8

Simplicial Methods in Homological Algebra

By now, the reader has seen several examples of chain complexes in which the boundary maps $C_n \to C_{n-1}$ are alternating sums $d_0 - d_1 + \cdots \pm d_n$. The primordial example is the singular chain complex of a topological space X; elements of $C_n(X)$ are formal sums of maps f from the n-simplex Δ_n into X, and $d_i(f)$ is the composition of f with the inclusion $\Delta_{n-1} \subset \Delta_n$ of the i^{th} face of the simplex (1.1.4). Other examples of this phenomenon include Koszul complexes (4.5.1), the bar resolution of a group (6.5.1), and the Chevalley-Eilenberg complex of a Lie algebra (7.7.1). Complexes of this form arise from simplicial modules, which are the subject of this chapter.

8.1 Simplicial Objects

Let Δ be the category whose objects are the finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for integers $n \geq 0$, and whose morphisms are nondecreasing monotone functions. If $\mathcal{A}$ is any category, a *simplicial object* A in $\mathcal{A}$ is a contravariant functor from Δ to $\mathcal{A}$, that is, $A\colon \Delta^{\text{op}} \to \mathcal{A}$. For simplicity, we write A_n for $A([n])$. Similarly, a *cosimplicial object* C in $\mathcal{A}$ is a covariant functor $C\colon \Delta \to \mathcal{A}$, and we write A^n for $A([n])$. A morphism of simplicial objects is a natural transformation, and the category $\mathcal{SA}$ of all simplicial objects in $\mathcal{A}$ is just the functor category $\mathcal{A}^{\Delta^{\text{op}}}$.

Example 8.1.1 (Constant simplicial objects) Let A be a fixed object of $\mathcal{A}$. The constant functor $\Delta \to \mathcal{A}$ sending every object to A is called the *constant simplicial object* in $\mathcal{A}$ at A. We have $A_n = A$ for all n, and $\alpha^* =$ identity morphism for every α in Δ.

We want to give a more combinational description of simplicial (and cosimplicial) objects, and for this we need to study the simplicial category Δ directly. The reader interested in more details about simplicial sets may want to read [May].

It is easy to see that for each n there are $n+1$ maps $[0] \to [n]$ but only one map $[n] \to [0]$. There are $\binom{n+2}{2}$ maps $[1] \to [n]$ and more generally $\binom{n+i+1}{i+1}$ maps $[i] \to [n]$ in Δ. In order to make sense out of this chaos, it is useful to introduce the *face maps* ε_i and *degeneracy maps* η_i. For each n and $i = 0, \cdots, n$ the map $\varepsilon_i : [n-1] \to [n]$ is the unique injective map in Δ whose image misses i and the map $\eta_i : [n+1] \to [n]$ is the unique surjective map in Δ with two elements mapping to i. Combinationally, this means that

$$\varepsilon_i(j) = \begin{Bmatrix} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{Bmatrix}, \quad \eta_i(j) = \begin{Bmatrix} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{Bmatrix}.$$

Exercise 8.1.1 Verify the following identities in Δ:

$$\varepsilon_j \varepsilon_i = \varepsilon_i \varepsilon_{j-1} \quad \text{if } i < j$$

$$\eta_j \eta_i = \eta_i \eta_{j+1} \quad \text{if } i \leq j$$

$$\eta_j \varepsilon_i = \begin{cases} \varepsilon_i \eta_{j-1} & \text{if } i < j \\ \text{identity} & \text{if } i = j \text{ or } i = j+1 \\ \varepsilon_{i-1} \eta_j & \text{if } i > j+1. \end{cases}$$

Lemma 8.1.2 *Every morphism $\alpha: [n] \to [m]$ in Δ has a unique epi-monic factorization $\alpha = \varepsilon\eta$, where the monic ε is uniquely a composition of face maps*

$$\varepsilon = \varepsilon_{i_1} \cdots \varepsilon_{i_s} \quad \textit{with} \quad 0 \leq i_s \leq \cdots \leq i_1 \leq m$$

and the epi η is uniquely a composition of degeneracy maps

$$\eta = \eta_{j_1} \cdots \eta_{j_t} \quad \textit{with} \quad 0 \leq j_1 < \cdots < j_t < n.$$

Proof Let $i_s < \cdots < i_1$ be the elements of $[m]$ not in the image of α and $j_1 < \cdots < j_t$ be the elements of $[n]$ such that $\alpha(j) = \alpha(j+1)$. Then if $p = n - t = m - s$, the map α factors as

$$[n] \xrightarrow{\eta} [p] \xhookrightarrow{\varepsilon} [m].$$

The rest of the proof is straightforward. (Check this!) ◇

Proposition 8.1.3 *To give a simplicial object A in $\mathcal{A}$, it is necessary and sufficient to give a sequence of objects $A_0, A_1, \cdots$ together with* face operators $\partial_i\colon A_n \to A_{n-1}$ *and* degeneracy operators $\sigma_i\colon A_n \to A_{n+1}$ $(i = 0, 1, \cdots, n)$, *which satisfy the following "simplicial" identities*

$$\partial_i\partial_j = \partial_{j-1}\partial_i \quad \text{if } i < j$$

$$\sigma_i\sigma_j = \sigma_{j+1}\sigma_i \quad \text{if } i \le j$$

$$\partial_i\sigma_j = \begin{cases} \sigma_{j-1}\partial_i & \text{if } i < j \\ \text{identify} & \text{if } i = j \text{ or } i = j+1 \\ \sigma_j\partial_{i-1} & \text{if } i > j+1. \end{cases}$$

Under this correspondence $\partial_i = A(\varepsilon_i)$ and $\sigma_i = A(\eta_i)$.

Proof If A is simplicial, we obtain the above data by setting $A_n = A([n])$ and considering only faces and degeneracies. Conversely, given the data and a map in Δ written in the standard form $\alpha = \varepsilon_{i_1} \cdots \eta_{j_t}$ of the lemma, we set $A(\alpha) = \sigma_{j_t} \cdots \partial_{i_1}$. Since the simplicial identities control composition in Δ, this makes A into a contravariant functor, that is, a simplicial object. $\diamond$

If we dualize the above discussion, we get cosimplicial objects. Recall that a cosimplicial object is a covariant functor $A\colon \Delta \to \mathcal{A}$.

Corollary 8.1.4 *To give a cosimplicial object A in $\mathcal{A}$, it is necessary and sufficient to give a sequence of objects $A^0, A^1, \cdots$ together with* coface operators $\partial^i\colon A^{n-1} \to A^n$ *and* codegeneracy operators $\sigma^i\colon A^{n+1} \to A^n$ $(i = 0, \cdots, n)$ *which satisfy the "cosimplicial" identities*

$$\partial^j\partial^i = \partial^i\partial^{j-i} \quad \text{if } i < j$$

$$\sigma^j\sigma^i = \sigma^i\sigma^{j+1} \quad \text{if } i \le j$$

$$\sigma^j\partial^i = \begin{cases} \partial^i\sigma^{j-1} & \text{if } i < j \\ \text{identity} & \text{if } i = j \text{ or } i = j+1 \\ \partial^{i-1}\sigma^j & \text{if } i > j+1. \end{cases}$$

Example 8.1.5 (Simplices) The geometric n-simplex Δ^n is the subspace of $\mathbb{R}^{n+1}$

$$\Delta^n = \{(t_0, \cdots, t_n)\colon 0 \le t_i \le 1, \sum t_i = 1\}.$$

If we identify the elements of $[n]$ with the vertices $v_0 = (1, 0, \cdots, 0), \cdots,$ $v_n = (0, \cdots, 0, 1)$ of Δ^n, then a map $\alpha\colon [n] \to [p]$ in Δ sends the vertices of

Δ^n to the vertices of Δ^p by the rule $\alpha(v_i) = v_{\alpha(i)}$. Extending linearly gives a map $\alpha_*\colon \Delta^n \to \Delta^p$ and makes the sequence $\Delta^0, \Delta^1, \cdots, \Delta^n, \cdots$ into a cosimplicial topological space. Geometrically, the face map ε_i induces the inclusion of Δ^{n-1} into Δ^n as the i^{th} face (the face opposite the vertex v_i), and the degeneracy map η_i induces the projection $\Delta^{n+1} \to \Delta^n$ onto the i^{th} face that identifies v_i and v_{i+1}. This geometric interpretation provided the historical origins of the terms face and degeneracy operators.

Geometric Realization 8.1.6 If X is a simplicial set, its *geometric realization* $|X|$ is a topological space constructed as follows. For each $n \geq 0$, topologize the product $X_n \times \Delta^n$ as the disjoint union of copies of the n-simplex Δ^n indexed by the elements x of X_n. On the disjoint union $\coprod X_n \times \Delta^n$, define the equivalence relation $\sim$ by declaring that $(x, s) \in X_m \times \Delta^m$ and $(y, t) \in X_n \times \Delta^n$ are equivalent if there is a map $\alpha\colon [m] \to [n]$ in Δ such that $\alpha^*(y) = x$ and $\alpha_*(s) = t$. That is,

$$(\alpha^*(y), s) \sim (y, \alpha_*(t)).$$

The identification space $\coprod(X_n \times \Delta^n)/\sim$ is the geometric realization $|X|$. It is easy to see that in forming $|X|$ we can ignore every n-simplex of the form $\sigma_i(y) \times \Delta^n$, so we say that the elements $\sigma_i(y)$ are *degenerate*. An element $x \in X_n$ is called *non-degenerate* if it is not of the form $\sigma_i(y)$ for some $i < n$ and $y \in X_{n-1}$; the nondegenerate elements of X_n index the n-cells of $|X|$, which implies that $|X|$ is a "CW complex." A more detailed discussion of the geometric realization may be found in [May].

Example 8.1.7 (Classifying space) Let G be a group and consider the simplicial set BG defined by $BG_0 = \{1\}$, $BG_1 = G, \cdots, BG_n = G^n, \cdots$. The face and degeneracy maps are defined by insertion, deletion, and multiplication:

$$\sigma_i(g_1, \cdots, g_n) = (g_1, \cdots, g_i, 1, g_{i+1}, \cdots, g_n)$$

$$\partial_i(g_1, \cdots, g_n) = \begin{cases} (g_2, \ldots, g_n) & \text{if } i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } 0 < i < n \\ (g_1, \ldots, g_{n-1}) & \text{if } i = n. \end{cases}$$

The geometric realization $|BG|$ of the simplicial set BG is called the *classifying space* of G. The name comes from the theory of fiber bundles; if X is a finite cell complex then the set $[X, |BG|]$ of homotopy classes of maps $X \to |BG|$ gives a complete classification of fiber bundles over X with structure group G. We will see in 8.2.3 and 8.3.3 that $|BG|$ is an Eilenberg-MacLane space whose homology is the same as the group homology $H_*(G)$ of Chapter 6. Thus we recover definition 6.10.4 as well as 6.10.5.

Example 8.1.8 (Simplicial complexes) A (combinational) *simplicial complex* is a collection K of nonempty finite subsets of some vertex set V such that if $\emptyset \neq \tau \subset \sigma \subset V$ and $\sigma \in K$ then $\tau \in K$. If the vertex set is ordered, we call K an *ordered* simplicial complex. To every such ordered simplicial complex we associate a simplicial set $SS(K)$ as follows. Let $SS_n(K)$ consist of all ordered $(n+1)$-tuples $(v_0, \cdots, v_n)$ of vertices, possibly including repetition, such that the underlying set $\{v_0, \cdots, v_n\}$ is in K. If $\alpha\colon [n] \to [p]$ is a map in Δ, define $\alpha_*\colon SS_p(K) \to SS_n(K)$ by $\alpha_*(v_0, \cdots, v_p) = (v_{\alpha(0)}, \cdots, v_{\alpha(n)})$. Note that $v_0 \leq \cdots \leq v_n$ and that

$$\partial_i(v_0, \cdots, v_n) = (v_0, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n)$$

$$\sigma^i(v_0, \cdots, v_n) = (v_0, \cdots, v_i, v_i, \cdots, v_n).$$

The following exercises explain how combinatorial simplicial complexes correspond to triangulated polyhedra. Clearly a triangulated polyhedron P gives rise to a combinatorial simplicial simplex K whose elements correspond to the faces of P, the vertices of P forming the vertex set V of K (see 1.1.3).

Exercise 8.1.2 Show that if K is an ordered combinatorial simplicial complex, then $SS(K)$ determines K, because there is a bijection between K and the subset of $SS(K)$ consisting of non-degenerate elements.

Exercise 8.1.3 Let K be the collection of all nonempty subsets of a vertex set V having $n+1$ elements. (K is the combinational simplicial complex arising from the polyhedron Δ^n.) Show that the geometric realization $|SS(K)|$ is homeomorphic to the geometric n-simplex Δ^n.

Exercise 8.1.4 (Geometric simplicial complexes) If K is a combinatorial simplicial complex (8.1.8), let $|K|$ denote the geometric realization $|SS(K)|$ of the simplicial set $SS(K)$ associated to some ordering of K. Show that $|K|$ is a triangulated polyhedron with one face e_σ for each $\sigma \in K$. (If σ has $n+1$ elements, then e_σ is homeomorphic to an n-simplex.) Therefore K is the combinational simplicial complex arising from $|K|$. The polyhedron $|K|$ is sometimes called the *geometric simplicial complex* associated to K.

Definition 8.1.9 (Semisimplicial objects) Let Δ_s denote the subcategory of Δ whose morphisms are the *injections* $\varepsilon\colon [i] \hookrightarrow [n]$. A *semi-simplicial* object K in a category $\mathcal{A}$ is a contravariant functor from Δ_s to $\mathcal{A}$.

For example, an ordered combinational simplicial complex K yields a semi-simplicial set with $K_n = \{\tau \in K\colon \tau \text{ has } n+1 \text{ elements}\}$. Every simplicial set becomes a semi-simplicial set by forgetting the degeneracies, but the degeneracies provide a richer combinatorial structure.

The forgetful functor from simplicial objects to semi-simplicial objects has a left adjoint L when $\mathcal{A}$ has finite coproducts; $(LK)_n$ is the coproduct $\coprod_{p\le n}\coprod_\eta K_p[\eta]$, where for each $p \le n$ the index η runs over all the surjections $[n] \to [p]$ in Δ and $K_p[\eta]$ denotes a copy of K_p. The maps defining the simplicial structure on LK are given in the following tedious exercise 8.1.5; LK is called the *left Kan extension* of K along $\Delta_s \subset \Delta$ in [MacCW, X.3]. When $\mathcal{A}$ is abelian we will give an alternate description of LK in exercise 8.4.3.

Exercise 8.1.5 (Left Kan extension) If $\alpha: [m] \to [n]$ is any morphism in Δ, define $LK(\alpha): LK_n \to LK_m$ by defining its restrictions to $K_p[\eta]$ for each surjection η as follows. Find the epi-monic factorization $\varepsilon\eta'$ of $\eta\alpha$ with $\eta': [m] \to [q]$ and $\varepsilon: [q] \to [n]$; the restriction of $LK(\alpha)$ to $K_p[\eta]$ is defined to be the map $K(\varepsilon)$ from K_p to the factor $K_q[\eta']$ of the coproduct $(LK)_m$. Show that these maps make LK into a simplicial object of $\mathcal{A}$.

Exercise 8.1.6 Show that a semi-simplicial object K is the same thing as a sequence of objects $K_0, K_1, \cdots$ together with *face operators* $\partial_i: K_n \to K_{n-1}$ $(i = 0, \cdots, n)$ such that if $i < j$ then $\partial_i\partial_j = \partial_{j-1}\partial_i$.

$$K_0 \overset{\partial_0}{\underset{\partial_1}{\leftleftarrows}} K_1 \overset{\partial_0}{\underset{\partial_2}{\Lleftarrow}} K_2 \;\Lleftarrow\; K_3 \cdots .$$

Historical Remark 8.1.10 Simplicial sets first arose in Eilenberg and Zilber's 1950 study [EZ] under the name "complete semi-simplicial sets" (c.s.s.). For them, semi-simplicial sets (defined as above) were more natural, and the adjective "complete" reflected the addition of degeneracies. By 1954, this adjective was often dropped, and "semi-simplicial set" was a common term for a c.s.s. By the late 1960s even the prefix "semi" was deleted, influenced by the book [May], and "simplicial set" is now universally used for c.s.s. In view of modern usage, we have decided to retain the original use of "semi-simplicial" in definition 8.1.9.

8.2 Operations on Simplicial Objects

Definition 8.2.1 Let A be a simplicial (or semi-simplicial) object in an abelian category $\mathcal{A}$. The *associated*, or *unnormalized, chain complex* $C =$

$C(A)$ has $C_n = A_n$, and its boundary morphism $d: C_n \to C_{n-1}$ is the alternating sum of the face operators $\partial_i: C_n \to C_{n-1}$:

$$d = \partial_0 - \partial_1 + \cdots + (-1)^n \partial_n.$$

The (semi-) simplicial identities for $\partial_i \partial_j$ imply that $d^2 = 0$. (Check this!)

Example 8.2.2 (Koszul complexes) Let $\boldsymbol{x} = (x_1, \cdots, x_m)$ be a sequence of central elements in a ring R. Then the sequence $R^m, \Lambda^2 R^m, \cdots, \Lambda^{n+1} R^m, \cdots$ of exterior products of R^m forms a semi-simplicial R-module with

$$\partial_i (e_{\alpha_0} \wedge \cdots \wedge e_{\alpha_n}) = x_{\alpha_i} e_{\alpha_0} \wedge \cdots \wedge \hat{e}_{\alpha_i} \wedge \cdots \wedge e_{\alpha_n}.$$

The Koszul complex $K(\boldsymbol{x})$ of 4.5.1 is obtained by augmenting the chain complex associated to the semi-simplicial module $\{\Lambda^{n+1} R^m\}$. If R is a k-algebra, this defines an action of the abelian Lie algebra $\mathfrak{g} = k^m$ on R, and $K(\boldsymbol{x})$ coincides with the Chevalley-Eilenberg complex 7.7.1 used to compute $H_*(\mathfrak{g}, R)$.

An extremely useful observation is that if we apply a functor $F: \mathcal{A} \to \mathcal{B}$ to a simplicial object A in $\mathcal{A}$, we obtain a simplicial object in $\mathcal{B}$. Similar remarks apply to semisimplicial and cosimplicial objects.

Example 8.2.3 (Simplicial homology) If R is a ring, the free module $R[X]$ on a set X is a functor **Sets** $\to R$–**mod**. Whenever $X = \{X_n\}$ is a (semi-) simplicial set, $R[X] = \{R[X_n]\}$ is a (semi-) simplicial R-module. The chain complex associated to $R[X]$ is the chain complex used to form the simplicial homology of the cellular complex $|X|$ with coefficients in R. (See 1.1.3.)

Motivated by this example, we define the *simplicial homology* $H_*(X; R)$ of any simplicial set X to be the homology of the chain complex associated to the simplicial module $R[X]$. Thus $H_*(X; R) = H_*(|X|; R)$.

For example, consider the classifying space $|BG|$ of a group G (8.1.7). The chain complex associated to $R[BG]$ is the canonical chain complex used in 6.5.4 to compute the group homology $H_*(G; R)$ of G with coefficients in the trivial G-module R. This yields the formula

$$H_*(G; R) \cong H_*(BG; R) = H_*(|BG|; R).$$

Example 8.2.4 (Singular chain complex) Let X be a topological space. Applying the contravariant functor $\mathrm{Hom}_{\mathbf{Top}}(-, X)$ to the cosimplicial space $\{\Delta^n\}$ gives a simplicial set $S(X)$ with $S_n(X) = \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X)$, called the *singular simplicial set* of X. The singular chain complex of X used to compute the singular homology of X with coefficients in R (1.1.4) is exactly the chain complex associated to the simplicial R-module $R[S(X)]$.

Remark There is a natural continuous map $|S(X)| \to X$, which is a homotopy equivalence if (and only if) X has the homotopy type of a CW complex. It is induced from the maps $S_n(X) \times \Delta^n \to X$ sending (f, t) to $f(t)$. In fact, S is the right adjoint to geometric realization: for every simplicial set K, $\text{Hom}_{\mathbf{Top}}(|K|, X) \cong \text{Hom}_{\mathcal{S}\mathbf{Sets}}(K, S(X))$. These assertions are proven in [May, section 16].

Example 8.2.5 For each $n \geq 0$ a simplicial set $\Delta[n]$ is given by the functor $\text{Hom}_\Delta(-, [n])$. These are universal in the following sense. For each simplicial set A, the Yoneda Embedding 1.6.10 gives a 1–1 correspondence between elements $a \in A_n$ and simplicial morphisms $f\colon \Delta[n] \to A$; f determines the element $a_f = f(\text{id}_{[n]})$ and conversely f_a is defined on $\lambda \in \text{Hom}_\Delta([m], [n])$ by $f_a(\lambda) = \lambda^*(a) \in A_m$.

Exercise 8.2.1 Show that $\Delta[n]$ is the simplicial set $SS(\Delta^n)$ associated (8.1.8) to the combinatorial simplicial complex underlying the geometric n-simplex Δ^n.

Cartesian Products 8.2.6 The cartesian product $A \times B$ of two simplicial objects A and B is defined as $(A \times B)_n = A_n \times B_n$ with face and degeneracy operators defined diagonally:

$$\partial_i(a, b) = (\partial_i a, \partial_i b) \quad \text{and} \quad \sigma_i(a, b) = (\sigma_i a, \sigma_i b).$$

If B is a simplicial set and A is a simplicial object in a category $\mathcal{A}$ having products, then we can also make sense out of $A \times B$ by defining $A_n \times B_n$ to be the product of B_n copies of A_n. This construction is most interesting when each B_n is finite, in which case $\mathcal{A}$ need only have finite products.

Exercise 8.2.2 If K and L are combinorial simplicial complexes (8.1.8), there is a combinational simplicial complex P with $|P| = |K| \times |L|$ as polyhedra, defined by $SS(P) = SS(K) \times SS(L)$; see [May, 14.3] or [EZ]. Verify this assertion by finding combinational simplicial complexes underlying the square $\Delta^1 \times \Delta^1$ and the prism $\Delta^2 \times \Delta^1$ whose associated simplicial sets are $\Delta[1] \times \Delta[1]$ and $\Delta[2] \times \Delta[1]$.

Fibrant Simplicial Sets 8.2.7 From the standpoint of homotopy theory, it is technically useful to restrict one's attention to those simplicial sets X that satisfy the following *Kan condition*:

For every n and k with $0 \le k \le n+1$, if $x_0, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n+1} \in X_n$ are such that $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j$ (i and j not equal to k), then there exists a $y \in X_{n+1}$ such that $\partial_i(y) = x_i$ for all $i \neq k$.

We call such simplicial sets *fibrant*; they are sometimes called *Kan complexes* after D. Kan, who first isolated this condition in 1955 and observed that the singular simplicial set $S(X)$ of a topological space X (8.2.4) is always fibrant. The class of fibrant simplicial sets includes all simplicial groups and all simplicial abelian groups by the following calculation.

Lemma 8.2.8 *If G is a simplicial group (a simplicial object in the category of groups), then the underlying simplicial set is fibrant. A fortiori every simplicial abelian group, and every simplicial R-module, is fibrant when considered as a simplicial set.*

Proof Suppose given $x_i \in G_n$ ($i \neq k$) such that $\partial_i x_j = \partial_{j-1} x_i$ for $i < j$. We use induction on r to find $g_r \in G_{n+1}$ such that $\partial_i(g_r) = x_i$ for all $i \le r$, $i \neq k$. We begin the induction by setting $g_{-1} = 1 \in G_{n+1}$ and suppose inductively that $g = g_{r-1}$ is given. If $r = k$, we set $g_r = g$. If $r \neq k$, we consider $u = x_r^{-1}(\partial_r g)$. If $i < r$ and $i \neq k$, then $\partial_i(u) = 1$ and hence $\partial_i(\sigma_r u) = 1$. Hence $g_r = g(\sigma_k u)^{-1}$ satisfies the inductive hypothesis. The element $y = g_n$ therefore has $\partial_i(y) = x_i$ for all $i \neq k$, so the Kan condition is satisfied. ◇

Exercise 8.2.3 Show that $\Delta[n]$ is not fibrant if $n \neq 0$. Then show that any fibrant simplicial set X is either constant (8.1.1) or has a non-degenerate "n-cell" $x \in X_n$ for every n (8.1.6).

Exercise 8.2.4 Show that BG is fibrant for every group G but that BG is a simplicial group if and only if G is abelian.

Fibrations 8.2.9 A map $\pi\colon E \to B$ of simplicial sets is called a (Kan) *fibration* if

for every n, $b \in B_{n+1}$ and $k \le n+1$, if $x_0, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n+1} \in E_n$ are such that $\partial_i b = \pi(x_i)$ and $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j$ ($i, j \neq k$), then there exists a $y \in E_{n+1}$ such that $\pi(y) = b$ and $\partial_i(y)$ for all $i \neq k$.

This notion generalizes that of a fibrant simplicial set X, which is after all just a simplicial set such that $X \to *$ is a fibration. The following two exercises give some important examples of fibrations.

Exercise 8.2.5 Show that every surjection $E \to B$ of simplicial groups is a fibration.

Exercise 8.2.6 (Principal G-fibrations) We say that a group G acts on a simplicial set X (or X is a *simplicial G-set*) if G acts on each X_n, and the action commutes with the face and degeneracy operators. The orbit spaces X_n/G fit together to form a simplicial set X/G; if G acts freely on X ($gx \neq x$ for every $g \neq 1$ and every x) we say that $X \to X/G$ is a *principal G-fibration*. Show that every principal G-fibration is a fibration.

Front-to-Back Duality 8.2.10 Simplicial constructions (e.g., homotopy in 8.3.11) always have a "front-to-back" dual formulation. Consider the involution $\check{}$ on Δ, which fixes every object $[n]$; it is defined on the morphisms in Δ by

$$\check{\partial_i} = \partial_{n-i} \colon [n-1] \to [n] \quad \text{and} \quad \check{\sigma_i} = \sigma_{n-i} \colon [n+1] \to [n].$$

We may think of it as reversing the ordering of $[n] = (0 < 1 < \cdots < n)$ to get the ordering $(n < \cdots < 1 < 0)$. That is, if $\alpha\colon [m] \to [n]$ then $\check{\alpha}(i) = n - \alpha(m-i)$. If A is a simplicial object in $\mathcal{A}$, then its *front-to-back dual* $\check{A}$ is the composition of A with this involution.

8.3 Simplicial Homotopy Groups

Given a fibrant simplicial set X (8.2.7) and a basepoint $* \in X_0$, we define $\pi_n(X)$ as follows. By abuse of notation, we write $*$ for the element $\sigma_0^n(*)$ of X_n and set $Z_n = \{x \in X_n : \partial_i(x) = * \text{ for all } i = 0, \cdots, n\}$. We say that two elements x and x' of Z_n are *homotopic*, and write $x \sim x'$, if there is a $y \in X_{n+1}$ (called a *homotopy* from x to x') such that

$$\partial_i(y) = \begin{cases} * & \text{if } i < n \\ x & \text{if } i = n \\ x' & \text{if } i = n+1. \end{cases}$$

Lemma/Definition 8.3.1 *If X is a fibrant simplicial set, then $\sim$ is an equivalence relation, and we set $\pi_n(X) = Z_n/\sim$.*

Proof The relation is reflexive since $y = (\sigma_n x)$ is a homotopy from x to itself. To see that $\sim$ is symmetric and transitive, suppose given homotopies y' and y'' from x to x' and from x to x''. The Kan condition 8.2.7 applied to the elements $*, \cdots, *, y', y''$ of X_{n+1} with $k = n+2$ yields an element $z \in X_{n+2}$ with $\partial_n z = y'$, $\partial_{n+1} z = y''$ and $\partial_i z = *$ for $i < n$. The element $y = \partial_{n+2} z$ is a homotopy from x' to x''. (Check this!) Therefore $x' \sim x''$. ◇

Remark If X is a fibrant simplicial set, $\pi_n(X)$ agrees with the topological homotopy group $\pi_n(|X|)$; see [May, 16.1]. Since $\pi_n(|X|) \cong \pi_n(|S(X)|)$, one usually defines $\pi_n(X)$ as $\pi_n S(X)$ when X is not fibrant. Thus $\pi_1(X)$ is a group, and $\pi_n(X)$ is an abelian group for $n \geq 2$.

Example 8.3.2 $\pi_0(X) = X_0/\sim$, where for each $y \in X_1$ we declare $\partial_0(y) \sim \partial_1(y)$.

Example 8.3.3 (Classifying space) Consider the classifying space BG of a group G. By inspection $Z_n = \{1\}$ for $n \neq 1$ and $Z_1 = G$. From this we deduce that

$$\pi_n(|BG|) = \pi_n(BG) = \begin{cases} G & \text{if } n = 1 \\ 1 & \text{if } n \neq 1. \end{cases}$$

Definition 8.3.4 If G is a group, then an *Eilenberg-MacLane space* of type $K(G, n)$ is a fibrant simplicial set K such that $\pi_n K = G$ and $\pi_i K = 0$ for $i \neq n$. Note that G must be abelian if $n \geq 2$. The previous example shows that BG is an Eilenberg-MacLane space of type $K(G, 1)$. In the next section (exercise 8.4.4), we will construct Eilenberg-MacLane spaces of type $K(G, n)$ for $n \geq 2$ as an application of the Dold-Kan correspondence 8.4.1. The term "space," rather than "simplicial set," is used for historical reasons as well as to avoid a nine-syllable name.

Exercise 8.3.1 If G is a simplicial group (or simplicial module), considered as a fibrant simplicial set, show that any two choices of basepoint lead to naturally isomorphic $\pi_n(G)$. *Hint:* G_0 acts on G.

If G is a simplicial group (or simplicial module), considered (by 8.2.8) as a fibrant simplicial set with basepoint $* = 1$, it is helpful to consider the subgroups

$$N_n(G) = \{x \in G_n : \partial_i x = 1 \quad \text{for all} \quad i \neq n\}.$$

Then $Z_n = \ker(\partial_n\colon N_n \to N_{n-1})$ and the image of the homomorphism $\partial_{n+1}\colon N_{n+1} \to N_n$ is $B_n = \{x : x \sim 1\}$. Hence $\pi_n(G)$ is the homology group Z_n/B_n of the (not necessarily abelian) chain complex N_*

$$1 \leftarrow N_0 \xleftarrow{\partial_1} N_1 \xleftarrow{\partial_2} N_2 \leftarrow \cdots.$$

Exercise 8.3.2 Show that B_n is a normal subgroup of Z_n, so that $\pi_n(G)$ is a group for all $n \geq 0$. Then show that $\pi_n(G)$ is abelian for $n \geq 1$. *Hint:* Consider $(\sigma_{n-1}x)(\sigma_n y)$ and $(\sigma_n x)(\sigma_{n-1}y)$ for $x, y \in G_n$.

Exercise 8.3.3 If $G \to G''$ is a surjection of simplicial groups with kernel G', show that there is a short exact sequence of (not necessarily abelian) chain complexes $1 \to NG' \to NG \to NG'' \to 1$. By modifying the discussion in Chapter 1, section 3 show that there is a natural connecting homomorphism $\partial\colon \pi_n G'' \to \pi_{n-1}G'$ fitting into a long exact sequence

$$\cdots \pi_{n+1}G'' \xrightarrow{\partial} \pi_n G' \to \pi_n G \to \pi_n G'' \xrightarrow{\partial} \pi_{n-1}G' \cdots .$$

Remark 8.3.5 More generally, suppose that $\pi\colon E \to B$ is a fibration with E and B fibrant. Suppose given basepoints $*_E \in E_0$ and $*_B = \pi(*_E) \in B_0$; the fibers $F_n = \pi^{-1}(\sigma_0^n(*))$ form a fibrant simplicial subset F of E. Given $b \in B_n$ with $\partial_i(b) = *$ for all i, the fibration condition yields $e \in E_n$ with $\pi(e) = b$ and $\partial_i(e) = *$ for all $i < n$. The equivalence class of $\partial_n(e)$ in $\pi_{n-1}(F)$ is independent of the choices of e and induces a map $\partial_n\colon \pi_n(B) \to \pi_{n-1}(F)$ fitting into a long "exact" sequence of homotopy "groups":

$$\cdots \pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F) \to \pi_n(E) \xrightarrow{\pi} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \cdots .$$

For more details, see [May].

This remark and exercise 8.3.3 show that the homotopy groups π_* form a (nonabelian) homological δ-functor. This observation forms the basis for the subject of nonabelian homological algebra. We shall not pursue this subject much, referring the reader to [DP] and [Swan 1]. Instead we use it as a model to generalize the definition of homology to any abelian category $\mathcal{A}$, even if the objects of $\mathcal{A}$ have no underlying set structure.

Definition 8.3.6 (Homotopy groups) Suppose that A is a simplicial object in an abelian category $\mathcal{A}$. The *normalized*, or *Moore*, chain complex $N(A)$ is the chain complex with

$$N_n(A) = \bigcap_{i=0}^{n-1} \ker(\partial_i\colon A_n \to A_{n-1})$$

and differential $d = (-1)^n \partial_n$. By construction, $N(A)$ is a chain subcomplex of the unnormalized complex $C(A)$ and we define

$$\pi_n(A) = H_n(N(A)).$$

If $\mathcal{A}$ is the category of abelian groups or R-modules, this recovers the definition 8.3.1 of $\pi_n(A)$ obtained by regarding A as a fibrant simplicial set and taking homotopy.

Exercise 8.3.4 Show that $N(A)$ is naturally isomorphic to its front-to-back dual $\check{N}(A) = N(\check{A})$, which has $\check{N}_n(A) = \{x \in A_n : \partial_i x = 0 \text{ for all } i \neq 0\}$ and differential ∂_0. (See 8.2.10.)

Now let $D(A)$ denote the "degenerate" chain subcomplex of $C(A)$ generated by the images of the degeneracies σ_i, so that $D_n(A) = \sum \sigma_i(C_{n-1}A)$.

Lemma 8.3.7 $C(A) = N(A) \oplus D(A)$. Hence $N(A) \cong C(A)/D(A)$.

Proof We will use an element-theoretic proof, which is valid by the Freyd-Mitchell Embedding Theorem 1.6.1. An element of $D_n(A)$ is a sum $y = \sum \sigma_j(x_j)$ with $x_i \in C_{n-1}(A)$. If $y \in N_n(A)$ and i is the smallest integer such that $\sigma_i(x_i) \neq 0$, then $\partial_i(y) = x_i$, which is a contradiction. Hence $D_n \cap N_n = 0$. To see that $D_n + N_n = C_n$, we pick $y \in C_n$ and use downward induction on the smallest integer j such that $\partial_j(y) \neq 0$. The element y is congruent modulo D_n to $y' = y - \sigma_j \partial_j(y)$, and for $i < j$ the simplicial identities yield

$$\partial_i(y') = \partial_i(y) - \sigma_{j-1}\partial_{j-1}\partial_i(y) = 0.$$

Since $\partial_j(y') = 0$ as well, y' is congruent modulo D_n to an element of N_n by induction, and hence $D_n + N_n = C_n$. ◇

Theorem 8.3.8 *In any abelian category $\mathcal{A}$, the homotopy $\pi_*(A)$ of a simplicial object A is naturally isomorphic to the homology $H_*(C)$ of the unnormalized chain complex $C = C(A)$:*

$$\pi_*(A) = H_*(N(A)) \cong H_*(C(A)).$$

Proof It suffices to prove that $D(A)$ is acyclic. Filter $D(A)$ by setting $F_0 D_n = 0$, $F_p D_n = D_n$ if $n \leq p$ and $F_p D_n = \sigma_0(C_{n-1}) + \cdots + \sigma_p(C_{n-1})$ otherwise. The simplicial identities show that each $F_p D$ is a subcomplex. (Check this!) Since this filtration is canonically bounded, we have a convergent first quadrant spectral sequence

$$E^1_{pq} = H_{p+q}(F_p D/F_{p-1}D) \Rightarrow H_{p+q}(D).$$

Therefore it suffices to show that each complex $F_pD/F_{p-1}D$ is acyclic. Note that $(F_pD/F_{p-1}D)_n$ is a quotient of $\sigma_p(C_{n-1})$ and is zero for $n < p$. In element-theoretic language, if $x \in C_{n-1}(A)$, the simplicial identities yield in $F_pD/F_{p-1}D$:

$$d\sigma_p(x) = \sum_{i=p+1}^{n} (-1)^i \sigma_p \partial_{i-1}(x),$$

$$d\sigma_p^2(x) - \sigma_{p-1} d\sigma_p(x) = \sum_{i=p+2}^{n+1} (-1)^i \sigma_p \partial_{i-1} \sigma_p(x) - \sum_{i=p+2}^{n} (-1)^i \sigma_p^2 \partial_{i-1}(x)$$

$$= (-1)^{p+1} \sigma_p(x).$$

Hence $\{s_n = (-1)^{p+1}\sigma_p\}$ forms a chain contraction of the identity map of $F_pD/F_{p-1}D$, which is therefore null homotopic and hence acyclic (1.4.5). ◇

Application 8.3.9 (Hurewicz homomorphism) Let X be a fibrant simplicial set, and $\mathbb{Z}[X]$ the simplicial abelian group that in degree n is the free abelian group with basis the set X_n (8.2.3). The simplicial set map $h: X \to \mathbb{Z}[X]$ sending X to the basis elements of $\mathbb{Z}[X]$ is called the *Hurewicz homomorphism*, since on homotopy groups it is the map

$$\pi_*(X) \to \pi_*(\mathbb{Z}[X]) \cong H_*C(\mathbb{Z}[X]) = H_*(X; \mathbb{Z})$$

corresponding via 8.2.4 and 8.3.1 to the topological Hurewicz homomorphism $\pi_*(|X|) \to H_*(|X|; \mathbb{Z})$. (To see this, represent an element φ of $\pi_n(|X|)$ by a map $f: \Delta^n \to |X|$ and consider f as an element of $S_n(|X|)$. The class of $h(f)$ in $H_n\mathbb{Z}[S(|X|)] = H_n(|X|); \mathbb{Z})$ is the topological Hurewicz element $h(\varphi)$.)

Proposition 8.3.10 *Let A be a simplicial abelian group. Then the Hurewicz map $h_*: \pi_*(A) \to H_*(A; \mathbb{Z}) = H_*(|A|; \mathbb{Z})$ is a split monomorphism.*

Proof There is a natural surjection from the free abelian group $\mathbb{Z}[G]$ onto G for every abelian group G, defined on the basis elements as the identity. Thus there is a natural surjection of simplicial abelian groups $j: \mathbb{Z}[A] \to A$. The composite simplicial set map $jh: A \to \mathbb{Z}[A] \to A$ is the identity, so on homotopy groups $j_*h_*: \pi_*(A) \to \pi_*(\mathbb{Z}[A]) \to \pi_*(A)$ is the identity homomorphism. ◇

Remark The above proposition is the key result used to prove that every simplicial abelian group has the homotopy type of a product of Eilenberg-MacLane spaces of type $K(\pi_n A, n)$; see [May, 24.5].

8.3.1 Simplicial Homotopies

8.3.11 Let A and B be simplicial objects in a category $\mathcal{A}$. Two simplicial maps $f, g\colon A \to B$ are said to be (*simplicially*) *homotopic* if there are morphisms $h_i\colon A_n \to B_{n+1}$ in $\mathcal{A}$ $(i = 0, \cdots, n)$ such that $\partial_0 h_0 = f$ and $\partial_{n+1} h_n = g$, while

$$\partial_i h_j = \left\{ \begin{array}{ll} h_{j-1}\partial_i & \text{if } i < j \\ \partial_i h_{i-1} & \text{if } i = j \neq 0 \\ h_j \partial_{i-1} & \text{if } i > j+1 \end{array} \right\},$$

$$\sigma_i h_j = \left\{ \begin{array}{ll} h_{j+1}\sigma_i & \text{if } i \le j \\ h_j \sigma_{i-1} & \text{if } i > j \end{array} \right\}.$$

We call $\{h_j\}$ a *simplicial homotopy* from f to g and write $f \simeq g$.

If $\mathcal{A}$ is an abelian category, or the category of sets, the next theorem gives a cleaner definition of simplicial homotopy using the Cartesian product $A \times \Delta[1]$ of 8.2.6 and the two maps $\varepsilon_0, \varepsilon_1\colon A = A \times \Delta[0] \to A \times \Delta[1]$ induced by the maps $\varepsilon_0, \varepsilon_1\colon [0] \to [1]$ in Δ.

Theorem 8.3.12 *Suppose that $\mathcal{A}$ is either an abelian category or the category of sets. Let A, B be simplicial objects and $f, g\colon A \to B$ two simplicial maps. There is a one-to-one correspondence between simplicial homotopies from f to g and simplicial maps $h\colon A \times \Delta[1] \to B$ such that the following diagram commutes.*

$$\begin{array}{ccccc} A & \xrightarrow{\varepsilon_0} & A \times \Delta[1] & \xleftarrow{\varepsilon_1} & A \\ & f \searrow & \downarrow h & \swarrow g & \\ & & B & & \end{array}$$

Proof We give the proof when $\mathcal{A}$ is an abelian category. The set $\Delta[1]_n$ consists of the maps $\alpha_i\colon [n] \to [1]$ $(i = -1, \cdots, n)$, where α_i is characterized by $\alpha_i^{-1}(0) = \{0, 1, \cdots, i-1\}$. Thus $(A \times \Delta[1])_n$ is the direct sum of $n+2$ copies of A_n indexed by the α_i. A map $h^{(n)}\colon (A \times \Delta[1])_n \to B_n$ is therefore equivalent to a family of maps $h_i^{(n)}\colon A_n \to B_n$ $(i = -1, \cdots, n)$. Given a simplicial homotopy $\{h_j\}$ we define $h^{(n)}_{-1} = g$, $h^{(n)}_n = f$ and $h^{(n)}_i = \partial_{i+1} h_i$ for $0 \le i < n$. It is easily verified that $\partial_i h^{(n)} = h^{(n-1)} \partial_i$ and $\sigma_i h^{(n)} = h^{(n+1)} \sigma_i$, so that the $h^{(n)}$ form a simplicial map h such that $h\varepsilon_0 = f$ and $h\varepsilon_1 = g$. (Exercise!) Conversely, given h the maps $h_i = h^{(n+1)}_i \sigma_i \colon A_n \to B_{n+1}$ define a simplicial homotopy from f to g. ◇

Exercise 8.3.5 (Swan) Show that the above theorem fails when $\mathcal{A}$ is the category of groups, but that the theorem will hold if $A \times \Delta[1]$ is replaced by the simplicial group $A * \Delta[1]$, which in degree n is the free product of $n+2$ copies of A_n indexed by the set $\Delta[1]_n$.

Exercise 8.3.6 In this exercise we show that simplicial homotopy is an additive equivalence relation when $\mathcal{A}$ is any abelian category. Let f, f', g, g' be simplicial maps A $\to B$, and show that:

1. $f \simeq f$.
2. if $f \simeq g$ and $f' \simeq g'$, then $(f + f') \simeq (g + g')$.
3. if $f \simeq g$, then $(-f) \simeq (-g)$, $(f - g) \simeq 0$ and $g \simeq f$.
4. if $f \simeq g$ and $g \simeq h$, then $f \simeq h$.

Lemma 8.3.13 *Let $\mathcal{A}$ be an abelian category and $f, g \colon A \to B$ two simplicially homotopic maps. Then $f_*, g_* \colon N(A) \to N(B)$ are chain homotopic maps between the corresponding normalized chain complexes.*

Proof By exercise 8.3.6 above we may assume that $f = 0$ (replace g by $g - f$). Define $s_n = \sum (-1)^j h_j$ as a map from A_n to B_{n+1}, where $\{h_j\}$ is a simplicial homotopy from 0 to g. The restriction of s_n to $Z_n(A)$ lands in $Z_n(B)$, and we have

$$\partial_{n+1} s_n - s_{n-1} \partial_n = (-1)^n g.$$

(Check this!) Therefore $\{(-1)^n s_n\}$ is a chain homotopy from 0_* to g_*. ◇

Path Spaces 8.3.14 There is a functor $P \colon \Delta \to \Delta$ with $P[n] = [n+1]$ such that the natural map $\varepsilon_0 \colon [n] \to [n+1] = P[n]$ is a natural transformation $id_\Delta \Rightarrow P$. It is obtained by formally adding an initial element $0'$ to each $[n]$ and then identifying $(0' < 0 < \cdots < n)$ with $[n+1]$. Thus $P(\varepsilon_i) = \varepsilon_{i+1}$ and $P(\eta_i) = \eta_{i+1}$. If A is a simplicial object in $\mathcal{A}$, the *path space* PA is the simplicial object obtained by composing A with P. Thus $(PA)_n = A_{n+1}$, the i^{th} face operator on PA is the ∂_{i+1} of A, and the i^{th} degeneracy operator on PA is the σ_{i+1} of A. Moreover, the maps $\partial_0 \colon A_{n+1} \to A_n$ form a simplicial map

$PA \to A$. The path space will play a key role in the proof of the Dold-Kan correspondence.

Exercise 8.3.7 ($PA \simeq A_0$) Let A be a simplicial object, and write A_0 for the constant simplicial object at A_0. The natural maps $\sigma_0^{n+1}: A_0 \to A_{n+1}$ form a simplicial map $\iota: A_0 \to PA$, and the maps $A_{n+1} \to A_0$ induced by the canonical inclusion of $[0] = \{0\}$ in $[n+1] = (0 < 1 < \cdots < n+1)$ form a simplicial map $\rho: PA \to A_0$ such that $\rho\iota$ is the identity on A_0. Use σ_0 to construct a homotopy from $\iota\rho$ to the identity on PA. This shows that PA is homotopy equivalent to the constant object A_0.

Exercise 8.3.8 If G is a group one usually writes EG for the simplicial set $P(BG)$. By the previous exercise 8.3.7, $EG \simeq \{1\}$. Show that the surjection $\partial_0: EG \to BG$ is a principal G-fibration (exercise 8.2.6). Then use the long exact homotopy sequence of a fibration (exercise 8.3.3) to recalculate $\pi_*(BG)$.

Exercise 8.3.9 (J. Moore) Let A be a simplicial object in an abelian category $\mathcal{A}$. Let ΛA denote the simplicial object of $\mathcal{A}$ which is the kernel of $\partial_0: PA \to A$; ΛA is a kind of brutal "loop space" of A. To see this, let $A_0[1]$ denote the chain complex that is A_0 concentrated in degree -1, and let cone(NA) be the mapping cone of the identity map of NA (1.5.1). Show that $N_n(\Lambda A) \cong N_{n+1}(A)$ for all $n \geq 0$ and that there are exact sequences:

$$0 \to A_0[1] \to NA[1] \to N(\Lambda A) \to 0,$$

$$0 \to A_0[1] \to \text{cone}(NA)[1] \to N(PA) \to 0.$$

That is, $N(\Lambda A)$ is the brutal truncation $\sigma_{\geq 0}NA[1]$ of $NA[1]$ and $N(PA)$ is the brutal truncation of cone(NA)[1], in the sense of 1.2.7 and 1.2.8.

8.4 The Dold-Kan Correspondence

Let $\mathcal{A}$ be an abelian category. The normalized chain complex $N(A)$ of a simplicial object A of $\mathcal{A}$ (8.3.6) depends naturally on A and forms a functor N from the category of simplicial objects in $\mathcal{A}$ to the category of chain complexes in $\mathcal{A}$. The following theorem, discovered independently by Dold and Kan in 1957, is called the *Dold-Kan correspondence*. (See [Dold].)

Dold-Kan Theorem 8.4.1 *For any abelian category $\mathcal{A}$, the normalized chain complex functor N is an equivalence of categories between $S\mathcal{A}$ and* $\mathbf{Ch}_{\geq 0}(\mathcal{A})$.

$$\mathcal{S}\mathcal{A} = \left\{ \begin{matrix} \text{simplicial} \\ \text{objects in } \mathcal{A} \end{matrix} \right\} \xrightarrow{N} \mathbf{Ch}_{\geq 0}(\mathcal{A}) = \left\{ \begin{matrix} \text{chain complexes } C \text{ in } \mathcal{A} \\ \text{with } C_n = 0 \text{ for } n < 0 \end{matrix} \right\}.$$

Under this correspondence, simplicial homotopy corresponds to homology (i.e., $\pi_*(A) \cong H_*(NA)$) and simplicially homotopic morphisms correspond to chain homotopic maps.

Corollary 8.4.2 (See 2.4.7) *The simplicial homotopy groups $\pi_* A$ of a simplicial object A of $\mathcal{A}$ form a universal δ-functor (the left derived functors of the functor π_0).*

Corollary/Definition 8.4.3 (Dual Dold-Kan correspondence) *For any abelian category $\mathcal{A}$, there is an equivalence*

$$\left\{ \begin{matrix} \text{cosimplicial} \\ \text{objects in } \mathcal{A} \end{matrix} \right\} \xrightarrow{N^*} \mathbf{Ch}^{\geq 0}(\mathcal{A}) = \left\{ \begin{matrix} \text{cochain complexes } C \text{ in } \mathcal{A} \\ \text{with } C^n = 0 \text{ for } n < 0 \end{matrix} \right\}.$$

N^*A is a summand of the unnormalized cochain CA of A. We define the cohomotopy *of a cosimplicial object A to be the cohomology of N^*A, that is, as $\pi^i A = H^i(N^*A)$. Then $\pi^i A \cong H^i(CA)$. Finally, if $\mathcal{A}$ has enough injectives, the cohomotopy groups $\pi^* A$ are the right derived functors of the functor π^0.*

8.4.4 The equivalence in the Dold-Kan Theorem is concretely realized by an inverse functor K:

$$\mathbf{Ch}_{\geq 0}(\mathcal{A}) \xrightarrow{K} \mathcal{S}\mathcal{A} = \left\{ \begin{matrix} \text{simplicial} \\ \text{objects in } \mathcal{A} \end{matrix} \right\}$$

which is constructed as follows. Given a chain complex C we define $K_n(C)$ to be the finite direct sum $\bigoplus_{p \leq n} \bigoplus_\eta C_p[\eta]$, where for each $p \leq n$ the index η ranges over all surjections $[n] \to [p]$ in Δ and $C_p[\eta]$ denotes a copy of C_p.

If $\alpha: [m] \to [n]$ is any morphism in Δ, we shall define $K(\alpha): K_n(C) \to K_m(C)$ by defining its restrictions $K(\alpha, \eta): C_p[\eta] \to K_m(C)$. For each surjection $\eta: [n] \to [p]$, find the epi-monic factorization $\varepsilon\eta'$ of $\eta\alpha$ (8.1.2):

$$\begin{array}{ccc} [m] & \xrightarrow{\alpha} & [n] \\ \eta' \downarrow & & \downarrow \eta \\ [q] & \stackrel{\varepsilon}{\hookrightarrow} & [p]. \end{array}$$

If $p = q$ (in which case $\eta\alpha = \eta'$) we take $K(\alpha, \eta)$ to be the natural identification of $C_p[\eta]$ with the summand $C_p[\eta']$ of $K_m(C)$. If $p = q + 1$ and $\varepsilon = \varepsilon_p$

(in which case the image of $\eta\alpha$ is the subset $\{0, \cdots, p-1\}$ of $[p]$), we take $K(\alpha, \eta)$ to be the map

$$C_p \xrightarrow{d} C_{p-1} = C_q[\eta'] \subseteq K_m(C).$$

Otherwise we define $K(\alpha, \eta)$ to be zero. Here is a picture of $K(C)$:

$$C_0 \Leftarrow C_0 \oplus C_1 \Lleftarrow C_0 \oplus C_1 \oplus C_1 \oplus C_2 \Lleftarrow C_0 \oplus (C_1)^3 \oplus (C_2)^3 \oplus C_3 \cdots.$$

Exercise 8.4.1 Show that $K(C)$ is a simplicial object of $\mathcal{A}$. Since it is clearly natural in C, this shows that K is a functor.

It is easy to see that $NK(C) \cong C$. Indeed, if $\eta: [n] \to [p]$ and $n \neq p$, then $\eta = \eta_{i_1} \cdots \eta_{i_t}$ and $C_p[\eta] = (\sigma_{i_t} \cdots \sigma_{i_1} C_p)[\mathrm{id}_p]$ lies in the degenerate subcomplex $D(K(C))$. If η is the identity map of $[n]$, then ∂_i restricted to $C_n[\mathrm{id}_n]$ is $K(\varepsilon_i, \mathrm{id}_n)$, which is 0 if $i \neq n$ and d if $i = n$. Hence $N_n(KC) = C_n[\mathrm{id}_n]$ and the differential is d. Therefore in order to prove the Dold-Kan Theorem we must show that $KN(A)$ is naturally isomorphic to A for every simplicial object A in $\mathcal{A}$.

We first construct a natural simplicial map $\psi_A: KN(A) \to A$. If $\eta: [n] \to [p]$ is a surjection, the corresponding summand of $KN_n(A)$ is $N_p(A)$, and we define the restriction of ψ_A to this summand to be $N_p(A) \subset A_p \xrightarrow{\eta} A_n$. Given $\alpha: [m] \to [n]$ in Δ, and the epi-monic factorization $\varepsilon\eta'$ of $\eta\alpha$ in Δ (8.1.2) with $\eta': [m] \to [q]$, the diagram

$$\begin{array}{ccccccc}
KN_n(A) & \supset & N_p(A) & \subset & A_p & \xrightarrow{\eta} & A_n \\
\alpha\downarrow & & \downarrow\varepsilon & & \downarrow\varepsilon & & \downarrow\alpha \\
KN_m(A) & \supset & N_q(A) & \subset & A_q & \xrightarrow{\eta'} & A_m
\end{array}$$

commutes because $\varepsilon: N_p(A) \to N_q(A)$ is zero unless $\varepsilon = \varepsilon_p$. (Check this!) Hence ψ_A is a simplicial morphism from $KN(A)$ to A and is natural in A. We have to show that ψ_A is an isomorphism for all A. From the definition of ψ_A it follows that $N\psi_A: NKN(A) \to N(A)$ is the above isomorphism $NK(NA) \cong NA$. The following lemma therefore implies that ψ_A is an isomorphism, proving that N and K are inverse equivalences.

Lemma 8.4.5 *If $f: B \to A$ is a simplicial morphism such that $Nf: N(B) \to N(A)$ is an isomorphism, then f is an isomorphism.*

Proof We prove that each $f_n: B_n \to A_n$ is an isomorphism by induction on n, the case $n = 0$ being the isomorphism $B_0 = N_0B \cong N_0A = A$. Recall from exercise 8.3.9 that the brutal loop space ΛA is the kernel of $\partial_0: PA \to A$, $(PA)_n = A_{n+1}$, and that $N(\Lambda A)$ is the translate $((NA)/A_0)[1]$. Therefore $N\Lambda f: N(\Lambda B) \to N(\Lambda A)$ is an isomorphism. By induction both f_n and $(\Lambda f)_n$ are isomorphisms. From the 5-lemma applied to the following diagram, we deduce that f_{n+1} is an isomorphism. ◇

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\Lambda B)_n & \longrightarrow & B_{n+1} & \xrightarrow{\partial_0} & B_n & \longrightarrow & 0 \\
 & & {\scriptstyle \Lambda f_n}\downarrow\cong & & \downarrow{\scriptstyle f_{n+1}} & & {\scriptstyle f_n}\downarrow\cong & & \\
0 & \longrightarrow & (\Lambda A)_n & \longrightarrow & A_{n+1} & \xrightarrow{\partial_0} & A_n & \longrightarrow & 0.
\end{array}$$

Exercise 8.4.2 Show that N and K are adjoint functors. That is, if A is a simplicial object and C is a chain complex, show that ψ induces a natural isomorphism:

$$\mathrm{Hom}_{\mathcal{SA}}(K(C), A) \cong \mathrm{Hom}_{\mathbf{Ch}}(C, NA).$$

Exercise 8.4.3 Given a semi-simplicial object B in $\mathcal{A}$, $KC(B)$ is a simplicial object. Show that KC is left adjoint to the forgetful functor from simplicial objects to semi-simplicial objects. (Cf. exercise 8.1.5.) *Hint:* Show that if A is a simplicial object, then there is a natural split surjection $KC(A) \to A$.

To conclude the proof of the Dold-Kan Theorem 8.4.1, we have to show that simplicially homotopic maps correspond to chain homotopic maps. We saw in 8.3.13 that if $f \simeq g$ then Nf and Ng were chain homotopic. Conversely suppose given a chain homotopy $\{s_n\}$ from f to g for two chain maps $f, g: C \to C'$. Define $h_i: K(C)_n \to K(C')_{n+1}$ as follows. On the summand C_n of $K(C)_n$ corresponding to $\eta = id$, set

$$h_i \text{ on } C_n = \begin{cases} \sigma_i f & \text{if } i < n-1 \\ \sigma_{n-1} f - \sigma_n s_{n-1} d & \text{if } i = n-1 \\ \sigma_n (f - s_{n-1} d) - s_n & \text{if } i = n. \end{cases}$$

On the summand $C_p[\eta]$ of $K(C)_n$ corresponding to $\eta: [n] \to [p]$, $n \neq p$, we define h_i by induction on $n - p$. Let j be the largest element of $[n]$ such that $\eta(j) = \eta(j+1)$ and write $\eta = \eta'\eta_j$. Then σ_j maps $C_p[\eta']$ isomorphically onto $C_p[\eta]$, and we have already defined the maps h_i on $C_p[\eta']$. Writing h'_i for the composite of $C_p[\eta] \cong C_p[\eta']$ with h_i restricted to $C_p[\eta']$, we define

$$h_i \text{ on } C_p[\eta] = \begin{cases} \sigma_j h'_{i-1} & \text{if } j < i \\ \sigma_{j+1} h'_i & \text{if } j \geq i. \end{cases}$$

A straightforward calculation (exercise!) shows that $\{h_i\}$ form a simplicial homotopy from $K(f)$ to $K(g)$. ◇

Exercise 8.4.4 (Eilenberg-MacLane spaces) Let G be an abelian group, and write $G[-n]$ for the chain complex that is G concentrated in degree n (1.2.8).

1. Show that the simplicial abelian group $K(G[-n])$ is an Eilenberg-MacLane space of type $K(G,n)$ in the sense of 8.3.4 and that the loop space of exercise 8.3.9 satisfies $\Lambda K(G[-n-1]) \cong K(G[-n])$ for $n \geq 0$.
2. Suppose that a simplicial abelian group A is an Eilenberg-MacLane space of type $K(G,n)$. Use the truncation $\tau_{\geq n} NA$ (1.2.7) to show that there are simplicial maps $A \leftarrow B \rightarrow K(G[-n])$ that induce isomorphisms on homotopy groups. Hence A has the same simplicial homotopy type as $K(G[-n])$. A similar result holds for all Eilenberg-MacLane spaces, and is given in [May, section 23].

Exercise 8.4.5 Suppose that $\mathcal{A}$ has enough projectives, so that the category of $\mathcal{SA}$ of simplicial objects in $\mathcal{A}$ has enough projectives (exercise 2.2.2). Show that a simplicial object P is projective in $\mathcal{SA}$ if and only if (1) each P_n is projective in $\mathcal{A}$, and (2) the identity map on P is simplicially homotopic to the zero map.

Augmented Objects 8.4.6 An *augmented simplicial object* in a category $\mathcal{A}$ is a simplicial object A_* together with a morphism $\varepsilon: A_0 \rightarrow A_{-1}$ to a fixed object A_{-1} such that $\varepsilon\partial_0 = \varepsilon\partial_1$. If $\mathcal{A}$ is an abelian category, this allows us to augment the associated chain complexes $C(A)$ and $N(A)$ by adding A_{-1} in degree -1.

$$0 \leftarrow A_{-1} \xleftarrow{\varepsilon} A_0 \xleftarrow{\partial_0 - \partial_1} A_1 \xleftarrow{d} A_2 \xleftarrow{d} \cdots.$$

An augmented simplicial object $A_* \rightarrow A_{-1}$ is called *aspherical* if $\pi_n(A_*) = 0$ for $n \neq 0$ and $\varepsilon: \pi_0(A_*) \cong A_{-1}$. In an abelian category, this is equivalent to the assertion that the associated augmented chain complexes are exact, that is, that $C(A)$ and $N(A)$ are resolutions for A_{-1} in $\mathcal{A}$. For this reason, A_* is sometimes called a *simplicial resolution* of A_{-1}. We will use aspherical simplicial objects to construct canonical resolutions in 8.6.8. The following exercise gives a useful criterion for $A_* \xrightarrow{\varepsilon} A_{-1}$ to be aspherical.

An augmented simplicial object $A_* \xrightarrow{\varepsilon} A_{-1}$ is called (*right*) *contractible* if there are morphisms $f_n\colon A_n \to A_{n+1}$ for all n (including $f_{-1}\colon A_{-1} \to A_0$) such that $\varepsilon f_{-1} = \text{id}$, $\partial_{n+1} f_n = \text{id}$ for $n \geq 0$, $\partial_0 f_0 = f_{-1}\varepsilon$, and $\partial_i f_n = f_{n-1}\partial_i$ for all $0 \leq i \leq n$. (It is called *left contractible* if its dual $\check{A} \xrightarrow{\varepsilon} A_{-1}$ (8.2.10) is right contractible, that is, if $\varepsilon f_{-1} = \text{id}$, $\partial_0 f_n = \text{id}$, $\partial_{-1} f_0 = f_{-1}\varepsilon$, and $\partial_i f_n = f_{n-1}\partial_{i-1}$.)

Exercise 8.4.6 (Gersten)

1. If $\mathcal{A}$ is an abelian category, prove that every contractible augmented simplicial object is aspherical, and that the associated augmented chain complexes are split exact.
2. Now suppose that $\mathcal{A}$ is the category of sets. Let X be a fibrant simplicial set with basepoint $*$ and $\varepsilon\colon X \to X_{-1}$ an augmentation. Prove that if $X \to X_{-1}$ is (left or right) contractible and $f_n(*) = *$ for all n, then X is aspherical. *Hint:* Set $y = f_n(x)$ in 8.3.1.

8.5 The Eilenberg-Zilber Theorem

A *bisimplicial object* in a category $\mathcal{A}$ is a contravariant functor A from $\Delta \times \Delta$ to $\mathcal{A}$. Alternatively, it is a bigraded sequence of objects A_{pq} $(p, q \geq 0)$, together with horizontal face and degeneracy operators $\partial_i^h\colon A_{pq} \to A_{p-1,q}$ and $\sigma_i^h\colon A_{pq} \to A_{p+1,q}$ as well as vertical face and degeneracy operators $\partial_i^v\colon A_{pq} \to A_{p,q-1}$ and $\sigma_i^v\colon A_{pq} \to A_{p,q+1}$. These operators must satisfy the simplicial identities (horizontally and vertically), and in addition every horizontal operator must commute with every vertical operator.

There is an (unnormalized) first quadrant double complex $CA = \{A_{pq}\}$ associated to any bisimplicial object A. The horizontal maps d^h are $\sum(-1)^i\partial_i^h$ and we use the sign trick (1.2.5) to define the vertical maps $d^v\colon A_{pq} \to A_{p,q-1}$ to be $(-1)^p\sum(-1)^i\partial_i^v$.

Clearly we may regard a bisimplicial object as a simplicial object in the catagory $\mathcal{SA}$ of simplicial objects in $\mathcal{A}$. The Dold-Kan correspondence implies that the category of bisimplicial objects is equivalent to the category of first quadrant double chain complexes, the normalized double complex corresponding to A being quasi-isomorphic to CA.

The *diagonal* $\text{diag}(A)$ of a bisimplicial object A is the simplicial object obtained by composing the diagonal functor $\Delta \to \Delta \times \Delta$ with the functor A.

Thus $\operatorname{diag}(A)_n = A_{nn}$, the face operators are $\partial_i = \partial_i^h \partial_i^v$, and the degeneracy operators are $\sigma_i = \sigma_i^h \sigma_i^v$.

Eilenberg-Zilber Theorem 8.5.1 *Let A be a bisimplicial object in an abelian category $\mathcal{A}$. Then there is a natural isomorphism*

$$\pi_*\operatorname{diag}(A) \cong H_* \operatorname{Tot}(CA).$$

Moreover there is a convergent first quadrant spectral sequence

$$E^1_{pq} = \pi^v_q(A_{p*}), \quad E^2_{pq} = \pi^h_p \pi^v_q(A) \Rightarrow \pi_{p+q}\operatorname{diag}(A).$$

Proof We first observe that $\pi_0 \cong H_0$. By inspection, we have decompositions $A_{10} = \sigma_0^h(A_{00}) \oplus N_{10}$, $A_{01} = \sigma_0^v(A_{00}) \oplus N_{01}$, and $A_{11} = \sigma_0^v \sigma_0^h(A_{00}) \oplus \sigma_0^v(N_{10}) \oplus \sigma_0^h(N_{01}) \oplus N_{11}$. Now $H_0 \operatorname{Tot}(CA) = A_{00}/(\partial_1^h(N_{10}) + \partial_1^v(N_{01}))$ and $\pi_0\operatorname{diag}(A)$ is the quotient of A_{00} by

$$\partial_1^h \partial_1^v(\sigma_0^v N_{10} \oplus \sigma_0^h N_{01} \oplus N_{11}) = \partial_1^h(N_{10}) + \partial_1^v(N_{01}) + 0.$$

Hence there is a natural isomorphism $\pi_0\operatorname{diag}(A) \cong H_0 \operatorname{Tot}(A)$.

Now the functors $\operatorname{diag}(A)$ and $\operatorname{Tot}(CA)$ are exact, while π_* and H_* are δ-functors, so both $\pi_*\operatorname{diag}(A)$ and $H_* \operatorname{Tot}(CA)$ are homological δ-functors on the category of bisimplicial objects in $\mathcal{A}$. We will show that they are both universal δ-functors, which will imply that they are naturally isomorphic. (The isomorphisms are given explicitly in 8.5.4.) This will finish the proof, since canonical first quadrant spectral sequence associated to the double complex CA has $E^1_{pq} = H^v_q(C_{p*}) = \pi^v_q(A_{p*})$ and $E^2_{pq} = H^h_p(C(\pi^v_q(A_{p*}))) = \pi^h_p \pi^v_q(A)$ and converges to $H_{p+q} \operatorname{Tot}(CA) \cong \pi_{p+q}\operatorname{diag}(A)$.

To see that π_* diag and H_* Tot C are universal δ-functors, we may assume (using the Freyd-Mitchell Embedding Theorem 1.6.1 if necessary) that $\mathcal{A}$ has enough projectives. (Why?) We saw in exercise 2.2.2 that this implies that the category of double complexes—and hence the category of bisimplicial objects by the Dold-Kan correspondence—has enough projectives. By the next lemma, diag and Tot C preserve projectives. Therefore we have the desired result:

$$\pi_*\operatorname{diag} = (L_*\pi_0)\operatorname{diag} = L_*(\pi_*\operatorname{diag}),$$
$$H_* \operatorname{Tot}\ C = (L_* H_0) \operatorname{Tot}\ C = L_*(H_0 \operatorname{Tot}\ C). \qquad \diamond$$

Lemma 8.5.2 *The functors diag and Tot C preserve projectives.*

Proof Fix a projective bisimplicial object P. We see from exercise 8.4.5 that any bisimplicial object A is projective if and only if each A_{pq} is projective in $\mathcal{A}$, each row and column is simplicially null-homotopic, and the vertical homotopies h_i^v are simplicial maps. Therefore diag(P) is a projective simplicial object, since each diag$(P)_n = P_{nn}$ is projective and the maps $h_i = h_i^h h_i^v$ form a simplicial homotopy (8.3.11) from the identity of diag(P) to zero. Now Tot(CP) is a non-negative chain complex of projective objects, so it is projective in $\mathbf{Ch}_{\geq 0}(\mathcal{A})$ if and only if it is split exact if and only if it is exact. But every column of Tot(CP) is acyclic, since $H_*(CP_{p*}) = \pi_*(P_{p*}) = 0$, so Tot$(CP)$ is exact by the Acyclic Assembly lemma 2.7.3 (or a spectral sequence argument). $\diamond$

Application 8.5.3 (Künneth formula) Let A and B be simplicial right and left R-modules, respectively. Their tensor product $(A \otimes_R B) = A_p \otimes_R B_q$ is a bisimplicial abelian group, and the associated double complex $C(A \otimes B)$ is the total tensor product Tot $C(A) \otimes_R C(B)$ of 2.7.1. The Eilenberg-Zilber Theorem 8.5.1 states that

$$\pi_* \text{diag}(A \otimes_R B) \cong H_*(\text{Tot } C(A) \otimes_R C(B)).$$

This is the form in which Eilenberg and Zilber originally stated their theorem in 1953. Now suppose that X and Y are simplicial sets and set $A = R[X]$, $B = R[Y]$ 8.2.3. Then diag$(A \otimes B) \cong R[X \times Y]$, and the computation of the homology of the product $X \times Y$ (8.2.6) with coefficients in R is

$$H_n(X \times Y; R) = \pi_n \text{diag}(A \otimes B) \cong H_n(\text{Tot } C(X) \otimes C(Y)).$$

The Künneth formula 3.6.3 yields $H_n(X \times Y) \cong \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y)$ when R is a field. If $R = \mathbb{Z}$ there is an extra Tor term, as described in 3.6.4.

The Alexander-Whitney Map 8.5.4 For many applications it is useful to have an explicit formula for the isomorphisms in the Eilenberg-Zilber Theorem 8.5.1. If $p + q = n$, we define $f_{pq}\colon A_{nn} \to A_{pq}$ to be the map

$$\partial_{p+1}^h \cdots \partial_n^h \partial_0^v \cdots \partial_0^v.$$

The sum over p and q yields a map $f_n\colon A_{nn} \to \text{Tot}_n(CA)$, and the f_n assemble to yield a chain complex map f from $C(\text{diag}(A))$ to Tot(CA). (Exercise!) The map f is called the *Alexander-Whitney map*, since these two mathematicians discovered it independently while constructing the cup product in topology. Since f is defined by face operators, it is natural and induces a morphism of universal δ-functors $f_*: \pi_*\text{diag}A \to H_*\,\text{Tot}(CA)$. Moreover, $f_0: A_{00} = A_{00}$, so f_* induces the natural isomorphism $\pi_0\text{diag}\,A \cong H_0\,\text{Tot}(CA)$. Therefore the Alexander-Whitney map is the unique chain map (up to equivalence) inducing the isomorphism f_* of the Eilenberg-Zilber Theorem.

The inverse map $\nabla: \text{Tot}(CA) \to C(\text{diag}A)$ is related to the shuffle product on the bar complex (6.5.11). The component $\nabla_{pq}: A_{pq} \to A_{nn}$ $(n = p+q)$ is the sum

$$\sum_{\mu}(-1)^{\mu}\sigma^h_{\mu(n)}\cdots\sigma^h_{\mu(p+1)}\sigma^v_{\mu(p)}\cdots\sigma^v_{\mu(1)}$$

over all (p,q)-shuffles μ. The proof that ∇ is a chain map is a tedious but straightforward exercise. Clearly, ∇ is natural, and it is easy to see that ∇_* induces the natural isomorphism $H_0\,\text{Tot}(CA) \cong \pi_0\text{diag}\ A$. Therefore ∇_* is the unique isomorphism of universal δ-functors given by the Eilenberg-Zilber Theorem. In particular, ∇_* is the inverse of the Alexander-Whitney map f_*.

Remark The analogue of the Eilenberg-Zilber Theorem for semi-simplicial simplicial objects is false; the degeneracies are necessary. For example, if A_{pq} is zero for $p \neq 1$, then $\pi_1\text{diag}(A) = A_{11}$ need not equal $H_1\,\text{Tot}(CA) = \pi_1(A_{1*})$.

8.6 Canonical Resolutions

To motivate the machinery of this section, we begin with a simplicial description of the (unnormalized) bar resolution of a group G. By inspecting the construction in 6.5.1 we see that the bar resolution

$$0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} B^u_0 \xleftarrow{d} B^u_1 \xleftarrow{d} B^u_2 \xleftarrow{d} \cdots$$

is exactly the augmented chain complex associated to the augmented simplicial G-module

$$\mathbb{Z} \xleftarrow{\varepsilon} B^u_0 \Leftarrow B^u_1 \Lleftarrow B^u_2 \Lleftarrow B^u_3 \cdots,$$

in which B^u_n is the free $\mathbb{Z}G$-module on the set G^n. In fact, we can construct the simplicial module B^u_* directly from the trivial G-module $\mathbb{Z}$ using only the functor $F = \mathbb{Z}G\otimes_{\mathbb{Z}}: G\text{–}\mathbf{mod} \to G\text{–}\mathbf{mod}$; B^u_n is $F^{n+1}\mathbb{Z} = \mathbb{Z}G \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}G$, the face operators are formed from the natural map $\varepsilon: \mathbb{Z}G \otimes_{\mathbb{Z}} M \to M$, and the degeneracy operators are formed from the natural map $\eta: M = \mathbb{Z} \otimes_{\mathbb{Z}} M \to \mathbb{Z}G \otimes_{\mathbb{Z}} M$.

In this section we formalize the above process (see 8.6.11) so that it yields augmented simplicial objects whose associated chain complexes provide canonical resolutions in a wide variety of contexts. To begin the formalization, we introduce the dual concepts of triple and cotriple. (The names "triple" and "cotriple" are unfortunate because nothing occurs three times. Nonetheless it is the traditional terminology. Some authors use "monad" and "comonad", which is not much better.)

Definition 8.6.1 A *triple* $(\top, \eta, \mu)$ on a category $\mathcal{C}$ is a functor $\top\colon \mathcal{C} \to \mathcal{C}$, together with natural transformations $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow \top$ and $\mu\colon \top\top \Rightarrow \top$, such that the following diagrams commute for every object C.

$$
\begin{array}{ccccc}
\top\top(\top C) = \top(\top\top C) & \xrightarrow{\top\mu} & \top(\top C) \\
\downarrow{\scriptstyle \mu_{\top C}} & & \downarrow{\scriptstyle \mu_C} \\
\top(\top C) & \xrightarrow{\mu C} & \top C
\end{array}
\qquad
\begin{array}{ccccc}
\top C & \xrightarrow{\top\eta_C} & \top(\top C) & \xleftarrow{\eta_{\top C}} & \top C \\
& {=}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{=} & \\
& & \top C & &
\end{array}
$$

Symbolically, we may write these as $\mu(\top\mu) = \mu(\mu\top)$ and $\mu(\top\eta) = \mathrm{id} = \mu(\eta\top)$.

Dually, a *cotriple* $(\perp, \varepsilon, \delta)$ in a category $\mathcal{A}$ is a functor $\perp\colon \mathcal{A} \to \mathcal{A}$, together with natural transformations $\varepsilon\colon \perp \Rightarrow \mathrm{id}_{\mathcal{A}}$ and $\delta\colon \perp \Rightarrow \perp\perp$, such that the following diagrams commute for every object A.

$$
\begin{array}{ccccc}
\perp A & \xrightarrow{\delta_A} & \perp(\perp A) \\
\downarrow{\scriptstyle \delta_A} & & \downarrow{\scriptstyle \delta_{\perp A}} \\
\perp(\perp A) & \xrightarrow{\perp\delta} \perp(\perp\perp A) = & \perp\perp(\perp A)
\end{array}
\qquad
\begin{array}{ccccc}
& & \perp A & & \\
& {=}\swarrow & \downarrow{\scriptstyle \delta} & \searrow{=} & \\
\perp A & \xleftarrow[\perp\varepsilon_A]{} & \perp(\perp A) & \xrightarrow[\varepsilon_{\perp A}]{} & \perp A
\end{array}
$$

Symbolically, we may write these as $(\perp\delta)\delta = (\delta\perp)\delta$ and $(\perp\varepsilon)\delta = \mathrm{id} = (\varepsilon\perp)\delta$. Note the duality: a cotriple in $\mathcal{A}$ is the same as a triple in $\mathcal{A}^{\mathrm{op}}$.

Exercise 8.6.1 Provided that they exist, show that any product $\Pi\top_\alpha$ of triples $\top_\alpha$ is a triple and that any coproduct $\amalg \perp_\alpha$ of cotriples $\perp_\alpha$ is again a cotriple.

Exercise 8.6.2 Show that the natural transformation ε of a cotriple satisfies the identity $\varepsilon(\varepsilon\perp) = \varepsilon(\perp\varepsilon)$. That is, for every A the following diagram commutes:

$$\begin{array}{ccc} \perp(\perp A) & \xrightarrow{\perp\varepsilon_A} & \perp A \\ {\scriptstyle \varepsilon_{\perp A}}\downarrow & & \downarrow{\scriptstyle \varepsilon_A} \\ \perp A & \xrightarrow{\varepsilon} & A. \end{array}$$

Main Application 8.6.2 (Adjoint functors) Suppose we are given an adjoint pair of functors (F, U) with F left adjoint to U.

$$\mathcal{B} \underset{U}{\overset{F}{\leftrightarrows}} \mathcal{C}$$

That is, $\mathrm{Hom}_{\mathcal{B}}(FC, B) \cong \mathrm{Hom}_{\mathcal{C}}(C, UB)$ for every C in $\mathcal{C}$ and B in $\mathcal{B}$. We claim that $\top = UF: \mathcal{C} \to \mathcal{C}$ is part of a triple $(\top, \eta, \mu)$ and that $\perp = FU: \mathcal{B} \to \mathcal{B}$ is part of a cotriple $(\perp, \varepsilon, \delta)$.

Recall from A.6.1 of the Appendix that such an adjoint pair determines two natural transformations: the *unit* of the adjunction $\eta: \mathrm{id}_{\mathcal{B}} \to UF$ and the *counit* of the adjunction $\varepsilon: FU \Rightarrow \mathrm{id}_{\mathcal{C}}$. We define δ and μ by

$$\delta_B = F(\eta_{UB}): F(UB) \to F(UF(UB)), \quad \mu_C = U(\varepsilon_{FC}): U(FU(FC)) \to U(FC).$$

In the Appendix, A.6.2 and exercise A.6.3, we see that $(\varepsilon F) \circ (F\eta): FC \to FC$ and $(U\varepsilon) \circ (\eta U): UB \to UB$ are the identity maps and that $\varepsilon \circ (FU\varepsilon) = \varepsilon \circ (\varepsilon FU): FU(FU(B)) \to B$. From these we deduce the triple axioms for $(\top, \eta, \mu)$:

$$\mu(\top\eta) = U((\varepsilon F) \circ (F\eta)) = \mathrm{id}, \quad \mu(\eta\top) = ((U\varepsilon) \circ (\eta U))F = \mathrm{id},$$
$$\mu(\top\mu) = (U\varepsilon F) \circ (UFU\varepsilon F) = U(\varepsilon \circ UF\varepsilon)F = U(\varepsilon \circ \varepsilon UF)F = \mu(\mu\top).$$

By duality applied to the adjoint pair $(U^{\mathrm{op}}, F^{\mathrm{op}})$, $(\perp, \varepsilon, \delta)$ is a cotriple on $\mathcal{B}$.

Example 8.6.3 The forgetful functor $U: G\text{–}\mathbf{mod} \to \mathbf{Ab}$ has for its left adjoint the functor $F(C) = \mathbb{Z}G \otimes_{\mathbb{Z}} C$. The resulting cotriple on G–**mod** has $\perp = FU$, and $\perp(\mathbb{Z}) \cong \mathbb{Z}G$. The following construction of a simplicial object out of the cotriple $\perp$ on the trivial G-module $\mathbb{Z}$ will yield the simplicial G-module used to form the unnormalized bar resolution described at the beginning of this section; see 8.6.11.

Simplicial Object of a Cotriple 8.6.4 Given a cotriple $\perp$ on $\mathcal{A}$ and an object A, set $\perp_n A = \perp^{n+1} A$ and define face and degeneracy operators

$$\partial_i = \perp^i \varepsilon \perp^{n-i}: \quad \perp^{n+1} A \to \perp^n A,$$
$$\sigma_i = \perp^i \delta \perp^{n-i}: \quad \perp^{n+1} A \to \perp^{n+2} A.$$

We claim that $\perp_* A$ is a simplicial object in $\mathcal{A}$. To see this, note that

$$\partial_i \sigma_i = \perp^i (\varepsilon \perp)\delta \perp^{n-i} = \perp^i (1) \perp^{n-i} = \text{ identity, and}$$

$$\partial_{i+1} \sigma_i = \perp^i (\perp \varepsilon)\delta \perp^{n-i} = \perp^i (1) \perp^{n-i} = \text{identity}.$$

Similarly, we have

$$\partial_i \partial_{i+1} = \perp^i (\varepsilon(\perp \varepsilon)) \perp^{n-i} = \perp^i (\varepsilon(\varepsilon \perp)) \perp^{n-i} = \partial_i \partial_i,$$

$$\sigma_{i+1} \sigma_i = \perp^i ((\perp \delta)\delta) \perp^{n-i} = \perp^i ((\delta \perp)\delta) \perp^{n-i} = \sigma_i \sigma_i.$$

The rest of the simplicial identities are formally valid. The map $\varepsilon_A: \perp A \to A$ satisfies $\varepsilon\partial_0 = \varepsilon\partial_1$ (because $\varepsilon(\varepsilon \perp) = \varepsilon(\perp \varepsilon)$), so in fact $\perp_* A \to A$ is an augmented simplicial object.

Dually, given a triple $\top$ on $\mathcal{C}$, we define $L^n = \top^{n+1} C$ and $\partial^i = \top^i \eta \top^{n-i}$, $\sigma^i = \top^i \mu \top^{n-i}$. Since a triple $\top$ on $\mathcal{C}$ is the same as a cotriple $\top^{\mathrm{op}}$ on $\mathcal{C}^{\mathrm{op}}$, $L^* = \top^{*+1} C$ is a cosimplicial object in $\mathcal{C}$ for every object C of $\mathcal{C}$.

Definition 8.6.5 *Let* $\perp$ be a cotriple in a category $\mathcal{A}$. An object A is called $\perp$*-projective* if $\varepsilon_A: \perp A \to A$ has a section $f: A \to \perp A$ (i.e., if $\varepsilon_A f = \mathrm{id}_A$). For example, if $\perp = FU$ for an adjoint pair (F, U), then every object FC is $\perp$-projective because $F\eta: FC \to F(UFC) = \perp (FC)$ is such a section.

Paradigm 8.6.6 (Projective R-modules) If R is a ring, the forgetful functor $U: R$–**mod** $\to$ **Sets** has the free R-module functor F as its left adjoint; we call FU the *free module cotriple*. Since $FU(P)$ is a free module, an R-module P is FU-projective if and only if P is a projective R-module. This paradigm explains the usage of the suggestive term "$\perp$-projective." It also shows that a cotriple on R–**mod** need not be an additive functor.

$\perp$-Projective Lifting Property 8.6.7 *Let $U: \mathcal{A} \to \mathcal{C}$ have a left adjoint F, and set $\perp = FU$. An object P is $\perp$-projective if and only if it satisfies the following lifting property: given a map $g: A_1 \to A_2$ in $\mathcal{A}$ such that $UA_1 \to UA_2$ is a split surjection and a map $\gamma: P \to A_2$, there is a map $\beta: P \to A_1$ such that $\gamma = g\beta$.*

Proof The lifting property applied to $FU(P) \to P$ shows that P is $\perp$-projective. For the converse we may replace P by $FU(P)$ and observe that since $\mathrm{Hom}_{\mathcal{A}}(FU(P), A) \cong \mathrm{Hom}_{\mathcal{C}}(UP, UA)$, the map $\mathrm{Hom}_{\mathcal{A}}(FU(P), A_1) \to \mathrm{Hom}_{\mathcal{A}}(FU(P), A_2)$ is a split surjection. ◇

Exercise 8.6.3 Show that an object P is $\perp$-projective if and only if there is an A in $\mathcal{A}$ such that P is a *retract* of $\perp A$. (That is, there are maps $i: P \to \perp A$ and $r: \perp A \to P$ so that $ri = \mathrm{id}_P$.)

Proposition 8.6.8 (Canonical resolution) *Let $\perp$ be a cotriple in an abelian category $\mathcal{A}$. If A is any $\perp$-projective object, then the augmented simplicial object $\perp_* A \xrightarrow{\varepsilon} A$ is aspherical, and the associated augmented chain complex is exact.*

$$0 \leftarrow A \xleftarrow{\varepsilon} \perp A \xleftarrow{\partial_0 - \partial_1} \perp^2 A \xleftarrow{d} \perp^3 A \xleftarrow{d} \cdots.$$

Proof For $n \geq 0$, set $f_n = \perp^{n+1} f: \perp^{n+1} A \to \perp^{n+2} A$, and set $f_{-1} = f$. By definition, $\partial_{n+1} f_n = \perp^{n+1} (\varepsilon f) =$ identity and $\partial_0 f_0 = (\varepsilon \perp)(\perp f) = f\varepsilon$. If $n \geq 1$ and $0 \leq i < n+1$, then (setting $j = n - i$ and $B = \perp^j A$) naturality of ε with respect to $g = \perp^j f$ yields

$$\partial_i f_n = (\perp^i \varepsilon_{\perp B})(\perp^i \perp g) = (\perp^i g)(\perp^i \varepsilon_B) = f_{n-1} \partial_i.$$

We saw (in 8.4.6 and exercise 8.4.6) that such a family of morphisms $\{f_n\}$ makes $\perp_* A \to A$ "contractible," hence aspherical. ◇

Corollary 8.6.9 *If $\mathcal{A}$ is abelian and $U: \mathcal{A} \to \mathcal{C}$ is a functor having a left adjoint $F: \mathcal{C} \to \mathcal{A}$, then for every C in $\mathcal{C}$ the augmented simplicial object $\perp_*(FC) \to FC$ is contractible, hence aspherical in $\mathcal{A}$.*

Proposition 8.6.10 *Suppose that $U: \mathcal{A} \to \mathcal{C}$ has a left adjoint $F: \mathcal{C} \to \mathcal{A}$. Then for every A in $\mathcal{A}$ the augmented simplicial object $U(\perp_* A) \xrightarrow{U\varepsilon} UA$ is left contractible in $\mathcal{C}$ and hence aspherical.*

Proof Set $f_{-1} = \eta U: UA \to UFUA = U(\perp A)$ and $f_n = \eta U \perp^n$. Then the $\{f_n\}$ make $U(\perp_* A)$ left contractible in the sense of 8.4.6. (Check this!) ◇

8.6.1 Applications

Group Homology 8.6.11 If G is a group, the forgetful functor $U: G$–**mod** $\to$ **Ab** has a left adjoint $F(C) = \mathbb{Z}G \otimes_{\mathbb{Z}} C$. For every G-module M, the resulting simplicial G-module $\perp_* M \to M$ is aspherical because its underlying simplicial abelian group $U(\perp_* M) \to UM$ is aspherical by 8.6.10. Moreover by Shapiro's Lemma 6.3.2 the G-modules $\perp^{n+1} M = F(C)$ are acyclic for $H_*(G; -)$ in the sense of 2.4.3. Therefore the associated chain complex $C(\perp_* M)$ is a resolution by $H_*(G; -)$-acyclic G-modules. It follows from 2.4.3 that we can compute the homology of the G-module M according to the formula

$$H_*(G; M) = H_*(C(\perp_* M)_G) = \pi_*((\perp_* M)_G),$$

using the homotopy groups of the simplicial abelian group $(\perp_* M)_G$.

If we take $M = \mathbb{Z}$, $C(\perp_* \mathbb{Z})$ is exactly the unnormalized bar resolution of 6.5.1. The proof given in 6.5.3 that the bar resolution is exact amounts to a paraphrasing of the proof of proposition 8.6.10.

The Bar Resolution 8.6.12 Let $k \to R$ be a ring homomorphism. The forgetful functor $U\colon R$–**mod** $\to k$–**mod** has $F(M) = R \otimes_k M$ as its left adjoint, so we obtain a cotriple $\perp = FU$ on R–**mod**. Since the homotopy groups of the simplicial R-module $\perp_* M$ may be computed using the underlying simplicial k-module $U(\perp_* M)$, it follows that $\perp_* M \to M$ is aspherical 8.4.6 ($\perp_* M$ is a simplicial resolution of M). The associated augmented chain complexes are not only exact in R–**mod**, they are split exact when considered as a complex of k-modules by 8.6.10. The unnormalized chain complex $\beta(R, M)$ associated to $\perp_* M$ is called the (unnormalized) *bar resolution* of a left R-module M. Thus $\beta(R, M)_0 = R \otimes_k M$, and $\beta(R, M)_n$ is $R^{\otimes(n+1)} \otimes_k M$. Note that $\beta(R, M) = \beta(R, R) \otimes_R M$:

$$0 \leftarrow M \xleftarrow{\varepsilon} R \otimes_k M \leftarrow R \otimes_k R \otimes_k M \leftarrow \cdots.$$

The *normalized bar resolution* of M, written $B(R, M)$, is the normalized chain complex associated to $\perp_* M$ and is described in the following exercise.

Exercise 8.6.4 Write $\bar{R}$ for the cokernel of the k-module homomorphism $k \to R$ sending 1 to 1, and write $\otimes$ for $\otimes_k$. Show that the normalized bar resolution has $B_n(R, M) = R \otimes \bar{R} \otimes \cdots \otimes \bar{R} \otimes M$ with n factors $\bar{R}$, with (well-defined) differential

$$\begin{aligned} d(r_0 \otimes \bar{r}_1 \otimes \cdots \bar{r}_n \otimes m) = {} & r_0 r_1 \otimes \bar{r}_2 \otimes \cdots \otimes \bar{r}_n \otimes m \\ & + \sum_{i=1}^{n-1} (-1)^i r_0 \otimes \cdots \otimes \bar{r}_i \bar{r}_{i+1} \otimes \cdots \otimes m \\ & + (-1)^n r_0 \otimes \bar{r}_1 \otimes \cdots \otimes \bar{r}_{n-1} \otimes r_n m. \end{aligned}$$

Proposition 8.6.13 *Suppose k is commutative. If M (resp. M') is a left module over a k-algebra R (resp. R'), then there is a chain homotopy equivalence of bar resolutions of the $R \otimes R'$-module $M \otimes M'$:*

$$\mathrm{Tot}(\beta(R, M) \otimes_k \beta(R', M')) \xrightarrow{\nabla} \beta(R \otimes_k R', M \otimes_k M').$$

Proof Let A (resp. A') denote the simplicial k-module $R^{\otimes *} \otimes M$ (resp. $R'^{\otimes *} \otimes M'$), where $\otimes$ denotes $\otimes_k$. The diagonal of the bisimplicial k-module $A \otimes A'$ is the simplicial k-module $[p] \mapsto (R^{\otimes p} \otimes M) \otimes (R'^{\otimes p} \otimes M) \cong (R \otimes R')^{\otimes p} \otimes (M \otimes M')$ whose associated chain complex is $\beta(R \otimes R', M \otimes M')$. The Eilenberg-Zilber Theorem (in the Künneth formula incarnation 8.5.3) gives a chain homotopy equivalence ∇ from the total tensor product Tot $C(A \otimes A') \cong \mathrm{Tot}\, C(A) \otimes C(A') = \mathrm{Tot}\, \beta(R, M) \otimes \beta(R', M')$ to C diag $(A \otimes A') \cong \beta(R \otimes R', M \otimes M')$. ◇

Remark The homotopy equivalence Tot $\beta(R, R) \otimes \beta(R', R') \xrightarrow{\nabla} \beta(R \otimes R', R \otimes R')$ is fundamental; applying $\otimes_{R \otimes R'}(M \otimes M')$ to it yields the proposition.

Exercise 8.6.5 (Shuffle product) Use the explicit formula for the shuffle map ∇ of 6.5.11 and 8.5.4 to establish the explicit formula (where μ ranges over all (p, q)-shuffles):

$$\nabla((r_0 \otimes \cdots \otimes r_p \otimes m) \otimes (r'_0 \otimes \cdots \otimes r'_q \otimes m')) =$$
$$\sum_{\mu} (-1)^{\mu} (r_0 \otimes r'_0) \otimes w_{\mu(1)} \otimes \cdots \otimes w_{\mu(p+q)} \otimes (m \otimes m').$$

Here the r_i are in R, the r'_j are in R', $m \in M$, $m' \in M'$, and $w_1, \cdots, w_{p+q}$ is the ordered sequence of elements $r_1 \otimes 1, \cdots, r_p \otimes 1, 1 \otimes r'_1, \cdots, 1 \otimes r'_q$ of $R \otimes R'$.

Free Resolutions 8.6.14 Let R be a ring and FU the free module cotriple, where $U\colon R$–**mod** $\to$ **Sets** is the forgetful functor whose left adjoint $F(X)$ is the free module on X. For every R-module M, we claim that the augmented simplicial R-module $(FU)_*M \to M$ is aspherical (8.4.6). This will prove that FU_*M is a simplicial resolution of M, and that the associated chain complex $C = C(FU_*M)$ is a canonical free resolution of M because

$$H_i(C) = \pi_i(FU_*M) = \pi_i(UFU_*M) = \begin{cases} M & i = 0 \\ 0 & i \neq 0. \end{cases}$$

Indeed, the underlying augmented simplicial set $U(FU)_*M \to UM$ is fibrant and contractible by 8.6.10. If we choose $[0] = \eta(0)$ as basepoint instead of 0, then the contraction satisfies $f_n([0]) = [0]$ for all n, and therefore $U(FU)_*(M)$ is aspherical (by exercise 8.4.6). As the sets $\pi_n U(FU)_*M$ are independent of the choice of basepoint (exercise 8.3.1), the augmented simplicial R-module $FU_*(M) \to M$ is also aspherical, as claimed.

Sheaf Cohomology 8.6.15 Let X be a topological space and Sheaves(X) the category of sheaves of abelian groups on X (1.6.5). If $\mathcal{F}$ is a sheaf we can form the stalks $\mathcal{F}_x$ and take the product $\top(\mathcal{F}) = \prod_{x\in X} x_*(\mathcal{F}_x)$ of the corresponding skyscraper sheaves as in 2.3.12. As $F_x = x_*$ and $U_x(\mathcal{F}) = \mathcal{F}_x$ are adjoint, each $F_xU_x(\mathcal{F}) = x_*(\mathcal{F}_x)$ is a triple. Hence their product $\top$ is a triple on Sheaves(X). Thus we obtain a coaugmented cosimplicial sheaf $\mathcal{F} \xrightarrow{\eta} (\top^{*+1}\mathcal{F})$ and a corresponding augmented cochain complex

$$0 \longrightarrow \mathcal{F} \xrightarrow{\eta} \top(\mathcal{F}) \xrightarrow{\partial^0-\partial^1} \top^2(\mathcal{F}) \xrightarrow{d} \cdots.$$

The resulting resolution of $\mathcal{F}$ by the Γ-acyclic sheaves $\top^{*+1}(\mathcal{F})$ is called the *Godement resolution* of $\mathcal{F}$, since it first appeared in [Gode]. (The proof that the Godement resolution is an exact sequence of sheaves involves interpreting $\prod U_x(\mathcal{F})$ as a sheaf on the disjoint union X^δ of the points of X.)

Example 8.6.16 (Commutative algebras) Let k be a commutative ring and **Commalg** the category of commutative k-algebras. Let $P_* \to R$ be an augmented simplicial object of **Commalg**; if its underlying augmented simplicial set is aspherical we say that P_* is a *simplicial resolution* of R.

The forgetful functor U: **Commalg** $\to$ **Sets** has a left adjoint taking a set X to the polynomial algebra $k[X]$ on the set X; the resulting cotriple $\perp$ on **Commalg** sends R to the polynomial algebra on the set underlying R. As with free resolutions 8.6.14, $U(\perp_* R) \to UR$ is aspherical, so $\perp_* R$ is a simplicial resolution of R. This resolution will be used in 8.8.2 to construct André-Quillen homology.

Another cotriple $\perp^S$ on arises from the left adjoint Sym of the forgetful functor U': **Commalg** $\to$ k**–mod**. The *Symmetric Algebra* Sym(M) of a k-module M is defined to be the quotient of the tensor algebra $T(M)$ by the 2-sided ideal generated by all $(x \otimes y - y \otimes x)$ with $x, y \in M$ (under the identification $i\colon M \hookrightarrow T(M)$). From the presentation of $T(M) \cong k \oplus M \oplus \cdots \oplus M^{\otimes m} \oplus \cdots$ in 7.3.1 it follows that Sym(M) is the free commutative algebra on generators $i(x)$, $x \in M$, subject only to the two k-module relations on M:

$$\alpha\, i(x) = i(\alpha x) \quad \text{and} \quad i(x) + i(y) = i(x+y) \quad (\alpha \in k;\, x, y \in M).$$

Thus any k-module map $M \to R$ into a commutative k-algebra extends uniquely to an algebra map $\mathrm{Sym}(M) \to R$. This gives a natural isomorphism $\mathrm{Hom}_k(M, R) \cong \mathrm{Hom}_{\mathbf{Commalg}}(\mathrm{Sym}(M), R)$, proving that Sym is left adjoint U. The resulting cotriple on **Commalg** sends R to the symmetric algebra $\perp^S(R) = \mathrm{Sym}(U'R)$ and we have a canonical adjunction $\varepsilon\colon \mathrm{Sym}(U'R) \to R$. As the simplicial k-module $U'(\perp_* R) \to U'R$ is aspherical, $\perp_*^S R \to R$ is another simplicial resolution of R in **Commalg**, and there is a simplicial map $\perp_* R \to \perp_*^S R$, natural in R.

Exercise 8.6.6 Let X be a set and M the free k-module with basis X. Show that $\mathrm{Sym}(M)$ is the commutative polynomial ring $k[X]$. Then show that the map $\perp_* k[X] \to \perp_*^S k[X]$ is a simplicial homotopy equivalence.

Exercise 8.6.7 In general, show that $\mathrm{Sym}(M) = k \oplus M \oplus S^2(M) \oplus \cdots \oplus S^n(M) \oplus \cdots$, where $S^n(M)$ is the module $(M \otimes \cdots \otimes M)_{\Sigma_n}$ of coinvariants for the evident permutation action of the n^{th} symmetric group $\sum_n$ on the n-fold tensor product of M.

8.7 Cotriple Homology

Suppose that $\mathcal{A}$ is a category equipped with a cotriple $\perp = (\perp, \varepsilon, \delta)$ as described in the previous section, and suppose given a functor $E\colon \mathcal{A} \to \mathcal{M}$ with $\mathcal{M}$ some abelian category. For each object A in $\mathcal{A}$ we can apply E to the augmented simplicial object $\perp_* A \to A$ to obtain the augmented simplicial object $E(\perp_* A) \to E(A)$ in $\mathcal{M}$.

Definition 8.7.1 (Barr and Beck [BB]) The *cotriple homology of A with coefficients in E* (relative to the cotriple $\perp$) is the sequence of objects $H_n(A; E) = \pi_n E(\perp_* A)$. From the Dold-Kan correspondence, this is the same as the homology of the associated chain complex $C(E \perp_* A)$:

$$0 \leftarrow E(\perp A) \overset{d}{\longleftarrow} E(\perp^2 A) \overset{d}{\longleftarrow} E(\perp^3 A) \leftarrow \cdots.$$

Clearly cotriple homology is functorial with respect to maps $A \to A'$ in $\mathcal{A}$ and natural transformations of the "coefficient functors" $E \to E'$. The augmentation gives a natural transformation $\varepsilon_*^A\colon H_0(A; E) = \pi_0(E \perp_* A) \to E(A)$, but at this level of generality ε_*^A need not be an isomorphism. (Take $\perp = 0$.)

Dually, if $(\top, \eta, \mu)$ is a triple on a category $\mathcal{C}$ and $E\colon \mathcal{C} \to \mathcal{M}$ is a functor, the *triple cohomology* of an object C with coefficients in E is the sequence of objects $H^n(C; E) = \pi^n E(\top^{*+1}C)$, which by definition is the cohomology of the associated cochain complex

$$0 \to E(\top C) \xrightarrow{d} E(\top^2 C) \xrightarrow{d} E(\top^3 C) \to \cdots$$

associated to the cosimplicial object $E(\top^{*+1}C)$ of $\mathcal{M}$. By duality, $H^n(C; E)$ is the object $H_n(C; E^{\mathrm{op}})$ in the opposite category $\mathcal{M}^{\mathrm{op}}$ corresponding to $E^{\mathrm{op}}: \mathcal{C}^{\mathrm{op}} \to \mathcal{M}^{\mathrm{op}}$; we shall not belabor the dual development of triple cohomology.

Another variant occurs when we are given a *contravariant* functor E from $\mathcal{A}$ to $\mathcal{M}$. In this case $E(\perp_* A)$ is a cosimplicial object of $\mathcal{M}$. We set $H^n(A; E) = \pi^n E(\perp_* A)$ and call it the *cotriple cohomology* of A with coefficients in E. Of course if we consider $\perp$ to be a triple on $\mathcal{A}^{\mathrm{op}}$ and take as coefficients $E: \mathcal{A}^{\mathrm{op}} \to \mathcal{M}$, then cotriple cohomology is just triple cohomology in disguise.

Example 8.7.2 (Tor and Ext) Let R be a ring and $\perp$ the free module cotriple on **mod**–R (8.6.6). We saw in 8.6.14 that the chain complex $C(\perp_* M)$ is a free resolution of M for every R-module M. If N is a left R-module and we take $E(M) = M \otimes_R N$, then homology of the chain complex associated to $E(\perp_* M) = (\perp_* M) \otimes_R N$ computes the derived functors of E. Therefore

$$H_n(M; \otimes_R N) = \mathrm{Tor}_n^R(M, N).$$

Similarly, if N is a right R-module and $E(M) = \mathrm{Hom}_R(M, N)$, then the cohomology of the cochain complex associated to $E(\perp_* M) = \mathrm{Hom}_R(\perp_* M, N)$ computes the derived functors of E. Therefore

$$H^n(M; \mathrm{Hom}_R(-, N)) = \mathrm{Ext}_R^n(M, N).$$

Definition 8.7.3 (Barr-Beck [BB]) Let $\perp$ be a fixed cotriple on $\mathcal{A}$ and $\mathcal{M}$ an abelian category. A *theory of $\perp$-left derived functors* (L_n, λ, ∂) is the assignment to every functor $E: \mathcal{A} \to \mathcal{M}$ a sequence of functors $L_n E: \mathcal{A} \to \mathcal{M}$, natural in E, together with a natural transformation $\lambda: L_0 E \Rightarrow E$ such that

1. $\lambda : L_0(E\perp) \cong E\perp$ and $L_n(E\perp) = 0$ for $n \neq 0$ and every E.
2. Whenever $\mathcal{E}: 0 \to E' \to E \to E'' \to 0$ is an exact sequence of functors such that $0 \to E'\perp \to E\perp \to E''\perp \to 0$ is also exact, there are "connecting" maps $\partial: L_n E'' \to L_{n-1} E'$, natural in $\mathcal{E}$, such that the following sequence is exact:

$$\cdots L_n E' \to L_n E \to L_n E'' \xrightarrow{\partial} L_{n-1} E' \to L_{n-1} E \cdots.$$

Uniqueness Theorem 8.7.4 *Cotriple homology $H_*(-;\, E)$ is a theory of $\perp$-left derived functors. Moreover, if (L_n, λ, ∂) is any other theory of $\perp$-left derived functors then there are isomorphisms $L_n E \cong H_n(-;\, E)$, natural in E, under which λ corresponds to ε and ∂ corresponds to the connecting map for $H_*(-;\, E)$.*

Proof A theory of left derived functors is formally similar to a universal (homological) δ-functor on the functor category $\mathcal{M}^{\mathcal{A}}$, the $E\perp$ playing the role of projectives. The proof in 2.4.7 that left derived functors form a universal δ-functor formally goes through, mutatis mutandis, to prove this result as well. ◇

8.7.1 Relative Tor and Ext

8.7.5 Fix an associative ring k and let $k \to R$ be a ring map. The forgetful functor $U\colon$ **mod**–$R \to$ **mod**–k has a left adjoint, the base-change functor $F(M) = M \otimes_k R$. If N is a left R-module, the *relative Tor groups* are defined to be the cotriple homology with coefficients in $\otimes_R N$ (relative to the cotriple $\perp = FU$):

$$\operatorname{Tor}_p^{R/k}(M, N) = H_p(M; \otimes_R N) = \pi_p((\perp_* M) \otimes_R N),$$

which is the homology of the associated chain complex $C(\perp_* M \otimes N)$ (8.3.8). Since $(\perp^{p+1} M) \otimes_R N = (\perp^p M) \otimes_k R \otimes_R N \cong \perp^p M \otimes_k N$, we can give an alternate description of this chain complex as follows. Write $\otimes$ for $\otimes_k$ and $R^{\otimes p}$ for $R \otimes R \otimes \cdots \otimes R$, so that $\perp^p M = M \otimes R^{\otimes p}$. Then $(\perp_* M \otimes N)$ is the simplicial abelian group $[p] \mapsto M \otimes R^{\otimes p} \otimes N$ with face and degeneracy operators

$$\partial_i(m \otimes r_1 \otimes \cdots \otimes r_p \otimes n) = \begin{cases} mr_1 \otimes r_2 \otimes \cdots \otimes r_p \otimes n & \text{if } i = 0 \\ m \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes n & \text{if } 0 < i < p \\ m \otimes r_1 \otimes \cdots \otimes r_{p-1} \otimes r_p n & \text{if } i = p; \end{cases}$$

$$\sigma_i(m \otimes r_1 \otimes \cdots \otimes r_p \otimes n) = m \otimes \cdots \otimes r_{i-1} \otimes 1 \otimes r_i \otimes \cdots \otimes n.$$

(Check this!) Therefore $\operatorname{Tor}_*^{R/k}(M, N)$ is the homology of the chain complex

$$0 \leftarrow M \otimes N \xleftarrow{\partial_0 - \partial_1} M \otimes R \otimes N \xleftarrow{d} M \otimes R^{\otimes 2} \otimes N \leftarrow \cdots M \otimes R^{\otimes p} \otimes N \leftarrow \cdots.$$

As in 2.7.2, one could also start with left modules and form the cotriple homology of the functor $M \otimes_R\colon R$–**mod** $\to$ **Ab** relative to the cotriple $\perp'(N) = R \otimes_k$

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N on R–**mod**. The resulting simplicial abelian group $[p] \mapsto M \otimes R^{\otimes p} \otimes N$ is just the front-to-back dual (8.2.10) of the one described above. This proves that relative Tor is a "balanced" functor in the sense that

$$\mathrm{Tor}_p^{R/k}(M, N) = H_p(M; \otimes_R N) = H_p(N; M \otimes_R).$$

If N is a right R-module we define the *relative Ext groups* to be the cotriple cohomology with coefficients in the contravariant functor $\mathrm{Hom}_R(-, N)$:

$$\mathrm{Ext}^p_{R/k}(M, N) = H^p(M; \mathrm{Hom}_R(-, N)) = \pi^p \mathrm{Hom}_R(\perp_* M, N),$$

which is the same as the cohomology of the associated cochain complex $C(\mathrm{Hom}_R(\perp_* M, N))$. Since $\mathrm{Hom}_R(M \otimes_k R, N) \cong \mathrm{Hom}_k(M, N)$ by 2.6.3, $\mathrm{Hom}_R(\perp_* M, N)$ is naturally isomorphic to the cosimplicial abelian group $[p] \mapsto \mathrm{Hom}_k(M \otimes R^{\otimes p}, N) = \{k\text{-multilinear maps } M \times R^p \to N\}$ with

$$(\partial^i f)(m, r_0, \cdots, r_p) = \begin{cases} f(mr_0, r_1, \ldots, r_p) & \text{if } i = 0 \\ f(m, \ldots, r_{i-1}r_i, \ldots) & \text{if } 0 < i < p \\ f(m, r_0, \ldots, r_{p-1})r_p & \text{if } i = p; \end{cases}$$

$$(\sigma^i f)(m, r_1, \cdots, r_{p-1}) = f(m, \cdots, r_i, 1, r_{i+1}, \cdots, r_{p-1}).$$

Exercise 8.7.1 Show that $\mathrm{Tor}_0^{R/k}(M, N) = M \otimes_R N$ and $\mathrm{Ext}^0_{R/k}(M, N) = \mathrm{Hom}_R(M, N)$.

Example 8.7.6 Suppose that $R = k/I$ for some ideal I of k. Since $\perp M \cong M$ for all M, $(\perp_* M) \otimes N$ and $\mathrm{Hom}_R(\perp_* M, N)$ are the constant simplicial groups $M \otimes N$ and $\mathrm{Hom}(M, N)$, respectively. Therefore $\mathrm{Tor}_i^{R/k}(M, N) = \mathrm{Ext}^i_{R/k}(M, N) = 0$ for $i \neq 0$. This shows one way in which the relative Tor and Ext groups differ from the absolute Tor and Ext groups of Chapter 3.

Just as with the ordinary Tor and Ext groups, the relative Tor and Ext groups can be computed from $\perp$-projective resolutions. For this, we need the following definition.

Definition 8.7.7 A chain complex P_* of R-modules is said to be *k-split* if the underlying chain complex $U(P_*)$ of k-modules is split exact (1.4.1). A resolution $P_* \to M$ is called k-split if its augmented chain complex is k-split.

Lemma 8.7.8 *If $\mathcal{E}: 0 \to M' \to M \to M'' \to 0$ is a k-split exact sequence of R-modules, there are natural long exact sequences*

$$\cdots \mathrm{Tor}_*^{R/k}(M', N) \to \mathrm{Tor}_*^{R/k}(M, N) \to \mathrm{Tor}_*^{R/k}(M'', N) \xrightarrow{\delta} \mathrm{Tor}_{*-1}^{R/k}(M', N) \cdots$$

$$\cdots \mathrm{Ext}^*_{R/k}(M'', N) \to \mathrm{Ext}^*_{R/k}(M, N) \to \mathrm{Ext}^*_{R/k}(M', N) \xrightarrow{\delta} \mathrm{Ext}^{*+1}_{R/k}(M'', N) \cdots$$

Proof Since $U(\mathcal{E})$ is split exact, for every $p \geq 1$ the complexes $(\perp^{p+1} \mathcal{E}) \otimes_R N = U(\mathcal{E}) \otimes_k (R^{\otimes p} \otimes_R N)$ and $\mathrm{Hom}_R(\perp^{p+1} \mathcal{E}, N) = \mathrm{Hom}_k(U\mathcal{E} \otimes_k R^{\otimes p}, N)$ are exact. Taking (co-) homology yields the result. ◇

By combining adjectives, we see that a "k-split $\perp$-projective resolution" of an R-module M is a resolution $P_* \to M$ such that each P_i is $\perp$-projective and the augmented chain complex is k-split.

$$0 \leftarrow M \overset{\dashrightarrow}{\longleftarrow} P_0 \overset{\dashrightarrow}{\longleftarrow} P_1 \overset{\dashrightarrow}{\longleftarrow} P_2 \cdots.$$

For example, we saw in 8.6.12 that the augmented bar resolutions $B(R, M) \to M$ and $\beta(R, M) \to M$ are k-split $\perp$-projective resolutions for every R-module M.

Comparison Theorem 8.7.9 *Let $P_* \to M$ be a k-split $\perp$-projective resolution and $f' \colon M \to N$ an R-module map. Then for every k-split resolution $Q_* \to N$ there is a map $f \colon P_* \to Q_*$ lifting f'. The map f is unique up to chain homotopy equivalence.*

Proof The proof of the Comparison Theorem 2.2.6 goes through. (Check this!) ◇

Theorem 8.7.10 *If $P_* \to M$ is any k-split $\perp$-projective resolution of an R-module M, then there are canonical isomorphisms:*

$$\mathrm{Tor}_*^{R/k}(M, N) \cong H_*(P \otimes_R N),$$
$$\mathrm{Ext}^*_{R/k}(M, N) \cong H^* \mathrm{Hom}_R(P, N).$$

Proof Since $\otimes_R N$ is right exact and $\mathrm{Hom}_R(-, N)$ is left exact, we have isomorphisms $\mathrm{Tor}_0^{R/k}(M, N) \cong M \otimes_R N \cong H_0(P \otimes_R N)$ and $\mathrm{Ext}^0_{R/k}(M, N) \cong \mathrm{Hom}_R(M, N) \cong H^0 \mathrm{Hom}_R(P, N)$. Now the proof in 2.4.7 that derived functors form a universal δ-functor goes through to prove this result. ◇

Lemma 8.7.11 *Suppose R_1 and R_2 are algebras over a commutative ring k; set $\perp_i = R_i \otimes$ and $\perp_{12} = R_1 \otimes R_2 \otimes$. If P_1 is $\perp_1$-projective and P_2 is $\perp_2$-projective, then $P_1 \otimes P_2$ is $\perp_{12}$-projective.*

Proof In general P_i is a summand of $R_i \otimes P_i$, so $P_1 \otimes P_2$ is a summand of $\perp_{12}(P_1 \otimes P_2) \cong (R_1 \otimes P_1) \otimes (R_2 \otimes P_2)$. ◇

Application 8.7.12 (External products for Tor) Suppose k is commutative, and we are given right and left R_1-modules M_1 and N_1 (resp. R_2-modules M_2 and N_2). Choose k-split $\perp_i$-projective resolutions $P_i \to N_i$; $\mathrm{Tot}(P_1 \otimes P_2)$ is therefore a k-split $\perp_{12}$-projective resolution of the $R_1 \otimes R_2$-module $N_1 \otimes N_2$. (Why?) Tensoring with $M_1 \otimes M_2$ yields an isomorphism of chain complexes

$$\mathrm{Tot}\{(M_1 \otimes_{R_1} P_1) \otimes (M_2 \otimes_{R_2} P_2)\} \cong (M_1 \otimes M_2) \otimes_{R_1 \otimes R_2} \mathrm{Tot}(P_1 \otimes P_2).$$

Applying homology yields the external product for relative Tor:

$$\mathrm{Tor}_i^{R/k}(M_1, N_1) \otimes_k \mathrm{Tor}_j^{R_2/k}(M_2, N_2) \to \mathrm{Tor}_{i+j}^{(R_1 \otimes R_2)/k}(M_1 \otimes M_2, N_1 \otimes N_2).$$

As in 2.7.8, the (porism version of the) Comparison Theorem 2.2.7 shows that this product is independent of the choice of resolution. The external product is clearly natural in M_1, N_1, M_2, N_2 and commutes with the connecting homomorphism δ in all four arguments. (Check this!) When $i = j = 0$, it is just the interchange $(M_1 \otimes_{R_1} N_1) \otimes_k (M_2 \otimes_{R_2} N_2) \cong (M_1 \otimes M_2) \otimes_{R_1 \otimes R_2} (N_1 \otimes N_2)$.

The bar resolutions $\beta(R_i, N_i)$ of 8.6.12 are concrete choices of the P_i. The shuffle map $\nabla\colon \mathrm{Tot}\,\beta(R_1, N_1) \otimes \beta(R_2, N_2) \to \beta(R_1 \otimes R_2, N_1 \otimes N_2)$ of 8.6.13 and exercise 8.6.5 may be used in this case to simplify the construction (cf. [MacH, X.7]).

Exercise 8.7.2 (External product for Ext) Use the notation of 8.7.12 to produce natural pairings, commuting with connecting homomorphisms:

$$\mathrm{Ext}^i_{R_1/k}(M_1, N_2) \otimes_k \mathrm{Ext}^j_{R_2/k}(M_2, N_2) \to \mathrm{Ext}^{i+j}_{(R_1 \otimes R_2)/k}(M_1 \otimes M_2, N_1 \otimes N_2).$$

If $i = j = 0$, this is just the map

$$\mathrm{Hom}(M_1, N_1) \otimes \mathrm{Hom}(M_2, N_2) \to \mathrm{Hom}(M_1 \otimes M_2, N_1 \otimes N_2).$$

Example 8.7.13 Suppose that R is a flat commutative algebra over k. If I is an ideal of R generated by a regular sequence $\boldsymbol{x} = (x_1, \cdots, x_d)$, then $T = \mathrm{Tor}_1^{R/k}(R/I, R/I)$ is isomorphic to $(R/I)^d$ and

$$\mathrm{Tor}_i^{R/k}(R/I, R/I) \cong \Lambda^i T \quad \text{for } i \geq 0.$$

In particular these vanish for $i > d$. To see this, we choose the Koszul resolution $K(\boldsymbol{x}) \to R/I$ (4.5.5); each $K_i(\boldsymbol{x}) = \Lambda^i R^d$ is $\perp$-projective. Since every differential in $R/I \otimes_R K(\boldsymbol{x})$ is zero, we have

$$\mathrm{Tor}_i^{R/k}(R/I, R/I) \cong R/I \otimes_R K_i(\boldsymbol{x}) \cong \Lambda^i T.$$

More is true: we saw in exercise 4.5.1 that $K(\boldsymbol{x})$ is a graded-commutative DG-algebra, so $\mathrm{Tor}_*^{R/k}(R/I, R/I)$ is naturally a graded-commutative R/I-algebra, namely via the exterior algebra structure. This product may also be obtained by composing the external product

$$\mathrm{Tor}_*^{R/k}(R/I, R/I) \otimes \mathrm{Tor}_*^{R/k}(R/I, R/I) \to \mathrm{Tor}_*^{R\otimes R/k}(R/I \otimes R/I, R/I \otimes R/I)$$

with multiplication arising from $R \otimes R \to R$ and $R/I \otimes R/I \to R/I$. Indeed, the external product is given by $K(\boldsymbol{x}) \otimes K(\boldsymbol{x})$ and the multiplication is resolved by the Koszul product $K(\boldsymbol{x}) \otimes K(\boldsymbol{x}) \to K(\boldsymbol{x})$; see exercise 4.5.5.

Theorem 8.7.14 (Products of rings) *Let $k \to R$ and $k \to R'$ be ring maps. Then there are natural isomorphisms*

$$\mathrm{Tor}_*^{(R\times R')/k}(M \times M', N \times N') \cong \mathrm{Tor}_*^{R/k}(M, N) \oplus \mathrm{Tor}_*^{R'/k}(M', N'),$$

$$\mathrm{Ext}^*_{(R\times R')/k}(M \times M', N \times N') \cong \mathrm{Ext}^*_{R/k}(M, N) \oplus \mathrm{Ext}^*_{R/k}(M', N').$$

Here M and N are R-modules, M' and N' are R'-modules, and we consider $M \times M'$ and $N \times N'$ as $(R \times R')$-modules by taking products componentwise.

Proof Write $\perp$ and $\perp'$ for the cotriples $\otimes R$ and $\otimes R'$, so that $\perp \oplus \perp'$ is the cotriple $\otimes(R \times R')$. Since $(\perp \oplus \perp')(M \times M') \cong (\perp M) \oplus (\perp M') \oplus (\perp' M) \oplus (\perp' M')$, both $\perp M = M \otimes R$ and $\perp' M' = M' \otimes R'$ are $(\perp \oplus \perp')$-projective $(R \times R')$-modules (exercise 8.6.3). The bar resolutions $\beta(R, M) \to M$ and $\beta(R', M') \to M'$ are therefore k-split $(\perp \oplus \perp')$-projective resolutions; so is the product $\beta(R, M) \times \beta(R', M') \to M \times M'$. Using this resolution to compute relative Tor and Ext over $R \times R'$ yields the desired isomorphisms, in view of the natural k-module isomorphisms

$$(M \times M') \otimes_{(R\times R')} (N \times N') \cong (M \otimes_R N) \oplus (M' \otimes_{R'} N'),$$

$$\mathrm{Hom}_{R\times R'}(M \times M', N \times N') \cong \mathrm{Hom}_R(M, N) \oplus \mathrm{Hom}_{R'}(M', N'). \qquad \diamond$$

Call a right R-module P *relatively flat* if $P \otimes_R N_*$ is exact for every k-split exact sequence of left R-modules N_*. As in exercise 3.2.1 it is easy to see that P is relatively flat if and only if $\mathrm{Tor}_*^{R/k}(P, N) = 0$ for $* \neq 0$ and all left modules N.

Relatively Flat Resolution Lemma 8.7.15 *If $P \to M$ is a k-split resolution of M by relatively flat R-modules, then* $\mathrm{Tor}_*^{R/k}(M, N) \cong H_*(P \otimes_R N)$.

Proof The proof of the Flat Resolution Lemma 3.2.8 goes through in this relative setting. ◇

Corollary 8.7.16 (Flat base change for relative Tor) *Suppose $R \to T$ is a ring map such that T is flat as an R-module. Then for all T-modules M and all R-modules N:*

$$\mathrm{Tor}_*^{R/k}(M, N) \cong \mathrm{Tor}_*^{T/k}(M, T \otimes_R N).$$

Moreover, if R is commutative and $M = L \otimes_R T$ these are isomorphic to

$$\mathrm{Tor}^{R/k}(L \otimes_R T, N) \cong T \otimes_R \mathrm{Tor}_*^{R/k}(L, N).$$

Proof This is like the Flat base change 3.2.9 for absolute Tor. Write $P \to M$ for the k-split resolution associated to $\perp_* M \to M$, with $\perp = \otimes_R T$. The right side is the homology of the chain complex $P \otimes_T (T \otimes_R N) \cong P \otimes_R N$, so it suffices to show that each $P_n = (\perp^n M) \otimes_k T$ is a relatively flat R-module. Because k is commutative there is a natural isomorphism $P \otimes_R N \cong T \otimes_R N \otimes_k (\perp^n M)$ for every N. If N_* is a k-split exact sequence of left R-modules, so is $N_* \otimes_k (\perp^n M)$; since T is flat over R, this implies that $P \otimes_R N_* \cong T \otimes_R N_* \otimes_k (\perp^n M)$ is exact. ◇

Exercise 8.7.3 (Localization) Let S be a central multiplicative set in R, and M, N two R-modules. Show that

$$\mathrm{Tor}^{S^{-1}R/k}(S^{-1}M, S^{-1}N) \cong \mathrm{Tor}^{R/k}(S^{-1}M, N) \cong S^{-1}\,\mathrm{Tor}_*^{R/k}(M, N).$$

Vista 8.7.17 (Algebraic K-theory) Let $\mathcal{R}$ be the category of rings-without-unit. The forgetful functor $U{:}\,\mathcal{R} \to$ **Sets** has a left adjoint functor $F{:}\,$**Sets** $\to \mathcal{R}$, namely the free ring functor. The resulting cotriple $\perp{:}\,\mathcal{R} \to \mathcal{R}$ takes a ring R to the free ring-without-unit on the underlying set of R. For each ring R, the augmented simplicial ring $\perp_* R \to R$ is aspherical in the sense of 8.4.6: the underlying (based, augmented) simplicial set $U(\perp_* R) \to UR$ is aspherical. (To see this, recall from 8.6.10 that $U(\perp_* R)$ is fibrant and left contractible, hence aspherical). If $G{:}\,\mathcal{R} \to$ **Groups** is any functor, the $\perp$-*left derived functors* of G (i.e., derived with respect to $\perp$) are defined to be $L_nG(R) = \pi_nG(\perp_* R)$, the homotopy groups of the simplicial group $G(\perp_* R)$. This is one type of non-abelian homological algebra (see 8.3.5).

Classical examples of such a functor G are the general linear groups $GL_m(R)$, defined for a ring-without-unit R as the kernel of the augmentation $GL_m(\mathbb{Z} \oplus R) \to GL_m(\mathbb{Z})$. The inclusion of $GL_m(R)$ in $GL_{m+1}(R)$ by $M \mapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$ allows us to form the infinite general linear group $GL(R)$ as the union $\cup GL_m(R)$. By inspection, $L_nGL(R) = \lim_{m\to\infty} L_nGL_m(R)$.

One of the equivalent definitions of the higher K-theory of a ring R, due to Gersten and Swan, is

$$K_n(R) = L_{n-2}GL(R) = \pi_{n-2}GL(\perp_* R) \quad \text{for } n \geq 3,$$

while K_1 and K_2 are defined by the exact sequence

$$0 \to K_2(R) \to L_0GL(R) \to GL(R) \to K_1(R) \to 0.$$

If R is a free ring, then $K_n(R) = 0$ for $n \geq 1$, because $GL(\perp_* R) \to GL(R)$ is contractible (8.6.9). If R has a unit, then $L_0GL(R)$ is the infinite Steinberg group $St(R) = \varinjlim St_n(R)$ of 6.9.13; $St(R)$ is the universal central extension of the subgroup $E(R)$ of $GL(R)$ generated by the elementary matrices (6.9.12). For details we refer the reader to [Swan1].

8.8 André-Quillen Homology and Cohomology

In this section we fix a commutative ring k and consider the category **Commalg** $=$ k-**Commalg** of commutative k-algebras R. We begin with a few definitions, which will be discussed further in Chapter 9, section 2.

8.8.1 The *Kähler differentials* of R over k is the R-module $\Omega_{R/k}$ having the following presentation: There is one generator dr for every $r \in R$, with $d\alpha = 0$ if $\alpha \in k$. For each $r, s \in R$ there are two relations:

$$d(r+s) = (dr) + (ds) \quad \text{and} \quad d(rs) = r(ds) + s(dr).$$

If M is a k-module, a *k-derivation* $D: R \to M$ is a k-module homomorphism satisfying $D(rs) = r(Ds) + s(Dr)$; the map $d: R \to \Omega_{R/k}$ (sending r to dr) is an example of a k-derivation. The set $\mathrm{Der}_k(R, M)$ of all k-derivations is an R-module in an obvious way.

Exercise 8.8.1 Show that the k-derivation $d: R \to \Omega_{R/k}$ is universal in the sense that $\mathrm{Der}_k(R, M) \cong \mathrm{Hom}_R(\Omega_{R/k}, M)$.

Exercise 8.8.2 If $R = k[X]$ is a polynomial ring on a set X, show that $\Omega_{k[X]/k}$ is the free R-module with basis $\{dx : x \in X\}$. If K is a k-algebra, conclude that $\Omega_{K[X]/K} \cong K \otimes_k \Omega_{k[X]/k}$. These results will be generalized in exercise 9.1.3 and theorem 9.1.7, using 9.2.2.

Recall from 8.6.16 that there is a cotriple $\perp$ on **Commalg**, $\perp R$ being the polynomial algebra on the set underlying R. If we take the resulting augmented simplicial k-algebra $\perp_* R \to R$, we have canonical maps from $\perp_n R = \perp^{n+1} R$ to R for every n. This makes an R-module M into a $\perp_n R$-module. The next definitions were formulated independently by M. André and D. Quillen in 1967; see [Q].

Definitions 8.8.2 The *André-Quillen cohomology* $D^n(R/k, M)$ of R with values in an R-module M is the cotriple cohomology of R with coefficients in $\mathrm{Der}_k(-, M)$:

$$D^n(R/k, M) = \pi^n \mathrm{Der}_k(\perp_* R, M) = H^n(R; \mathrm{Der}_k(-, M)).$$

The *cotangent complex* $\mathbb{L}_{R/k} = \mathbb{L}_{R/k}(\perp_* R)$ of the k-algebra R is defined to be the simplicial R-module $[n] \mapsto R \otimes_{(\perp_n R)} \Omega_{(\perp_n R)/k}$. The *André-Quillen homology* of R with values in an R-module M is the sequence of R-modules

$$D_n(R/k, M) = \pi_n(M \otimes_R \mathbb{L}_{R/k}).$$

When $M = R$, we write $D_*(R/k)$ for the R-modules $D_*(R/k, R) = \pi_* \mathbb{L}_{R/k}$.

There is a formal analogy: D_* resembles Tor_* and D^* resembles Ext^*. Indeed, the cotangent complex is constructed so that $\mathrm{Hom}_R(\mathbb{L}_{R/k}, M) \cong \mathrm{Der}_k(\perp_* R, M)$ and hence that $D^*(R/k, M) \cong \pi^* \mathrm{Hom}_R(\mathbb{L}_{R/k}, M)$. To see this, note that for each n we have

$$\mathrm{Hom}_R(R \otimes_{(\perp_n R)} \Omega_{(\perp_n R)/k}, M) \cong \mathrm{Hom}_{\perp_n R}(\Omega_{(\perp_n R)/k}, M) \cong \mathrm{Der}_k(\perp_n R, M).$$

Exercise 8.8.3 Show that $D^0(R/k, M) \cong \mathrm{Der}_k(R, M)$ and $D_0(R/k, M) \cong M \otimes_R \Omega_{R/k}$.

Exercise 8.8.4 (Algebra extensions [EGA, IV]) Let $\mathrm{Exalcomm}_k(R, M)$ denote the set of all commutative k-algebra extensions of R by M, that is, the equivalence classes of commutative algebra surjections $E \to R$ with kernel M, $M^2 = 0$. Show that

$$\mathrm{Exalcomm}_k(R, M) \cong D^1(R, M).$$

Hint: Choose a set bijection $E \cong R \times M$ and obtain an element of the module $\mathrm{Hom}_{\mathbf{Sets}}(\perp R, M) \cong \mathrm{Der}_k(\perp^2 R, M)$ by evaluating formal polynomials $f \in \perp R$ in the algebra E.

Exercise 8.8.5 Polynomial k-algebras are $\perp$-projective objects of **Commalg** (8.6.7). Show that if R is a polynomial algebra then for every M and $i \neq 0$ $D^i(R/k, M) = D_i(R/k, M) = 0$. We will see in exercise 9.4.4 that this vanishing also holds for smooth k-algebras.

Exercise 8.8.6 Show that for each M there are universal coefficient spectral sequences

$$E^2_{pq} = \mathrm{Tor}^R_p(D_q(R/k), M) \Rightarrow D_{p+q}(R/k, M);$$
$$E_2^{pq} = \mathrm{Ext}^p_R(D_q(R/k), M) \Rightarrow D^{p+q}(R/k, M).$$

If k is a field, conclude that

$$D_q(R/k, M) \cong D_q(R/k) \otimes_R M \quad \text{and} \quad D^q(R/k, M) \cong \mathrm{Hom}_R(D_q(R/k), M).$$

In order to give the theory more flexibility, we need an analogue of the fact that $\perp$-projective resolutions may be used to compute cotriple homology. We say that an augmented simplicial k-algebra $P_* \to R$ is a *simplicial polynomial resolution* of R if each P_i is a polynomial k-algebra and the underlying augmented simplicial set is aspherical. The polynomial resolution $\perp_* R \to R$ is the prototype of this concept. Since polynomial k-algebras are $\perp$-projective, there is a simplicial homotopy equivalence $P_* \xrightarrow{\sim} \perp_* R$ (2.2.6, 8.6.7). Therefore $\mathrm{Der}_k(P_*, M) \simeq \mathrm{Der}_k(\perp_* R, M)$ and $D^*(R/k, M) \cong \pi^n \mathrm{Der}_k(P_*, M)$. Similarly, there is a chain homotopy equivalence between the cotangent complex $\mathbb{L}_{R/k}$ and the simplicial module $\mathbb{L}_{R/k}(P_*)\colon [n] \mapsto R \otimes_{P_n} \Omega_{P_n/k}$. (Exercise!) Therefore we may also compute homology using the resolution P_*.

8.8.3 Here is one useful application. Suppose that k is noetherian and that R is a finitely generated k-algebra. Then it is possible to choose a simplicial polynomial resolution $P_* \to R$ so that each P_n has finitely many variables. Consequently, if M is a finitely generated R-module, the R-modules $D^q(R/k, M)$ and $D_q(R/k, M)$ are all finitely generated.

8.8.4 (Flat base change) As another application, suppose that R and K are k-algebras such that $\mathrm{Tor}^k_i(K, R) = 0$ for $i \neq 0$. This is the case if K is flat over k. Because these Tors are the homology of the k-module chain complex $C(K \otimes_k \perp_* R)$, it follows that $K \otimes_k \perp_* R \to K \otimes_k R$ is a simplicial polynomial resolution (use 8.4.6). Therefore

$$D^*(K \otimes_k R\,/K, M) \cong \pi^* \mathrm{Der}_K(K \otimes_k \perp_* R, M)$$
$$\cong \pi^* \mathrm{Der}_k(\perp_* R, M) = D^*(R/k, M)$$

for every $K \otimes R$-module M. Similarly, from the fact that $\Omega_{K[X]/k} \cong K \otimes_k \Omega_{k[X]/k}$ for a polynomial ring $k[X]$ it follows that $\mathbb{L}_{K\otimes R/K} \simeq K \otimes_k \mathbb{L}_{R/k}$ and hence that $D_*(K \otimes_k R\ /K) \cong K \otimes_k D_*(R/k)$. This family of results is called *Flat base change*.

Exercise 8.8.7 Show that $D^*(R/k, M) = D_*(R/k, M) = 0$ if R is any localization of k.

8.8.5 As a third application, suppose that R is free as a k-module. This will always be the case when k is a field. We saw in 8.6.16 that the forgetful functor U': **Commalg** $\to$ k–**mod** has a left adjoint Sym; the resulting cotriple $\perp^S(R) = \mathrm{Sym}(U'R)$ is somewhat different than the cotriple $\perp$. Our assumption that R is free implies that $\mathrm{Sym}(U'R)$ is a polynomial algebra, and free as a k-module. Hence $\perp_*^S(R) \to R$ is also a simplicial polynomial resolution of R. Therefore $D^*(R/k, M)$ is isomorphic to the cotriple cohomology $\pi^*(\perp_*^S R, M)$ of R with respect to the cotriple $\perp^S$. Similarly, $\mathbb{L}_{R/k}$ and $\mathbb{L}^S_{R/k} = \{R \otimes_{(\perp_n^s R)} \Omega_{(\perp_n^s R)/k}\}$ are homotopy equivalent, and $D_*(R/k, M) \cong \pi_*(M \otimes_R \mathbb{L}^S_{R/k})$.

8.8.6 (Transitivity) A fourth basic structural result, which we cite from [*Q*], is *Transitivity*. This refers to the following exact sequences for every k-algebra map $K \to R$ and every R-module M:

$$0 \to \mathrm{Der}_K(R, M) \to \mathrm{Der}_k(R, M) \to \mathrm{Der}_k(K, M) \xrightarrow{\delta} \mathrm{Exalcomm}_K(R, M) \to$$

$$\mathrm{Exalcomm}_k(R, M) \to \mathrm{Exalcomm}_k(K, M) \xrightarrow{\delta} D^2(R/K, M) \to \cdots$$

$$\cdots \to D^n(R/K, M) \to D^n(R/k, M) \to D^n(K/k, M) \xrightarrow{\delta} D^{n+1}(R/K, M) \to \cdots,$$

and its homology analogue:

$$\cdots \to D_{n+1}(R/K) \xrightarrow{\partial} R \otimes_K D_n(K/k) \to D_n(R/k) \to D_n(R/K) \xrightarrow{\partial} D_{n-1}(R/k) \to \cdots.$$

The end of this sequence is the first fundamental sequence 9.2.6 for $\Omega_{R/k}$.

Exercise 8.8.8 Suppose that k is a noetherian local ring with residue field $F = R/\mathfrak{m}$. Show that $D^1(F/k) \cong D_1(F/k) \cong \mathfrak{m}/\mathfrak{m}^2$, and conclude that if R is a k/I-algebra we may have $D^*(R/k, M) \neq D^*(R/(k/I), M)$.

Exercise 8.8.9 (Barr) In this exercise we interpret André-Quillen homology as a cotriple homology. For a commutative k-algebra R, let **Commalg/R** be the "comma" category whose objects are k-algebras P equipped with an algebra map $P \to R$, and whose morphisms $P \to Q$ are algebra maps such that $P \to R$ factors as $P \to Q \to R$. Let Diff: **Commalg/**$R \to R$–**mod** be the functor $\text{Diff}(P) = \Omega_{P/k} \otimes_P R$. Show that $\perp$ induces a cotriple on **Commalg/**R, and that if we consider R as the terminal object in **Commalg/**R, then the cotriple homology groups (8.7.1) are André-Quillen homology:

$$D_n(R/k) = H_n(R; \text{Diff}) \quad \text{and} \quad D_n(R/k, M) = H_n(R; \text{Diff} \otimes_R M).$$

8.8.1 Relation to Hochschild Theory

When k is a field of characteristic zero, there is a much simpler way to calculate $D^*(R/k, M)$ and $D_*(R/k, M)$, due to M. Barr [Barr].

Barr's Theorem 8.8.7 *Suppose $C_*(R)$ is an R-module chain complex, natural in R for each R in* **Commalg**, *such that*

1. $H_0(C_*(R)) \cong \Omega_{R/k}$ *for each R.*
2. *If R is a polynomial algebra, $C_*(R) \to \Omega_{R/k}$ is a split exact resolution.*
3. *For each p there is a functor F_p: k–***mod** $\to k$–**mod** *such that $C_p(R) \cong R \otimes_k F_p(UR)$, where UR is the k-module underlying R.*

Then there are natural isomorphisms

$$D^q(R/k, M) \cong H^q \operatorname{Hom}_R(C_*(R), M) \text{ and}$$
$$D_q(R/k, M) \cong H_q(M \otimes_R C_*(R)).$$

Proof We give the proof for cohomology, the proof for homology being similar but more notationally involved. Form the first quadrant double complex

$$E_0^{pq} = \operatorname{Hom}_R(C_p(\perp_q^S R), M)$$

with horizontal differentials coming from C_* and vertical differentials coming from the naturality of the C_p. We shall compute $H^* \operatorname{Tot}(E_0)$ in two ways.

If we fix q, the ring $\perp_q^S R$ is polynomial, so by (2) $C_*(\perp_q^S R) \to \Omega_{\perp_q^S R/k}$ is split exact. Hence $H^p \operatorname{Hom}_R(C_*(\perp_q^S R), M) = 0$ for $p \neq 0$, while

$$H^0 \operatorname{Hom}_R(C_*(\perp_q^S R), M) \cong \operatorname{Hom}_R(\Omega_{\perp_q^S R/k}, M) \cong \operatorname{Der}_k(\perp_q^S R, M).$$

Thus the spectral sequence 5.6.2 associated to the row-filtration on E_0 degenerates at E_2 to yield $H^q \operatorname{Tot}(E_0) \cong H^q \operatorname{Der}_k(\perp_*^S R, M) = D^q(R/k, M)$.

On the other hand, if we fix p and set $G(L) = \operatorname{Hom}_k(F_p(L), M)$ we see by condition (3) that $E_0^{p*} = G(U \perp_*^S R)$. But the augmented simplicial k-module $U \perp_*^S R \to UR$ is left contractible (8.4.6), because $\perp^S R = \operatorname{Sym}(UR)$

(see 8.6.10). As G is a functor, $E_0^{p*} \to G(UR) = \mathrm{Hom}_R(C_p(R), M)$ is also left contractible, hence aspherical. Thus $H^q(E_0^{p*}) = 0$ for $q \neq 0$, and $H^0(E_0^{p*}) \cong \mathrm{Hom}_R(C_p(R), M)$. Thus the spectral sequence 5.6.1 associated to the column filtration degenerates at E_2 as well, yielding $H^p\,\mathrm{Tot}(E_0) \cong H^p\,\mathrm{Hom}_R(C_*(R), M)$. ◇

Preview 8.8.8 In the next chapter, we will construct the Hochschild homology $H_*(R, R)$ of a commutative k-algebra R as the homology of a natural R-module chain complex $C_*^h(R)$ with $C_p^h(R) = R \otimes_k F_p(UR)$, $F_p(L)$ being the p-fold tensor product $(L \otimes_k L \otimes_k \cdots \otimes_k L)$. There is a natural isomorphism $H_1(R, R) \cong \Omega_{R/k}$ and the map $C_1^h(R) \to C_0^h(R)$ is zero. We will see in 9.4.7 that if R is a polynomial algebra, then $H_n(R, R) \cong \Omega^n_{R/k}$, so C_*^h does not quite satisfy condition (2) of Barr's Theorem.

To remedy this, we need the Hodge decomposition of Hochschild homology from 9.4.15. When $\mathbb{Q} \subseteq k$ there are natural decompositions $F_p(L) = \oplus F_p(L)^{(i)}$ such that each $C_*^h(R)^{(i)} = R \otimes_k F_*(UR)^{(i)}$ is a chain subcomplex of $C_*^h(R)$ and $C_*^h(R) = \oplus C_*^h(R)^{(i)}$. If M is an R-module (an $R{-}R$ bimodule via $mr = rm$), set $H_n^{(i)}(R, M) = H_n(M \otimes_R C_*^h(R)^{(i)})$ and $H^n_{(i)}(R, M) = H^n\,\mathrm{Hom}_R(C_*^h(R)^{(i)}, M)$. The Hodge decomposition is

$$H_n(R, M) = \oplus H_n^{(i)}(R, M) \quad \text{and} \quad H^n(R, M) = \oplus H^n_{(i)}(R, M).$$

If R is a polynomial algebra, then $H_n^{(i)}(R, R) = 0$ for $i \neq n$, and $H_n^{(n)}(R, R) \cong \Omega^n_{R/k}$ is a free R-module (exercise 9.4.4). In particular, since $C_n^h(R)^{(i)} = 0$ for $i > n$ the augmented complex $C_*^h(R)^{(i)} \to \Omega^i_{R/k}[-i]$ is split exact for all i.

If we let $C_p(R)$ be $C_{p+1}^h(R)^{(1)}$, then the above discussion show that C_* satisfies the conditions of Barr's Theorem 8.8.7. In summary, we have proven the following.

Corollary 8.8.9 *Suppose that k is a field of characteristic zero. Then André-Quillen homology is a direct summand of Hochschild homology, and André-Quillen cohomology is a direct summand of Hochschild cohomology:*

$$D_q(R/k, M) \cong H_{q+1}^{(1)}(R, M) \quad \text{and} \quad D^q(R/k, M) \cong H^{q+1}_{(1)}(R, M).$$