H_0(R, M) \cong M/[M, R]$.

Similarly, we obtain a cosimplicial k-module with $[n] \mapsto \mathrm{Hom}_k(R^{\otimes n}, M) = \{k\text{-multilinear maps } f\colon R^n \to M\}$ ($\mathrm{Hom}(R^{\otimes 0}, M) = M$) by declaring

$$(\partial^i f)(r_0, \cdots, r_n) = \begin{cases} r_0 f(r_1, \ldots, r_n) & \text{if } i = 0 \\ f(r_0, \ldots, r_{i-1}r_i, \ldots) & \text{if } 0 < i < n \\ f(r_0, \ldots, r_{n-1})r_n & \text{if } i = n \end{cases}$$

$$(\sigma^i f)(r_1, \cdots, r_{n-1}) = f(r_1, \ldots, r_i, 1, r_{i+1}, \ldots, r_n).$$

The *Hochschild cohomology* $H^*(R, M)$ of R with coefficients in M is defined to be the k-modules

$$H^n(R, M) = \pi^n(\mathrm{Hom}_k(R^{\otimes *}, M)) = H^n C(\mathrm{Hom}_k(R^{\otimes *}, M)).$$

Here $C\ \mathrm{Hom}_k(R^*, M)$ is the associated cochain complex

$$0 \longrightarrow M \xrightarrow{\partial^0 - \partial^1} \mathrm{Hom}_k(R, M) \xrightarrow{d} \mathrm{Hom}_k(R \otimes R, M) \xrightarrow{d} \cdots.$$

For example, it follows immediately that

$$H^0(R, M) = \{m \in M : rm = mr \quad \text{for all } r \in R\}.$$

Exercise 9.1.1 If R is a commutative k-algebra, show that $M \otimes R^{\otimes *}$ is a simplicial R-module via $r \cdot (m \otimes r_1 \otimes \cdots) = (rm) \otimes r_1 \otimes \cdots$. Conclude that each $H_n(R, M)$ is an R-module. Similarly, show that $\mathrm{Hom}_R(R^{\otimes *}, M)$ is a cosimplicial R-module, and conclude that each $H^n(R, M)$ is an R-module.

Exercise 9.1.2 If $0 \to M_0 \to M_1 \to M_2 \to 0$ is a k-split exact sequence of bimodules (8.7.7), show that there is a long exact sequence

$$\cdots \xrightarrow{\partial} H_i(R, M_0) \to H_i(R, M_1) \to H_i(R, M_2) \xrightarrow{\partial} H_{i-1}(R, M_0) \cdots.$$

Example 9.1.2 (Group rings) Let R be the group ring $k[G]$ of a group G, and M a right G-module. Write ${}_\varepsilon M$ for M considered as a $G{-}G$ bimodule with trivial left G-module structure ($gm = m$ for all $g \in G$, $m \in M$). If B_*^u denotes the unnormalized bar resolution of 6.5.1, then $H_*(G; M)$ is the homology of $M \otimes_{\mathbb{Z}G} B_*^u$, the chain complex that in degree i is $M \otimes (\mathbb{Z}G)^{\otimes i}$. By inspection, this is the same complex used in 9.1.1 to define the Hochschild homology of $\mathbb{Z}G$, provided that we take coefficients in the bimodule ${}_\varepsilon M$. Similarly, $H^*(G; M)$ is the cohomology of the chain complex $\mathrm{Hom}_G(B_*^u, M)$, which is the same as the complex $\mathrm{Hom}_k((\mathbb{Z}G)^{\otimes *}, {}_\varepsilon M)$ used to define Hochschild cohomology. Thus

$$H_*(G; M) \cong H_*(\mathbb{Z}G; {}_\varepsilon M) \quad \text{and} \quad H^*(G; M) \cong H^*(\mathbb{Z}G; {}_\varepsilon M).$$

The above definitions, originally given by G. Hochschild in 1945, have the advantage of being completely natural in R and M. In order to put them into a homological framework, it is necessary to consider the *enveloping algebra* $R^e = R \otimes_k R^{\mathrm{op}}$ of R. Here R^{op} is the "opposite ring"; R^{op} has the same underlying abelian group structure as R, but multiplication in R^{op} is the opposite of that in R (the product $r \cdot s$ in R^{op} is the same as the product sr in R). The main feature of R^{op} is this: A right R-module M is the same thing as a left R^{op}-module via the product $r \cdot m = mr$ because associativity requires that

$$(r \cdot s) \cdot m = (sr) \cdot m = m(sr) = (ms)r = r \cdot (ms) = r \cdot (s \cdot m).$$

Similarly a left R-module N is the same thing as a right R^{op}-module via $n \cdot r = rn$. Consequently, the main feature of R^e is that an $R-R$ bimodule M is the same thing as a left R^e-module via the product $(r \otimes s) \cdot m = rms$, or as a right R^e-module via the product $m \cdot (r \otimes s) = smr$. (Check this!) This gives a slick way to consider the category R–**mod**–R of $R-R$ bimodules as the category of left R^e-modules or as the category of right R^e-modules. In particular, the canonical $R-R$ bimodule structure on R makes R into both a left and right R^e-module.

Lemma 9.1.3 *Hochschild homology and cohomology are isomorphic to relative Tor and Ext for the ring map $k \to R^e = R \otimes R^{\mathrm{op}}$:*

$$H_*(R, M) \cong \mathrm{Tor}_*^{R^e/k}(M, R) \quad \text{and} \quad H^*(R, M) \cong \mathrm{Ext}^*_{R^e/k}(R, M).$$

Proof Consider the unnormalized bar resolution $\beta(R, R)$ of R as a left R-module (8.6.12). Each term $\beta(R, R)_n = R^{\otimes n+1} \otimes R$ is isomorphic as an $R-R$ bimodule to $R \otimes R^{\otimes n} \otimes R \cong (R \otimes R^{\mathrm{op}}) \otimes R^{\otimes n}$ and hence is $\perp$-projective (8.6.5), where $\perp = R^e \otimes$. Since $\beta(R, R)$ is a k-split $\perp$-projective resolution of the R^e-module R, 8.7.10 yields

$$\mathrm{Tor}_*^{R^e/k}(M, R) = H_*(M \otimes_{R^e} \beta(R, R)) \quad \text{and}$$
$$\mathrm{Ext}^*_{R^e/k}(R, M) = H^* \operatorname{Hom}_{R^e}(\beta(R, R), M).$$

On the other hand, the isomorphism $M \otimes_{R^e} (R \otimes R^{\otimes n} \otimes R) \to M \otimes R^n$ sending $m \otimes (r_0 \otimes \cdots \otimes r_{n+1})$ to $(r_{n+1} m r_0) \otimes (r_1 \otimes \cdots \otimes r_n)$ identifies $M \otimes_{R^e}$

$\beta(R, R)$ with the chain complex $C(M \otimes R^{\otimes *})$ used to define Hochschild homology. Similarly, the isomorphism $\mathrm{Hom}_{R^e}(R \otimes R^{\otimes n} \otimes R, M) \to \mathrm{Hom}_k(R^{\otimes n}, M)$ sending f to $f(1, -, 1)$ identifies $\mathrm{Hom}_{R^e}(\beta(R, R), M)$ with the cochain complex $C(\mathrm{Hom}_k(R^{\otimes *}, M))$ used to define Hochschild cohomology. $\diamond$

Next we show that in good cases, such as when k is a field, we can identify Hochschild homology and cohomology with the absolute Tor and Ext over the ring R^e.

Lemma 9.1.4 *If P and Q are flat (resp. projective) k-modules, then so is $P \otimes Q$.*

Proof Let $\mathcal{E}$ be an exact sequence of k-modules. If P and Q are flat, then by definition $\mathcal{E} \otimes P$ and hence $\mathcal{E} \otimes P \otimes Q$ are exact; hence $P \otimes Q$ is flat. If P and Q are projective, then $\mathrm{Hom}(Q, \mathcal{E})$ and hence $\mathrm{Hom}(P, \mathrm{Hom}(Q, \mathcal{E})) \cong \mathrm{Hom}(P \otimes Q, \mathcal{E})$ are exact; as we saw in 2.2.3, this implies that $P \otimes Q$ is projective. $\diamond$

Corollary 9.1.5 *If R is flat as a k-module, then $H_*(R, M) \cong \mathrm{Tor}_*^{R^e}(M, R)$. If R is projective as a k-module, then $H^*(R, M) \cong \mathrm{Ext}_{R^e}^*(R, M)$.*

Proof If R is flat (resp. projective), then each $R^{\otimes n}$ is a flat (resp. projective) k-module, and hence each $\beta(R, R)_n \cong R^e \otimes R^{\otimes n}$ is a flat (resp. projective) R^e-module. Thus $\beta(R, R)$ is a resolution of R by flat (resp. projective) R^e-modules. It follows that the relative Tor (resp. relative Ext) modules are isomorphic to the absolute Tor (resp. absolute Ext) modules. $\diamond$

Here are three cases in which $H_*(R, M)$ is easy to compute. First, let us recall from 7.3.1 that the tensor algebra of a k-module V is the graded algebra

$$T(V) = k \oplus V \oplus (V \otimes V) \oplus \cdots \oplus V^{\otimes j} \oplus \cdots .$$

Proposition 9.1.6 *Let $T = T(V)$ be the tensor algebra of a k-module V, and let M be a T–T bimodule. Then $H_i(T, M) = 0$ for $i \neq 0, 1$ and there is an exact sequence*

$$0 \to H_1(T, M) \to M \otimes V \xrightarrow{b} M \to H_0(T, M) \to 0$$

where b is the usual map $b(m \otimes v) = mv - vm$. In particular, if σ denotes the cyclic permutation $\sigma(v_1 \otimes \cdots \otimes v_j) = v_j \otimes v_1 \otimes \cdots v_{j-1}$ of $V^{\otimes j}$ and we write

$(V^{\otimes j})^\sigma$ and $(V^{\otimes j})_\sigma$ for the invariants and covariants of this group action, then we have

$$H_0(T,T) = k \oplus \bigoplus_{j=1}^{\infty} (V^{\otimes j})_\sigma, \quad H_1(T,T) = \bigoplus_{j=1}^{\infty} (V^{\otimes j})^\sigma.$$

Proof The formula $d(t \otimes v \otimes t') = tv \otimes t' - t \otimes vt'$ defines a $T-T$ bimodule map from $T \otimes V \otimes T$ to $T \otimes T$. As the kernel I of the multiplication $\mu: T \otimes T \to T$ is generated by the elements $v \otimes 1 - 1 \otimes v = d(1 \otimes v \otimes 1)$ and $\mu d = 0$, the image of d is I. As d is a direct sum (over p and q) of maps from $V^{\otimes p} \otimes V \otimes V^q$ to $V^{\otimes p+1} \otimes V^{\otimes q}$ and to $V^{\otimes p} \otimes V^{\otimes q+1}$, each of which is an isomorphism, d is an injection. (Check this!) Hence

$$0 \to T \otimes V \otimes T \xrightarrow{d} T \otimes T \xrightarrow{\mu} T \to 0$$

is a $\perp$-projective resolution of the T^e-module T; μ is k-split by the map $id \otimes 1: T \to T \otimes T$. Hence we can compute $\mathrm{Tor}_*^{T^e/k}(M,T)$ using this resolution. Tensoring with M yields $H_i(T,M) = 0$ for $i \neq 0, 1$ and the advertised exact sequence for H_1 and H_0. ◇

Exercise 9.1.3 (Polynomials) If $R = k[x_1, \cdots, x_m]$, show that R^e is isomorphic to the polynomial ring $k[y_1, \cdots, y_n, z_1, \cdots, z_m]$ and that the kernel of $R^e \to R$ is generated by the regular sequence $x = (y_1 - z_1, \cdots, y_m - z_m)$. Using the Koszul resolution $K(\mathbf{x})$ of 4.5.5, show that $H_p(R,R) \cong H^p(R,R) \cong \Lambda^p(R^n)$ for $p = 0, \cdots, n$, while $H_p(R,M) = H^p(R,M) = 0$ for $p > n$ and all bimodules M. This is a special case of Theorem 9.4.7 below.

Exercise 9.1.4 (Truncated polynomials) If $R = k[x]/(x^{n+1} = 0)$, let $u = x \otimes 1 - 1 \otimes x$ and $v = x^n \otimes 1 + x^{n-1} \otimes x + \cdots + x \otimes x^{n-1} + 1 \otimes x^n$ as elements in R^e. Show that

$$0 \leftarrow R \leftarrow R^e \xleftarrow{u} R^e \xleftarrow{v} R^e \xleftarrow{u} R^e \xleftarrow{v} R^e \xleftarrow{u} \cdots$$

is a periodic R^e-resolution of R, and conclude that $H_i(R,M)$ and $H^i(R,M)$ are periodic of period 2 for $i \geq 1$. Finally, show that when $\frac{1}{n+1} \in R$ we have $H_i(R,R) \cong H^i(R,R) \cong R/(x^n R)$ for all $i \geq 1$.

Let $k \to \ell$ be a commutative ring map. If R is a k-algebra, then $R_\ell = R \otimes_k \ell$ is an ℓ-algebra. If M is an $R_\ell - R_\ell$ bimodule then via the ring map $R \to R_\ell$ $(r \mapsto r \otimes 1)$ we can also consider M to be an $R-R$ bimodule. We would like to compare the Hochschild homology $H_*^k(R, M)$ of the k-algebra R with the Hochschild homology $H_*^\ell(R_\ell, M)$ of the ℓ-algebra $R_\ell = R \otimes \ell$.

Theorem 9.1.7 (Change of ground ring) *Let R be a k-algebra and $k \to \ell$ a commutative ring map. Then there are natural isomorphisms for every $R_\ell - R_\ell$ bimodule M:*

$$H_*^k(R, M) \cong H_*^\ell(R_\ell, M) \quad \text{and} \quad H_k^*(R, M) \cong H_\ell^*(R_\ell, M).$$

Proof The unnormalized chain complexes used for computing homology are isomorphic by the isomorphisms $M \otimes_k R \otimes_k \cdots \otimes_k R \cong M \otimes_\ell (R \otimes_k \ell) \otimes_\ell \cdots \otimes_\ell (R \otimes_k \ell)$. Similarly, the unnormalized cochain complexes used for computing cohomology are isomorphic, by the bijection between k-multilinear maps $R^n \to M$ and ℓ-multilinear maps $(R_\ell)^n \to M$. $\diamond$

Theorem 9.1.8 (Change of rings) *Let R be a k-algebra and M an $R-R$ bimodule.*

1. *(Product) If R' is another k-algebra and M' an $R'-R'$ bimodule, then*

$$H_*(R \times R', M \times M') \cong H_*(R, M) \oplus H_*(R', M')$$
$$H^*(R \times R', M \times M') \cong H^*(R, M) \oplus H^*(R', M').$$

2. *(Flat base change) If R is a commutative k-algebra and $R \to T$ is a ring map such that T is flat as a (left and right) R-module, then*

$$H_*(T, T \otimes_R M \otimes_R T) \cong T \otimes_R H_*(R, M).$$

3. *(Localization) If S is a central multiplicative set in R, then*

$$H_*(S^{-1}R, S^{-1}R) \cong H_*(R, S^{-1}R) \cong S^{-1}H_*(R, R).$$

Proof For (1), note that $(R \times R')^e \cong R^e \times R'^e \times (R \otimes R'^{\mathrm{op}}) \times (R' \otimes R^{\mathrm{op}})$; since M and M' are left R^e and R'^e-modules, respectively, this is a special case of relative Tor and Ext for products of rings (8.7.14). For (2), note that $R^e \to T^e$ makes T^e flat as an R^e-module (because $T^e \otimes_{R^e} M = T \otimes_R M \otimes_R T$). By flat base change for relative Tor (8.7.16) we have

$$\mathrm{Tor}_*^{T^e/k}(T, T^e \otimes M) \cong \mathrm{Tor}_*^{R^e/k}(T, M) \cong T \otimes_R \mathrm{Tor}_*^{R^e/k}(R, M).$$

The first part of (3) is also flat base change for relative Tor 8.7.16 with $T = S^{-1}R$, and the isomorphism $H_*(R, S^{-1}R) \cong S^{-1}H_*(R, R)$ is a special case of the isomorphism $\mathrm{Tor}^{R^e/k}(S^{-1}M, N) \cong S^{-1}\,\mathrm{Tor}_*^{R^e/k}(M, N)$ for localization (3.2.10 or exercise 8.7.3). $\diamond$

Here is one way to form $R{-}R$ bimodules. If M and N are left R-modules, $\mathrm{Hom}_k(M, N)$ becomes an $R{-}R$ bimodule by the rule $rfs : m \mapsto rf(sm)$. The Hochschild cohomology of this bimodule is just the relative Ext of 8.7.5:

Lemma 9.1.9 *Let M and N be left R-modules. Then*

$$H^n(R, \mathrm{Hom}_k(M, N)) \cong \mathrm{Ext}^n_{R/k}(M, N).$$

Proof Let $B = B(R, R)$ be the bar resolution of R. Thinking of M as an $R{-}k$ bimodule, we saw in 2.6.2 that the functor $\otimes_R M\colon R\text{–}\mathbf{mod}\text{–}R \to R\text{–}\mathbf{mod}\text{–}k$ is left adjoint to the functor $\mathrm{Hom}_k(M, -)$. Naturality yields an isomorphism of chain complexes:

$$\mathrm{Hom}_R(B \otimes_R M, N) \cong \mathrm{Hom}_{R-R}(B, \mathrm{Hom}_k(M, N)).$$

As $B \otimes_R M$ is the bar resolution $B(R, M)$, the homology of the left side is the relative Ext. Since the homology of the right side is the Hochschild cohomology of R with coefficients in $\mathrm{Hom}(M, N)$, we are done. $\diamond$

9.2 Derivations, Differentials, and Separable Algebras

It is possible to give simple interpretations to the low-dimensional Hochschild homology and cohomology modules. We begin by observing that the kernel of the map $d\colon \mathrm{Hom}_k(R, M) \to \mathrm{Hom}_k(R \otimes R, M)$ is the set of all k-linear functions $f\colon R \to M$ satisfying the identity

$$f(r_0 r_1) = r_0 f(r_1) + f(r_0) r_1.$$

Such a function is called a *k-derivation* (or *crossed homomorphism*); the k-module of all k-derivations is written $\mathrm{Der}_k(R, M)$ (as in 8.8.1). On the other hand, the image of the map $d\colon M \to \mathrm{Hom}_k(R, M)$ is the set of all k-derivations of the form $f_m(r) = rm - mr$; call f_m a *principal derivation* and write $\mathrm{PDer}(R, M)$ for the submodule of all principal derivations. Taking H^1, we find exactly the same situation as for the cohomology of groups (6.4.5):

Lemma 9.2.1 $H^1(R, M) = \mathrm{Der}_k(R, M)/\mathrm{PDer}_k(R, M)$.

Now suppose that R is commutative. Recall from 8.8.1 that the *Kähler differentials* of R over k is the R-module $\Omega_{R/k}$ defined by the presentation: There is one generator dr for every $r \in R$, with $d\alpha = 0$ if $\alpha \in k$. For each $r_1, r_2 \in R$ there are two relations:

$$d(r_0 + r_1) = d(r_0) + d(r_1) \quad \text{and} \quad d(r_0 r_1) = r_0(dr_1) + (dr_0)r_1.$$

We saw in exercise 8.8.1 that $\mathrm{Der}_k(R, M) \cong \mathrm{Hom}_R(\Omega_{R/k}, M)$ for every right R-module M. If we make M into a bimodule by setting $rm = mr$ for all $r \in R$, $m \in M$ then $H^1(R, M) \cong \mathrm{Der}_k(R, M)$. This makes the following result seem almost immediate from the Universal Coefficient Theorem (3.6.2), since the chain complex $C(M \otimes R^{\otimes *})$ is isomorphic to $M \otimes_R C(R \otimes R^{\otimes *})$.

Proposition 9.2.2 *Let R be a commutative k-algebra, and M a right R-module. Making M into an $R{-}R$ bimodule by the rule $rm = mr$, we have natural isomorphisms $H_0(R, M) \cong M$ and $H_1(R, M) \cong M \otimes_R \Omega_{R/k}$. In particular,*

$$H_1(R, R) \cong \Omega_{R/k}.$$

Proof Since $rm = mr$ for all m and r, the map $\partial_0 - \partial_1 \colon M \otimes R \to M$ is zero. Therefore $H_0 \cong M$ and $H_1(R, M)$ is the quotient of $M \otimes_k R$ by the relations that for all $m \in M$, $r_i \in R$ $\; mr_1 \otimes r_2 - m \otimes r_1 r_2 + r_2 m \otimes r_1 = 0$. It follows that there is a well-defined map $H_1(R, M) \to M \otimes_R \Omega_{R/k}$ sending $m \otimes r$ to $m \otimes dr$. Conversely, we see from the presentation of $\Omega_{R/k}$ that there is an R-bilinear map $M \times \Omega_{R/k} \to H_1(R, M)$ sending $(m, r_1 dr_2)$ to the class of $mr_1 \otimes r_2$; this induces a homomorphism $M \otimes_R \Omega_{R/k} \to H_1(R, M)$. By inspection, these maps are inverses. $\diamond$

Corollary 9.2.3 *If S is a multiplicatively closed subset of R, then*

$$\Omega_{(S^{-1}R)/k} \cong S^{-1}(\Omega_{R/k}).$$

Proof The Change of Rings Theorem (9.1.8) states that $H_1(S^{-1}R, S^{-1}R) \cong S^{-1}H_1(R, R)$. $\diamond$

Alternate Calculation 9.2.4 For any k-algebra R, let I denote the kernel of the ring map $\varepsilon \colon R \otimes R \to R$ defined by $\varepsilon(r_1 \otimes r_2) = r_1 r_2$. Since $r \mapsto r \otimes 1$ defines a k-module splitting of ε, the sequence $0 \to I \to R^e \xrightarrow{\varepsilon} R \to 0$ is k-split. As $H_1(R, R^e) = 0$, the long exact homology sequence (exercise 9.1.2) yields

$$H_1(R, M) \cong \ker(I \otimes_{R^e} M \to IM).$$

If R is commutative and $rm = mr$, then $IM = 0$ and $H_1(R, M) \cong I/I^2 \otimes_R M$. In particular, if we take $M = R$ this yields

$$\Omega_{R/k} \cong H_1(R, R) \cong I/I^2.$$

Explicitly, the generator $dr \in \Omega_{R/k}$ corresponds to $1 \otimes r - r \otimes 1 \in I/I^2$. (Check this!)

Example 9.2.5 Let k be a field and R a separable algebraic field extension of k. Then $\Omega_{R/k} = 0$. In fact, for any $r \in R$ there is a polynomial $f(x) \in k[x]$ such that $f(r) = 0$ and $f'(r) \neq 0$. Since $d\colon R \to \Omega_{R/k}$ is a derivation we have $f'(r)dr = d(f(r)) = 0$, and hence $dr = 0$. As $\Omega_{R/k}$ is generated by the dr's, we get $\Omega_{R/k} = 0$.

Exercise 9.2.1 Suppose that R is commutative and M is a bimodule satisfying $rm = mr$. Show that there is a spectral sequence

$$E^2_{pq} = \operatorname{Tor}^R_p(H_q(R, R), M) \Rightarrow H_{p+q}(R, M).$$

Use this to give another proof of proposition 9.2.2. Then show that if M (or every $H_*(R, R)$) is a flat R-module, then $H_n(R, M) \cong H_n(R, R) \otimes_R M$ for all n.

The following two sequences are very useful in performing calculations. They will be improved later (in 9.3.5) by adding a smoothness hypothesis.

First Fundamental Exact Sequence for Ω 9.2.6 Let $k \to R \to T$ be maps of commutative rings. Then there is an exact sequence of T-modules:

$$\Omega_{R/k} \otimes_R T \xrightarrow{\alpha} \Omega_{T/k} \xrightarrow{\beta} \Omega_{T/R} \to 0.$$

The maps in this sequence are given by $\alpha(dr \otimes t) = tdr$ and $\beta(dt) = dt$.

Proof Clearly β is onto. By the Yoneda Lemma (1.6.11), in order for this sequence of T-modules to be exact at $\Omega_{T/k}$, it is sufficient to show that for every T-module N the sequence

$$\operatorname{Hom}_T(\Omega_{R/k} \otimes_R T, N) \xleftarrow{\alpha^*} \operatorname{Hom}_T(\Omega_{T/k}, N) \xleftarrow{\beta^*} \operatorname{Hom}_T(\Omega_{T/R}, N)$$

be exact. But this is just the sequence of derivation modules

$$\operatorname{Der}_k(R, N) \leftarrow \operatorname{Der}_k(T, N) \leftarrow \operatorname{Der}_R(T, N),$$

and this is exact because any k-derivation $D: T \to N$ satisfying $D(R) = 0$ is an R-derivation. $\diamond$

Second Fundamental Exact Sequence for Ω 9.2.7 Let I be an ideal of a commutative k-algebra R. Then there is an R-module map $\delta: I/I^2 \to \Omega_{R/k} \otimes_R R/I$ defined by $\delta(x) = dx \otimes 1$, fitting into an exact sequence

$$I/I^2 \xrightarrow{\delta} \Omega_{R/k} \otimes_R R/I \xrightarrow{\alpha} \Omega_{(R/I)/k} \to 0.$$

Proof If $x \in I$ and $r \in R$, then $\delta(rx) = dx \otimes r$ as $dr \otimes x = 0$; if $r \in I$ then $rx \in I^2$ and $\delta(rx) = 0$. Hence δ is well defined and R-linear. Once more we use the Yoneda Lemma 1.6.11 to take an R/I-module N and consider

$$\mathrm{Hom}_{R/I}(I/I^2, N) \xleftarrow{\delta^*} \mathrm{Der}_k(R, N) \xleftarrow{\alpha^*} \mathrm{Der}_k(R/I, N) \leftarrow 0.$$

If $D: R \to N$ is a k-derivation, then $(\delta^* D)(x) = D(x)$, so if $\delta^* D = 0$, then $D(I) = 0$, and D may be considered as a k-derivation on R/I. $\diamond$

9.2.1 Finite Separable Algebras

A finite-dimensional semisimple algebra R over a field k is called *separable* if for every extension field $k \subseteq \ell$ the ℓ-algebra $R_\ell = R \otimes_k \ell$ is semisimple.

Lemma 9.2.8 *If K is a finite field extension of k, this definition agrees with the usual definition of separability: every element of K is separable over k.*

Proof If $x \in K$ is not separable, its minimal polynomial $f \in k[X]$ has multiple roots in any splitting field ℓ. Then $K \otimes \ell$ contains $k(x) \otimes \ell = \ell[X]/f$, which is not reduced, so $K \otimes \ell$ is not reduced. Otherwise we can write $K = k(x)$, where the minimal polynomial f of x has distinct roots in any field extension ℓ of k. Hence $K \otimes \ell = \ell[X]/(f)$ is reduced, hence semisimple. $\diamond$

Corollary 9.2.9 *A finite-dimensional commutative algebra over a field is separable if and only if it is a product of separable field extensions.*

Proof A finite commutative algebra R is semisimple if and only if it is a product of fields. R is separable if and only if these fields are separable. $\diamond$

The matrix rings $M_m(k)$ form another important class of separable algebras, since $M_m(k) \otimes_k \ell \cong M_m(\ell)$. More generally, Wedderburn's Theorem states that every semisimple ring R is a finite product of simple rings, each isomorphic to $M_m(\Delta)$ for some m and some division algebra Δ; R is separable if and only if each of its simple factors is separable.

Suppose that $M_m(\Delta)$ is separable. If F is the center of Δ, then $F \otimes \ell$ is a subring of $\Delta \otimes \ell$ and $M_m(\Delta) \otimes \ell$, so F must also be a finite separable extension of k. It is easy to see that if ℓ is a splitting field of F, then $F \otimes \ell$ is a finite product of copies of ℓ, so each of the simple factors of $M_m(\Delta) \otimes \ell$ has center ℓ. As we saw in 6.6.10 (see [BAII, 8.4]), there exists a finite extension L of ℓ such that $L \otimes_k M_m(\Delta) = L \otimes_\ell (\ell \otimes_k M_m(\Delta))$ is a product of matrix rings over L. In summary, we have proven that if R is separable over k, then there is a finite extension L of k such that $R \otimes L$ is a finite product of matrix rings $M_{m_i}(L)$.

Lemma 9.2.10 *If $R = M_m(k)$, then R is a projective R^e-module.*

Proof The element $e = \sum e_{i1} \otimes e_{1i}$ of $R^e = M_m(k) \otimes M_m(k)^{\mathrm{op}}$ is idempotent ($e^2 = e$) and the product map $\varepsilon \colon R \otimes R^{\mathrm{op}} \to R$ sends e to $\sum e_{ii} = 1$. Define $\alpha \colon R \to R^e$ by $\alpha(r) = re$. Since the basis elements e_{ij} of R satisfy $e_{ij}e = e_{i1} \otimes e_{1j} = ee_{ij}$, we have $re = er$ for all $r \in R$; hence α is an $R{-}R$ bimodule map. Since $\varepsilon\alpha$ is the identity on R, this shows that R is a summand of R^e. $\diamond$

Theorem 9.2.11 *Let R be an algebra over a field k. The following are equivalent:*

1. *R is a finite-dimensional separable k-algebra.*
2. *R is projective as a left R^e-module.*
3. *$H_*(R, M) = 0$ for all $* \neq 0$ and all bimodules M.*
4. *$H^*(R, M) = 0$ for all $* \neq 0$ and all bimodules M.*

Proof From the "pd" and "fd" lemmas of 4.1.6 and 4.1.10 we see that (2), (3), and (4) are equivalent. If R is separable, choose $k \subset \ell$ so that R_ℓ is a finite product of matrix rings $R_i = M_{m_i}(\ell)$. Since every $R{-}R$ bimodule is a product $M = \Pi M_i$ of $R_i{-}R_i$ bimodules M_i we have $H_*(R, M) = \Pi H_*(R_i, M_i) = 0$ by 9.1.8 and the above lemma. Thus (1) $\Rightarrow$ (3).

Now assume that (2) holds for R. Then (2), (3), and (4) hold for every $R \otimes \ell$ because $R \otimes \ell$ is projective over the ring

$$(R_\ell)^e = (R \otimes \ell) \otimes_\ell (R \otimes \ell)^{\mathrm{op}} = (R \otimes R^{\mathrm{op}}) \otimes \ell = (R^e) \otimes \ell.$$

We have isolated the proof that $\dim(R) < \infty$ in lemma 9.2.12 following this proof. Now each R_ℓ is semisimple if and only if R_ℓ has global dimension 0

(4.2.2). If M and N are left R_ℓ-modules, we saw in 9.1.3 and 9.1.9 that

$$\mathrm{Ext}^*_{R_\ell}(M, N) = \mathrm{Ext}^*_{R_\ell/k}(M, N) \cong H^*(R_\ell, \mathrm{Hom}_k(M, N)).$$

As (4) holds for R_ℓ, the right side is zero for $* \neq 0$ and all M, N; the Global Dimension Theorem (4.1.2) implies that R_ℓ has global dimension 0. Hence (2) $\Rightarrow$ (1). ◇

Lemma 9.2.12 (Villamayor-Zelinsky) *Let R be an algebra over a field k. If R is projective as an R^e-module, then R is finite-dimensional as a vector space over k.*

Proof Let $\{x_i\}$ be a basis for R as a vector space and $\{f^i\}$ a dual basis for $\mathrm{Hom}_k(R, k)$. As R^e is a free left R-module on basis $\{1 \otimes x_i\}$ with dual basis $\{1 \otimes f^i\} \subseteq \mathrm{Hom}_R(R^e, R)$, we have

$$u = \sum (1 \otimes f^i)(u) \otimes x_i \quad \text{for all } u \in R^e.$$

Now if R is a projective R^e-module, the surjection $\varepsilon: R^e \to R$ must be split. Hence there is an idempotent $e \in R^e$ such that $R^e \cdot e \cong R$ and $\varepsilon(e) = 1$. In particular, $(1 \otimes r - r \otimes 1)e = 0$ for all $r \in R$. Setting $u = (1 \otimes r)e = (r \otimes 1)e$ yields

$$(*) \qquad r = \varepsilon(u) = \sum (1 \otimes f_i)((r \otimes 1)e) \cdot x_i = r \sum (1 \otimes f_i)(e)x_i.$$

Therefore the sum in $(*)$ is over a finite indexing set independent of r. Writing $e = \sum e_{\alpha\beta} x_\alpha \otimes x_\beta$ with $e_{\alpha\beta} \in k$ allows us to rewrite $(*)$ as

$$r = \sum (1 \otimes f_i)(e_{\alpha\beta} x_\alpha \otimes r x_\beta) x_i = \sum e_{\alpha\beta} f(r x_\beta)(x_\alpha x_i).$$

Therefore the finitely many elements $x_\alpha x_i$ span R as a vector space. ◇

9.3 H^2, Extensions, and Smooth Algebras

From the discussion in Chapter 6, section 6 about extensions and factor sets we see that $H^2(R, M)$ should have something to do with extensions. By a (square zero) *extension* of R by M we mean a k-algebra E, together with a surjective ring homomorphism $\varepsilon: E \to R$ such that $\ker(\varepsilon)$ is an ideal of square zero (so that $\ker(\varepsilon)$ has the structure of an $R-R$ bimodule), and an R-module isomorphism of M with $\ker(\varepsilon)$. We call it a *Hochschild extension* if the short exact sequence $0 \to M \to E \to R \to 0$ is k-split, that is, split as a sequence of k-modules. Choosing such a splitting $\sigma: R \to E$ yields a k-module decomposition $E \cong R \oplus M$, with multiplication given by

$$(*) \qquad (r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2 + f(r_1, r_2)).$$

We call the function $f: R \otimes R \to M$ the *factor set* of the extension corresponding to the splitting σ. Since the product $(r_0, 0)(r_1, 0)(r_2, 0)$ is associative, the factor set must satisfy the cocycle condition

$$r_0 f(r_1, r_2) - f(r_0 r_1, r_2) + f(r_0, r_1 r_2) - f(r_0, r_1) r_2 = 0.$$

Conversely, any function satisfying this cocycle condition yields a Hochschild extension with multiplication defined by $(*)$. (Check this!) A different choice $\sigma': R \to E$ of a splitting yields a factor set f', and

$$\begin{aligned} f'(r_1, r_2) - f(r_1, r_2) &= \sigma'(r_1)\sigma'(r_2) - \sigma'(r_1 r_2) - \sigma(r_1)\sigma(r_2) + \sigma(r_1 r_2) \\ &= \sigma'(r_1)[\sigma'(r_2) - \sigma(r_2)] - [\sigma'(r_1 r_2) - \sigma(r_1 r_2)] \\ &\quad + [\sigma'(r_1) - \sigma(r_1)]\sigma(r_2), \end{aligned}$$

which is the coboundary of the element $(\sigma' - \sigma) \in \mathrm{Hom}(R, M)$. Hence a Hochschild extension determines a unique cohomology class, independent of the choice of splitting σ.

The *trivial extension* is obtained by taking $E \cong R \oplus M$ with product $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$. Since its factor set is $f = 0$, the trivial extension yields the cohomology class $0 \in H^2(R, M)$.

As with group extensions, we say that two extensions E and E' are *equivalent* if there is a ring isomorphism $\varphi : E \cong E'$ making the familiar diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & R & \longrightarrow & 0 \\ & & \| & & \varphi\downarrow & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & E' & \longrightarrow & R & \longrightarrow & 0. \end{array}$$

Since E and E' share the same factor sets, they determine the same cohomology class. We have therefore proven the following result.

Classification Theorem 9.3.1 *Given a k-algebra R and an $R-R$ bimodule M, the equivalence classes of Hochschild extensions are in 1–1 correspondence with the elements of the Hochschild cohomology module $H^2(R, M)$.*

Here is a variant of the Classification Theorem 9.3.1 when R is a commutative k-algebra. If a commutative algebra E is a Hochschild extension of R by an R–R bimodule M, then M must be *symmetric* in the sense that $rm = mr$ for every $m \in M$ and $r \in R$. A moment's thought shows that symmetric bimodules are the same thing as R-modules.

If we choose a k-splitting $\sigma : R \to E$ for a commutative Hochschild extension, then the corresponding factor set f must satisfy $f(r_1, r_2) = f(r_2, r_1)$, because $\sigma(r_1)$ and $\sigma(r_2)$ must commute in E. Let us call such a factor set *symmetric*. If f is a symmetric factor set, the equation $(*)$ shows that multiplication in E is commutative.

Let us write $H_s^2(R, M)$ for the submodule of $H^2(R, M)$ consisting of the equivalence classes of symmetric factor sets. With this notation, we can summarize the above discussion as follows

Commutative Extensions 9.3.1.1 *Let R be a commutative k-algebra and M an R-module. Then the equivalence classes of commutative Hochschild extensions of R by M are in 1–1 correspondence with the elements of the module $H_s^2(R, M)$.*

Remark Let k be a field. This classification, together with Exercise 8.8.4, proves that $H_s^2(R, M)$ is just the André-Quillen cohomology $D^1(R, M)$. The characteristic zero version of this was given in 8.8.9.

9.3.2 We say that a k-algebra is *quasi-free* (over k) if for every square-zero extension $0 \to M \to E \xrightarrow{\varepsilon} T \to 0$ of a k-algebra T by a T–T bimodule M and every algebra map $\nu : R \to T$, there exists a k-algebra homomorphism $u : R \to E$ lifting ν in the sense that $\varepsilon u = \nu$. For example, it is clear that every free algebra is quasi-free over k.

$$\begin{tikzcd}
 & & k \arrow[r] \arrow[d] & R \arrow[dl] \arrow[d, "\nu"] & \\
0 \arrow[r] & M \arrow[r] & E \arrow[r, "\varepsilon"'] & T \arrow[r] & 0
\end{tikzcd}$$

If R is quasi-free and J is a nilpotent ideal in another k-algebra E, then every algebra map $R \to E/J$ may be lifted to a map $R \to E$. In fact, we can lift it successively to $R \to E/J^2$, to $R \to E/J^3$, and so on. Since $J^m = 0$ for some m, we eventually lift it to $R \to E/J^m = E$.

Proposition 9.3.3 (J.H.C. Whitehead-Hochschild) *If k is a field, then a k-algebra R is quasi-free iff and only if $H^2(R, M) = 0$ for all R–R bimodules M.*

Proof If R is quasi-free, every extension of R by a bimodule M must be trivial, so $H^2(R, M) = 0$ by the Classification Theorem 9.3.1. Conversely, given an extension $0 \to M \to E \to T \to 0$ and $\nu: R \to T$, let D be the pullback $D = \{(r, e) \in R \times T : \nu(r) = \bar{e} \text{ in } T\}$. Then D is a subring of $R \times E$ and the kernel of $D \to R$ is a square zero ideal isomorphic to M.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & D & \longrightarrow & R & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \nu & & \\ 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & T & \longrightarrow & 0. \end{array}$$

Since k is a field, D is a Hochschild extension of R and is classified by an element of $H^2(R, M)$. So if $H^2(R, M) = 0$, then there is a k-algebra splitting $\sigma: R \to D$ of $D \to R$; the composite of σ with $D \to E$ is a lifting of $R \to T$. Quantifying over all such M proves that R is quasi-free. $\diamond$

Corollary 9.3.3.1 *If R is an algebra over a field k and $H^2(R, M) = 0$ for every $R - R$ bimodule M, then any k-algebra surjection $E \to R$ with nilpotent kernel must be split by a k-algebra injection $\sigma : R \to E$.*

Exercise 9.3.1 (Wedderburn's Principal Theorem) Let R be a finite-dimensional algebra over a field k, with Jacobson radical $J = J(R)$. It is well known that the quotient R/J is a semisimple ring ([BAII, 4.2]). Prove that if R/J is separable, then there is a k-algebra injection $R/J \subset R$ splitting the natural surjection $R \to R/J$. *Hint:* Use the General Version 4.3.10 of Nakayama's Lemma to show that J is nilpotent.

9.3.1 Smooth Algebras

For the rest of this section, all the algebras we consider will be commutative.

We say that a commutative k-algebra is *smooth* (over k) if for every square-zero extension $0 \to M \to E \xrightarrow{\varepsilon} T \to 0$ of commutative k-algebras and every algebra map $\nu : R \to T$, there exists a k-algebra homomorphism $u : R \to E$ lifting ν in the sense that $\varepsilon u = \nu$. For example, it is clear that every polynomial algebra $R = k[x_1, \dots, x_n]$ is smooth over k.

Proposition 9.3.4 (Whitehead-Hochschild) *Let R be an algebra over a field k. Then R is smooth if and only if $H_s^2(R, M) = 0$ for all R-modules M.*

If R is smooth, then any surjection $E \to R$ of commutative k-algebras with nilpotent kernel J must be split by a k-algebra injection $\sigma : R \to E$.

Proof The proof of the Whitehead-Hochschild result 9.3.3, and the arguments in 9.3.2, go through with no changes for commutative algebras. $\diamond$

Exercise 9.3.2

1. (Localization) If R is smooth over k and $S \subset R$ is a central multiplicative set, show that $S^{-1}R$ is smooth over k.
2. (Transitivity) If R is smooth over K and K is smooth over k, show that R is smooth over k.
3. (Base change) If R is smooth over k and $k \to \ell$ is any ring map, show that $R \otimes_k \ell$ is smooth over ℓ.
4. If k is a field, show that any filtered union of smooth algebras is smooth.

Exercise 9.3.3 Let $0 \to M \to E \xrightarrow{\varepsilon} T \to 0$ be a square zero algebra extension and $u: R \to E$ a k-algebra map. If $u': R \to E$ is any k-module map with $\varepsilon u' = \varepsilon u$, then $u' = u + D$ for some k-module map $D: R \to M$. Show that u' is a k-algebra map if and only if D is a k-derivation.

Fundamental Sequences for Ω with Smoothness 9.3.5 Let $k \to R \xrightarrow{f} T$ be maps of commutative rings.

1. If T is smooth over R, then the first fundamental sequence 9.2.6 becomes a *split* exact sequence by adding $0 \to$ on the left:
$$0 \to \Omega_{R/k} \otimes_R T \xrightarrow{\alpha} \Omega_{T/k} \xrightarrow{\beta} \Omega_{T/R} \to 0.$$
2. If $T = R/I$ and T is smooth over k, then the second fundamental sequence 9.2.7 becomes a *split* exact sequence by adding $0 \to$ on the left:
$$0 \to I/I^2 \xrightarrow{\delta} \Omega_{R/k} \otimes_R R/I \xrightarrow{\alpha} \Omega_{(R/I)/k} \to 0.$$

Proof For (1), let N be a T-module, and $D: R \to N$ a k-derivation. Define a ring map φ from R to the trivial extension $T \oplus N$ by $\varphi(r) = (f(r), Dr)$. By smoothness, the projection $T \oplus N \to T$ is split by an R-module homomorphism $\sigma: T \to T \oplus N$. Writing $\sigma(t) = (t, D't)$, then $D': T \to N$ is a k-derivation of T such that $D'f = D$. (Check this!) Now take N to be $\Omega_{R/k} \otimes_R T$; D' corresponds to a T-bilinear map $\gamma: \Omega_{T/k} \to \Omega_{R/k} \otimes_R T$. If D is the derivation $D(r) = dr \otimes 1$, then $\gamma\alpha$ is the identity on N and γ splits α.

For (2), note that smoothness of $T = R/I$ implies that the sequence $0 \to I/I^2 \to R/I^2 \xrightarrow{f} R/I \to 0$ is split by a k-algebra map $\sigma: R/I \to R/I^2$. The map $D = 1 - \sigma f: R \to R/I^2$ satisfies $\bar{f}D = f - (\bar{f}\sigma)f = 0$, so the image of D lies in I/I^2 and D is a derivation. Moreover the restriction of D to I is the natural projection $I \to I/I^2$. By universality, D corresponds to an R-module map $\theta: \Omega_{R/k} \to I/I^2$ sending $r\,ds$ to $rD(s)$. Thus θ kills $I\Omega_{R/k}$ and factors through $\Omega_{R/k} \otimes_R R/I$, with $\theta\delta$ the identity on I/I^2. ◇

We are going to characterize those field extensions K that are smooth over k. For this, we recall some terminology and results from field theory [Lang, X.6]. Let k be a field and K a finitely generated extension field. We say that K is *separately generated* over k if we can find a transcendence basis $(t_1, \cdots, t_r)$ of K/k such that K is separably algebraic over the purely transcendental field $k(t_1, \cdots, t_r)$. If $\mathrm{char}(k) = 0$, or if k is perfect, it is known that every finitely generated extension of k is separably generated.

Proposition 9.3.6 *If k is a field, every separably generated extension field K is smooth over k.*

Proof K is separably algebraic over some purely transcendental field $F = k(t_1, \cdots, t_r)$. As F is a localization of the polynomial ring $k[t_1, \cdots, t_r]$, which is smooth over k, F is smooth over k. By transitivity of smoothness, it suffices to prove that K is smooth over F. Since K is a finite separable algebraic extension of F, we may write $K = F(x)$, where $f(x) = 0$ for some irreducible polynomial f with $f'(x) \neq 0$. Suppose given a map $\nu: K \to T$ and a square zero extension $0 \to M \to E \to T \to 0$. Choosing any lift $y \in E$ of $\nu(x) \in T$, we have $f(y+m) = f(y) + f'(y)m$ for every $m \in M$. Since $\nu(f(x)) = 0$ and $\nu(f'(x))$ is a unit of T, $f(y) \in M$ and $f'(y)$ is a unit of E. If we put $m = -f(y)/f'(y)$, then $f(y+m) = 0$, so we may define a lift $K \to E$ by sending x to $y + m$. ◇

Corollary 9.3.7 *If k is a perfect field, every extension field K is smooth over k. In particular, every extension field is smooth when $char(k) = 0$.*

Proof If K_α is a finitely generated extension subfield of K, then K_α is separably generated and hence smooth. If M is a K-module, then $H_s^2(K_\alpha, M) = 0$. As tensor products and homology commute with filtered direct limits, we have $H_s^2(K, M) = \lim H_s^2(K_\alpha, M) = 0$. Hence K is smooth. ◇

When $\mathrm{char}(k) \neq 0$ and k is not perfect, the situation is as follows. Call K *separable* (over k) if every finitely generated extension subfield is separably generated. The proof of the above corollary shows that separable extensions are smooth; in fact the converse is also true [Mat, 20.L]:

Theorem 9.3.8 *Let $k \subset K$ be an extension of fields. Then*

$$K \text{ is separable over } k \Leftrightarrow K \text{ is smooth over } k.$$

Remark 9.3.9 One of the major victories in field theory was the discovery that a field extension $k \subset K$ is separable if and only if for any finite field extension $k \subset \ell$ the ring $K \otimes_k \ell$ is reduced. If $\mathrm{char}(k) = p$, separability is also equivalent to *MacLane's criterion for separability:* K is linearly disjoint from the field $\ell = k^{1/p^\infty}$ obtained from k by adjoining all p-power roots of elements of k. See [Mat, 27.F] and [Lang, X.6]. Here is the most important part of this relationship.

Lemma 9.3.10 *Let K be a separably generated extension of a field k. Then for every field extension $k \subset \ell$ the ring $K \otimes_k \ell$ is reduced.*

Proof It is enough to consider the case of a purely transcendental extension and the case of a finite separable algebraic extension. If $K = k(x)$ is purely transcendental, then each $K \otimes \ell = \ell(x)$ is a field. If K is a finite separable extension, we saw that $K \otimes \ell$ is reduced for every ℓ in 9.2.8 ◇

Exercise 9.3.4 A commutative algebra R over a field k is called *separable* if R is reduced and for any algebraic field extension $k \subset \ell$ the ring $R \otimes_k \ell$ is reduced. By the above remark, this agrees with the previous definition when R is a field. Show that

1. Every subalgebra of a separable algebra is again separable.
2. The filtered union of separable algebras is again separable.
3. Any localization of a separable algebra is separable.
4. If $\mathrm{char}(k) = 0$, or more generally if k is perfect, every reduced k-algebra is separable; this completely classifies separable algebras over k.
5. An artinian k-algebra R is separable if and only if R is a finite product of separable field extensions of k (see 9.2.9).
6. A finite-dimensional algebra R is separable in the sense of this exercise if and only if it is separable in the sense of section 9.2.1.

9.3.2 Smoothness and Regularity

For the next result, we shall need the *Hilbert-Samuel function* $h_R(n) =$ length of $R/\mathfrak{m}^n$ of a d-dimensional noetherian local ring R. There is a polynomial $H_R(t)$ of degree d, called the *Hilbert-Samuel polynomial*, such that $h_R(n) = H_R(n)$ for all large n; see [Mat, 12.C&H]. For example, if R is the localization of the polynomial ring $K[x_1, \cdots, x_d]$ at the maximal ideal $M = (x_1, \cdots, x_d)$, then $h_R(n) = H_R(n) = \binom{n+d-1}{d} = \frac{n(n+1)\cdots(n+d-1)}{d!}$ for all $n \geq 1$.

Theorem 9.3.11 *Let R be a noetherian local ring containing a field k. If R is smooth over k, then R is a regular local ring.*

Proof Set $d = \dim_K(\mathfrak{m}/\mathfrak{m}^2)$, and write S for the local ring of $K[x_1, \cdots, x_d]$ at the maximal ideal $M = (x_1, \cdots, x_d)$. Note that $S/M^2 \cong K \oplus \mathfrak{m}/\mathfrak{m}^2$. By replacing k by its ground field if necessary, we may assume that the residue field $K = R/\mathfrak{m}$ is also smooth over k. This implies that the square zero extension $R/\mathfrak{m}^2 \to K$ splits, yielding an isomorphism $R/\mathfrak{m}^2 \cong K \oplus (\mathfrak{m}/\mathfrak{m}^2) \cong S/M^2$. Since R is smooth, we can lift $R \to R/\mathfrak{m}^2 \cong S/M^2$ to maps $R \to S/M^n$ for every n. By Nakayama's Lemma 4.3.9, if R maps onto S/M^n, then R maps onto S/M^{n+1} (because $\mathfrak{m}(S/M^{n+1})$ contains M^n/M^{n+1}). Inductively, this proves that $R/\mathfrak{m}^n$ maps onto S/M^n for every n and hence that $h_R(n) \geq h_S(n)$ for all n. Therefore the Hilbert polynomial $H_R(t)$ has degree $\geq d$, and hence $\dim(R) \geq d$. Since we always have $\dim(R) \leq d$ (4.4.1), this yields $\dim(R) = d$, that is, R is a regular local ring. $\diamond$

Definition 9.3.12 A commutative noetherian ring R is called *regular* if the localization of R at any prime ideal is a regular local ring (see 4.4.1). We say that R is *geometrically regular* over a field k if R contains k, and for every finite field extension $k \subset \ell$ the ring $R \otimes_k \ell$ is also regular.

Corollary 9.3.13 *Let R be a commutative noetherian ring containing a field k. If R is smooth over k, then R is geometrically regular over k.*

Proof If R is smooth over k, then so is every localization of R. Hence R is regular. For each $k \subset \ell$, $R \otimes \ell$ is smooth over ℓ, hence regular. $\diamond$

Remark In fact the converse is true: Geometrically regular k-algebras are smooth over k; see [EGA, 0_{IV}(22.5.8)].

Theorem 9.3.14 *If R is a smooth k-algebra, then $\Omega_{R/k}$ is a projective R-module.*

Proof We will show that $\Omega_{R/k}$ satisfies the projective lifting property. Suppose given an R-module surjection $u: M \to N$ and a map $v: \Omega_{R/k} \to N$. If I is the kernel of $R^e \to R$, then the square zero algebra extension $R^e/I^2 \to R$ is trivial, that is, $R^e/I^2 \cong R \oplus I/I^2$ as a k-algebra. Moreover, $I/I^2 \cong \Omega_{R/k}$ by 9.2.4. We thus have a diagram of k-algebras

$$\begin{array}{ccc} R^e & \longrightarrow R^e/I^2 \cong & R \oplus \Omega_{R/k} \\ \exists \downarrow w & & \downarrow v \\ R \oplus M & \xrightarrow{(1,u)} & R \oplus N. \end{array}$$

The kernel of $R \oplus M \to R \oplus N$ is the square zero ideal $0 \oplus \ker(u)$. By base change (exercise 9.3.2) $R^e = R \otimes_k R$ is smooth over R, hence over k, so $R^e \to R \oplus N$ lifts to a k-algebra map $w: R^e \to R \oplus M$. Since $w(I)$ is contained in $0 \oplus M$ (why?), $w(I^2) = 0$. Thus w induces an R-module lifting $I/I^2 \to M$ of v. ◇

Remark The rank of $\Omega_{R/k}$ is given in 9.4.8.

Application 9.3.15 (Jacobian criterion) Suppose that $R = k[x_1, \cdots, x_n]/J$, where J is the ideal generated by polynomials $f_1, \cdots, f_m$. The second fundamental sequence 9.2.7 is

$$J/J^2 \xrightarrow{\delta} R^n \to \Omega_{R/k} \to 0,$$

where R^n denotes the free R-module on basis $\{dx_1, \cdots, dx_n\}$. Since J/J^2 is generated by $f_1, \cdots, f_m$ the map δ is represented by the $m \times n$ *Jacobian matrix* $(\partial f_i/\partial x_j)$. Now suppose that R is smooth, so that this sequence is split exact and J/J^2 is also a projective R-module. If M is a maximal ideal of $k[x_1, \cdots, x_n]$ with residue field $K = R/M$, and $d = \dim(R_M)$, then J_M is generated by a regular sequence of length $n - d$, so $(J/J^2) \otimes_R K$ is a vector space of dimension $n - d$. Therefore the Jacobian matrix $(\partial f_i/\partial x_j)$ has rank $n - d$ when evaluated over $K = R/M$. This proves the necessity of the following criterion; the sufficiency is proven in [EGA, 0_{IV}(22.6.4)], and in [Mat, section 29].

Jacobian criterion: R is smooth if and only if the Jacobian matrix $(\partial f_i/\partial x_j)$ has rank $n - \dim(R_M)$ when evaluated over R/M for every maximal ideal M.

9.4 Hochschild Products

There are external and internal products in Hochschild homology, just as there were for absolute Tor (and Ext) in 2.7.8 and exercise 2.7.5, and for relative Tor (and Ext) in 8.7.12 and exercise 8.7.2. All these external products involve two k-algebras R and R' and their tensor product algebra $R \otimes R'$. To obtain internal products in homology we need an algebra map $R \otimes R \to R$, which requires R to commutative. This situation closely resembles that of algebraic topology (pretend that R is a topological space X; the analogue of R being commutative is that X is an H-space). We shall not discuss the internal product for cohomology, since it is entirely analogous but needs an algebra map $R \to R \otimes R$, which requires R to be a Hopf algebra (or a bialgebra).

We begin with the external product for Hochschild homology. Let R and R' be k-algebras. Since the bar resolution $\beta(R, R)$ is an $R{-}R$ bimodule resolution of R and $\beta(R', R')$ is an $R'{-}R'$ bimodule resolution of R', their tensor product $\beta(R, R) \otimes \beta(R', R')$ comes from a bisimplicial object in the category **bimod** of $(R \otimes R')-(R \otimes R')$ bimodules. In 8.6.13 we showed that the shuffle product ∇ induces a chain homotopy equivalence in **bimod:**

$$\operatorname{Tot} \beta(R, R) \otimes \beta(R', R') \xrightarrow{\nabla} \beta(R \otimes R', R \otimes R').$$

If M is an $R{-}R$ bimodule and M' is an $R'{-}R'$ bimodule, then we can tensor over $(R \otimes R')^e$ with $M \otimes M'$ to obtain a chain homotopy equivalence

$$\operatorname{Tot}\{(M \otimes_{R^e} \beta(R, R)) \otimes (M' \otimes_{R'^e} \beta(R', R'))\} \xrightarrow{\nabla} (M \otimes M') \otimes_{(R \otimes R')^e} \beta(R \otimes R', R \otimes R').$$

Recall from 9.1.3 that the Hochschild chain complex $C(M \otimes R^{\otimes *})$ is isomorphic to $M \otimes_{R^e} \beta(R, R)$. Hence we may rewrite the latter equivalence as

$$\operatorname{Tot}\{C(M \otimes R^{\otimes *}) \otimes C(M' \otimes R'^{\otimes *})\} \xrightarrow{\nabla} C((M \otimes M') \otimes (R \otimes R')^{\otimes *}).$$

If we apply $\operatorname{Hom}_{\mathbf{bimod}}(-, M \otimes M')$ we get an analogous cochain homotopy equivalence

$$\operatorname{Tot} \operatorname{Hom}_{\mathbf{bimod}}(\beta(R, R) \otimes \beta(R', R'), M \otimes M') \xrightarrow{\nabla} C \operatorname{Hom}_k((R \otimes R')^{\otimes *}, M \otimes M'),$$

but the natural map from $\operatorname{Hom}_R(\beta, M) \otimes \operatorname{Hom}_{R'}(\beta', M')$ to $\operatorname{Hom}_{\mathbf{bimod}}(\beta \otimes \beta', M \otimes M')$ is not an isomorphism unless R or R' is a finite-dimensional algebra. The Künneth formula for complexes (3.6.3) yields the following result.

Proposition 9.4.1 (External products) *The shuffle product ∇ induces natural maps*

$$H_i(R, M) \otimes H_j(R', M') \xrightarrow{\nabla} H_{i+j}(R \otimes R', M \otimes M'),$$

$$H^i(R, M) \otimes H^j(R', M') \xrightarrow{\nabla} H^{i+j}(R \otimes R', M \otimes M').$$

For $i = j = 0$ these products are induced by the identity map on $M \otimes M'$. If k is a field, the direct sum of the shuffle product maps yields natural isomorphisms

$$\begin{aligned} H_n(R \otimes R', M \otimes M') &\cong [H_*(R, M) \otimes H_*(R', M')]_n \\ &= \bigoplus_{i+j=n} H_i(R, M) \otimes H_j(R', M'). \end{aligned}$$

Similarly, the shuffle product $\nabla\colon H^*(R, M) \otimes H^*(R', M') \to H^*(R \otimes R', M \otimes M')$ is an isomorphism when either R or R' is finite-dimensional over a field k.

Remark The explicit formula for ∇ in exercise 8.6.5 shows that the external product is associative from $H(R, M) \otimes H(R', M') \otimes H(R'', M'')$ to $H(R \otimes R' \otimes R'', M \otimes M' \otimes M'')$.

Exercise 9.4.1 Let $0 \to M_0 \to M_1 \to M_2 \to 0$ be a k-split exact sequence of $R-R$ bimodules. Show that ∇ commutes with the connecting homomorphism ∂ in the sense that there is a commutative diagram

$$\begin{array}{ccc} H_i(R, M_2) \otimes H_j(R', M') & \xrightarrow{\nabla} & H_{i+j}(R \otimes R', M_2 \otimes M') \\ {\scriptstyle \partial\otimes 1}\downarrow & & \downarrow{\scriptstyle \partial} \\ H_{i-1}(R, M_0) \otimes H_j(R', M') & \xrightarrow{\nabla} & H_{i+j-1}(R \otimes R', M_0 \otimes M'). \end{array}$$

9.4.1 Internal Product

Now suppose that R is a commutative k-algebra. Then the product $R \otimes R \to R$ is a k-algebra homomorphism. Composing the external products with this homomorphism yields a product in Hochschild homology

$$H_p(R, M) \otimes H_q(R, M') \to H_{p+q}(R, M \otimes_{R^e} M').$$

Here $M \otimes M'$ is an $R-R$ bimodule by $r(m \otimes m')s = (rm) \otimes (m's)$. When $M = M' = R$, the external products yield an associative product on $H_*(R, R)$.

In fact, more is true. At the chain level, the shuffle product 8.6.13 gives a map

$$\operatorname{Tot} C(R \otimes R^{\otimes *}) \otimes C(R \otimes R^{\otimes *}) \xrightarrow{\nabla} C((R \otimes R) \otimes (R \otimes R)^{\otimes *}) \to C(R \otimes R^{\otimes *}).$$

Proposition 9.4.2 *If R is a commutative k-algebra, then*

1. *$C(R \otimes R^{\otimes *}) = R \otimes_{R^e} \beta(R, R)$ is a graded-commutative differential graded k-algebra (4.5.2).*
2. *$H_*(R, R)$ is a graded-commutative k-algebra.*

Proof It suffices to establish the first point (see exercise 4.5.1). Write C_* for $C(R \otimes R^{\otimes *}) = R \otimes_{R^e} \beta(R, R)$. The explicit formula for ∇ (exercise 8.6.5) becomes

$$(r_0 \otimes r_1 \otimes \cdots \otimes r_p) \nabla (r_0' \otimes r_{p+1} \otimes \cdots \otimes r_{p+q}) =$$
$$\sum_{\mu} (-1)^{\mu} (r_0 r_0') \otimes r_{\mu^{-1}(1)} \otimes \cdots \otimes r_{\mu^{-1}(p+q)},$$

where μ ranges over all (p, q)-shuffles. The product ∇ is associative, because an (n, p, q)-shuffle may be written uniquely either as the composition of a (p, q)-shuffle and an $(n, p+q)$-shuffle, or as the composition of an (n, p)-shuffle and an $(n+p, q)$-shuffle. Interchanging p and q amounts to precomposition with the shuffle $\nu = (p+1, \cdots, p+q, 1, \cdots, p)$; since $(-1)^{\nu} = (-1)^{pq}$ the product ∇ is graded-commutative. Finally, we know that $\nabla \colon \operatorname{Tot}(C_* \otimes C_*) \to C_*$ is a chain map. Therefore if we set $\rho = (r_0, r_1, \cdots, r_p)$ and $\rho' = (r_0', r_{p+1}, \cdots, r_{p+q})$ and recall the sign trick 1.2.5 for d^v we have the Leibniz formula:

$$d(\rho \nabla \rho') = \nabla (d^h + d^v)(\rho \otimes \rho') = (d\rho) \nabla \rho' + (-1)^p \rho \nabla (d\rho'). \qquad \diamond$$

Corollary 9.4.3 *If R is commutative and M is an $R{-}R$ bimodule, then $H_*(R, M)$ is a graded $H_*(R, R)$-module.*

9.4.2 The Exterior Algebra $\Omega^*_{R/k}$

As an application, recall that $H_1(R, R)$ is isomorphic to the R-module $\Omega_{R/k}$ of Kähler differentials of R over k. If we write $\Omega^n_{R/k}$ for the n^{th} exterior product $\Lambda^n(\Omega_{R/k})$, then the exterior algebra $\Omega^*_{R/k}$ on $\Omega_{R/k}$ is

$$\Omega^*_{R/k} = R \oplus \Omega_{R/k} \oplus \Omega^2_{R/k} \oplus \cdots.$$

Note that $\Omega^0_{R/k} = R$ and $\Omega^1_{R/k} = \Omega_{R/k}$. $\Omega^*_{R/k}$ is the free graded-commutative R-algebra generated by $\Omega_{R/k}$; if K_* is a graded-commutative R-algebra, then any R-module map $\Omega_{R/k} \to K_1$ extends uniquely to an algebra map $\Omega^*_{R/k} \to K_*$.

Corollary 9.4.4 *If R is a commutative k-algebra, the isomorphism $\Omega^1_{R/k} \cong H_1(R, R)$ extends to a natural graded ring map $\psi\colon \Omega^*_{R/k} \to H_*(R, R)$. If $\mathbb{Q} \subset R$, this is an injection, split by a graded ring surjection $e\colon H_*(R, R) \to \Omega^*_{R/k}$.*

Proof Since $H_*(R, R)$ is graded-commutative, the first assertion is clear. For the second, define a map $e\colon R^{\otimes n+1} \to \Omega^n_{R/k}$ by the multilinear formula

$$e(r_0 \otimes r_1 \otimes \cdots \otimes r_n) = \frac{1}{n!} r_0 dr_1 \wedge \cdots \wedge dr_n.$$

The explicit formula for ∇ shows that $e(\rho \nabla \rho') = e(\rho) \wedge e(\rho')$ in $\Omega^*_{R/k}$. Therefore e is a graded R-algebra map from $R^{\otimes *+1}$ to $\Omega^*_{R/k}$. An easy calculation shows that $e(b(r_0 \otimes \cdots \otimes r_{n+1})) = 0$. (Check this!) Hence e induces an algebra map $HH_*(R, R) \to \Omega^*_{R/k}$. To see that e splits ψ, we compute that

$$\begin{aligned} e\psi(r_0 dr_1 \wedge \cdots \wedge dr_n) &= e((r_0 \otimes r_1)\nabla(1 \otimes r_2)\nabla \cdots \nabla(1 \otimes r_n)) \\ &= e(r_0 \otimes r_1) \wedge e(1 \otimes r_2) \wedge \cdots \wedge e(1 \otimes r_n) \\ &= r_0 dr_1 \wedge r_2 \wedge \cdots \wedge r_n. \end{aligned}$$ ◇

Definition 9.4.5 We say that a commutative k-algebra R is *essentially of finite type* if it is a localization of a finitely generated k-algebra. If k is noetherian, this implies that R and $R^e = R \otimes R$ are both noetherian rings (by the Hilbert Basis Theorem).

Proposition 9.4.6 *Suppose that R is a commutative algebra, essentially of finite type over a field k. If R is smooth over k, then R^e is a regular ring.*

Proof We saw in 9.3.13 that smooth noetherian k-algebras are regular. By smooth base change and transitivity (exercise 9.3.2), $R^e = R \otimes R$ is smooth over R and hence smooth over k. Since R^e is noetherian, it is regular. ◇

Theorem 9.4.7 (Hochschild-Kostant-Rosenberg) *Let R be a commutative algebra, essentially of finite type over a field k. If R is smooth over k, then ψ is an isomorphism of graded R-algebras:*

$$\psi\colon \Omega^*_{R/k} \xrightarrow{\cong} H_*(R, R).$$

Proof As with any R-module homomorphism, ψ is an isomorphism if and only if $\psi \otimes_R R_\mathfrak{m}$ is an isomorphism for every maximal ideal $\mathfrak{m}$ of R. The Change of Rings Theorem (9.1.8) states that $H_*(R, R) \otimes_R R_\mathfrak{m} \cong H_*(R_\mathfrak{m}, R_\mathfrak{m})$. Since $\Omega^*_{R/k} \otimes_R R_\mathfrak{m} = \Omega^*_{R_\mathfrak{m}/k}$, $\psi \otimes_R R_\mathfrak{m}$ is obtained by replacing R by $R_\mathfrak{m}$. Hence we may assume that R is a local ring.

Let I be the kernel of $R \otimes R \to R$ and M the pre-image of $\mathfrak{m}$ in $R^e = R \otimes R$. M is a maximal ideal in the regular ring R^e, so $S = (R^e)_M$ is a regular local ring. By flat base change (8.7.16) $H_*(R, R) \cong \mathrm{Tor}_*^{S/k}(R, R)$. Since S and $R = S/I_M$ are regular local rings, I_M is generated by a regular sequence of length $d = \dim(R) = \dim(S) - \dim(R)$; see exercise 4.4.2. We also saw in 8.7.13 that the external product makes $\mathrm{Tor}_*^{S/k}(R, R)$ isomorphic to $\Lambda^*\Omega_{R/k} = \Omega^*_{R/k}$ as a graded-commutative R-algebra. Since the external product can also be computed via the bar resolution and the shuffle product (8.7.12), the above product agrees with the internal product on $H_*(R, R) \cong \mathrm{Tor}_*^{S/k}(R, R)$. $\diamond$

Remark 9.4.8 We saw in 9.3.14 and 8.7.13 that $\Omega_{R/k}$ is a projective module whose localization at a maximal ideal $\mathfrak{m}$ of R is a free module of rank $\dim(R_\mathfrak{m})$. Hence for $d = \dim(R) = \max\{\dim(R_\mathfrak{m})\}$ we have $\Omega^d_{R/k} \neq 0$ and $H_n(R, R) = \Omega^n_{R/k} = 0$ for $n > d$. The converse holds: If $H_n(R, R) = 0$ for all large n, then R is smooth over k. See L. Avramov and M. Vigué-Poirrier, "Hochschild homology criteria for smoothness," *International Math. Research Notices* (1992, No.1), 17–25.

Exercise 9.4.2 Extend the Hochschild-Kostant-Rosenberg Theorem to the case in which k is a commutative noetherian ring; if R is smooth over k and essentially of finite type, then $\psi : \Omega^*_{R/k} \cong H_*(R, R)$. *Hint:* Although S and $R = S/I$ may not be regular local rings, the ideal I is still generated by a regular sequence of length d.

9.4.3 Hodge Decomposition

When $\mathbb{Q} \subset R$ and R is commutative, we shall show (in 9.4.15) that the Hochschild chain complex $C_*^h(R) = C(R \otimes R^{\otimes *})$ decomposes as the direct sum of chain complexes $C_*^h(R)^{(i)}$. The resulting decompositions $H_*(R, R) = \oplus H_*^{(i)}(R, R)$ and $H^*(R, R) = \oplus H^*_{(i)}(R, R)$ are called the Hodge decompositions of Hochschild homology and cohomology in order to reflect a relationship with the Hodge decomposition of the cohomology of complex analytic manifolds. (This relationship was noticed by Gerstenhaber and Schack [GS]; see Remark 9.8.19 for more details.) In the process, we will establish the facts needed to apply Barr's Theorem (8.8.7), showing that the summand $H_*^{(1)}(R, R)$ may be identified with the André-Quillen homology modules $D_{*-1}(R/k)$.

If R does not contain $\mathbb{Q}$, there is a filtration on $H_*(R, R)$ but need be no decomposition [Q]. This filtration may be based on certain operations λ^k; see [Loday, 4.5.15]. When $\mathbb{Q} \subset R$ the eigenspaces of the λ^k give the decomposition; λ^k acts as multiplication by $\pm k^i$ on $C_*^h(R)^{(i)}$ and hence on $H_*^{(i)}(R, R)$ and $H_{(i)}^*(R, R)$. For this reason, the Hodge decomposition is often called the λ- *decomposition*.

The symmetric group Σ_n acts on the n-fold tensor product $R^{\otimes n}$ and hence on $M \otimes R^{\otimes n}$ by permuting coordinates: $\sigma(m \otimes r_1 \otimes \cdots \otimes r_n) = m \otimes r_{\sigma^{-1}1} \otimes \cdots \otimes r_{\sigma^{-1}n}$. Consider, for example, the effect of the *signature idempotent* $\varepsilon_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (-1)^\sigma \sigma$ of $\mathbb{Q}\Sigma_n$; the definition of the shuffle product ∇ shows that in $R \otimes R^{\otimes n}$ we have the identity:

$$n!\varepsilon_n(r_0 \otimes r_1 \otimes \cdots \otimes r_n) = r_0(1 \otimes r_1)\nabla \cdots \nabla(1 \otimes r_n).$$

This element is an n-cycle in the Hochschild complex representing the element $\psi(r_0 dr_1 \wedge \cdots \wedge dr_n)$ of $H_n(R, R)$, where $\psi : \Omega_{R/k}^* \hookrightarrow H_*(R, R)$ is the injection discussed in 9.4.4. The formula for the chain-level splitting e: $R \otimes R^{\otimes *} \to \Omega_{R/k}^*$ of ψ is skew-symmetric, so we also have $e(r_0 \otimes r_1 \otimes \cdots \otimes r_n) = e(\varepsilon_n(r_0 \otimes r_1 \otimes \cdots \otimes r_n))$. Hence e factors through $\varepsilon_n(R \otimes R^{\otimes n})$.

The following criterion for recognizing the signature idempotent will be handy. Consider the action of Σ_n on the module $R \otimes R^{\otimes n}$.

Barr's Lemma 9.4.9 *If $u \in \mathbb{Q}\Sigma_n$ satisfies $bu(1 \otimes r_1 \otimes \cdots \otimes r_n) = 0$ for all algebras R, then $u = c\varepsilon_n$ for some $c \in \mathbb{Q}$.*

Proof Write $u = \sum c_\sigma \sigma$ with $c_\sigma \in \mathbb{Q}$. We consider its action on the element $x = (1 \otimes r_1 \otimes \cdots \otimes r_n)$ of $R \otimes R^{\otimes n}$, where R is the polynomial ring $k[r_1, \ldots, r_n]$. In $b(ux) = \sum c_\sigma b(1 \otimes r_{\sigma^{-1}1} \otimes \cdots \otimes r_{\sigma^{-1}n})$ the term

$$1 \otimes r_{\sigma^{-1}1} \otimes \cdots \otimes r_{\sigma^{-1}i} r_{\sigma^{-1}(i+1)} \otimes \cdots \otimes r_{\sigma^{-1}n}$$

occurs once with coefficient $(-1)^i c_\sigma$ and once with coefficient $(-1)^i c_{\tau\sigma}$, where τ is the transposition $(i, i+1)$. Since these terms form part of a basis for the free k-module $R \otimes R^{\otimes n}$, we must have $c_\sigma = -c_{\tau\sigma}$ for all σ and all $\tau = (i, i+1)$. Hence $c_\sigma = (-1)^\sigma c_1$ for all $\sigma \in \Sigma_n$, and therefore $u = c_1 \sum (-1)^\sigma \sigma = c_1 \varepsilon_n$. $\diamond$

To fit this into a broader context, fix $n > 1$ and define the "shuffle" elements

s_{pq} of $\mathbb{Z}\Sigma_n$ to be the sum $\sum(-1)^{\mu}\mu$ over all (p,q)-shuffles in Σ_n (so by convention $s_{pq}=0$ unless $p+q=n$). Let s_n be the sum of the s_{pq} for $0<p<n$.

Lemma 9.4.10 *$bs_n = s_{n-1}b$ for every n.*

Proof If $p+q=n$, $x=r_0\otimes\cdots\otimes r_p$ and $y=1\otimes r_{p+1}\otimes\cdots\otimes r_n$, then $x\nabla y = s_{pq}(r_0\otimes\cdots\otimes r_n)$. Since $R^{\otimes *+1}$ is a DG-algebra (9.4.2), we have

$$\begin{aligned} bs_{pq}(r_0\otimes\cdots\otimes r_n) &= b(x\nabla y) = (bx)\nabla y + (-1)^p x\nabla(by) \\ &= s_{p-1,q}((bx)\otimes y) + (-1)^p s_{p,q-1}(x\otimes(by)). \end{aligned}$$

Summing over p gives $bs_n = s_{n-1}b$. $\diamond$

Proposition 9.4.11 ([GS]) *The minimal polynomial for $s_n\in\mathbb{Q}\Sigma_n$ is*

$$f_n(x) = x(x-\lambda_2)\cdots(x-\lambda_n), \text{ where } \lambda_i = 2^i - 2.$$

Therefore the commutative subalgebra $\mathbb{Q}[s_n]$ of $\mathbb{Q}\Sigma_n$ contains n uniquely determined idempotents $e_n^{(i)}$, $i=1,\dots,n$ such that $s_n=\sum\lambda_i e_n^{(i)}$ and $\mathbb{Q}[s_n]=\prod\mathbb{Q}e_n^{(i)}$. In particular, $e_n^{(i)}e_n^{(j)}=0$ for $i\neq j$. $\diamond$

Definition 9.4.12 The idempotents $e_n^{(i)}$ are called the *Eulerian idempotents* of $\mathbb{Q}\Sigma_n$. Because s_n has only n eigenvalues, $e_n^{(i)}=0$ for $i>n$. By convention, $e_0^{(0)}=1$ and $e_n^{(0)}=0$ otherwise.

Proof If $n=1$ then $s_1=0$, while if $\tau=(1,2)$ then $s_2=1-\tau$ satisfies $x(x-2)$. For $n\geq 3$ we proceed by induction. Since $bs_n=s_{n-1}b$, we have $bf_{n-1}(s_n)=f_{n-1}(s_{n-1})b=0$. By Barr's Lemma, $f_{n-1}(s_n)=c\varepsilon_n$ for some constant c. To evaluate c, note that $\varepsilon_n s_n=\lambda_n\varepsilon_n$ because s_n has λ_n terms and $\varepsilon_n\sigma=(-1)^n\varepsilon_n$ for every $\sigma\in\Sigma_n$. Thus

$$f_{n-1}(s_n)=\varepsilon_n f_{n-1}(s_n)=f_{n-1}(\varepsilon_n s_n)=f_{n-1}(\lambda_n\varepsilon_n)=c\varepsilon_n\neq 0,$$

where $c=\lambda_n f_{n-1}(1)\neq 0$. Thus $f_n(s_n)=c\varepsilon_n(s_n-\lambda_n)=0$. $\diamond$

Corollary 9.4.13 *$e_n^{(n)}$ is the signature idempotent ε_n.*

Proof $\mathbb{Q}[s_n]$ contains $\varepsilon_n=f_{n-1}(s_n)/c$, and $\varepsilon_n s_n=\lambda_n\varepsilon_n$. $\diamond$

Corollary 9.4.14 *$be_n^{(i)}=e_{n-1}^{(i)}b$ for $i<n$, and $be_n^{(n)}=0$.*

Proof For all i, let $p_i(x)$ be the product of the terms $(x-\lambda_j)/(\lambda_i-\lambda_j)$ for $j\neq i$, $j\leq n$, so that $p_i(s_n)=e_n^{(i)}$ and $p_i(s_{n-1})=e_{n-1}^{(i)}$; this is the Lagrange interpolation formula for diagonalizable operators and is most easily checked using $\mathbb{Q}[s_n]=\prod\mathbb{Q}e_n^{(i)}$. Since $bs_n=s_{n-1}b$, we have

$$be_n^{(i)}=bp_i(s_n)=p_i(s_{n-1})b=e_{n-1}^{(i)}b.$$

As a special case, we have the formula $be_n^{(n)}=e_{n-1}^{(n)}b=0$. ◇

Definition 9.4.15 Suppose that R is a commutative k-algebra containing $\mathbb{Q}$. For $i\geq 1$, let $C_n^h(R)^{(i)}$ denote the summand $e_n^{(i)}R\otimes R^{\otimes n}$ of $C_n^h(R)=R\otimes R^{\otimes n}$. By 9.4.14, each $C_*^h(R)^{(i)}$ is a chain subcomplex of $C_*^h(R)$. For $i=0$ we let $C_*^h(R)^{(0)}$ denote the complex that is R, concentrated in degree zero, so that $C_*^h(R)$ is the direct sum of the chain subcomplexes $C_*^h(R)^{(i)}$ for $i\geq 0$. We define $H_n^{(i)}(R,R)$ to be $H_n(C_*^h(R)^{(i)})$. The resulting formula

$$H_n(R,R)=H_n^{(1)}(R,R)\oplus\cdots\oplus H_n^{(n)}(R,R),\ n\neq 0,$$

is called *the Hodge decomposition* of Hochschild homology. Similarly, we define $H^n_{(i)}(R,R)$ to be $H^n\operatorname{Hom}_R(C_*^h(R)^{(i)},R)$ and call the resulting formula

$$H_n(R,R)=H^n_{(1)}(R,R)\oplus\cdots\oplus H^n_{(n)}(R,R),\ n\neq 0,$$

the *Hodge decomposition* of Hochschild cohomology.

The Hodge decomposition (or λ-decomposition) arose implicitly in [Barr] (via 9.4.9 and 8.8.7) and [Q] and was made explicit in [GS].

Exercise 9.4.3 Let $\bar{C}_n^h(R)^{(i)}$ denote the summand $e_n^{(i)}R\otimes(R/k)^{\otimes n}$ of the normalized Hochschild complex $R\otimes(R/k)^{\otimes n}$. Show that $H_n^{(i)}(R,R)=H_n\bar{C}_n^h(R)^{(i)}$.

Exercise 9.4.4 Show that $H_n^{(n)}(R,R)\cong\Omega^n_{R/R}$ for every R. Conclude that if R is smooth and essentially of finite type over k, then $H_n(R,R)=H_n^{(n)}(R,R)$.

9.5 Morita Invariance

Definition 9.5.1 Two rings R and S are said to be *Morita equivalent* if there is an $R{-}S$ bimodule P and an $S{-}R$ bimodule Q such that $P\otimes_S Q\cong R$ as $R{-}R$ bimodules and $Q\otimes_R P\cong S$ as $S{-}S$ bimodules. It follows that the functors $\otimes_R P$: **mod**–$R \to$ **mod**–S and $\otimes_S Q$: **mod**–$S \to$ **mod**–R are inverse equivalences, because for every right R-module M we have $(M \otimes_R P) \otimes_S Q \cong M \otimes_R (P \otimes_S Q) \cong M$ and similarly for right S-modules.

Exercise 9.5.1 Show that

1. Morita equivalence is an equivalence relation.
2. If R and S are Morita equivalent, so are R^{op} and S^{op}.
3. If R and S are Morita equivalent, then the bimodule categories R–**mod**–R and S–**mod**–S are equivalent (via $Q \otimes_R - \otimes_R P$).

Proposition 9.5.2 *The matrix rings $M_m(R)$ are Morita equivalent to R.*

Proof Let P be the module of row vectors $(r_1, \cdots, r_m)$ of length m and Q the module of column vectors of length m. The matrix ring $S = M_m(R)$ acts on the right of P and the left of Q by the usual matrix multiplication, so P is an $R-S$ bimodule and Q is an $S-R$ bimodule. Matrix multiplication yields bimodule maps $P \otimes_S Q \to R$ and $Q \otimes_R P \to S$: if $p = (p_1, \cdots, p_m)$ and $q = (q_1, \cdots, q_m)^T$, then $p \otimes q$ maps to $\sum p_i q_i$ and $q \otimes p$ maps to the matrix $(q_i p_j)$. It is easy to check that these maps are isomorphisms (do so!). $\diamond$

Corollary 9.5.3 *The isomorphism R–**mod**–$R \to M_m(R)$–**mod**–$M_m(R)$ associates to an $R-R$ bimodule M the $M_m(R)-M_m(R)$ bimodule $M_m(M)$ of all $m \times m$ matrices with entries in M.*

Lemma 9.5.4 *If P and Q define a Morita equivalence between R and S, then P is a finitely generated projective left R-module. P is also a finitely generated projective right S-module.*

Proof Given $p \in P$ and $q \in Q$ we write $p \cdot q$ and $q \cdot p$ for the elements of R and S corresponding to $p \otimes q \in P \otimes_S Q$ and $q \otimes p \in Q \otimes_R P$, respectively. As $Q \otimes_R P \cong S$, we can write $1 = q_1 \cdot p_1 + \cdots + q_m \cdot p_m$ for some m. Define $e: P \to R^m$ by $e(p) = (p \cdot q_1, \cdots, p \cdot q_m)$ and $h: R^m \to P$ by $h(r_1, \cdots, r_m) = \sum r_i p_i$; e and h are left R-module homomorphisms. Since $he(p) = \sum (p \cdot q_i) p_i = \sum p(q_i \cdot p_i) = p$, this expresses P as a summand of R^m in R–**mod**. The proof that P is a summand of some S^n in **mod**–S is similar. $\diamond$

Exercise 9.5.2 Show that the bimodule structures induce ring isomorphisms

$$\mathrm{End}_R(Q) \cong S \cong \mathrm{End}_R(P)^{\mathrm{op}}.$$

Conclude that if all projective R-modules are free, then any ring which is Morita equivalent to R must be a matrix ring $M_m(R)$.

Lemma 9.5.5 *If L is a left R-module and Q is a projective right R-module then $H_i(R, L \otimes Q) = 0$ for $i \neq 0$ and $H_0(R, L \otimes Q) \cong Q \otimes_R L$.*

Proof By additivity, it suffices to prove the result with $Q = R$. The standard chain complex (9.1.1) used to compute $H_*(R, L \otimes R)$ is isomorphic to the bar resolution $\beta(R, L)$ of the left R-module L (8.6.12), which has $H_i(\beta) = 0$ for $i \neq 0$ and $H_0(\beta) \cong R \otimes_R L$. ◇

Theorem 9.5.6 (R. K. Dennis) *Hochschild homology is Morita invariant. That is, if R and S are Morita equivalent rings and M is an $R-R$ bimodule, then*

$$H_*(R, M) \cong H_*(S, Q \otimes_R M \otimes_R P).$$

Proof Let L denote the $S-R$ bimodule $Q \otimes_R M$. Consider the bisimplicial k-module $X_{ij} = S^{\otimes i} \otimes L \otimes R^{\otimes j} \otimes P$, where the j^{th} row is the standard complex 9.1.1 for the Hochschild homology over S of the $S-S$ bimodule $L \otimes R^{\otimes j} \otimes P$ and the i^{th} column is the standard complex for the Hochschild homology of the $R-R$ bimodule $P \otimes S^{\otimes i} \otimes L$ (with the P rotated). Using the sign trick 1.2.5, form a double complex C_{**}. We will compute the homology of $\mathrm{Tot}(C)$ in two ways.

$$\begin{array}{ccccc}
\cdots & & & & \\
L \otimes R \otimes R \otimes P & \longleftarrow & \cdots & & \\
\downarrow & & \downarrow & & \\
L \otimes R \otimes P & \longleftarrow & S \otimes L \otimes R \otimes P & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow \\
L \otimes P & \longleftarrow & S \otimes L \otimes P & \longleftarrow & S \otimes S \otimes L \otimes P \ \cdots
\end{array}$$

Since P is a projective right S-module, the j^{th} row is exact except at $i = 0$, where $H_0(C_{*j}) = P \otimes_S (L \otimes R^{\otimes j}) \cong M \otimes R^{\otimes j}$ (9.5.5). The vertical differentials of the chain complex $H_0(C_{*j})$ make it isomorphic to the standard complex for the Hochschild homology of M. Thus $H_i \,\mathrm{Tot}(C) \cong H_i(R, M)$ for all i. On the other hand, since P is a projective left R-module, the i^{th} column is exact except at $j = 0$, where $H_0(C_{i*}) = S^{\otimes i} \otimes L \otimes_S P$ (9.5.5). The horizontal differentials of $H_0(C_{i*})$ make it isomorphic to the standard complex for the Hochschild homology of $L \otimes_S P = Q \otimes_R M \otimes_S P$. Thus $H_i \,\mathrm{Tot}(C) \cong H_i(S, Q \otimes_R M \otimes_S P)$ for all i. ◇

Definition 9.5.7 (Trace) The usual trace map from $M_m(R)$ to R is the map sending a matrix $g = (g_{ij})$ to its trace $\sum g_{ii}$. More generally, given an $R-R$ bimodule M we can define maps trace_n from $M_m(M) \otimes M_m(R)^{\otimes n}$ to $M \otimes R^{\otimes n}$ by the formula

$$\text{trace}_n(m \otimes g^1 \otimes \cdots \otimes g^n) = \sum_{i_0,\ldots,i_n=1}^{n} m_{i_0 i_1} \otimes g^1_{i_1 i_2} \otimes \cdots \otimes g^r_{i_r i_{r+1}} \otimes \cdots \otimes g^n_{i_n i_0}.$$

These maps are compactible with the simplicial operators ∂_i and σ_i (check this!), so they assemble to yield a simplicial module homomorphism from $M_m(M) \otimes M_m(R)^{\otimes *}$ to $M \otimes R^{\otimes *}$. They therefore induce a map on Hochschild homology, called the *trace map*.

Corollary 9.5.8 *The natural isomorphism of theorem 9.5.6 is given by the trace map* $H_*(M_m(R), M_m(M)) \to H_*(R, M)$.

Proof Let us write $F = F(R,S,P,Q,M)$ for the natural isomorphism $H_*(R, M) \to H_*(S, Q \otimes M \otimes P)$ given by the bisimplicial k-module X of theorem 9.5.6. Fixing R, set $S' = R$ and $S = M_m(R)$, $P' = R$ and $P = R^m$, $Q' = R$ and $Q = (R^m)^T$. The diagonal map $\Delta\colon R \to M_m(R)$ sending $r \in R$ to the diagonal matrix $\begin{bmatrix} r & 0 \\ \cdots & \cdots \\ 0 & r \end{bmatrix}$ is compatible with the maps $P' \to P$ and $Q' \to Q$ sending $p \in P'$ and $q \in Q'$ to $(p, 0, \cdots, 0)^T$ and $(q, 0, \cdots, 0)$, respectively. It therefore yields a map $\Delta\colon X(R, S', P', Q') \to X(R, S, P, Q)$. (Check this!) This yields a commutative square

$$\begin{array}{ccccc} H_n(R, M) & \xrightarrow{F'} & H_n(R, R \otimes_R M \otimes_R R) & = & H_n(R, M) \\ \| & & & & \downarrow \Delta \\ H_n(R, M) & \xrightarrow{F} & H_n(M_m(R), Q \otimes M \otimes P) & = & H_n(M_m(R), M_m(M)). \end{array}$$

It follows that Δ is an isomorphism. At the chain level, we have

$$\Delta(m \otimes r_1 \otimes \cdots \otimes r_m) = \begin{bmatrix} m & 0 & 0 \\ 0 & \cdots & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} r_1 & 0 \\ \cdots & \cdots \\ 0 & r_1 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} r_n & 0 \\ \cdots & \cdots \\ 0 & r_n \end{bmatrix}.$$

Clearly $\text{trace}_n(\Delta(m \otimes r_1 \otimes \cdots \otimes r_n)) = m \otimes r_1 \otimes \cdots \otimes r_n$, so the trace map $H_*(M_m(R), M_m(M)) \to H_*(R, M)$ is the inverse isomorphism to Δ. ◇

Exercise 9.5.3 For $m < n$, consider the (nonunital) inclusion $\iota\colon M_m(R) \hookrightarrow M_n(R)$ sending g to $\begin{bmatrix} g & 0 \\ 0 & 0 \end{bmatrix}$. Show that ι induces a chain map ι_* from the complex $M_m(M) \otimes M_m(R)^{\otimes *}$ to the complex $M_n(M) \otimes M_n(R)^{\otimes *}$ for every R-module M. Then show that this chain map is compatible with the trace maps (i.e., that trace $=$ trace $\circ\iota_*$), and conclude that ι_* induces the Morita invariance isomorphism

$$H_*(M_m(R), M_m(M)) \cong H_*(M_n(R), M_n(M)).$$

Exercise 9.5.4 Let $e_{ij}(r)$ denote the matrix with exactly one nonzero entry, namely r, occurring in the (i, j) spot. Show that

$$\text{trace } e_{12}(r_1) \otimes e_{23}(r_2) \otimes \cdots \otimes e_{n1}(r_n) = r_1 \otimes \cdots \otimes r_n.$$

Then show that for any permutation σ of $\{1, 2, \cdots, n\}$

$$\text{trace } e_{\sigma 1,\sigma 2}(r_1) \otimes e_{\sigma n,\sigma 1}(r_n) = \begin{cases} r_1 \otimes \cdots \otimes r_n & \text{if } \sigma \in C_n \\ 0 & \text{if not,} \end{cases}$$

where C_n is the subgroup of the symmetric group generated by $(12 \cdots n)$.

9.6 Cyclic Homology

The simplicial k-module $ZR = R \otimes R^{\otimes *}$ used to construct the Hochschild homology modules $H_*(R, R)$ has a curious "cyclic" symmetry, which is suggested by writing a generator $r_0 \otimes r_1 \otimes \cdots \otimes r_n$ of $R \otimes R^{\otimes n}$ in the circular form illustrated here.

$$\begin{array}{cccccc}
 & & r_1 & \otimes & \ddots & \\
 & \otimes & & & & \otimes \\
\longrightarrow & r_0 & & & & r_i \\
 & \otimes & & & & \otimes \\
 & & r_n & \otimes & \cdot^{\cdot^{\cdot}} &
\end{array}$$

The arrow $\rightarrow$ serves as a place marker, and there are $n + 1$ of the symbols $\otimes$. The $n + 1$ face and degeneracy operators replace the appropriate symbol $\otimes$ by a product or a "$\otimes 1 \otimes$," respectively. This symmetry defines an action of the cyclic group C_{n+1} on $R \otimes R^{\otimes n}$; the generator t of C_{n+1} acts as the operator

$t(r_0 \otimes \cdots \otimes r_n) = r_n \otimes r_0 \otimes \cdots \otimes r_{n-1}$. We may visualize t as a rotation of the above circular representation (with the place marker fixed). Clearly $\partial_i t = t\partial_{i-1}$ and $\sigma_i t = t\sigma_{i-1}$ for $i > 0$; for $i = 0$ we have $\partial_0 t = \partial_n$ and $\sigma_0 t = t^2 \sigma_n$. (Check this!) This leads to the notion of an abstract cyclic k-module: a simplicial k-module with this extra cyclic symmetry. After giving the definition in this fashion, we shall construct a category ΔC such that a cyclic k-module is a contravariant functor from ΔC to k–**mod**, paralleling the definition in Chapter 8 of a simplicial object.

Definition 9.6.1 A *cyclic object* A in a category $\mathcal{A}$ is a simplicial object together with an automorphism t_n of order $n+1$ on each A_n such that $\partial_i t = t\partial_{i-1}$ and $\sigma_i t = t\sigma_{i-1}$ for $i \neq 0$, $\partial_0 t_n = \partial_n$ and $\sigma_0 t_n = t_{n+1}^2 \sigma_n$. (Writing t instead of t_n is an abuse of notation we shall often employ for legibility.)

We will use the term "cyclic module" for a cyclic object in the category of modules. For example, there is a cyclic k-module ZR associated to every k-algebra R; $Z_n R$ is $R^{\otimes n+1}$ and the rest of the structure was described above.

Example 9.6.2 We will also use the term "cyclic set" for a cyclic object in the category of sets. For example, let G be a group. The simplicial set BG (8.1.7) may be considered as a cyclic set by defining t on $BG_n = G^n$ to be $t(g_1, \cdots, g_n) = (g_0, g_1, \cdots, g_{n-1})$, where $g_0 = (g_1 \cdots g_n)^{-1}$. Another cyclic set is ZG, which has $(ZG)_n = G^{n+1}$,

$$\partial_i(g_0, \cdots, g_n) = \begin{cases} (g_0, \cdots, g_i g_{i+1}, \cdots, g_n) & \text{if } i < n \\ (g_n g_0, g_1, \cdots, g_{n-1}) & \text{if } i = n \end{cases}$$

$$\sigma_i(g_0, \cdots, g_n) = (g_0, \cdots, g_i, 1, g_{i+1}, \cdots)$$

$$t(g_0, \cdots, g_n) = (g_n, g_0, \cdots, g_{n-1}).$$

As the notation suggests, there is a natural inclusion $BG \subset ZG$ and the free k-modules $k(ZG)_n$ fit together to form the cyclic k-module $Z(kG)$.

We now propose to construct a category ΔC containing Δ such that a cyclic object in $\mathcal{A}$ is the same thing as a contravariant functor from ΔC to $\mathcal{A}$. Recall from Chapter 8, section 1 that the simplicial category Δ has for its objects the finite (ordered) sets $[n] = \{0, 1, \cdots, n\}$, morphisms being nondecreasing monotone functions. Let t_n be the "cyclic" automorphism of the set $[n]$ defined by $t_n(0) = n$ and $t_n(j) = j-1$ for $j \neq 0$.

Definition 9.6.3 Let $\mathrm{Hom}_{\Delta C}([n], [p])$ denote the family of formal pairs (α, t^i), where $0 \leq i \leq n$ and $\alpha\colon [n] \to [p]$ is a nondecreasing monotone function. Let $\mathrm{Hom}_C([n], [p])$ denote the family of all set maps $\varphi\colon [n] \to [p]$

which factor as $\varphi = \alpha t_n^i$ for some pair (α, t^i) in $\mathrm{Hom}_{\Delta C}([n], [p])$. Note that $\varphi(i) \leq \varphi(i+1) \leq \cdots \leq \varphi(i-1)$ in this case. Therefore the obvious surjection from $\mathrm{Hom}_{\Delta C}([n], [p])$ to $\mathrm{Hom}_{\mathcal{C}}([n], [p])$ is almost a bijection—that is, φ uniquely determines (α, t^i) such that $\varphi = \alpha t^i$ unless φ is a constant map, in which case φ determines α $(\alpha = \varphi)$ but all $n+1$ of the pairs (φ, t^i) yield the set map φ. We identify $\mathrm{Hom}_{\Delta}([n], [p])$ as the subset of all pairs $(\alpha, 1)$ in $\mathrm{Hom}_{\Delta C}([n], [p])$.

There is a subcategory $\mathcal{C}$ of **Sets**, containing Δ, whose objects are the sets $[n]$, $n \geq 0$, and whose morphisms are the functions in $\mathrm{Hom}_{\mathcal{C}}([n], [p])$. To see this we need only check that the composition of $\psi = \beta t_m^j$ and $\varphi = \alpha t_n^i$ is in $\mathcal{C}$, and this follows from the following identities of set functions for the functions $\varepsilon_i \colon [n-1] \to [n]$ and $\eta_j \colon [n+1] \to [n]$ generating Δ (see exercise 8.1.1)

$$t_n \varepsilon_i = \begin{Bmatrix} \varepsilon_n & i = 0 \\ \varepsilon_{i-1} t_{n-1} & i > 0 \end{Bmatrix} \quad \text{and} \quad t_n \eta_i = \begin{Bmatrix} \eta_n t_{n+1}^2 & i = 0 \\ \eta_{i-1} t_{n+1} & i > 0 \end{Bmatrix}.$$

Proposition 9.6.4 (A. Connes) *The formal pairs in* $\mathrm{Hom}_{\Delta C}([n], [p])$ *form the morphisms of a category* ΔC *containing* Δ*, the objects being the sets* $[n]$ *for* $n \geq 0$*. Moreover, a cyclic object in a category* $\mathcal{A}$ *is the same thing as a contravariant functor from* ΔC *to* $\mathcal{A}$*.*

Proof We need to define the composition (γ, t^k) of $(\beta, t^j) \in \mathrm{Hom}_{\Delta C}([m], [n])$ and $(\alpha, t^i) \in \mathrm{Hom}_{\Delta C}([n], [p])$ in such a way that if $i = j = 0$, then $(\gamma, t^k) = (\alpha\beta, 1)$. If β is not a constant set map, then the composition $t^i \beta t^j$ in $\mathcal{C}$ is not constant, so there is a unique (β', t^k) such that $t^i \beta t^j = \beta' t^k$; we set $(\gamma, t^k) = (\alpha\beta', t^k)$. If β is constant, we set $(\gamma, t^k) = (\alpha\beta, t^j)$. By construction, the projections from $\mathrm{Hom}_{\Delta C}$ to $\mathrm{Hom}_{\mathcal{C}}$ are compatible with composition; as $\mathcal{C}$ is a category, it follows that the (id,1) are 2-sided identity maps and that composition in ΔC is associative (except possibly for the identity $(\varphi \circ (\beta, t^j)) \circ \psi = \varphi \circ ((\beta, t^j) \circ \psi)$ when β is constant, which is easily checked). Thus ΔC is a category and $\Delta \to \Delta C \to \mathcal{C}$ are functors. The final assertion is easily checked using the above identities for $t\varepsilon_i$ and $t\eta_j$. ◇

Remark The original definition given by A. Connes in [Connes] is that $\mathrm{Hom}_{\Delta C}([n], [p])$ is the set of equivalence classes of continuous increasing maps of degree 1 from $S^1 = \{z \in \mathbb{C} \colon |z| = 1\}$ to S^1 sending the $(n+1)^{st}$ roots of unity to $(p+1)^{st}$ roots of unity. Connes also observed that ΔC is isomorphic to its opposite category $(\Delta C)^{\mathrm{op}}$. See [Loday] for more details.

Exercise 9.6.1 Show that the automorphisms of $[n]$ in ΔC form the cyclic group C_{n+1} of order $n+1$.

Definitions 9.6.5 Let A be a cyclic object in an abelian category $\mathcal{A}$. The chain complex $C_*^h(A)$ associated to the underlying simplicial object of A (8.2.1) is called the *Hochschild complex* of A. It is traditional to write b for the differential of $C_*^h(A)$, so that $b = \partial_0 - \partial_1 + \cdots \pm \partial_n$ goes from $C_n^h(A) = A_n$ to $C_{n-1}^h(A) = A_{n-1}$. The *Hochschild homology* $HH_*(A)$ of A is the homology of $C_*^h(A)$; when $A = ZR$ (9.6.1) we will write $HH_*(R)$ for $HH_*(ZR) = H_*(R, R)$. The *acyclic complex* of A, $C_*^a(A)$, is the complex obtained from $C_*^h(A)$ by omitting the last face operator. Thus $C_n^a(A) = A_n$, and we write b' for the resulting differential $\partial_0 - \partial_1 + \cdots \mp \partial_{n-1}$ from A_n to A_{n-1}.

Exercise 9.6.2 Show the "acyclic" complex $C_*^a(A)$ is indeed acyclic. *Hint:* The path space PA (8.3.14) is a simplicial resolution of A_0.

Definition 9.6.6 (Tsygan's double complex) If A is a cyclic object in an abelian category, there is an associated first quadrant double complex $CC_{**}(A)$, first found by B. Tsygan in [Tsy], and independently by Loday and Quillen in [LQ]. The columns are periodic of order two: If p is even, the p^{th} column is the Hochschild complex C_*^h of A; if p is odd, the p^{th} column is the acyclic complex C_*^a of A with differential $-b'$. (The minus sign comes from the sign trick of 1.2.5.) Thus $CC_{pq}(A)$ is A_q, independently of p. The q^{th} row of $CC_{**}(A)$ is the periodic complex associated to the action of the cyclic group C_{q+1} on A_q, in which the generator acts as multiplication by $(-1)^q t$. Thus the differential $A_q \to A_q$ is multiplication by $1-(-1)^q t$ when p is odd; when p is even it is multiplication by the norm operator

$$N = 1 + (-1)^q t + \cdots + (-1)^{iq} t^i + \cdots + (-1)^q t^q.$$

$$\begin{array}{ccccccccc}
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
A_2 & \xleftarrow{1-t} & A_2 & \xleftarrow{N} & A_2 & \xleftarrow{1-t} & A_2 & \xleftarrow{N} & \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
A_1 & \xleftarrow{1+t} & A_1 & \xleftarrow{N} & A_1 & \xleftarrow{1+t} & A_1 & \xleftarrow{N} & \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
A_0 & \xleftarrow{1-t} & A_0 & \xleftarrow{N} & A_0 & \xleftarrow{1-t} & A_0 & \xleftarrow{N} &
\end{array}$$

Tsygan's double complex $CC_{**}(A)$

Definition 9.6.7 The *cyclic homology* $HC_*(A)$ of a cyclic object A is the homology of Tot $CC_{**}(A)$. The cyclic homology $HC_*(R)$ of an k-algebra R is the cyclic homology of the cyclic object ZR $(= R \otimes R^{\otimes *})$ of 9.6.1. In particular, $HC_0(A) = HH_0(A)$ and $HC_0(R) = R/[R, R]$.

One of the advantages of generalizing from algebras to cyclic objects is that a short exact sequence $0 \to A \to B \to C \to 0$ of cyclic objects gives rise to short exact sequences of Hochschild complexes as well as Tsygan complexes, which in turn give rise to long exact sequences

$$\cdots HH_n(A) \to HH_n(B) \to HH_n(C) \to HH_{n+1}(A) \cdots$$
$$\cdots HC_n(A) \to HC_n(B) \to HC_n(C) \to HC_{n-1}(A) \cdots.$$

Lemma 9.6.8 *$CC_{**}(A)$ is a double complex.*

Proof Set $\eta = (-1)^q$. We have to see that $b(1 - \eta t) = (1 + \eta t)b'$ and $Nb = b'N$ as maps from A_q to A_{q-1}. Now $b - b' = \eta \partial_q$ and the cyclic relations imply that $bt = \partial_q - tb'$, yielding the first relation. The cyclic relations also imply that

$$b' = \sum_{i=0}^{q-1} (-t)^i \partial_q t^{q-i} \quad \text{and} \quad b = \sum_{i=0}^{q} (-t)^{q-i} \partial_q t^i.$$

(Check this!) Since $(1 - \eta t)N = 0$, we have $t^i N = \eta^i N$ on A_q. Since $N(1 + \eta t) = 0$, we have $Nt^i = (-\eta)^i N$ on A_{q-1}. Thus

$$\eta Nb = \eta \sum_{i=0}^{q} N(\eta)^{q-i} \partial_q t^i = \eta^{q+1} N \partial_q \sum (\eta t)^i = N \partial_q N,$$

$$\eta b' N = \eta \sum_{i=0}^{q-1} (-t)^i \partial_q \eta^{q-i} N = \eta^{q+1} \sum (-\eta t)^i \partial_q N = N \partial_q N.$$

This yields the second relation, $Nb = b'N$. ◇

Corollary 9.6.9 *Let $A_n/\sim$ denote the quotient of A_n by the action of the cyclic group. These form a quotient chain complex $A_*/\sim$ of the Hochschild complex $C_*^h(A)$:*

$$0 \leftarrow A_0 \xleftarrow{b} A_1/\sim \xleftarrow{b} A_2/\sim \xleftarrow{b} \cdots.$$

Indeed, $A_*/\sim$ is the cokernel of the chain map $CC_{1*} \to CC_{0*}$, so there is a natural map from $H_n(A_*/\sim)$ to $HC_n(A)$.

Remark Some authors define the cyclic homology of R to be $H_n(R^{\otimes *+1}/\sim)$, especially when $k = \mathbb{C}$. The following lemma states that their definition is equivalent to ours.

Lemma 9.6.10 *If k contains $\mathbb{Q}$, then $HC_*(A)$ may be computed as the homology of the quotient complex $A_*/\sim$ of the Hochschild complex.*

Proof Filtering Tsygan's double complex 9.6.6 by rows yields a spectral sequence starting with group homology of the cyclic groups:

$$E^1_{pq} = H_p(C_{q+1}; A_q) \Rightarrow HC_{p+q}(A).$$

The edge map from $HC_*(A)$ to the homology of $E^1_{0q} = H_0(C_{q+1}; A_q) = A_q/\sim$ arises from the augmentation $CC_{0q} \to A_q/\sim$, so the E^2 edge map maps $H_n(A_*/\sim)$ to $HC_n(A)$. In characteristic zero the group homology vanishes (6.1.10) and the spectral sequence degenerates at E^2. ◇

Remark Filtering Tsygan's double complex by columns yields the even more interesting spectral sequence 9.8.6 (see exercise 9.8.2).

The three basic homomorphisms S, B, and I relating cyclic and Hochschild homology are obtained as follows. The inclusion of $C^h_*(A)$ as the column $p = 0$ in $CC_{**}(A)$ yields a map $I\colon HH_n(A) \to HC_n(A)$. Now let CC^{01}_{**} denote the double subcomplex of $CC_{**}(A)$ consisting of the columns $p = 0, 1$; the inclusion of $C^h_*(A)$ into CC^{01}_{**} induces an isomorphism $HH_n(A) \cong H_n \operatorname{Tot}(CC^{01}_{**})$ because the quotient is the acyclic complex $C^a_*(A)$. The quotient double complex $CC[-2] = CC/CC^{01}$, which consists of the columns $p \geq 2$, is isomorphic to CC_{**} except that it has been translated 2 columns to the right. The quotient map $\operatorname{Tot}(CC_{**}) \to \operatorname{Tot}(CC[-2])$ therefore yields a map $S\colon HC_n(A) \to HC_{n-2}(A)$. The short exact sequence of double complexes

$$0 \to CC^{01} \xrightarrow{I} CC(A) \xrightarrow{S} CC[-2] \to 0$$

yields the map $B\colon HC_{n-1}(A) \to HH_n(A)$ and the following "SBI" sequence.

Proposition 9.6.11 (SBI sequence) *For any cyclic object A there is a long exact "SBI" sequence*

$$\cdots HC_{n+1}(A) \xrightarrow{S} HC_{n-1}(A) \xrightarrow{B} HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \cdots.$$

In particular, there is a long exact sequence for every algebra R:

$$\cdots HC_{n+1}(R) \xrightarrow{S} HC_{n-1}(R) \xrightarrow{B} H_n(R, R) \xrightarrow{I} HC_n(R) \xrightarrow{S} HC_{n-2}(R) \cdots.$$

Remark In the literature the "SBI" sequence is also called "Connes' sequence" and the "Gysin" sequence. See exercise 9.7.4 for an explanation.

Corollary 9.6.12 *If $A \to A'$ is a morphism of cyclic objects with $HH_n(A) \xrightarrow{\cong} HH_n(A')$, then the induced maps $HC_n(A) \to HC_n(A')$ are all isomorphisms.*

Proof This follows from induction on n via the 5-lemma and 9.6.7. ◇

Application 9.6.13 Let R be a k-algebra. The explicit formula in 9.5.7 for the trace map $Z(M_m R) \to Z(R)$ shows that it is actually a map of cyclic k-modules. Since it induces isomorphisms on Hochschild homology, it also induces isomorphisms

$$HC_*(M_m R) \xrightarrow{\cong} HC_*(R).$$

Exercise 9.6.3 For $m < n$, show that the nonunital inclusion $\iota\colon M_m(R) \hookrightarrow M_n(R)$ of exercise 9.5.3 induces a cyclic map $ZM_m(R) \to ZM_n(R)$, which in turn induces isomorphisms

$$\iota_*\colon HC_*M_m(R) \cong HC_*M_n(R).$$

Example 9.6.14 Since $H_n(k, k) = 0$ for $n \neq 0$, the SBI sequence quickly yields

$$HC_n(k) = \begin{cases} k & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

with the maps $S\colon HC_{n+2}(k) \to HC_n(k)$ all isomorphisms. The same calculation applies for any finite separable algebra R over a field k because we saw in 9.2.11 that $H_n(R, R) = 0$ for all $n \neq 0$.

$\mathbf{HC_1}$ 9.6.15 The SBI sequence interprets $HC_1(R)$ as a quotient of $H_1(R, R)$:

$$H_0(R, R) \xrightarrow{B} H_1(R, R) \to HC_1(R) \to 0.$$

Now suppose that R is commutative, so that $H_0(R, R) = R$ and $H_1(R, R) = \Omega_{R/k}$. The map $B\colon R \to \Omega_{R/k}$ maps $r \in R$ to dr. (Check this!) Therefore we may identify B with d and make the identification

$$HC_1(R) \cong \Omega_{R/k}/(dR).$$

Example 9.6.16 Since $H_n(k[x], k[x]) = 0$ for $n \geq 2$, the $S\colon HC_{n+2}(k[x]) \to HC_n(k[x])$ are isomorphisms for all $n \geq 1$ and there is an exact sequence

$$0 \to HC_2(k[x]) \xrightarrow{S} k[x] \xrightarrow{d} \Omega_{k[x]/k} \xrightarrow{I} HC_1(k[x]) \to 0.$$

If k contains $\mathbb{Q}$, then $x^n dx = d(x^{n+1}/n+1)$ for all $n \geq 0$, so d is onto and $HC_1(k[x]) = 0$. This yields the calculation

$$HC_n(k[x]) = \begin{cases} k[x] & \text{if } n = 0 \\ k & \text{if } n \geq 2 \text{ is even} \\ 0 & \text{if } n \geq 1 \text{ is odd.} \end{cases}$$

Similar remarks pertain to the Laurent polynomial ring $k[x, x^{-1}]$, except that the map $d\colon k[x, x^{-1}] \to \Omega_{k[x,x^{-1}]/k} \cong k[x, x^{-1}]$ has cokernel k (on dx/x) when $\mathbb{Q} \subseteq k$. Thus when $\mathbb{Q} \subseteq k$ we have

$$HC_n(k[x, x^{-1}]) \cong k \quad \text{for all} \quad n \geq 1.$$

Remark We will compute $HC_*(R)$ for a smooth algebra R in 9.8.11 and 9.8.12 in terms of de Rham cohomology.

Exercise 9.6.4 Consider the truncated polynomial ring $R = k[x]/(x^{n+1})$ over a field k of characteristic 0. We saw in exercise 9.1.4 that $\dim_k H_i(R, R) = n$ for $i > 0$. Show explicitly that $HC_1(R) = 0$. Then use the SBI sequence to show that $HC_i(R) = 0$ for all odd i, while for even $i \neq 0$ $HC_i(R) \cong HC_i(k) \oplus H_i(R, R) \cong k^{n+1}$. Another approach will be given in exercise 9.9.2.

9.6.1 Variations: HP and HN

9.6.17 We may use the periodicity of Tsygan's first quadrant double complex $CC_{**}(A)$ to extend it to the left, obtaining an upper half-plane double complex $CC^P_{**}(A)$. (See 9.6.6.) The *periodic cyclic homology* of A is the homology of the product total complex

$$HP_*(A) = H_* \operatorname{Tot}^{\Pi}(CC^P_{**}(A)).$$

If we truncate CC^P_{**} to the left of the $2p^{th}$ column, we obtain Tsygan's double complex 9.6.6 translated $2p$ times. These truncations $\{CC_{**}[-2p]\}$ form a tower of double chain complexes in the sense of Chapter 3, section 5. The homology of this tower of double complexes is the tower of k-modules

$$\cdots \xrightarrow{S} HC_{n+4}(A) \xrightarrow{S} HC_{n+2}(A) \xrightarrow{S} HC_n(A).$$

As we saw in 3.5.8, this means that there is an exact sequence

$$0 \to \varprojlim{}^1 HC_{n+2p+1}(A) \to HP_n(A) \to \varprojlim HC_{n+2p}(A) \to 0.$$

Moreover, it is visually clear from the periodicity of $CC^P_{**}(A)$ that each map $S\colon HP_{n+2}(A) \to HP_n(A)$ is an isomorphism. This accounts for the name "periodic cyclic homology": the modules $HP_n(A)$ are periodic of order 2.

Similarly, we can consider the "negative" subcomplex $CC^N_{**}(A)$ of the periodic complex $CC^P_{**}(A)$ consisting of the columns with $p \le 0$. This is a second quadrant double complex. The *negative cyclic homology* of A is defined to be the homology of the product total complex of $CC^N_{**}(A)$:

$$HN_*(A) = H_* \operatorname{Tot}^{\Pi}(CC^N_{**}(A)).$$

We leave it to the reader to check that there is an SBI exact sequence 9.6.11 for $I\colon HN_* \to HP_*$ fitting into the following commutative diagram:

$$\begin{tikzcd}
HP_{n+1}(A) \arrow[r, "S"] \arrow[d, "S"] & HC_{n-1}(A) \arrow[r, "B"] \arrow[d, "="] & HN_n(A) \arrow[r, "I"] \arrow[d] & HP_n(A) \arrow[r, "S"] \arrow[d, "S"] & HC_{n-2}(A) \arrow[d, "="] & \cdots \\
HC_{n+1}(A) \arrow[r, "S"] \arrow[d, "B"] & HC_{n-1}(A) \arrow[r, "B"] & HH_n(A) \arrow[r, "I"] \arrow[d, "I"] & HC_n(A) \arrow[r, "S"] \arrow[d, "B"] & HC_{n-2}(A) & \cdots \\
HN_n(A) & & HN_{n-1}(A) \arrow[r, phantom, "="] & HN_{n-1}(A) & &
\end{tikzcd}$$

9.7 Group Rings

In this section we fix a commutative ring k and a group G. Our goal is to compute HH_* and HC_* of the group ring kG (9.7.5 and 9.7.9). To prepare for this we calculate HC_* of kBG, which we call $HC_*(G)$.

In 9.6.2 we saw that BG could be regarded as a cyclic set by defining $t(g_1, \cdots, g_n) = ((g_1 \cdots g_n)^{-1}, g_1, \cdots, g_{n-1})$. Applying the free k-module functor to BG therefore yields a cyclic k-module kBG. If we adopt the notation $HH_*(G) = HH_*(kBG)$, $HC_*(G) = HC_*(kBG)$, and so on, then we see (using 8.2.3) that

$$HH_n(G) = \pi_n(kBG) = H_n(BG; k) = H_n(G; k).$$

Theorem 9.7.1 (Karoubi) *For each group G,*

$$HC_n(G) \cong H_n(G;k) \oplus H_{n-2}(G;k) \oplus H_{n-4}(G;k) \oplus \cdots.$$

Moreover, the maps $S\colon HC_n(G) \to HC_{n-2}(G)$ are the natural projections with kernel $H_n(G;k)$, and the maps B are zero.

Remark It is suggested to write Karoubi's Theorem in the form $HC_*(G) \cong H_*(G;k) \otimes HC_*(k)$.

Proof Consider the path space $EG = P(BG)$ of BG (8.3.14 and exercise 8.3.8), which as a simplicial set has $(EG)_n = G^{n+1}$ and $\partial_i(g_0, \cdots, g_n) = (\cdots, g_i g_{i+1}, \cdots)$ for $i \neq n$ and $\partial_n(g_0, \cdots, g_n) = (g_0, \cdots, g_{n-1})$. If we define

$$t(g_0, \cdots, g_n) = (g_0 \cdots g_n, (g_1 \cdots g_n)^{-1}, g_1, g_2, \cdots, g_{n-1}),$$

then the cyclic identities ($t^{n+1} = 1$, $\partial_i t = t\partial_{i-1}$, etc.) are readily verified. (Do so!) Therefore EG is also a cyclic set, and the projection π: EG $\to BG$, which forgets g_0, is a morphism of cyclic sets. Applying the free k-module functor, $\pi\colon\ kEG \to kBG$ is a morphism of cyclic k-modules. More is true: The group G acts on EG by $g(g_0, g_1, \cdots) = (gg_0, g_1, \cdots)$ in a way that makes kEG into a cyclic left kG-module, and $kBG = k \otimes_{kG} kEG$. In particular, Tsygan's double complex $CC_{**}(kEG)$ is a double complex of free kG-modules and $CC_{**}(kBG) = k \otimes_{kG} CC_{**}(kEG)$. It follows that $HC_*(G) = H_*\,\mathrm{Tot}(CC_{**}(kBG))$ is the hyperhomology $\mathbb{H}_*(G; \mathrm{Tot}CC_{**}(kEG))$ of the group G (6.1.15), because each summand $CC_{pq}(kEG)$ of $\mathrm{Tot}CC_{**}(kEG)$ is a free (hence flat) kG-module.

We saw in exercise 8.3.7 that the augmentation $EG \to 1$ is a simplicial homotopy equivalence. Applying the free module functor, the augmentation $kEG \to k$ is a simplicial homotopy equivalence. Hence $C^h_*(kEG)$ is a resolution of the trivial kG-module k, just as $C^a_*(kEG)$ is a resolution of the kG-module 0. Fitting these together, Tsygan's double complex $CC_{**}(kEG)$ is a "resolution" (in the sense of hyperhomology) of the trivial chain complex

$$K_*\colon 0 \leftarrow k \leftarrow 0 \leftarrow k \leftarrow 0 \leftarrow k \leftarrow \cdots$$

which has $K_i = 0$ for $i < 0$ or i odd and $K_i = k$ for i even, $i \geq 0$. But the hyperhomology of K_* is easy to compute:

$$HC_n(G) = \mathbb{H}_n(G; K_*) = \bigoplus_{i=0}^{\infty} \mathbb{H}_{n-2i}(G;k) = \bigoplus H_{n-2i}(G;k).$$

The assertions that the maps $S\colon HC_n(G) \to HC_{n-2}(G)$ are the natural projections with kernel $HH_n(G) = H_n(G; k)$, and that the maps $B\colon HC_{n-1}(G) \to HH_n(G)$ are thus all zero, follow from a visual inspection of $\mathbb{H}_*(G; K_*)$. $\diamond$

Corollary 9.7.2

$$HP_n(G) = \varprojlim HC_{n+2i}(G) = \begin{cases} \prod_{i=0}^{\infty} H_{2i}(G; k), & n \text{ even} \\ \prod_{i=0}^{\infty} H_{2i+1}(G; k), & n \text{ odd} \end{cases}$$

Exercise 9.7.1 When $\mathbb{Q} \subset k$, use $kBG/\sim$ to compute $HC_*(G)$.

We now turn to the Hochschild homology of the group ring kG. Let $<G>$ denote the set of conjugacy classes of elements of G. Our first step is to find a decomposition of the cyclic set ZG of 9.6.2 and the cyclic module $Z(kG) = k(ZG)$ which is indexed by $<G>$. There is a cyclic set map from ZG to the trivial cyclic set $<G>$, which sends $(g_0, g_1, \cdots, g_n) \in (ZG)_n = G^{n+1}$ to the conjugacy class of the product $g_0 \cdots g_n$ in $<G>$. (Check this!) For $n = 0$ this yields an isomorphism

$$HC_0(kG) = HH_0(kG) \xrightarrow{\cong} k<G> = \bigoplus_{<G>} k.$$

Indeed, the kernel of the surjection $kG \to k<G>$ is generated by the elements $x - gxg^{-1} = g^{-1}(gx) - (gx)g^{-1} = b(g^{-1} \otimes gx)$, and $HC_0(k<G>) = k<G>$.

Definition 9.7.3 For $x \in G$, let $Z_n(G, x)$ denote the subset of $G^{n+1} = Z_nG$ consisting of all $(g_0, \cdots, g_n)$ such that $g_0 \cdots g_n$ is conjugate to x, that is, $Z_n(G, x)$ is the inverse image of $<x> \in <G>$. As n varies, these form a cyclic subset $Z(G, x)$ of ZG. Note that $Z(G, 1)$ is isomorphic to the cyclic set BG (forget g_0). Applying the free k-module functor gives cyclic k-submodules $kZ(G, x)$ of $kZ(G)$, one for each conjugacy class. We shall write $HH_*(G, x)$ for $HH_*(kZ(G, x))$, $HC_*(G, x)$ for $HC_*(kZ(G, x))$, etc. for simplicity. As $Z(G)$ is the disjoint union of the cyclic sets $Z(G, x)$, $kZ(G)$ is the direct sum of the $kZ(G, x)$. Therefore $HH_*(kG) \cong \bigoplus_x HH_*(G, x)$ and $HC_*(kG) \cong \bigoplus_x HC_*(G, x)$.

To describe $HH_*(G, x)$ etc. we recall that the *centralizer subgroup* of $x \in G$ is the subgroup $C_G(x) = \{g \in G : gxg^{-1} = x\}$. If x' is conjugate to x, then $C_G(x')$ and $C_G(x)$ are conjugate subgroups of G. In fact, if we let G act on itself by conjugation, then $C_G(x)$ is the stabilizer subgroup of x; if we choose a set $\{y\}$ of coset representatives for $G/C_G(x)$, then for each x' conjugate to x there is a unique coset representative y such that $yx'y^{-1} = x$.

Proposition 9.7.4 *For each $x \in G$ the inclusion $C_G(x) \subseteq G$ induces isomorphisms $HH_*(C_G(x), x) \cong HH_*(G, x)$ and $HC_*(C_G(x), x) \cong HC_*(G, x)$.*

Proof Write H for $C_G(x)$, and choose a set $\{y\}$ of coset representatives for G/H, the coset of H being represented by $y = 1$. Given $(g_0, \cdots, g_n) \in Z_n(G, x)$, let y_i be the (unique) coset representative such that $y_i(g_{i+1} \cdots g_n g_0 \cdots g_i)y_i^{-1} = x$ and set

$$\rho(g_0, \cdots, g_n) = (y_n g_0 y_0^{-1}, y_0 g_1 y_1^{-1}, \cdots, y_{i-1} g_i y_i^{-1}, \cdots, y_{n-1} g_n y_n^{-1}).$$

Each $y_{i-1} g_i y_i^{-1}$ is in H (check this!), so $\rho(g_0, \cdots, g_n) \in Z_n(H, x)$. By inspection, $\rho: Z(G, x) \to Z(H, x)$ is a cyclic morphism splitting the inclusion $\iota : Z(H, x) \hookrightarrow Z(G, x)$. There is a simplicial homotopy h from the identity map of $Z(G, x)$ to $\iota\rho$ defined by

$$h_j(g_0, \cdots, g_n) = (g_0 y_0^{-1}, y_0 g_1 y_1^{-1}, \cdots, y_{j-1} g_j y_j^{-1}, y_j, g_{j+1}, \cdots, g_n),$$

$j = 0, \cdots, n$. (Check this!) Hence the inclusion $Z(H, x) \subseteq Z(G, x)$ is a simplicial homotopy equivalence. This implies that $kZ(H, x) \subseteq kZ(G, x)$ is also a homotopy equivalence. Hence $HH_*(H, x) = \pi_* kZ(H, x)$ is isomorphic to $HH_*(G, x) = \pi_* kZ(G, x)$, which in turn implies that $HC_*(H, x) \cong HC_*(G, x)$. ◇

Corollary 9.7.5 *For each $x \in G$, $HH_*(G, x) \cong H_*(C_G(x); k)$. Hence*

$$HH_*(kG) = \bigoplus_{x \in \langle G \rangle} H_*(C_G(x); k).$$

Proof We have to show that $HH_*(C_G(x), x)$ is isomorphic to $H_*(C_G(x); k)$ for each x, so suppose x is in the center of G. There is an isomorphism $Z(G, 1) \to Z(G, x)$ of simplicial sets given by $(g_0, \cdots, g_n) \mapsto (xg_0, g_1, \cdots, g_n)$. Therefore $H_*(G; k) = HH_*(kBG) \cong HH_*(G, 1)$ is isomorphic to $HH_*(G, x)$. ◇

Remark One might naively guess from the above calculation that $HC_*(kG)$ would be the sum of the modules $HC_*(C_G(x)) = H_*(C_G(x); k) \otimes HC_*(k)$. However, when G is the infinite cyclic group T and $\mathbb{Q} \subseteq k$, we saw in 9.6.16 that for $n \geq 1$

$$HC_n(kT) = HC_n(k[t, t^{-1}]) \cong k \cong HC_n(T).$$

Therefore if $\mathbb{Q} \subseteq k$, then for all $x \neq 1$ in T we have $HC_n(T, x) = 0, n \neq 0$.

Exercise 9.7.2 Show that $t^{n-1} \otimes t \in Z_1(kT, t^n)$ represents the differential $t^{n-1}dt$ in $HH_1(kT) \cong \Omega_{kT/k}$, and use this to conclude that for general k.

$$HC_i(T, t^n) \cong \begin{cases} k, & i = 0 \\ k/nk, & i \geq 1 \text{ odd} \\ \mathrm{Tor}(k, \mathbb{Z}/n), & i \geq 2 \text{ even.} \end{cases}$$

Lemma 9.7.6 *If $\mathbb{Q} \subseteq k$ and $x \in G$ is a central element of finite order, then*

$$HC_*(G, x) \cong HC_*(G) \cong H_*(G; k) \otimes HC_*(k).$$

Proof Let $\bar{G}$ denote the quotient of G by the subgroup $\{x\}$ generated by x, and write $\bar{g}$ for the image of $g \in G$ in $\bar{G}$. The map of cyclic sets $Z(G, x) \to Z(\bar{G}, 1)$ sending $(g_0, \cdots, g_n)$ to $(\bar{g}_0, \cdots, \bar{g}_n)$ induces the natural map from $H_*(G; k) \cong HH_*(G, x)$ to $H_*(\bar{G}; k) \cong HH_*(\bar{G}, 1)$, because its composition with the simplicial isomorphism $Z(G, 1) \to Z(G, x)$ is the natural quotient map. The Hochschild-Serre spectral sequence $E^2_{pq} = H_p(\bar{G}; H_q(\{x\}; k)) \Rightarrow H_{p+q}(G; k)$ degenerates since $\mathbb{Q} \subset k$ (6.1.10) to show that the natural map $H_p(G; k) \to H_p(\bar{G}; k)$ is in fact an isomorphism. This yields $HC_*(G) \cong HC_*(\bar{G})$ by Karoubi's Theorem 9.7.1, as well as $HC_*(G, x) \cong HC_*(\bar{G}, 1) \cong HC_*(\bar{G})$. ◇

Corollary 9.7.7 *If $\mathbb{Q} \subseteq k$ and G is a finite group, then*

$$HC_*(kG) \cong \bigoplus_{x \in <G>} HC_*(C_G(x)) \cong k<G> \otimes HC_*(k).$$

Remark When k is a field of characteristic zero, Maschke's Theorem states that kG is a semisimple (hence separable) k-algebra. In 9.2.11 we saw that this implied that $HH_n(kG) = 0$ for $n \neq 0$, so the SBI sequence yields an alternate proof of this corollary.

Example 9.7.8 ($G = C_2$) Things are more complicated for general k, even when G is the cyclic group $C_2 = \{1, x\}$ of order 2. For example, when $k = \mathbb{Z}$ the group $HC_n(C_2, x)$ is $\mathbb{Z}$ for n even and 0 for n odd, which together with Karoubi's Theorem for $HC_*(C_2)$ yields

$$HC_n(\mathbb{Z}C_2) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & n \text{ even} \\ (\mathbb{Z}/2)^{(n+1)/2}, & n \text{ odd .} \end{cases}$$

This calculation may be found in {G. Cortiñas, J. Guccione, and O. Villamayor, "Cyclic homology of $K[\mathbb{Z}/p\mathbb{Z}]$," *K-theory* 2 (1989), 603–616}.

Exercise 9.7.3 (Kassel) Set $k = \mathbb{Z}$ and show that $HP_n(\mathbb{Z}C_2)$ is not the inverse limit of the groups $HC_{n+2i}(\mathbb{Z}C_2)$ by showing that

$$HP_0(C_2, x) \cong \varprojlim{}^1 HC_{2i+1}(C_2, x) \cong \hat{\mathbb{Z}}_2/\mathbb{Z},$$

where $\hat{\mathbb{Z}}_2$ denotes the 2-adic integers. *Hint:* Show that the SBI sequence breaks up, conclude that S is multiplication by 2, and use 3.5.5.

Theorem 9.7.9 (Burghelea) *Suppose that $\mathbb{Q} \subset k$. Then $HC_*(kG)$ is the direct sum of*

$$\bigoplus_{\substack{x \in <G> \\ \text{finite order}}} HC_*(C_G(x)) \cong \bigoplus_{\substack{x \in <G> \\ \text{finite order}}} H_*(C_G(x); k) \otimes HC_*(k)$$

and

$$\bigoplus_{\substack{x \in <G> \\ \text{infinite order}}} H_*(W(x); k).$$

Here $W(x)$ denotes the quotient group $C_G(x)/\{x^n\}$.

Proof We have already seen that $HC_*(kG)$ is the direct sum over all x in $<G>$ of the groups $HC_*(C_G(x), x)$, and that if x has finite order this equals $HC_*(C_G(x))$. Therefore it remains to suppose that $x \in G$ is a central element of infinite order and prove that $HC_*(G, x) \cong H_*(G/T; k)$, where T is the subgroup of G generated by x. For this, we pull back the path space $E(G/T)$ of 9.7.1 to $Z(G, x)$.

Let E be the cyclic subset of $E(G/T) \times Z(G, x)$ consisting of all pairs (e, z) which agree in $B(G/T)$. Forgetting the redundant first coordinates of e and z, we may identify E_n with $(G/T) \times G^n$ in such a way that (for $\bar{g}_0 \in G/T$, $g_1 \in G$):

$$\partial_i(\bar{g}_0, g_1, \cdots, g_n) = \begin{cases} (\bar{g}_0\bar{g}_1, g_2, \ldots, g_n), & i = 0 \\ (\bar{g}_0, \ldots, g_i g_{i+1}, \ldots), & 0 < i < n \\ (\bar{g}_0, g_1, \ldots, g_{n-1}), & i = n \end{cases}$$

$$t(\bar{g}_0, g_1, \cdots, g_n) = (\bar{g}_0 \cdots \bar{g}_n, (g_1 \cdots g_n)^{-1}, g_1, \cdots, g_{n-1}).$$

As in the proof of Karoubi's theorem 9.7.1, the action of G/T on the $\bar{g}_0$ coordinate makes E into a cyclic G/T-set and makes the morphism of cyclic sets $\pi: E \to Z(G, x)$ into a principal G/T-fibration (exercise 8.2.6). Therefore $kZ(G, x) = k \otimes_{kG/T} kE$, Tsygan's double complex $CC_{**}(kE)$ consists of free kG/T-modules and $CC_{**}kZ(G,x) = k \otimes_{kG/T} CC_{**}(kE)$. We will prove that $\mathrm{Tot}CC_{**}(kE)$ is a free kG/T-module resolution of k, so that

$$HC_*(G,x) = H_*(k \otimes_{kG/T} \mathrm{Tot}CC_{**}(kE)) \cong H_*(G/T;k).$$

The homotopy sequence for the principal G/T-fibration $E \to Z(G,x)$ (exercise 8.2.6 and 8.3.5) shows that $\pi_i(E) = 0$ for $i \neq 1$ and $\pi_1(E) \cong T$. The natural cyclic map $Z(T,x) \to E$, which sends $(t_0, \cdots, t_n) \in T^{n+1}$ to $(1, t_1, \cdots, t_n) \in E_n = (G/T) \times G^n$ induces isomorphisms on simplicial homotopy groups and therefore on simplicial homology (see 8.2.3). That is, $HH_*(T,x) \cong HH_*(kE)$. It follows that if $\mathbb{Q} \subseteq k$, then

$$HC_n(kE) \cong HC_n(T,x) = \begin{cases} k & n = 0 \\ 0 & n \neq 0. \end{cases}$$

Hence the natural map from $CC_{00}(kE) = kG/T$ to $k = HC_0(kE)$ provides the augmentation making Tot $CC_{**}(kE) \to k$ into a free kG/T-resolution of k, as claimed. $\diamond$

Exercise 9.7.4 Show that the SBI sequence for $Z(G,x)$ may be identified with the Gysin sequence of 6.8.6:

$$\cdots H_n(G;k) \xrightarrow{\text{coinf}} H_n(G/T;k) \to H_{n-2}(G/T;k) \to H_{n-1}(G;k) \cdots.$$

Hint: Compare $C_*^h(G,x) \to CC_{**}(G,x)$ to the coinflation map for $G \to G/T$.

9.8 Mixed Complexes

We can eliminate the odd (acyclic) columns in Tsygan's double complex 9.6.6 $CC_{**}(A)$, and obtain a double complex $\mathcal{B}_{**}(A)$ due to A. Connes. To do this, fix the chain contraction $s_n = t\sigma_n \colon A_n \to A_{n+1}$ of the acyclic complex $C_*^a(A)$ and define $B\colon A_n \to A_{n+1}$ to be the composite $(1 + (-1)^n t)sN$, where N is the norm operator on A_n. (Exercise: Show that s is a chain contraction.) Setting $\eta = (-1)^n$, we have

$$\begin{aligned} B^2 &= (1-\eta t)sN(1+\eta t)sN = 0 \\ bB + Bb &= b(1+\eta t)sN + (1-\eta t)sNb = (1-\eta t)(b's + sb')N \\ &= (1-\eta t)N = 0. \end{aligned}$$

Connes' double complex $\mathcal{B}_{**}(A)$ is formed using b and B as vertical and horizontal differentials, with $\mathcal{B}_{pq} = A_{q-p}$ for $p \geq 0$. We can formalize this construction as follows.

$$\begin{array}{ccccccc}
\cdots & & & & & & \cdots \\
M_3 & \xleftarrow{B} & M_2 & \xleftarrow{B} & M_1 & \xleftarrow{B} & M_0 \\
\downarrow b & & \downarrow b & & \downarrow b & & \\
M_2 & \xleftarrow{B} & M_1 & \xleftarrow{B} & M_0 & & \\
\downarrow b & & \downarrow b & & & & \\
M_1 & \xleftarrow{B} & M_0 & & & & \\
\downarrow b & & & & & & \\
M_0 & & & & & &
\end{array}$$

Definition 9.8.1 (Kassel) A *mixed complex* (M, b, B) in an abelian category $\mathcal{A}$ is a graded object $\{M_m : m \geq 0\}$ endowed with two families of morphisms $b: M_m \to M_{m-1}$ and $B: M_m \to M_{m+1}$ such that $b^2 = B^2 = bB + Bb = 0$. Thus a mixed complex is both a chain and a cochain complex.

The above calculation shows that every cyclic object A gives rise to a mixed complex (A, b, B), where A is considered as a graded object, b is the Hochschild differential on A and B is the map constructed as above.

Definition 9.8.2 (Connes' double complex) Let (M, b, B) be a mixed complex. Define a first quadrant double chain complex $\mathcal{B}_{**}(M)$ as follows. $\mathcal{B}_{pq}$ is M_{q-p} if $0 \leq p \leq q$ and zero otherwise. The vertical differentials are the b maps, and the horizontal differentials are the B maps.

We write $H_*(M)$ for the homology of the chain complex (M, b), and $HC_*(M)$ for the homology of the total complex $\mathrm{Tot}(\mathcal{B}_{**}(M))$. $HC_*(M)$ is called the *cyclic homology* of the mixed complex (M, b, B), a terminology which is justified by the following result.

Proposition 9.8.3 *If A is a cyclic object, then $HC_*(A)$ is naturally isomorphic to the cyclic homology of the mixed complex* (A, b, B).

Proof For each $0 \leq p \leq q$, set $t = q - p$ and map $\mathcal{B}_{pq} = A_t$ to $CC_{2p,t} \oplus CC_{2p-1,t+1} = A_t \oplus A_{t+1}$ by the map $(1, sN)$. The direct sum over p, q gives a morphism of chain complexes $\mathrm{Tot}(\mathcal{B}_{**}) \to \mathrm{Tot}(CC_{**})$. (Check this!) These two complexes compute $HC_*(A, b, B)$ and $HC_*(A)$, respectively by 9.8.2 and 9.6.6; we have to see that this morphism is a quasi-isomorphism. For this we filter $\mathcal{B}_{**}$ by columns and select the "double column" filtration for $CC_{**} : F_pCC = \oplus\{CC_{st} : t \le 2p\}$. The morphism $\mathrm{Tot}(\mathcal{B}_{**}) \to \mathrm{Tot}(CC_{**})$ is filtration-preserving, so it induces a morphism of the corresponding spectral sequences 5.4.1. To compare these spectral sequences we must compute the E^1 terms. Clearly $E^1_{pq}(\mathcal{B}) = H_{q-p}(A)$. Let T_p denote the total complex of the 2-column double complex obtained from the $(2p-1)^{st}$ and $(2p)^{th}$ columns of CC_{**}; the degree $p+q$ part of T_p is $CC_{2p,q-p} \oplus CC_{2p-1,q-p+1}$. The translates (1.2.8) of $C^a_*(A)$ and $C^h_*(A)$ fit into a short exact sequence $0 \to C^a_*(A)[1-2p] \to T_p \to C^h_*(A)[-2p] \to 0$, so the spectral sequence 5.4.1 associated to the double column filtration of CC has $E^0_{pq} = (T_p)_{p+q}$ and

$$E^1_{pq}(CC) = H_{p+q}(T_p) \cong H_{p+q}(C^h_*(A)[-2p]) \cong H_{q-p}(A).$$

By inspection, the map $E^1_{pq}(\mathcal{B}) \to E^1_{pq}(CC)$ is an isomorphism for all p and q. By the Comparison Theorem (5.2.12), $\mathrm{Tot}(\mathcal{B}) \to \mathrm{Tot}(CC)$ is a quasi-isomorphism. ◇

Remark If A is a cyclic object, any other choice of the chain contraction s, such as $s_n = (-1)^n\sigma_n$, will yield a slightly different mixed complex $M = (A, b, B')$. The proof of the above proposition shows that we would still have $HC_*(M) \cong HC_*(A)$. Our choice is dictated by the next application and by the historical selection $s(r_0 \otimes \cdots \otimes r_n) = 1 \otimes r_0 \otimes \cdots \otimes r_n$ for $A = ZR$ in [LQ].

Application 9.8.4 (Normalized mixed complex) By the Dold-Kan Theorem 8.4.1, the Hochschild homology of a cyclic k-module A may be computed using either the unnormalized chain complex $C^h_*(A)$ or the normalized chain complex $\bar{C}_*(A) = C^h_*(A)/D_*(A)$, obtained by modding out by the degenerate subcomplex $D_*(A)$. Since $D_*(A)$ is preserved by t (why?) as well as our choice of s, it is preserved by $B = (1 \pm t)s(\sum \pm t^i)$. Hence B passes to the quotient complex $\bar{C}_*(A)$, yielding a mixed complex $(\bar{C}_*(A), b, B)$. Since the morphism of mixed complexes from (A, b, B) to $(\bar{C}_*(A), b, B)$ induces an isomorphism on homology, it follows (say from the SBI sequence 9.8.7 below) that it also induces an isomorphism on cyclic homology: $HC_*(A) \cong HC_*(\bar{C}_*(A))$.

One advantage of the normalized mixed complex is that it simplifies the expression for $B = (1 \pm t)sN$. Since $ts = t^2\sigma_n = \sigma_0 t = 0$ on $\bar{C}_n(A)$, we have

$$B = t\sigma_n N = t\sigma_n + (-1)^n t^2\sigma_{n-1} + \cdots + (-1)^{ni} t^{i+1}\sigma_{n-i} + \cdots + (-1)^n t^{n+1}\sigma_0.$$

In particular, if R is a k-algebra and $A = ZR$, then in $\bar{C}_n(A) = B_n(R, R)$:

$$B(r_0 \otimes \cdots \otimes r_n) = \sum_{i=0}^{n} (-1)^{in} \otimes r_i \otimes \cdots \otimes r_n \otimes r_0 \otimes \cdots \otimes r_{i-1}.$$

Example 9.8.5 (Tensor algebra) Let $T = T(V)$ be the tensor algebra (7.3.1) of a k-module V. If $v_1, \cdots, v_j \in V$, write $(v_1 \cdots v_j)$ for their product in the degree j part $V^{\otimes j}$ of T; the generator σ of the cyclic group C_j acts on $V^{\otimes j}$ by $\sigma(v_1 \cdots v_j) = (v_j v_1 \cdots v_{j-1})$. In 9.1.6 we saw that $H_i(T, T) = 0$ for $i \neq 0$, so to use Connes' double complex 9.8.2 it suffices to describe the map

$$B \colon H_0(T, T) = \bigoplus (V^{\otimes j})_\sigma \to \bigoplus (V^{\otimes j})^\sigma = H_1(T, T).$$

Of course the definition of $B \colon T \to T \otimes T$ yields $B(r) = 1 \otimes r + r \otimes 1$ for every $r \in R$. If we modify this by elements of the form $b(r_0 \otimes r_1 \otimes r_2) = r_0 r_1 \otimes r_2 - r_0 \otimes r_1 r_2 + r_2 r_0 \otimes r_1$ we obtain a different representative of the same element of $H_1(T, T)$. Thus for $r = (v_1 \cdots v_j)$ we have

$$\begin{aligned} B(r) = r \otimes 1 + 1 \otimes r &\sim v_1 \otimes (v_2 \cdots v_j) + (v_2 \cdots v_j) \otimes v_1 + r \otimes 1 \\ &\sim (v_1 v_2) \otimes (v_3 \cdots v_j) + (v_3 \cdots v_j v_1) \otimes v_2 \\ &\quad + (v_2 \cdots v_j) \otimes v_1 + r \otimes 1 \\ &\sim \sum (v_{i+1} \cdots v_j v_1 \cdots v_{i-1}) \otimes v_i + r \otimes 1. \end{aligned}$$

Upon identifying the degree j part of $T \otimes V$ with $V^{\otimes j}$ and ignoring the degenerate term $r \otimes 1$ by passing to $\bar{C}_*$, we see that $B(r) = (1 + \sigma + \cdots + \sigma^{j-1})r$ as a map from $(V^{\otimes j})_\sigma$ to $(V^{\otimes j})^\sigma$. Identifying B with the norm map for the action of C_j on $V^{\otimes j}$, we see from Connes' complex and 6.2.2 that

$$HC_n(T) = HC_n(k) \oplus \bigoplus_{j=1}^{\infty} H_n(C_j; V^{\otimes j}).$$

In particular, if $\mathbb{Q} \subseteq k$, then $HC_n(T) = HC_n(k)$ for all $n \neq 0$.

Exercise 9.8.1 If R has an ideal I with $I^2 = 0$ and $R/I \cong k$, show that

$$HC_n(R) = HC_n(k) \oplus \bigoplus_{j=1}^{n+1} H_{n+1-j}(C_j; I^{\otimes j}).$$

Connes' Spectral Sequence 9.8.6 The increasing filtration by columns on $\mathcal{B}_{**}(M)$ gives a spectral sequence converging to $HC_*(M)$, as in 5.6.1. Since the p^{th} column is the translate $M[-p]$ of $(M, \pm b)$, we have

$$E^1_{pq} = H_{q-p}(M) \Rightarrow HC_{p+q}(M)$$

with d^1 differential $H_i(M) \to H_{i+1}(M)$ induced by Connes' operator B. This quickly yields $HC_0(M) = H_0(M)$, $HC_1(M) = H_1(M)/B(M_0)$ and a sequence of low degree terms

$$H_1(M) \xrightarrow{B} H_2(M) \xrightarrow{I} HC_2(M) \to H_0(M) \xrightarrow{B} H_1(M) \xrightarrow{I} HC_1(M) \to 0.$$

In order to extend this sequence to the left, it is convenient to proceed as follows. The inclusion of M_* as the column $p = 0$ of $\mathcal{B} = \mathcal{B}_{**}(M)$ yields a short exact sequence of chain complexes

$$0 \to M_* \xrightarrow{I} \mathrm{Tot}(\mathcal{B}) \xrightarrow{S} \mathrm{Tot}(\mathcal{B})[-2] \to 0,$$

since $\mathcal{B}/M_*$ is the double complex obtained by translating $\mathcal{B}$ up and to the right. The associated long exact sequence in homology is what we sought:

$$\cdots HC_{n+1}(M) \xrightarrow{S} HC_{n-1}(M) \xrightarrow{B} H_n(M) \xrightarrow{I} HC_n(M) \xrightarrow{S} HC_{n-2}(M) \cdots.$$

(9.8.7)

We call this the "SBI sequence" of the mixed complex M, since the proof of 9.8.3 above shows that when $M = (A, b, B)$ is the mixed complex of a cyclic object A this sequence is naturally isomorphic to the SBI sequence of A constructed in 9.6.11. As in *loc. cit.*, if $M \to M'$ is a morphism of mixed complexes such that $H_*(M) \cong H_*(M')$, then $HC_*(M) \cong HC_*(M')$ as well.

Exercise 9.8.2 Show that the spectral sequence 5.6.1 arising from Tsygan's double complex $CC_{**}(A)$, which has $E^2_{2p,q} = HH_q(A)$, has for its d^2 differential the map $HH_q(A) \to HH_{q+1}(A)$ induced by Connes' operator B. Then show that this spectral sequence is isomorphic (after reindexing) to Connes' spectral sequence 9.8.6. *Hint:* Show that the exact couple 5.9.3 of the filtration on $\mathcal{B}_{**}$ is the derived couple of the exact couple associated to $CC_{**}(A)$.

Notational consistency Our uses of the letter "B" are compatible. The map $B\colon M_m \to M_{m+1}$ defining the mixed complex M induces the d^1 differentials $B\colon H_m(M) \to H_{m+1}(M)$ in Connes' spectral sequence because it is used for the horizontal arrows in Connes' double complex 9.8.2. This is the same map as the composition $BI: H_m(M) \to HC_m(M) \to H_{m+1}(M)$ in the SBI sequence (9.8.7). (Exercise!)

Trivial Mixed Complexes 9.8.8 If (C_*, b) is any chain complex, we can regard it as a trivial mixed complex $(C_*, b, 0)$ by taking $B = 0$. Since the horizontal differentials vanish in Connes' double complex we have

$$HC_n(C_*, b, 0) = H_n(C) \oplus H_{n-2}(C) \oplus H_{n-4}(C) \oplus \cdots.$$

Similarly, if (C^*, B) is any cochain complex, we can regard it as the trivial mixed complex $(C^*, 0, B)$. Since the rows of Connes' double complex are the various brutal truncations (1.2.7) of C, we have

$$HC_n(C^*, 0, B) = C^n/B(C^{n-1}) \oplus H^{n-2}(C) \oplus H^{n-4}(C) \oplus \cdots.$$

The de Rham complex 9.8.9 provides us with an important example of this phenomenon.

9.8.1 de Rham Cohomology

9.8.9 Let R be a commutative k-algebra and $\Omega^*_{R/k}$ the exterior algebra of Kähler differentials discussed in sections 9.2 and 9.4. The *de Rham differential* $d\colon \Omega^n_{R/k} \to \Omega^{n+1}_{R/k}$ is characterized by the formula

$$d(r_0 dr_1 \wedge \cdots \wedge dr_n) = dr_0 \wedge dr_1 \wedge \cdots \wedge dr_n \quad (r_i \in R).$$

We leave it to the reader to check (using the presentation of $\Omega_{R/k}$ in 8.8.1; see [EGA, IV.16.6.2]) that d is well defined. Since $d^2 = 0$, we have a cochain complex $(\Omega^*_{R/k}, d)$ called the *de Rham complex*; the cohomology modules $H^*_{dR}(R) = H^*(\Omega^*_{R/k})$ are called the (algebraic) *de Rham cohomology* of R. All this is an algebraic parallel to the usual construction of de Rham cohomology for manifolds in differential geometry and has applications to algebraic geometry that we will not pursue here. The material here is based on [LQ].

Exercise 9.8.3 Show that d makes $\Omega^*_{R/k}$ into a differential graded algebra (4.5.2), and conclude that $H^*_{dR}(R)$ is a graded-commutative k-algebra.

If we consider $(\Omega^*_{R/k}, d)$ as a trivial mixed complex with $b = 0$, then by 9.8.8

$$HC_n(\Omega^*_{R/k}, 0, d) = \Omega^n_{R/k}/d\Omega^{n-1}_{R/k} \oplus H^{n-2}_{dR}(R) \oplus \cdots.$$

In many ways, this serves as a model for the cyclic homology of R. For example, in 9.4.4 we constructed a ring homomorphism $\psi\colon \Omega^*_{R/k} \to H_*(R, R)$, which was an isomorphism if R is smooth over k (9.4.7). The following result allows us to interpret the d^1 differentials in Connes' spectral sequence.

Lemma 9.8.10 *The following square commutes:*

$$\begin{array}{ccc} \Omega^n_{R/k} & \xrightarrow{\psi} & H_n(R, R) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle B} \\ \Omega^{n+1}_{R/k} & \xrightarrow{\psi} & H_{n+1}(R, R). \end{array}$$

Proof Given a generator $\omega = r_0 dr_1 \wedge \cdots \wedge dr_n$ of $\Omega^n_{R/k}$, $\psi(\omega)$ is the class of

$$\begin{aligned}(r_0 \otimes r_1)\nabla(1 \otimes r_2)\nabla \cdots \nabla(1 \otimes r_n) &= n!\varepsilon_n(r_0 \otimes \cdots \otimes r_n) \\ &= \sum_\sigma (-1)^\sigma r_0 \otimes r_{\sigma^{-1}(1)} \otimes \cdots \otimes r_{\sigma^{-1}(n)}\end{aligned}$$

where σ ranges over all permutations of $\{1, \cdots, n\}$ and ∇ denotes the shuffle product on $\beta(R, R)$ given in 9.4.2. Passing to the normalized complex $B_n(R, R)$, defining $\sigma(0) = 0$ and applying B, the description in 9.8.4 gives us

$$\sum_\sigma (-1)^\sigma \sum_t (-1)^t \otimes r_{\sigma^{-1}t^{-1}(0)} \otimes r_{\sigma^{-1}t^{-1}(1)} \otimes \cdots \otimes r_{\sigma^{-1}t^{-1}(n)}$$

where t ranges over the cyclic permutations $p \mapsto p + i$ of $\{0, 1, \cdots, n\}$. Since every permutation μ of $\{0, 1, \cdots, n\}$ can be written uniquely as a composite $t\sigma$, this expression equals the representative of $\psi(dr_0 \wedge dr_1 \wedge \cdots \wedge dr_n)$:

$$(n + 1)!\varepsilon_n(1 \otimes r_0 \otimes \cdots \otimes r_n) = \sum_\mu (-1)^\mu \otimes r_{\mu^{-1}(0)} \otimes \cdots \otimes r_{\mu^{-1}(n)}. \quad \diamond$$

Porism Suppose that $1/(n + 1)! \in R$. The above proof shows that

$$\begin{aligned}B(n!\varepsilon_n)(r_0 \otimes \cdots \otimes r_n) &= (n + 1)!\varepsilon_{n+1}(1 \otimes r_0 \otimes \cdots \otimes r_n) \\ &= n!\varepsilon_{n+1}B(r_0 \otimes \cdots \otimes r_n).\end{aligned}$$

Dividing by $n!$ gives the identity $B\varepsilon_n = \varepsilon_{n+1}B$.

Corollary 9.8.11 *If R is smooth over k, the E^2 terms of Connes' spectral sequence are*

$$E^2_{pq} = \begin{cases} \Omega^q_{R/k}/d\Omega^{q-1}_{R/k} & \text{if } p = 0 \\ H^{q-p}_{dR}(R) & \text{if } p > 0. \end{cases}$$

We will now show that in characteristic zero this spectral sequence collapses at E^2; we do not know if it collapses in general. Of course, when R is smooth, the sequence of low-degree terms always yields the extension (split if $1/2 \in R$):

$$0 \to \Omega^2_{R/k}/d\Omega_{R/k} \to HC_2(R) \to H^0_{dR}(R) \to 0.$$

9.8.12 Assuming that R is commutative and $\mathbb{Q} \subset R$, we saw in 9.4.4 that the maps $e\colon R^{\otimes n+1} \to \Omega^n_{R/k}$ defined by $e(r_0 \otimes \cdots) = r_0 dr_1 \wedge \cdots \wedge dr_n/n!$ satisfied $eb = 0$ and $e\psi =$ identity. In fact, e is a morphism of mixed complexes from $(R^{\otimes *+1}, b, B)$ to $(\Omega^*_{R/k}, 0, d)$ because by 9.8.4

$$eB(r_0 \otimes \cdots) = \sum \frac{(-1)^{in}}{(n+1)!} dr_i \wedge \cdots \wedge dr_n \wedge dr_0 \wedge \cdots \wedge dr_{i-1} = de(r_0 \otimes \cdots).$$

Therefore e induces natural maps

$$HC_n(R) \to HC_n(\Omega^*_{R/k}) = \Omega^n_{R/k}/d\Omega^{n-1}_{R/k} \oplus H^{n-2}_{dR}(R) \oplus H^{n-4}_{dR}(R) \oplus \cdots.$$

Theorem 9.8.13 *If R is a smooth commutative algebra, essentially of finite type over a field k of characteristic 0, then e induces natural isomorphisms*

$$HC_n(R) \cong \Omega^n_{R/k}/d\Omega^{n-1}_{R/k} \oplus H^{n-2}_{dR}(R) \oplus H^{n-4}_{dR}(R) \oplus \cdots,$$

$$HP_n(R) \cong \prod_{i \in \mathbb{Z}} H^{n+2i}_{dR}(R).$$

Proof On Hochschild homology, e induces maps $H_n(R,R) \to HH_n(\Omega^*_{R/k}) = \Omega^n_{R/k}$. When R is smooth, the Hochschild-Kostant-Rosenberg Theorem 9.4.7 states that these are isomorphisms. It follows (9.8.7) that e induces isomorphisms on HC_* and HP_* as well. ◇

Exercise 9.8.4 When R is commutative and $\mathbb{Q} \subset R$, show that $\Omega^n_{R/k}/d\Omega^{n-1}_{R/k}$ and $H^{n-2}_{dR}(R)$ are always direct summands of $HC_n(R)$. I do not know if the other $H^{n-2i}_{dR}(R)$ are direct summands.

Exercise 9.8.5 Show that the SBI sequence for a trivial mixed complex $(C^*, 0, B)$ is not split in general. Conclude that the SBI sequence of a smooth algebra R need not split in low degrees. Of course, if R is smooth and finitely generated, we observed in 9.4.8 that $H_n(R, R) = 0$ for $n > d = \dim(R)$, so the first possible non-split map is $S\colon HC_{d+1}(R) \to HC_{d-1}(R)$.

9.8.2 Hodge Decomposition

There is a decomposition for cyclic homology analogous to that for Hochschild homology. To construct it we consider Connes' double complex $\mathcal{B}_{**}$ (9.8.2) for the normalized mixed complex $(\bar{C}^h_*(R), b, B)$. Lemma 9.8.15 below shows that B sends $\bar{C}^h_n(R)^{(i)}$ to $\bar{C}^h_{n+1}(R)^{(i+1)}$. Therefore there is a double subcomplex $\mathcal{B}^{(i)}_{**}$ of $\mathcal{B}_{**}$ whose p^{th} column is the complex $\bar{C}^h_*(R)^{(i-p)}$ shifted p places vertically.

$$
\begin{array}{ccccccccccc}
\cdots & & \cdots & & & & \cdots & & & & \\
\bar{C}^{(i)}_n & \xleftarrow{B} & \bar{C}^{(i-1)}_{n-1} & \xleftarrow{B} & \cdots & \xleftarrow{B} & \bar{C}^{(1)}_{n-i+1} & \longleftarrow & 0 & & \\
\cdots & & \cdots & & & & \cdots & & & & \\
\downarrow b & & \downarrow b & & & & \downarrow b & & & & \\
\bar{C}^{(i)}_{i+1} & \xleftarrow{B} & \bar{C}^{(i-1)}_{i} & \xleftarrow{B} & \cdots & \xleftarrow{B} & \bar{C}^{(1)}_{2} & \longleftarrow & 0 & & \\
\downarrow b & & \downarrow b & & & & \downarrow b & & & & \\
\bar{C}^{(i)}_{i} & \xleftarrow{B} & \bar{C}^{(i-1)}_{i-1} & \xleftarrow{B} & \cdots & \xleftarrow{B} & \bar{C}^{(1)}_{1} & \xleftarrow{B} & R & \longleftarrow & 0 \\
\downarrow b & & \downarrow b & & & & \downarrow b & & & & \\
0 & & 0 & & & & 0 & & & &
\end{array}
$$

Definition 9.8.14 (Loday) If $i \geq 1$, then $HC^{(i)}_n(R) = H_n \operatorname{Tot} \mathcal{B}^{(i)}_{**}$. Because $e^{(0)}_n = 0$ for $n \neq 0$, $HC^{(0)}_*(R) = HC^{(0)}_0(R) = R$. The *Hodge decomposition* of HC_n for $n \geq 1$ is

$$HC_n(R) = HC^{(1)}_n(R) \oplus HC^{(2)}_n \oplus \cdots \oplus HC^{(n)}_n(R).$$

Lemma 9.8.15 $e^{(i+1)}_{n+1} B = B e^{(i)}_n$ *for every n and $i \leq n$.*

Proof When $n = i = 1$ we have $Be^{(1)}_1(r_0 \otimes r_1) = B(r_0 \otimes r_1) = 1 \otimes r_0 \otimes r_1 - 1 \otimes r_1 \otimes r_0$, which is $\varepsilon_2 B(r_0 \otimes r_1)$. More generally, if $i = n$, the equality

$\varepsilon_{n+1}B = B\varepsilon_n$ was established in the porism to lemma 9.8.10. For $i < n$, we proceed by induction. Set $F = e_{n+1}^{(i+1)}B - Be_n^{(i)}$. The following calculation shows that $b(F) = 0$:

$$be_{n+1}^{(i+1)}B = e_n^{(i+1)}bB = -e_n^{(i+1)}Bb = -Be_{n-1}^{(i)}b = -Bbe_n^{(i)} = +bBe_n^{(i)}.$$

Now observe that there is an element u of $\mathbb{Q}\Sigma_{n+1}$ such that

$$u(1 \otimes r_0 \otimes \cdots \otimes r_n) = e_{n+1}^{(i+1)}B(r_0 \otimes \cdots \otimes r_n) - Be_n^{(i)}(r_0 \otimes \cdots \otimes r_n).$$

By Barr's Lemma 9.4.9, $u = c\varepsilon_n$ and it suffices to evaluate the constant c. Because $i < n$ we have $\varepsilon_{n+1}e_{n+1}^{(i+1)} = 0$ and $\varepsilon_n e_n^{(i)} = 0$. Therefore

$$\begin{aligned}\varepsilon_{n+1}u(1 \otimes r_0 \otimes \cdots \otimes r_n) &= -\varepsilon_{n+1}Be_n^{(i)}(r_0 \otimes \cdots \otimes r_n)\\ &= -B\varepsilon_n e_n^{(i)}(r_0 \otimes \cdots \otimes r_n)\\ &= 0.\end{aligned}$$

This gives the desired relation $u = \varepsilon_{n+1}u = 0$. ◇

Corollary 9.8.16 $HC_n^{(n)}(R) = \Omega_{R/k}^n / d\Omega_{R/k}^{n-1}$.

Proof Filtering $\mathcal{B}_{**}^{(i)}$ by columns and looking in the lower left-hand corner, we see that $HC_n^{(n)}(R)$ is the cokernel of the map $B = d\colon H_{n-1}^{(n-1)}(R, R) \to H_n^{(n)}(R, R)$. ◇

Theorem 9.8.17 *When $\mathbb{Q} \subseteq R$, the SBI sequence breaks up into the direct sum of exact sequences*

$$\cdots HC_{n+1}^{(i)}(R) \xrightarrow{S} HC_{n-1}^{(i-1)}(R) \xrightarrow{B} H_n^{(i)}(R, R) \xrightarrow{I} HC_n^{(i)}(R) \xrightarrow{S} HC_{n-1}^{(i-1)}(R) \cdots.$$

Proof The quotient double complex $\mathcal{B}_{**}^{(i)}/\bar{C}_*^h(R)^{(i)}$ is a translate of $\mathcal{B}_{**}^{(i-1)}$. ◇

Corollary 9.8.18 *Let k be a field of characteristic zero. Then*

$$HC_n^{(1)}(R) \cong H_n^{(1)}(R, R) \cong D_{n-1}(R/k)$$

(André-Quillen homology) for $n \geq 3$, while for $n = 2$ there is an exact sequence

$$0 \to D_1(R/k) \to HC_2^{(1)} \to H_{dR}^0(R/k) \to 0.$$

Exercise 9.8.6 Show that if R is smooth over k, then $HC_n^{(i)}(R) = 0$ for $i < n/2$, while if $n/2 \leq i < n$ we have $HC_n^{(i)}(R) \cong H_{dR}^{2i-n}(R/k)$.

Exercise 9.8.7 Show that there is also a Hodge decomposition for $HP_*(R)$:

$$HP_*(R) = \Pi HP_*^{(i)}(R).$$

If R is smooth, show that $HP_*^{(i)}(R) \cong H_{dR}^{2i-n}(R/k)$.

Remark 9.8.19 (Schemes) It is possible to extend Hochschild and cyclic homology to schemes over k by formally replacing R by $\mathcal{O}_X$ and $R^{\otimes n}$ by $\mathcal{O}_X^{\otimes n}$ to get chain complexes of sheaves on X, and then taking hypercohomology (Chapter 5, section 7). For details, see [G-W]. If X is smooth over k and contains $\mathbb{Q}$, it turns out that $HH_n^{(i)}(X) \cong H^{i-n}(X, \Omega_X^i)$ and $HP_n^{(i)}(X) = H_{dR}^{2i-n}(X)$. If X is a smooth projective scheme and $p = i - n$, then $HC_n^{(i)}(X)$ is the p^{th} level $F^p H_{dR}^{2i-n}(X)$ of the classical Hodge filtration on $H_{dR}^*(X) \cong H^*(X(\mathbb{C}); k)$. This direct connection to the classical Hodge filtration of $H_{dR}^*(X)$ justifies our use of the term "Hodge decomposition."

9.9 Graded Algebras

Let $R = \oplus R_i$ be a graded k-algebra. If $r_0, \cdots, r_p$ are homogeneous elements, define the *weight* of $r_0 \otimes \cdots \otimes r_p \in R^{\otimes p+1}$ to be $w = \sum |r_i|$, where $|r_i| = j$ means that $r_i \in R_j$. This makes the tensor product $R^{\otimes p+1}$ into a graded k-module, $(R^{\otimes p+1})_w$ being generated by elements of weight w. Since the face and degeneracy maps, as well as the cyclic operator t, all preserve weight, the $\{(R^{\otimes p+1})_w\}$ form a cyclic submodule $(ZR)_w$ of $ZR = R^{\otimes *+1}$ and allows us to view $ZR = \oplus (ZR)_w$ as a graded cyclic module or cyclic object in the abelian category of graded k-modules (9.6.1). As our definitions work in any abelian category, this provides each $HH_p(R) = HH_p(ZR)$ and $HC_p(R) = HC_p(ZR)$ with the structure of graded k-modules: $HH_p(R)_w = HH_p((ZR)_w)$ and $HC_p(R)_w = HC_p((ZR)_w)$. We are going to prove the following theorem, due to T. Goodwillie [Gw].

Goodwillie's Theorem 9.9.1 *If R is a graded k-algebra, then the image of S: $HC_p(R)_w \to HC_{p-2}(R)_w$ is annihilated by multiplication by w. In particular, if $\mathbb{Q} \subset R$, then $S = 0$ on $HC_*(R)_w$ for $w \neq 0$, and the SBI sequence splits up into short exact sequences*

$$0 \to HC_{p-1}(R)_w \xrightarrow{B} HH_p(R)_w \xrightarrow{I} HC_p(R)_w \to 0.$$

If R is positively graded ($R = R_0 \oplus R_1 \oplus \cdots$), then clearly $(ZR)_0 = Z(R_0)$, so that the missing piece $w = 0$ of the theorem has $HC(R)_0 = HC(R_0)$.

Corollary 9.9.2 *If R is positively graded and $\mathbb{Q} \subset R$, then $HP_*(R) \cong HP_*(R_0)$.*

Corollary 9.9.3 (Poincaré Lemma) *If R is commutative, positively graded, and $\mathbb{Q} \subset R$, then*

$$H^*_{dR}(R) \cong H_{dR}(R_0).$$

Proof It suffices to show that the weight w part of the de Rham complex $(\Omega^*_{R/k}, d)$ of 9.8.9 is zero for $w \neq 0$. This is a direct summand (by 9.4.4, exercise 9.4.4) of the chain complex $(HH_*(R)_w, BI)$, which is exact because the kernel of $BI\colon HH_p(R)_w \to HH_{p+1}(R)_w$ is $HC_{p-1}(R)_w$. ◇

Example 9.9.4 The tensor algebra $T = T(V)$ of a k-module V may be graded by setting $T_i = V^{\otimes i}$. We saw in 9.1.6 that $HH_n(T) = 0$ for $n \neq 0, 1$. If $\mathbb{Q} \subseteq k$, this immediately yields $HC_n(T)_w = 0$ for $n \neq 0$ and $w \neq 0$, and hence we have $HC_n(T) = HC_n(k)$ for $n \neq 0$. If $\mathbb{Q} \not\subset k$, the explicit calculation in 9.8.5 shows that $HC_n(T)_w \cong H_n(C_w;\, V^{\otimes w})$, which is a group of exponent w as the cyclic group C_w has order w.

Exercise 9.9.1 Given a k-module V we can form the ring $R = k \oplus V$ with $V^2 = 0$. If we grade R with $R_1 = V$ and fix $w \neq 0$, show that

$$HC_n(R)_w \cong H_{n+1-w}(C_w;\, V^{\otimes w}).$$

Exercise 9.9.2 Let R be the truncated polynomial ring $k[x]/(x^{m+1})$, and suppose that $\mathbb{Q} \subset k$. We saw that $HH_n(R) \cong k^m$ for all $n \neq 0$ in exercise 9.1.4. Show that $HC_n(R) = 0$ for n odd, while for n even $HC_n(R) \cong k^{m+1}$. Compare this approach with that of exercise 9.6.4.

Exercise 9.9.3 (Generating functions) Let k be a field of characteristic zero, and suppose that R is a positively graded k-algebra with each R_i finite-dimensional. Show that $h(n, w) = \dim\ HH_n(R)_w$ is finite and that for every $w \neq 0$ we have

$$\dim\ HC_n(R)_w = (-1)^n \sum_{i=0}^{n} (-1)^i h(i, w).$$

Now set $h_w(t) = \sum h(n,w)t^n$, $f_w(t) = \sum \dim HC_n(R)_w t^n$, and show that $h_w(t) = (1+t)f_w(t)$.

In order to prove Goodwillie's Theorem, we work with the normalized mixed complex $\bar{C}_*(R)$ of R. First we describe those maps $F: R^{\otimes m+1} \to \bar{C}_n(R)$ which are natural with respect to the graded ring R (and k). For each sequence of weights $w = (w_0, \cdots, w_m)$ we must give a map F_w from $R_{w_0} \otimes \cdots \otimes R_{w_m}$ to $\bar{C}_n(R)$. Let T_w denote the free k-algebra on elements $x_0, \cdots, x_m$, graded so that x_i has weight w_i. Given $r_i \in R_{w_i}$ there is a graded algebra map $T_w \to R$ sending x_i to r_i; the map $\bar{C}_n(T_w) \to \bar{C}_n(R)$ must send $y = F_w(x_0 \otimes \cdots \otimes x_m)$ to $F_w(r_0 \otimes \cdots \otimes r_m)$. Thus F_w is determined by the element $y = y(x_0, \cdots, x_m)$ of $\bar{C}_n(T_w) = T_w \otimes \bar{T}_w \otimes \cdots$, that is, by a k-linear combination of terms $M_0 \otimes \cdots \otimes M_n$, where the M_j are noncommutative monomials in the x_i, and $M_j \neq 1$ for $i \neq 0$. In order for y to induce a natural map F_w we must have multilinearity:

$$\lambda y(x_0, \cdots, x_m) = y(x_0, \cdots, \lambda x_i, \cdots, x_m)$$

for all i and all $\lambda \in k$. Changing k if necessary (so that for each j there is a $\lambda \in k$ such that $\lambda^j \neq \lambda$), this means there can be at most one occurrence of each x_i in each monomial $M_0 \otimes \cdots \otimes M_n$ in $y(x_0, \cdots, x_m)$.

If $n \geq m+2$, then at least two of the monomials M_i must be one in each term $M_0 \otimes \cdots \otimes M_n$ of y. This is impossible unless $y = 0$. If $n = m+1$, then we must have $M_0 = 1$ in each term, and y must be a linear combination of the monomials $1 \otimes x_{\sigma 0} \otimes \cdots \otimes x_{\sigma m}$ as σ runs over all permutations of $\{0, \cdots, m\}$. An example of such a natural map is B; the universal formula in this case is given by $y = B(x_0 \otimes \cdots \otimes x_m)$, where only cyclic permutations are used. From this we make the following deduction.

Lemma 9.9.5 *Any natural map $F: R^{\otimes m+1} \to \bar{C}_{m+1}(R)$ must satisfy $FB = BF = 0$, and induces a map $\bar{F}: \bar{C}_m(R) \to \bar{C}_{m+1}(R)$.*

Examples 9.9.6 If $m = n$, there is a natural map $D: \bar{C}_m(R) \to \bar{C}_m(R)$ which is multiplication by $w = \sum w_i$ on $R_{w_0} \otimes \cdots \otimes R_{w_m}$. When $m = 0$, D is the map from $R = \bar{C}_0(R)$ to itself sending $r \in R_w$ to wr. The formula

$$e(r_0 \otimes \cdots \otimes r_m) = (-1)^{m-1}(Dr_m)r_0 \otimes r_1 \otimes \cdots \otimes r_{m-1}$$

gives a natural map $e: \bar{C}_m(R) \to \bar{C}_{m-1}(R)$. This map is of interest because $eb + be = 0$ (check this!), and also because of its resemblance to the face map ∂_n (which is natural on $R^{\otimes m+1}$ but does not induce a natural map $\bar{C}_m \to \bar{C}_{m-1}$).

Proof of Theorem 9.9.1 Since D commutes with B and b, it is a map of mixed complexes and induces an endomorphism of $HC_*(R)$ — namely, it is multiplication by w on $HC_*(R)_w$. We must show that $DS = 0$. To do this we construct a chain contraction $Se + SE$ of $DS\colon \mathrm{Tot}_n \mathcal{B}_{**} \to \mathrm{Tot}_{n-2} \mathcal{B}_{**}$, where $\mathcal{B}_{**}$ is Connes' double complex for the normalized complex $\bar{C}_*(R)$ and S is the periodicity map $\mathcal{B}_{pq} \to \mathcal{B}_{p-1,q-1}$. The map $e\colon \mathcal{B}_{pq} \to \mathcal{B}_{p+1,q}$ is the map $\bar{C}_m \to \bar{C}_{m-1}$ given in 9.9.6, and E will be a map $\mathcal{B}_{pq} \to \mathcal{B}_{p,q+1}$ induced by natural maps $E_m\colon \bar{C}_m \to \bar{C}_{m+1}$. If we choose E so that D equals

$$(*) \qquad (e + E)(B + b) + (B + b)(e + E) = eB + Be + Eb + bE$$

$$\begin{tikzcd}
 & \bar{C}_{m+1} & \\
\bar{C}_{m+1} & \bar{C}_m = \mathcal{B}_{pq} \arrow[l, "B"'] \arrow[r, "e"] \arrow[u, "E"] \arrow[d, "b"] & \bar{C}_{m-1} \\
 & \bar{C}_{m-1} &
\end{tikzcd}$$

on $\bar{C}_m(R)$, then $S(e + E)$ will be a chain contraction of DS. Note that the term eB of $(*)$ does not make sense on $\mathcal{B}_{0q}$, but the term SeB does.

All that remains is to construct $E_m\colon \bar{C}_m(R) \to \bar{C}_{m+1}(R)$, and we do this by induction on m, starting with $E_0 = 0$ and $E_1(r_0 \otimes r_1) = 1 \otimes Dr_1 \otimes r_0$. Because

$$\begin{aligned}
(eB + Be)(r_0) &= e(1 \otimes r_0) = Dr_0, \\
(eB + Be + bE_1)(r_0 \otimes r_1) &= e(1 \otimes r_0 \otimes r_1 - 1 \otimes r_1 \otimes r_0) \\
&\quad + B(Dr_1)r_0 + bE_1(r_0 \otimes r_1) \\
&= -Dr_1 \otimes r_0 + Dr_0 \otimes r_1 \\
&\quad + 1 \otimes (Dr_1)r_0 + b(1 \otimes Dr_1 \otimes r_0) \\
&= Dr_0 \otimes r_1 + r_0 \otimes Dr_1 \\
&= D(r_0 \otimes r_1),
\end{aligned}$$

the expression $(*)$ equals D on $\bar{C}_0(R)$ and $\bar{C}_1(R)$. For $m \geq 2$, we assume E_{m-1}, E_{m-2} constructed; for each w we need to find elements $y \in \bar{C}_{m+1}(T_w)$ such that

$$by + (eB + Be + E_{m-1}b)(x_0 \otimes \cdots \otimes x_m) = D(x_0 \otimes \cdots \otimes x_m)$$

in $\bar{C}_m(T_w)$. Set $z = (D - eB - Be - E_{m-1}b)(x_0 \otimes \cdots \otimes x_m)$; by induction and $(*)$,

$$\begin{aligned} bz &= (Db + ebB + Bbe - bE_{m-1}b - E_{m-2}b^2)(x_0 \otimes \cdots \otimes x_m) \\ &= (D - eB - Be - bE_{m-1} - E_{m-2}b)b(x_0 \otimes \cdots \otimes x_m) \\ &= 0. \end{aligned}$$

We saw in 9.1.6 that $H_m(T_w, T_w) = 0$ for $m \geq 2$, so the normalized complex $\bar{C}_*(T_w)$ and hence its summand $\bar{C}_*(T_w)_w$ of weight w are exact at m. Thus there is an element y in $\bar{C}_{m+1}(T_w)_w$ such that $by = z$. Since y has weight w_i with respect to each x_i, there can be at most one occurrence of each x_i in each monomial in $y(x_0, \cdots, x_m)$. Hence if we define

$$E_m(r_0 \otimes \cdots \otimes r_m) = y(r_0, \cdots, r_m),$$

then E_m is a natural map from $\bar{C}_m(R)$ to $\bar{C}_{m+1}(R)$ such that $(*)$ equals D on $\bar{C}_m(R)$. This finishes the construction of E and hence the proof of Goodwillie's Theorem. $\diamond$

Remark 9.9.7 The "weight" map $D: R \to R$ is a derivation, and Goodwillie's Theorem 9.9.1 holds more generally for any derivation acting on a k-algebra R; see [Gw]. All the basic formulas in the proof—such as the formula $(*)$ for D—were discovered by G. Rinehart 20 years earlier; see sections 9, 10 of "Differential forms on general commutative algebras, *Trans. AMS* 108 (1963), 195–222.

As an application of Goodwillie's Theorem, suppose that I is an ideal in a k-algebra R. Let $Z(R, I)$ denote the kernel of the surjection $Z(R) \to Z(R/I)$; we define the cyclic homology modules $HC_*(R, I)$ to be the cyclic homology of the cyclic module $Z(R, I)$. Since cyclic homology takes short exact sequences of cyclic modules to long exact sequences, we have a long exact sequence

$$\cdots HC_{n+1}(R) \to HC_{n+1}(R/I) \to HC_n(R, I) \to HC_n(R) \to HC_n(R/I) \cdots.$$

Thus $HC_*(R, I)$ measures the difference between $HC_*(R)$ and $HC_*(R/I)$.

We can filter each module $Z_pR = R^{\otimes p+1}$ by the submodules F_p^i generated by all the $I^{i_0} \otimes \cdots \otimes I^{i_p}$ with $i_0 + \cdots + i_p = i$. Since the structure maps ∂_i, σ_i, t preserve this filtration, the F_*^i are cyclic submodules of ZR. As F_*^1 is $Z(R, I)$, we have $F_*^0/F_*^1 = Z(R/I)$.

Exercise 9.9.4 If k is a field, show that the graded cyclic vector spaces $\oplus F_*^i/F_*^{i+1}$ and $Z(grR)$ are isomorphic, where $gr(R) = R/I \oplus I/I^2 \oplus \cdots \oplus I^m/I^{m+1} \oplus \cdots$ is the associated graded algebra of $I \subset R$.

Proposition 9.9.8 *Let k be a field of characteristic zero. If $I^{m+1} = 0$, then the maps $S^i\colon HC_{p+2i}(R, I) \to HC_p(R, I)$ are zero for $i \geq m(p+1)$.*

Proof By the above exercise, $HC_*(grR)_w \cong HC_*(F_*^w/F_*^{w+1})$. Since $gr(R)$ is graded, the map S is zero on all but the degree zero part of $HC_*(grR)$. Hence $S^i = 0$ on $HC_*(F^1/F^{i+1})$. Since $F_p^{i+1} = 0$ for $i \geq m(p+1)$, the map S^i factors as

$$HC_{p+2i}(R, I) \to HC_{p+2i}(F_*^1/F_*^{i+1}) \xrightarrow{S^i} HC_p(F_*^1/F_*^{i+1}) = HC_p(R, I),$$

which is the zero map. $\diamond$

Corollary 9.9.9 *If I is a nilpotent ideal of R, then $HP_*(R, I) = 0$ and $HP_*(R) \cong HP_*(R/I)$.*

Proof The tower $\{HC_{*+2i}(R, I)\}$ satisfies the trivial Mittag-Leffler condition. $\diamond$

Exercise 9.9.5 If I is a nilpotent ideal of R and k is a field with char$(k) = 0$, show that $H_{dR}^*(R) \cong H_{dR}^*(R/I)$. *Hint:* Study the complex $(HH_*(R), BI)$.

9.9.1 Homology of DG-Algebras

9.9.10 It is not hard to extend Hochschild and cyclic homology to DG-algebras, that is, graded algebras with a differential $d\colon R_n \to R_{n-1}$ satisfying the Leibnitz identity $d(r_0 r_1) = (dr_0)r_1 + (-1)^{|r_0|} r_0(dr_1)$; see 4.5.2. (Here $|r_0| = j$ if $r_0 \in R_j$.) If we forget the differential, we can consider ZR (9.6.1) as a graded cyclic module as in Goodwillie's Theorem 9.9.1. If we lay out the Hochschild complex in the plane with $(R^{\otimes q+1})_p$ in the (p, q) spot, then there is also a "horizontal" differential given by

$$d(r_0 \otimes \cdots \otimes r_q) = \sum_{i=0}^{q} (-1)^{|r_0| + \cdots + |r_{i-1}|} r_0 \otimes \cdots \otimes dr_i \otimes \cdots \otimes r_q.$$

Thus the Hochschild complex becomes a double complex $C_*^h(R, d)_*$; we define the Hochschild homology $HH_*^{DG}(R)$ to be the homology of $\mathrm{Tot}^{\oplus} C_*^h(R)_*$. If R is positively graded, then $C^h(R, d)$ lies in the first quadrant and there is a spectral sequence converging to $HH_*^{DG}(R)$ with $E_{pq}^2 = H_p^h(HH_q(R)_*)$. *Warning:* If R is a graded algebra endowed with differential $d = 0$, then $HH_n^{DG}(R)$ is the sum of the $HH_q(R)_p$ with $p + q = n$ and not $HH_n(R)$.

In the literature (e.g., in [MacH, X]) one often considers DG-algebras to have a differential $d\colon R^n \to R^{n+1}$ and $R^n = 0$ for $n < 0$. If we reindex R^n as R_{-n} this is a negatively graded DG-algebra. It is more natural to convert $C_*^h(R,d)_*$ into a cochain double complex in the fourth quadrant and to write $HH^n_{DG}(R)$ for $HH^{DG}_{-n}(R)$.

Exercise 9.9.6 If $R^0 = k$ and $R^1 = 0$, construct a convergent fourth quadrant spectral sequence converging to $HH^*_{DG}(R)$ with $E_2^{pq} = H^p HH_{-q}(R)$.

Exercise 9.9.7 Let (R_*, d) be a DG-algebra and M a chain complex that is also a graded R-module in such a way that the Leibnitz identity holds with $r_0 \in M$, $r_1 \in R$. Define $H_*^{DG}(R, M)$ to be the homology of the total complex $(M \otimes R^{\otimes q})_p$ obtained by taking $r_0 \in M$ in 9.9.10. If M and R are positively graded, show that there is a spectral sequence

$$E^2_{pq} = H^h_p H_q(R, M) \Rightarrow H^{DG}_{p+q}(R, M).$$

We now return to the cyclic viewpoint. The chain complexes $Z_q(R)_* = (R^{\otimes q+1})_*$ fit together to form a cyclic object $Z(R, d)$ in $\mathbf{Ch}(k\text{–}\mathbf{mod})$, the abelian category of chain complexes, provided that we use the sign trick to insert a sign of $(-1)^{|r_q|(|r_0|+\cdots+|r_{q-1}|)}$ in the formulas for ∂_q and t. (Check this!) As in any abelian category, we can form HH_* and HC_* in $\mathbf{Ch}(k\text{–}\mathbf{mod})$. However, since $C_*^h(Z(R,d))$ is really a double complex whose total complex yields $HH_*^{DG}(R)$ it makes good sense to imitate 9.6.7 and define $HC_*^{DG}(R)$ as $H_* \operatorname{Tot}^{\oplus} CC_{**}Z(R,d)$. If R is positively graded, then we can define $HP_*^{DG}(R)$ using the product total complex of $CC^P_{**}Z(R,d)$. All the major structural results for ordinary cyclic homology clearly carry over to this DG-setting.

Proposition 9.9.11 *If $f\colon (R,d) \to (R', d')$ is a homomorphism of flat DG-algebras such that $H_*(R) \cong H_*(R')$, then f induces isomorphisms*

$$HH_*^{DG}(R) \cong HH_*^{DG}(R') \quad \text{and} \quad HC_*^{DG}(R) \cong HC_*^{DG}(R').$$

Proof As each $R^{\otimes n}$ is also flat as a k-module, the chain maps

$$f^{\otimes n+1}\colon R^{\otimes n+1} \to R^{\otimes n} \otimes R' \to (R')^{\otimes n+1}$$

are quasi-isomorphisms for all n. Filtering by rows 5.6.2 yields a convergent spectral sequence

$$E^1_{pq} = H_q(R^{\otimes p+1}) \Rightarrow HH^{DG}_{p+q}(R).$$

By the Comparison Theorem 5.2.12, we have $HH_*^{DG}(R,d) \cong HH_*^{DG}(R',d')$. The isomorphism on HC_*^{DG} follows formally using the 5-lemma and the SBI sequence 9.6.11. ◇

Vista 9.9.12 (Free loop spaces) Suppose that X is a fixed simply connected topological space, and write $C^*(X)$ for the DG-algebra of singular chains on X with coefficients in a field k; the singular cohomology $H^*(X)$ of X is the cohomology of $C^*(X)$. Let X^I denote the space of all maps $f\colon I \to X$, I denoting the interval [0,1]; the *free loop space* ΛX is $\{f \in X^I\colon f(0) = f(1)\}$ and if $* \in X$ is fixed, the *loop space* ΩX is $\{f \in X^I\colon f(0) = f(1) = *\}$. The general machinery of the "Eilenberg-Moore spectral sequence" [Smith] for the diagram

$$\begin{array}{ccccc} \Omega X & \longrightarrow & \Lambda X & \longrightarrow & X^I \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & X & \xrightarrow{\Delta} & X \times X \end{array}$$

yields isomorphisms:

$$H^n(\Omega X) \cong HH^n_{DG}(C^*(X), k) \cong HH^{DG}_{-n}(C^*(X), k);$$
$$H^n(\Lambda X) \cong HH^n_{DG}(C^*(X)) \cong HH^{DG}_{-n}(C^*(X)).$$

We say that a space X is *formal* (over k) if there are DG-algebra homomorphisms $C^*(X) \leftarrow R \rightarrow H^*(X)$ that induce isomorphisms in cohomology. Here we regard the graded ring $H^*(X)$ as a DG-algebra with $d = 0$, either positively graded as a cochain complex or negatively graded as a chain complex. Proposition 9.9.11 above states that for formal spaces we may replace $C^*(X)$ by $H^*(X)$ in the above formulas for $H^n(\Omega X)$ and $H^n(\Lambda X)$.

All this has an analogue for cyclic homology, using the fact that the topological group S^1 acts on ΛX by rotating loops. The equivariant homology $H_*^{S^1}(\Lambda X)$ of the S^1-space ΛX is defined to be $H_*(\Lambda X \times_{S^1} ES^1)$, the singular homology of the topological space $\Lambda X \times_{S^1} ES^1 = \{(\lambda, e) \in \Lambda X \times ES^1\colon \lambda(1) = \pi(e)\}$. Several authors (see [Gw], for example) have identified $H_*^{S^1}(\Lambda X)$ with the cyclic homology $HC_*^{DG}(R_*)$ of the DG-algebra R_* whose homology is $H_*(\Omega X)$.

9.10 Lie Algebras of Matrices

In this section we fix a field k of characteristic zero and an associative k-algebra with unit R. Our goal is to relate the homology of the Lie algebra $\mathfrak{gl}_m(R) = \mathrm{Lie}(M_m(R))$ of $m \times m$ matrices, described in Chapter 7, to the cyclic homology of R. This relationship was discovered in 1983 by J.-L. Loday and D. Quillen [LQ], and independently by B. Feigin and B. Tsygan. We shall follow the exposition in [LQ].

The key to this relationship is the map

$$H_*^{Lie}(\mathfrak{gl}_m(R); k) \xrightarrow{\lambda_*} HC_*(M_m(R)) \cong HC_*(R)$$

constructed as follows. Recall from 7.7.3 that the homology of a Lie algebra $\mathfrak{g}$ can be computed as the homology of the Chevalley-Eilenberg complex $\Lambda^*\mathfrak{g} = k \otimes_{U\mathfrak{g}} V_*(\mathfrak{g})$, with differential

$$d(x_1 \wedge \cdots \wedge x_p) = \sum_{i<j} (-1)^{i+j}[x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_p.$$

On the other hand, we saw in 9.6.10 that the cyclic homology of R may be computed using the quotient complex $C_*(R) = C_*^h(R)/\sim$ of the Hochschild complex $C_*^h(R)$. Define $\lambda\colon \Lambda^{p+1}\mathfrak{gl}_m(R) \to C_*(M_m(R))$ by

$$\lambda(x_0 \wedge \cdots \wedge x_p) = (-1)^p \sum_\sigma (-1)^\sigma x_0 \otimes x_{\sigma 1} \otimes \cdots \otimes x_{\sigma p},$$

where the sum is over all possible permutations σ of $\{1, \cdots, p\}$. (Exercise: Why is λ well defined?)

Lemma 9.10.1 *λ is a morphism of chain complexes, and induces maps*

$$\lambda_*\colon H_{p+1}(\mathfrak{gl}_m(R); k) \to HC_p(R).$$

Moreover λ is compatible with the usual nonunital inclusion $\iota\colon M_m(R) \hookrightarrow M_{m+1}(R)$, $\iota(g) = \begin{bmatrix} g & 0 \\ 0 & 0 \end{bmatrix}$, in the sense that the following diagram commutes.

$$\begin{array}{ccccc}
\Lambda^{*+1}\mathfrak{gl}_m(R) & \xrightarrow{\lambda} & C_*(M_m(R)) & \xrightarrow{\text{trace}} & C_*(R) \\
{\scriptstyle \mathrm{Lie}(\iota)}\downarrow & & \downarrow{\scriptstyle \iota_*} & & \| \\
\Lambda^{*+1}\mathfrak{gl}_{m+1}(R) & \xrightarrow{\lambda} & C_*(M_{m+1}(R)) & \xrightarrow{\text{trace}} & C_*(R).
\end{array}$$

Proof Commutativity of the right square amounts to the assertion that ι_* is compatible with the trace maps, and was established in exercise 9.5.3. Now set $\omega = x_0 \wedge \cdots \wedge x_p$ with $x_i \in \mathfrak{gl}_m(R)$. The formula for λ shows that $\iota_*(\lambda\omega) = \lambda(\iota x_0 \wedge \cdots \wedge \iota x_p) = \lambda(\iota\omega)$, which gives commutativity of the left square. It also shows that

$$b\lambda(\omega) = (-1)^n \sum (-1)^\nu (x_{\nu 0} x_{\nu 1}) \otimes x_{\nu 2} \otimes \cdots \otimes x_{\nu p},$$

the sum being over all permutations ν of $\{0, 1, \cdots, p\}$. Since

$$\omega = (-1)^{i+j+1} x_i \wedge x_j \wedge x_0 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_p$$

for $i < j$, it is readily verified (do so!) that $\lambda(d\omega) = b(\lambda\omega)$. This proves that λ is a morphism of complexes. ◇

Primitive Elements 9.10.2 An element x in a coalgebra H (6.7.13) is called *primitive* if $\Delta(x) = x \otimes 1 + 1 \otimes x$. The primitive elements form a submodule $\mathrm{Prim}(H)$ of the k-module underlying H. If H is a graded coalgebra and Δ is a graded map, the homogeneous components of any primitive element must be primitive, so $\mathrm{Prim}(H)$ is a graded submodule of H.

We saw in exercise 7.3.8 that the homology $H = H_*(\mathfrak{g}; k)$ of any Lie algebra $\mathfrak{g}$ is a graded coalgebra with coproduct $\Delta \colon H \to H \otimes H$ induced by the diagonal $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$. When $\mathfrak{g}$ is the Lie algebra $\mathfrak{gl}(R) = \cup \mathfrak{gl}_m(R)$, we are going to prove in 9.10.10 that $\mathrm{Prim}\, H_i(\mathfrak{g}; k) \cong HC_{i-1}(R)$.

The first step in the proof is to recall from exercise 7.7.6 that any Lie group $\mathfrak{g}$ acts on $\Lambda^n \mathfrak{g}$ by the formula $[x_1 \wedge \cdots \wedge x_n, g] = \sum x_1 \wedge \cdots \wedge [x_i g] \wedge \cdots \wedge x_n$. This makes the Chevalley-Eilenberg complex $\Lambda^* \mathfrak{g}$ into a chain complex of right $\mathfrak{g}$-modules, and $\mathfrak{g}$ acts trivially on $H_*(\mathfrak{g}; k) = H_*(\Lambda^* \mathfrak{g})$, again by exercise 7.7.6. Applying this to $\mathfrak{gl}_m(R)$, we observe that $\Lambda^* \mathfrak{gl}_m(R)$ is a chain complex of modules over $\mathfrak{gl}_m(R)$ and hence over the simple Lie algebra $\mathfrak{sl}_m = \mathfrak{sl}_m(k)$ of matrices over k with trace 0 (7.1.3, 7.8.1). Therefore we may take coinvariants to form the chain complex $H_0(\mathfrak{sl}_m; \Lambda^* \mathfrak{gl}_m(R))$.

Proposition 9.10.3 *Taking coinvariants gives a quasi-isomorphism of complexes*

$$\Lambda^* \mathfrak{gl}_m(R) \to H_0(\mathfrak{sl}_m; \Lambda^* \mathfrak{gl}_m(R)).$$

Proof Weyl's Theorem 7.8.11 states that, like every finite-dimensional $\mathfrak{sl}_m$-module, $\Lambda^n \mathfrak{gl}_m(k)$ is a direct sum of simple modules. As R is a free k-module, each $\Lambda^n \mathfrak{gl}_m(R) = \Lambda^n \mathfrak{gl}_m(k) \otimes R$ is also a direct sum of simple modules. Write

Q^n for the direct sum of the simple modules on which $\mathfrak{sl}_m$ acts non-trivially, so that $\Lambda^*\mathfrak{gl}_m(R) = Q^* \oplus H_0(\mathfrak{sl}_m; \Lambda^*\mathfrak{gl}_m(R))$ as an $\mathfrak{sl}_m$-module complex. As $\mathfrak{sl}_m$ acts trivially on the homology of $\Lambda^*\mathfrak{gl}_m(R)$ by exercise 7.7.6, the complex Q^* has to be acyclic, proving the proposition. $\diamond$

Corollary 9.10.4 *If $m \geq n$ the maps $H_n(\mathfrak{gl}_m(R); k) \to HC_{n-1}(R)$ are split surjections.*

Proof Let $e_{ij}(r)$ denote the matrix which is r in the (i, j) spot and zero elsewhere. Exercise 9.5.4 showed that if we set

$$\omega = \omega(r_1, \cdots, r_n) = e_{12}(r_1) \wedge e_{23}(r_2) \wedge \cdots \wedge e_{n-1,n}(r_{n-1}) \wedge e_{n1}(r_n),$$

then $\omega \in \Lambda^n\mathfrak{gl}_m(R)$ satisfies $\mathrm{trace}(\lambda\omega) = (-1)^{n-1} r_1 \otimes \cdots \otimes r_n$. Moreover

$$\begin{aligned} -d\omega = e_{13}(r_1r_2) \wedge \cdots + e_{12}(r_1) \wedge e_{24}(r_2r_3) \wedge \cdots \\ + (-1)^{n+1} e_{n2}(r_nr_1) \wedge e_{23}(r_2) \wedge \cdots. \end{aligned}$$

Modulo coinvariants this equals $-\omega(b(r_1 \otimes \cdots \otimes r_n))$. Therefore ω defines a chain complex homomorphism from the translated cyclic complex $R^{\otimes *}/\sim\, = (R^{\otimes *+1}/\sim)[-1]$ to $H_0(\mathfrak{sl}_m; \Lambda^*\mathfrak{gl}_m(R))$. As ω is split by $\mathrm{trace}(\lambda)$, the result follows upon taking homology. $\diamond$

Invariant Theory Calculation 9.10.5 Let Σ_n be the symmetric group of permutations of $\{1, \cdots, n\}$ and (sgn) the 1-dimensional Σ_n-module on which $\sigma \in \Sigma_n$ acts as multiplication by its signature $(-1)^\sigma$. If Σ_n acts on $V^{\otimes n}$ by permuting coordinates, then $\Lambda^n V = V^{\otimes n} \otimes_{k\Sigma_n} (sgn)$. In particular,

$$\Lambda^n\mathfrak{gl}_m(R) = (\mathfrak{gl}_m(k) \otimes R)^{\otimes n} \otimes_{k\Sigma_n} (sgn) = (\mathfrak{gl}_m(k)^{\otimes n} \otimes R^{\otimes n}) \otimes_{k\Sigma_n} (sgn).$$

To compute the coinvariants, we pull a rabbit out of the "hat" of classical invariant theory. The action of Σ_n on $V^{\otimes n}$ gives a homomorphism from $k\Sigma_n$ to $\mathrm{End}(V^{\otimes n}) = \mathrm{End}(V)^{\otimes n}$; the Lie algebra $\mathfrak{g}$ associated (7.1.2) to the associative algebra $\mathrm{End}(V)$ also acts on $V^{\otimes n}$ and the action of Σ_n is $\mathfrak{g}$-invariant, so the image of $k\Sigma_n$ belongs to the invariant submodule $(\mathrm{End}(V)^{\otimes n})^{\mathfrak{g}} = (\mathfrak{g}^{\otimes n})^{\mathfrak{g}}$. The classical invariant theory of [Weyl] asserts that $k\Sigma_n \cong (\mathfrak{g}^{\otimes n})^{\mathfrak{g}}$ whenever $\dim(V) \geq n$. If $\dim(V) = m$, then $\mathfrak{g} \cong \mathfrak{gl}_m(k) \cong k \times \mathfrak{sl}_m(k)$ and the abelian Lie algebra k acts trivially on $(\mathfrak{g}^{\otimes n})$. By Weyl's Theorem (7.8.11), $\mathfrak{g}^{\otimes n}$ is a direct sum of simple $\mathfrak{sl}_m(k)$-modules, so

$$k\Sigma_n \cong (\mathfrak{g}^{\otimes n})^{\mathfrak{sl}_m} \cong (\mathfrak{g}^{\otimes n})_{\mathfrak{sl}_m(k)}, \quad m \geq n.$$

Tensoring with the trivial $\mathfrak{g}$-module $R^{\otimes n}$ therefore yields (for $m \geq n$):

$$\begin{aligned} H_0(\mathfrak{sl}_m;\ \Lambda^n \mathfrak{gl}_m(k)) &= H_0(\mathfrak{sl}_m;\ (\mathfrak{gl}_m^{\otimes n} \otimes R^{\otimes n}) \otimes_{k\Sigma_n} (sgn)) \\ &= (H_0(\mathfrak{sl}_m;\ \mathfrak{gl}_m^{\otimes n}) \otimes R^{\otimes n}) \otimes_{k\Sigma_n} (sgn) \\ &= (k\Sigma_n \otimes R^{\otimes n}) \otimes_{k\Sigma_n} (sgn). \end{aligned}$$

The action of Σ_n on $k\Sigma_n$ in the final term is by conjugation.

Corollary 9.10.6 (Stabilization) *For every associative k-algebra R and every n the following stabilization homomorphisms are isomorphisms:*

$$H_n(\mathfrak{gl}_n(R); k) \cong H_n(\mathfrak{gl}_{n+1}(R); k) \cong \cdots \cong H_n(\mathfrak{gl}(R); k).$$

Proof The invariant theory calculation shows that the first $n+1$ terms (resp. n terms) of the chain complex $H_0(\mathfrak{sl}_m;\ \Lambda^* \mathfrak{gl}_m(R))$ are independent of m, as long as $m \geq n+1$ (resp. $m \geq n$). This yields a surjection $H_n(\mathfrak{gl}_n(R); k) \to H_n(\mathfrak{gl}_{n+1}(R); k)$ and stability for $m \geq n+1$. For the more subtle invariant theory needed to establish stability for $m = n$, we cite [Loday, 10.3.5]. $\diamond$

Remark 9.10.7 (Loday-Quillen) It is possible to describe the obstruction to improving the stability result to $m = n-1$. If R is commutative, we have a naturally split exact sequence

$$H_n(\mathfrak{gl}_{n-1}(R); k) \to H_n(\mathfrak{gl}_n(R); k) \xrightarrow{\lambda} \Omega^{n-1}_{R/k}/d\Omega^{n-2}_{R/k} \to 0.$$

The right-hand map is the composite of λ_*: $H_n(\mathfrak{gl}_n(R); k) \to HC_{n-1}(R)$, defined in 9.10.1, and the projection $HC_i(R) \to \Omega^i_{R/k}/d\Omega^{i-1}_{R/k}$ of 9.8.12. The proof of this assertion uses slightly more invariant theory and proposition 9.10.9 below; see [LQ, 6.9]. If R is not commutative, we only need to replace $\Omega^{n-1}_{R/k}/d\Omega^{n-2}_{R/k}$ by a suitable quotient of $\Lambda^n R$; see [Loday, 10.3.3 and 10.3.7] for details.

9.10.8 In order to state our next proposition, we need to introduce some standard facts about DG-coalgebras, expanding upon the discussion of graded coalgebras in 6.7.13 and 9.10.2.

If V is any vector space, the exterior algebra $\Lambda^*(V)$ is a graded coalgebra with counit ε: $\Lambda^*(V) \to \Lambda^*(0) = k$ induced by $V \to 0$ and coproduct

$$\Delta\colon \Lambda^*(V) \to \Lambda^*(V \times V) \cong (\Lambda^* V) \otimes (\Lambda^* V)$$

induced by the diagonal $V \to V \times V$. (Check this!) In particular, $\Lambda^* \mathfrak{g}$ is a graded coalgebra for every Lie algebra $\mathfrak{g}$. Since $\mathfrak{g} \to 0$ and $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ are

Lie algebra maps, $H_0(\mathfrak{h}; \Lambda^*\mathfrak{g})$ is a coalgebra for every Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. (Check this!) In particular, $H_0(\mathfrak{sl}_m(k); \Lambda^*\mathfrak{gl}_m(R))$ is a graded coalgebra for each m.

A *differential graded coalgebra* (or *DG-coalgebra*) C is a graded coalgebra endowed with a differential d making it into a chain complex in such a way that $\varepsilon: C_* \to k$ and $\Delta: C \to C \otimes C$ are morphisms of complexes. For example, $\Lambda^*\mathfrak{g}$ and $H_0(\mathfrak{sl}_m(k); \Lambda^*\mathfrak{gl}_m(R))$ are DG-coalgebras because ε and Δ arise from Lie algebra homomorphisms. By the Künneth formula 3.6.3, Δ induces a map

$$H_*(C) \to H_*(C \otimes C) \cong H_*(C) \otimes H_*(C),$$

making the homology of a DG-coalgebra C again into a graded coalgebra. Moreover, if $x \in C_n$ is primitive (9.10.2), then $dx \in C_{n-1}$ is primitive, because

$$\Delta(dx) = d\Delta(x) = d(x \otimes 1 + 1 \otimes x) = (dx) \otimes 1 + 1 \otimes (dx).$$

Therefore the graded submodule $\operatorname{Prim}(C)$ is a chain subcomplex of C.

Proposition 9.10.9 *The chain complex $L_* = H_0(\mathfrak{sl}(k); \Lambda^*\mathfrak{gl}(R))$ is a DG-coalgebra whose primitive part Prim(L_*) is the translate $C_{*-1}(R) = R^{\otimes *}/\sim$ of the chain complex for cyclic homology.*

Proof Recall from the discussion 9.10.5 on invariant theory that we have

$$L_n \cong (k\Sigma_n \otimes R^{\otimes n}) \otimes_{k\Sigma_n} (sgn).$$

This Σ_n-module splits into a direct sum of modules, one for each conjugacy class of elements of Σ_n. Let U_n be the conjugacy class of the cyclic permutation $\tau = (12\cdots n)$; we first prove that $\operatorname{Prim}(L_n)$ is $(kU_n \otimes R^n) \otimes_{k\Sigma_n} (sgn)$. If $\sigma \in \Sigma_n$ and $r_i \in R$, then consider the element $x = \sigma \otimes (r_1 \otimes \cdots \otimes r_n)$ of L_n. We have

$$\Delta(x) = \sum_{I,J} (\sigma_I \otimes (\cdots \otimes r_i \otimes \cdots)) \otimes (\sigma_J \otimes (\cdots \otimes r_j \otimes \cdots)),$$

where the sum is over all partitions (I, J) of $\{1, \cdots, n\}$ such that $\sigma(I) = I$ and $\sigma(J) = J$, and where σ_I (resp. σ_J) denotes the restriction of σ to I (resp. to J). (Check this!) By inspection, x is primitive if and only if σ admits no nontrivial partitions (I, J), that is, if and only if $\sigma \in U_n$.

Now Σ_n acts on U_n by conjugation, the stabilizer of τ being the cyclic group C_n generated by τ. Hence U_n is isomorphic to the coset space $\Sigma_n/C_n = \{C_n\sigma\}$

and $k[\Sigma_n/C_n] = k \otimes_{kC_n} k\Sigma_n$. From this we deduce the following sequence of isomorphisms:

$$\begin{aligned}\mathrm{Prim}(L_n) &\cong (kU_n \otimes R^{\otimes n}) \otimes_{k\Sigma_n} (sgn)\\ &\cong (k[\Sigma_n/C_n] \otimes R^{\otimes n}) \otimes_{k\Sigma_n} (sgn)\\ &\cong R^{\otimes n} \otimes_{kC_n} (sgn)\\ &\cong R^{\otimes n}/\sim,\end{aligned}$$

because $R^{\otimes n} \otimes_{kC_n} (sgn)$ is the quotient of $R^{\otimes n}$ by $1-(-1)^n\tau$. Note that this sequence of isomorphism sends the class of

$$\omega = e_{12}(r_1) \wedge e_{23}(r_2) \wedge \cdots \wedge e_{n1}(r_n) \in \Lambda^n \mathfrak{gl}_n(R)$$

to $(-1)^{n-1} r_1 \otimes \cdots \otimes r_n$. We leave it as an exercise for the reader to show that the class of $d\omega \in \Lambda^{n-1}\mathfrak{gl}_n(R)$ is sent to $b(r_1 \otimes \cdots \otimes r_n)$. This identifies the differential d on $\mathrm{Prim}(L_*)$ with the differential b of $R^{\otimes *}/\sim$ up to a sign. ◇

Theorem 9.10.10 (Loday-Quillen, Tsygan) *Let k be a field of characteristic zero and R an associative k-algebra. Then*

1. *The restriction of trace(λ) to primitive elements is an isomorphism*

$$\mathrm{Prim}\ H_n(\mathfrak{gl}(R); k) \cong HC_{n-1}(R).$$

2. *$H_*(\mathfrak{gl}(R); k)$ is a graded Hopf algebra, isomorphic to the tensor product*

$$\mathrm{Sym}\left(\bigoplus_{i=1}^{\infty} HC_{2i-1}(R)\right) \otimes_k \Lambda^*\left(\bigoplus_{i=0}^{\infty} HC_{2i}(R)\right).$$

Proof The direct sums $\oplus\colon \mathfrak{gl}_m(R) \times \mathfrak{gl}_n(R) \to \mathfrak{gl}_{m+n}(R)$ sending (x, y) to $x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ yield chain complex homomorphisms

$$\mu_{mn}\colon H_0(\mathfrak{sl}_m; \Lambda^*\mathfrak{gl}_m(R)) \otimes H_0(\mathfrak{sl}_n; \Lambda^*\mathfrak{gl}_n(R)) \to H_0(\mathfrak{sl}_{m+n}; \Lambda^*\mathfrak{gl}_{m+n}(R)).$$

Because we have taken coinvariants, which allow us to move the indices of $\mathfrak{gl}_{m+n}$ around inside $\mathfrak{gl}_{m+n+1}$, the maps μ_{mn}, $\mu_{m,n+1}$, and $\mu_{m+1,n}$ are compatible. Taking the limit as $m, n \to \infty$ yields an associative product μ on $L_* = H_0(\mathfrak{sl}; \Lambda^*\mathfrak{gl}(R))$. This makes L_* into a DG-algebra as well as a

DG-coalgebra. In fact L_* is a graded Hopf algebra (6.7.15) because the formula $(x, x) \oplus (y, y) \sim (x \oplus y, x \oplus y)$ in $\mathfrak{gl}_{m+n}(R) \times \mathfrak{gl}_{m+n}(R)$ shows that $\Delta\colon L_* \to L_* \otimes L_*$ is an algebra map. It follows that $H_*(\mathfrak{gl}(R); k) = H_*(L_*)$ is also a Hopf algebra.

The classification of graded-commutative Hopf algebras H_* over a field k of characteristic zero is known [MM]. If $H_0 = k$, then $H_* = \operatorname{Sym}(P_e) \otimes \Lambda^*(P_o)$, where P_e (resp. P_o) is the sum of the $\operatorname{Prim}(H_i)$ with i even (resp. i odd). Thus (1) implies (2). Applying this classification to L_*, a simple calculation (exercise!) shows that Prim $H_n(L_*) \cong H_n\operatorname{Prim}(L_*)$. But $H_n\operatorname{Prim}(L_*) = HC_{n-1}(R)$ by Proposition 9.10.9. $\diamond$

Exercise 9.10.11 (Bloch, Kassel-Loday) Use the Hochschild-Serre spectral sequence (7.5.2) for $\mathfrak{sl} \subset \mathfrak{gl}$ to show that $H_2(\mathfrak{sl}_2(R); k) \cong HC_1(R)$.