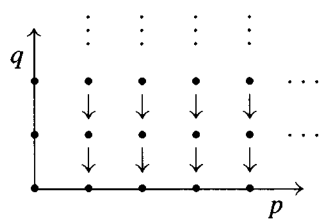
5
Spectral Sequences

5.1 Introduction

Spectral sequences were invented by Jean Leray, as a prisoner of war during World War II, in order to compute the homology (or cohomology) of a chain complex [Leray]. They were made algebraic by Koszul in 1945.

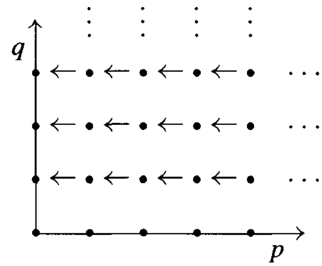
In order to motivate their construction, consider the problem of computing the homology of the total chain complex T_* of a first quadrant double complex E_{**}. As a first step, it is convenient to forget the horizontal differentials and add a superscript zero, retaining only the vertical differentials d^v along the columns E^0_{p*}.

E^0_{**}:



If we write E^1_{pq} for the vertical homology $H_q(E^0_{p*})$ at the (p, q) spot, we may once again arrange the data in a lattice, this time using the horizontal diffentials d^h.

E^1_{**}:



Now we write E^2_{pq} for the horizontal homology $H_p(E^1_{*q})$ at the (p, q) spot. In a sense made clearer by the following exercises, the elements of E^2_{pq} are a second-order approximation of the homology of $T_* = \mathrm{Tot}(E_{**})$.

Exercise 5.1.1 Suppose that the double complex E consists solely of the two columns p and $p-1$. Fix n and set $q = n - p$, so that an element of $H_n(T)$ is represented by an element $(a, b) \in E_{p-1,q+1} \times E_{pq}$. Show that we have calculated the homology of $T = \mathrm{Tot}(E)$ up to extension in the sense that there is a short exact sequence

$$0 \to E^2_{p-1,q+1} \to H_{p+q}(T) \to E^2_{pq} \to 0.$$

Exercise 5.1.2 (Differentials at the E^2 stage)

1. Show that E^2_{pq} can be presented as the group of all pairs (a, b) in $E_{p-1,q+1} \times E_{pq}$ such that $0 = d^v b = d^v a + d^h b$, modulo the relation that these pairs are trivial: $(a, 0)$; $(d^h x, d^v x)$ for $x \in E_{p,q+1}$; and $(0, d^h c)$ for all $c \in E_{p+1,q}$ with $d^v c = 0$.
2. If $d^h(a) = 0$, show that such a pair (a, b) determines an element of $H_{p+q}(T)$.
3. Show that the formula $d(a, b) = (0, d^h(a))$ determines a well-defined map

$$d: E^2_{pq} \to E^2_{p-2,q+1}.$$

Exercise 5.1.3 (Exact sequence of low degree terms) Recall that we have assumed that E^0_{pq} vanishes unless both $p \geq 0$ and $q \geq 0$. By diagram chasing, show that $E^2_{00} = H_0(T)$ and that there is an exact sequence

$$H_2(T) \to E^2_{20} \xrightarrow{d} E^2_{01} \to H_1(T) \to E^2_{10} \to 0.$$

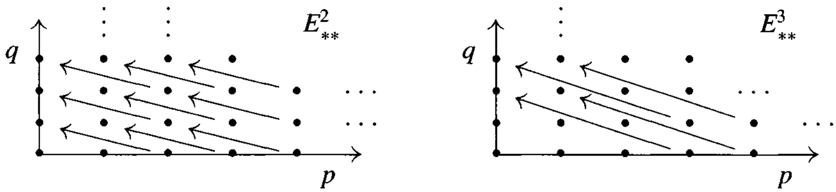


Figure 5.1. The steps E^2 and E^3 of the spectral sequence.

There is an algorithm for computing $H_*(T)$ up to extension, called a *spectral sequence*, and we have just performed the first two steps of this algorithm. The next two steps are illustrated in Figure 5.1.

5.2 Terminology

Definition 5.2.1 A *homology spectral sequence* (*starting with* E^a) in an abelian category $\mathcal{A}$ consists of the following data:

1. A family $\{E^r_{pq}\}$ of objects of $\mathcal{A}$ defined for all integers p, q, and $r \geq a$
2. Maps $d^r_{pq}: E^r_{pq} \to E^r_{p-r,q+r-1}$ that are differentials in the sense that $d^r d^r = 0$, so that the "lines of slope $-(r+1)/r$" in the lattice E^r_{**} form chain complexes (we say the differentials go "to the left")
3. Isomorphisms between E^{r+1}_{pq} and the homology of E^r_{**} at the spot E^r_{pq}:

$$E^{r+1}_{pq} \cong \ker(d^r_{pq})/\text{image } (d^r_{p+r,q-r+1})$$

Note that E^{r+1}_{pq} is a subquotient of E^r_{pq}. The *total degree* of the term E^r_{pq} is $n = p + q$; the terms of total degree n lie on a line of slope -1, and each differential d^r_{pq} decreases the total degree by one.

There is a category of homology spectral sequences; a morphism $f: E' \to E$ is a family of maps $f^r_{pq}: E'^r_{pq} \to E^r_{pq}$ in $\mathcal{A}$ (for r suitably large) with $d^r f^r = f^r d^r$ such that each f^{r+1}_{pq} is the map induced by f^r_{pq} on homology.

Example 5.2.2 A *first quadrant (homology) spectral sequence* is one with $E^r_{pq} = 0$ unless $p \geq 0$ and $q \geq 0$, that is, the point (p, q) belongs to the first quadrant of the plane. (If this condition holds for $r = a$, it clearly holds for all r.) If we fix p and q, then $E^r_{pq} = E^{r+1}_{pq}$ for all large r ($r > \max\{p, q + 1\}$ will do), because the d^r landing in the (p, q) spot come from the fourth quadrant, while the d^r leaving E^r_{pq} land in the second quadrant. We write E^∞_{pq} for this stable value of E^r_{pq}.

Dual Definition 5.2.3 A *cohomology spectral sequence* (*starting with* E_a) in $\mathcal{A}$ is a family $\{E_r^{pq}\}$ of objects ($r \geq a$), together with maps d_r^{pq} going "to the right":

$$d_r^{pq}: E_r^{pq} \to E_r^{p+r,q-r+1},$$

which are differentials in the sense that $d_r d_r = 0$, and isomorphisms between E_{r+1} and the homology of E_r. In other words, it is the same thing as a homology spectral sequence, reindexed via $E_r^{pq} = E^r_{-p,-q}$, so that d_r *increases* the total degree $p+q$ of E^r_{pq} by one.

There is a category of cohomology spectral sequences; a morphism $f: E' \to E$ is a family of maps $f_r^{pq}: E_r'^{pq} \to E_r^{pq}$ in $\mathcal{A}$ (for r suitably large) with $d_r f_r = f_r d_r$ such that each f_{r+1}^{pq} is the map induced by f_r^{pq}.

Mapping Lemma 5.2.4 *Let $f: \{E^r_{pq}\} \to \{E'^r_{pq}\}$ be a morphism of spectral sequences such that for some fixed r, $f^r: E^r_{pq} \cong E'^r_{pq}$ is an isomorphism for all p and q. The 5-lemma implies that $f^s: E^s_{pq} \cong E'^s_{pq}$ for all $s \geq r$ as well.*

Bounded Convergence 5.2.5 A homology spectral sequence is said to be *bounded* if for each n there are only finitely many nonzero terms of total degree n in E^a_{**}. If so, then for each p and q there is an r_0 such that $E^r_{pq} = E^{r+1}_{pq}$ for all $r \geq r_0$. We write E^∞_{pq} for this stable value of E^r_{pq}.

We say that a bounded spectral sequence *converges to* H_* if we are given a family of objects H_n of $\mathcal{A}$, each having a *finite* filtration

$$0 = F_s H_n \subseteq \cdots \subseteq F_{p-1}H_n \subseteq F_p H_n \subseteq F_{p+1}H_n \subseteq \cdots \subseteq F_t H_n = H_n,$$

and we are given isomorphisms $E^\infty_{pq} \cong F_p H_{p+q}/F_{p-1}H_{p+q}$. The traditional symbolic way of describing such a bounded convergence is like this:

$$E^a_{pq} \Rightarrow H_{p+q}.$$

Similarly, a cohomology spectral sequence is called *bounded* if there are only finitely many nonzero terms in each total degree in E_a^{**}. In a bounded cohomology spectral sequence, we write E_∞^{pq} for the stable value of the terms E_r^{pq} and say the (bounded) spectral sequence *converges to* H^* if there is a *finite* filtration

$$0 = F^t H^n \subseteq \cdots F^{p+1}H^n \subseteq F^p H^n \cdots \subseteq F^s H^n = H^n \text{ so that}$$
$$E_\infty^{pq} \cong F^p H^{p+q}/F^{p+1}H^{p+q}.$$

Example 5.2.6 If a first quadrant homology spectral sequence converges to H_*, then each H_n has a finite filtration of length $n+1$:

$$0 = F_{-1}H_n \subseteq F_0H_n \subseteq \cdots \subseteq F_{n-1}H_n \subseteq F_nH_n = H_n.$$

The bottom piece $F_0H_n = E^\infty_{0n}$ of H_n is located on the y-axis, and the top quotient $H_n/F_{n-1}H_n \cong E^\infty_{n0}$ is located on the x-axis. Note that each arrow landing on the x-axis is zero, and each arrow leaving the y-axis is zero. Therefore each E^∞_{0n} is a subobject of E^a_{0n}, and each E^∞_{n0} is a quotient of E^a_{n0}. The terms E^r_{0n} on the y-axis are called the *fiber* terms, and the terms E^r_{n0} on the x-axis are called the *base* terms for reasons that will become apparent in the next section. The resulting maps $E^a_{0n} \to E^\infty_{0n} \subset H_n$ and $H_n \to E^\infty_{n0} \subset E^a_{n0}$ are known as the *edge homomorphisms* of the spectral sequence for the obvious visual reason. Similarly, if a first quadrant cohomology spectral sequence converges to H^*, then H^n has a finite filtration:

$$0 = F^{n+1}H^n \subseteq F^nH^n \subseteq \cdots \subseteq F^1H^n \subseteq F^0H^n = H^n.$$

In this case, the bottom piece $F^nH^n \cong E^{n0}_\infty$ is located on the x-axis, and the top quotient $H^n/F^1H^n \cong E^{0n}_\infty$ is located on the y-axis. In this case, the edge homomorphisms are the maps $E^{n0}_a \to E^{n0}_\infty \subset H^n$ and $H^n \to E^{0n}_\infty \subset E^{0n}_a$.

Definition 5.2.7 A (homology) spectral sequence *collapses at* E^r $(r \geq 2)$ if there is exactly one nonzero row or column in the lattice $\{E^r_{pq}\}$. If a collapsing spectral sequence converges to H_*, we can read the H_n off: H_n is the unique nonzero E^r_{pq} with $p+q=n$. The overwhelming majority of all applications of spectral sequences involve spectral sequences that collapse at E^1 or E^2.

Exercise 5.2.1 (2 columns) Suppose that a spectral sequence converging to H_* has $E^2_{pq}=0$ unless $p=0, 1$. Show that there are exact sequences

$$0 \to E^2_{1,n-1} \to H_n \to E^2_{0n} \to 0.$$

Exercise 5.2.2 (2 rows) Suppose that a spectral sequence converging to H_* has $E^2_{pq}=0$ unless $q=0,1$. Show that there is a long exact sequence

$$\cdots H_{p+1} \to E^2_{p+1,0} \xrightarrow{d} E^2_{p-1,1} \to H_p \to E^2_{p0} \xrightarrow{d} E^2_{p-2,1} \to H_{p-1} \cdots.$$

If a spectral sequence is not bounded, everything is more complicated, and there is no uniform terminology in the literature. For example, a filtration in [CE] is "regular" if for each n there is an N such that $H_n(F_pC)=0$ for $p<N$,

and all filtrations are exhaustive. In [MacH] exhaustive filtrations are called "convergent above." In [EGA, $0_{\text{III}}(11.2)$] even the definition of spectral sequence is different, and "regular" spectral sequences are not only convergent but also bounded below. In what follows, we shall mostly follow the terminology of Bourbaki [BX, p.175].

E^∞ **Terms 5.2.8** Given a homology spectral sequence, we see that each E^{r+1}_{pq} is a subquotient of the previous term E^r_{pq}. By induction on r, we see that there is a nested family of subobjects of E^a_{pq}:

$$0 = B^a_{pq} \subseteq \cdots \subseteq B^r_{pq} \subseteq B^{r+1}_{pq} \subseteq \cdots \subseteq Z^{r+1}_{pq} \subseteq Z^r_{pq} \subseteq \cdots \subseteq Z^a_{pq} = E^a_{pq}$$

such that $E^r_{pq} \cong Z^r_{pq}/B^r_{pq}$. We introduce the intermediate objects

$$B^\infty_{pq} = \bigcup_{r=a}^{\infty} B^r_{pq} \quad \text{and} \quad Z^\infty_{pq} = \bigcap_{r=a}^{\infty} Z^r_{pq}$$

and define $E^\infty_{pq} = Z^\infty_{pq}/B^\infty_{pq}$. In a bounded spectral sequence both the union and intersection are finite, so $B^\infty_{pq} = B^r_{pq}$ and $Z^\infty_{pq} = Z^r_{pq}$ for large r. Thus we recover our earlier definition: $E^\infty_{pq} = E^r_{pq}$ for large r.

Warning: In an unbounded spectral sequence, we will tacitly assume that B^∞_{pq}, Z^∞_{pq}, and E^∞_{pq} exist! The reader who is willing to only work in the category of modules may ignore this difficulty. The queasy reader should assume that the abelian category $\mathcal{A}$ satisfies axioms $(AB4)$ and $(AB4^*)$.

Exercise 5.2.3 (Mapping Lemma for E^∞) Let $f: \{E^r_{pq}\} \to \{E'^r_{pq}\}$ be a morphism of spectral sequences such that for some r (hence for all large r by 5.2.4) $f^r : E^r_{pq} \cong E'^r_{pq}$ is an isomorphism for all p and q. Show that $f^\infty: E^\infty_{pq} \cong E'^\infty_{pq}$ as well.

Definition 5.2.9 (Bounded below) Bounded below spectral sequences have good convergence properties. A homology spectral sequence is said to be *bounded below* if for each n there is an integer $s = s(n)$ such that the terms E^a_{pq} of total degree n vanish for all $p < s$. Bounded spectral sequences are bounded below. Right half-plane homology spectral sequences are bounded below but not bounded.

Dually, a cohomology spectral sequence is said to be *bounded below* if for each n the terms of total degree n vanish for large p. A left half-plane cohomology spectral sequence is bounded below but not bounded.

Definition 5.2.10 (Regular) Regularity is the most useful general condition for convergence used in practice; bounded below spectral sequences are also regular. We say that a spectral sequence is *regular* if for each p and q the differentials d^r_{pq} (or d_r^{pq}) leaving E^r_{pq} (or E_r^{pq}) are zero for all large r. Note that a spectral sequence is regular iff for each p and q: $Z^\infty_{pq} = Z^r_{pq}$ for all large r.

Convergence 5.2.11 We say the spectral sequence *weakly converges to* H_* if we are given objects H_n of $\mathcal{A}$, each having a filtration

$$\cdots \subseteq F_{p-1}H_n \subseteq F_pH_n \subseteq F_{p+1}H_n \subseteq \cdots \subseteq H_n,$$

together with isomorphisms $\beta_{pq}: E^\infty_{pq} \cong F_pH_{p+q}/F_{p-1}H_{p+q}$ for all p and q. Note that a weakly convergent spectral sequence cannot detect elements of $\cap F_pH_n$, nor can it detect elements in H_n that are not in $\cup F_pH_n$.

We say that the spectral sequence $\{E^r_{pq}\}$ *approaches* H_* (or *abuts to* H_*) if it weakly converges to H_* and we also have $H_n = \cup F_pH_n$ and $\cap F_pH_n = 0$ for all n. Every weakly convergent spectral sequence approaches $\cup F_pH_*/\cap F_pH_*$.

We say that the spectral sequence *converges to* H_* if it approaches H_*, it is regular, and $H_n = \varprojlim(H_n/F_pH_n)$ for each n. A bounded below spectral sequence converges to H_* whenever it approaches H_*, because the inverse limit condition is always satisfied in a bounded below spectral sequence.

To show that our notion of convergence is a good one, we offer the following Comparison Theorem. If $\{E^r_{pq}\}$ and $\{E'^r_{pq}\}$ weakly converge to H_* and H'_*, respectively, we say that a map $h: H_* \to H'_*$ is *compatible* with a morphism $f: E \to E'$ if h maps F_pH_n to $F_pH'_n$ and the associated maps $F_pH_n/F_{p-1}H_n \to F_pH'_n/F_{p-1}H'_n$ correspond under β and β' to $f^\infty_{pq}: E^\infty_{pq} \to E'^\infty_{pq}$ $(q = n - p)$.

Comparison Theorem 5.2.12 *Let $\{E^r_{pq}\}$ and $\{E'^r_{pq}\}$ converge to H_* and H'_*, respectively. Suppose given a map $h: H_* \to H'_*$ compatible with a morphism $f: E \to E'$ of spectral sequences. If $f^r: E^r_{pq} \cong E'^r_{pq}$ is an isomorphism for all p and q and some r (hence for $r = \infty$ by the Mapping Lemma), then $h: H_* \to H'_*$ is an isomorphism.*

Proof Weak convergence gives exact sequences

$$\begin{tikzcd}
0 \arrow[r] & F_{p-1}H_n/F_sH_n \arrow[r] \arrow[d] & F_pH_n/F_sH_n \arrow[r] \arrow[d] & E^\infty_{p,n-p} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & F_{p-1}H'_n/F_sH'_n \arrow[r] & F_pH'_n/F_sH'_n \arrow[r] & E'^\infty_{p,n-p} \arrow[r] & 0.
\end{tikzcd}$$

Fixing s, induction on p shows that $F_pH_n/F_sH_n \cong F_pH'_n/F_sH'_n$ for all p. Since $H_n = \cup F_pH_n$, this yields $H_n/F_sH_n \cong H'_n/F_sH'_n$ for all s. Taking inverse limits yields the desired isomorphism $H_n \cong H_n$. ◇

Remark The same spectral sequence may converge to two different graded groups H_*, and it can be very difficult to reconstruct a picture of H_* from this data. For example, knowing that a first quadrant spectral sequence has $E^\infty_{pq} \cong \mathbb{Z}/2$ for all p and q does not allow us to determine whether H_3 is $\mathbb{Z}/16$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/8$, or even the group $(\mathbb{Z}/2)^4$. The Comparison Theorem 5.2.12 helps us reconstruct H_* without the need for convergence.

Multiplicative Structures 5.2.13 Suppose that for $r = a$ we are given a bigraded product

$$(*) \qquad E^r_{p_1q_1} \times E^r_{p_2q_2} \to E^r_{p_1+p_2,q_1+q_2}$$

such that the differential d^r satisfies the Leibnitz relation

$$(**) \qquad d^r(x_1x_2) = d^r(x_1)x_2 + (-1)^{p_1}x_1d^r(x_2), \qquad x_i \in E^r_{p_iq_i}.$$

Then the product of two cycles (boundaries) is again a cycle (boundary), and by induction we have $(*)$ and $(**)$ for every $r \geq a$. We shall call this a *multiplicative structure* on the spectral sequence. Clearly this can be a useful tool in explicit calculations.

5.3 The Leray-Serre Spectral Sequence

Before studying the algebraic aspects of spectral sequences, we shall illustrate their computational power by citing the topological applications that led to their creation by Leray. The material in this section is taken from [MacH, XI.2].

Definition 5.3.1 A sequence $F \xrightarrow{i} E \xrightarrow{\pi} B$ of based topological spaces is called a *Serre fibration* if F is the inverse image $\pi^{-1}(*_B)$ of the basepoint of B and if π has the following "homotopy lifting property": if P is any finite polyhedron and I is the unit interval $[0, 1]$, $g\colon P \to E$ is a map, and $H\colon P \times I \to B$ is a homotopy between $\pi g = H(-, 0)$ and $h_1 = H(-, 1)$,

$$\begin{array}{ccc} P & \xrightarrow{g} & E \\ {\scriptstyle \times 0}\downarrow & G\nearrow & \downarrow{\scriptstyle \pi} \\ P \times I & \xrightarrow[H]{} & B \end{array}$$

there is a homotopy $G: P \times I \to E$ between g and a map $g_1 = G(-, 1)$ which lifts H in the sense that $\pi G = H$. The spaces F, E, and B are called the *Fiber*, *total space* (*Espace totale* for Leray), and *Base space*, respectively. The importance of Serre fibrations lies in the fact (proven in Serre's thesis) that associated to each fibration is a long exact sequence of homotopy groups

$$\cdots \pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F) \to \pi_n(E) \to \pi_n(B) \xrightarrow{\partial} \cdots.$$

In order to simplify the presentation below, we shall assume that B is *simply connected*, that is, that $\pi_0(B) = \pi_1(B) = 0$. Without this assumption, we would have to introduce the action of $\pi_1(B)$ on the homology of F and talk about the homology of B with "local coefficients" in the twisted bundles $H_q(F)$.

Theorem 5.3.2 (Leray-Serre spectral sequence) *Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a Serre fibration such that B is simply connected. Then there is a first quadrant homology spectral sequence starting with E^2 and converging to $H_*(E)$:*

$$E^2_{pq} = H_p(B; H_q(F)) \Rightarrow H_{p+q}(E).$$

Addendum 1 $H_0(B) = \mathbb{Z}$, so along the y-axis we have $E^2_{0q} = H_q(F)$. Because $E^2_{pq} = 0$ for $p < 0$, the groups $E^3_{0q}, \cdots, E^{n+1}_{0q} = E^\infty_{0q}$ are successive quotients of E^2_{0q}. The theorem states that $E^\infty_{0q} \cong F_0 H_q(E)$, so there is an "edge map"

$$H_q(F) = E^2_{0q} \twoheadrightarrow E^\infty_{0q} \subseteq H_q(E).$$

This edge map is the map $i_*: H_q(F) \to H_q(E)$.

Addendum 2 Suppose that $\pi_0(F) = 0$, so that $H_0(F) = \mathbb{Z}$. Along the x-axis we then have $E^2_{p0} = H_p(B)$. Because $E^2_{pq} = 0$ for $q < 0$, the groups $E^3_{p0}, \cdots, E^{n+1}_{p0} = E^\infty_{p0}$ are successive subgroups of E^2_{p0}. The theorem states that $E^\infty_{p0} \cong H_p(E)/F_{p-1}H_p(E)$, so there is an "edge map"

$$H_p(E) \twoheadrightarrow E^\infty_{p0} \hookrightarrow E^2_{p0} = H_p(B).$$

This edge map is the map $\pi_*: H_p(E) \to H_p(B)$.

Remark The Universal Coefficient Theorem 3.6.4 tells us that

$$H_p(B; H_q(F)) \cong H_p(B) \otimes H_q(F) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(B), H_q(F)).$$

Therefore the terms E^2_{pq} are not hard to calculate. In particular, since $\pi_1(B) = 0$ we have $H_1(B) = H_1(B; H_q(F)) = 0$ for all q. By the Hurewicz homomorphism, $\pi_2(B) \cong H_2(B)$ and therefore $H_2(B; H_q(F)) \cong H_2(B) \otimes H_q(F)$ for all q as well.

Application 5.3.3 (Exact sequence of low degree terms) In the lower left corner of this spectral sequence we find

$$E^2_{**}: \quad \begin{tikzcd}
H_2(F) & 0 & \bullet \\
H_1(F) & 0 & \bullet \arrow[ull] & \bullet \\
\mathbb{Z} & 0 & H_2(B) \arrow[ull] & H_3(B) & H_4(B) \arrow[ull]
\end{tikzcd}$$

The kernel of the map $d^2 = d^2_{20}$ is the quotient E^∞_{20} of $H_2(E)$, because the maps d^r_{20} are zero for $r \geq 3$. Similarly, the cokernel of d^2 is the subgroup E^∞_{01} of $H_1(E)$. From this we obtain the exact homology sequence in the following diagram:

$$\begin{tikzcd}
\pi_3(B) \arrow[r] \arrow[d] & \pi_2(F) \arrow[r] \arrow[d] & \pi_2(E) \arrow[r] \arrow[d] & \pi_2(B) \arrow[r] \arrow[d, "\cong"] & \pi_1(F) \arrow[r] \arrow[d] & \pi_1(E) \arrow[r] \arrow[d] & 0 \\
X \arrow[r, hook] & H_2(F) \arrow[r] & H_2(E) \arrow[r] & H_2(B) \arrow[r, "d^2"] & H_1(F) \arrow[r] & H_1(E) \arrow[r] & 0.
\end{tikzcd}$$

Here the group labeled X contains the image in $H_2(F)$ of $E^2_{21} \cong H_2(B) \otimes H_1(F)$ and elements related to $E^3_{30} = H_3(B)$. Thus $H_2(B) \otimes H_1(F)$ is the first obstruction involved in finding a long exact sequence for the homology of a fibration.

Application 5.3.4 (Loop spaces) Let PB denote the space of *based paths* in B, that is, maps $[0, 1] \to B$ sending 0 to $*_B$. The subspace of *based loops* in B (maps $[0, 1] \to B$ sending 0 and 1 to $*_B$) is written ΩB. There is a fibration $\Omega B \to PB \xrightarrow{\pi} B$, where π is evaluation at $1 \in [0, 1]$. The space PB is contractible, because paths may be pulled back along themselves to the basepoint, so $H_n(PB) = 0$ for $n \neq 0$. Therefore, except for $E^\infty_{00} = \mathbb{Z}$, we have a spectral sequence converging to zero. From the low degree terms (assuming that $\pi_1(B) = 0$!), we see that $H_1(\Omega B) \cong H_2(B)$ and that

$$H_4(B) \xrightarrow{d^2} H_2(B) \otimes H_2(B) \xrightarrow{d^2} H_2(\Omega B) \to H_3(B) \to 0$$

is exact. We can use induction on n to estimate the size of $H_n(\Omega B)$.

Exercise 5.3.1 Show that if $n \geq 2$ the loop space ΩS^n has

$$H_p(\Omega S^n) \cong \begin{cases} \mathbb{Z} & \text{if } (n-1) \text{ divides } p,\ p \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

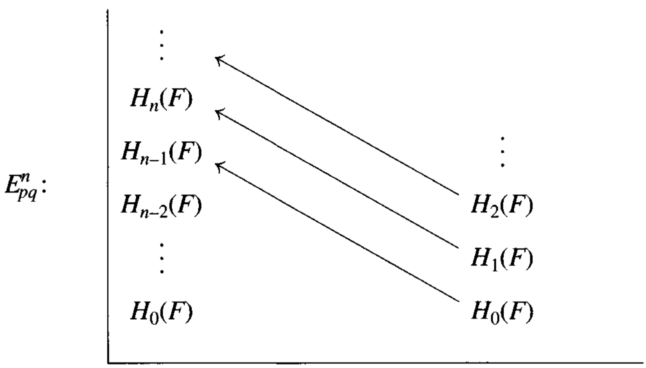
Application 5.3.5 (Wang sequence) If $F \xrightarrow{i} E \xrightarrow{\pi} S^n$ is a fibration whose base space is an n-sphere ($n \neq 0, 1$), there is a long exact sequence

$$\cdots \to H_q(F) \xrightarrow{i} H_q(E) \to H_{q-n}(F) \xrightarrow{d^n} H_{q-1}(F) \xrightarrow{i} H_{q-1}(E) \to \cdots.$$

In particular, $H_q(F) \cong H_q(E)$ if $0 \leq q \leq n-2$.

Proof $H_p(S^n) = 0$ for $p \neq 0, n$ and $H_n(S^n) = H_0(S^n) = \mathbb{Z}$. Therefore the nonzero terms E^2_{pq} all lie on the two vertical lines $p = 0, n$ and $E^2_{pq} = H_q(F)$ for $p = 0$ or n. All the differentials d^r_{pq} must therefore vanish for $r \neq n$, so $E^2_{pq} = E^n_{pq}$ and $E^{n+1}_{pq} = E^\infty_{pq}$. The description of E^{n+1} as the homology of E^n amounts to the exactness of the sequences

$$0 \longrightarrow E^\infty_{n,q} \longrightarrow H_q(F) \xrightarrow{d^n} H_{q+n-1}(F) \longrightarrow E^\infty_{0,q+n-1} \longrightarrow 0.$$



On the other hand, the filtration of $H_q(E)$ is given by the E^∞_{pq}, so it is determined by the short exact sequence

$$0 \to E^\infty_{0q} \to H_q(E) \to E^\infty_{n,q-n} \to 0.$$

The Wang sequence is now obtained by splicing together these two families of short exact sequences. $\diamond$

Example 5.3.6 The *special orthogonal group* $SO(3)$ is a 3-dimensional Lie group acting on $S^2 \subseteq \mathbb{R}^3$. This action gives rise to the Serre fibration

$$SO(1) \to SO(3) \to S^2.$$

Because $SO(1) = S^1$, we get $H_3(SO(3)) \cong \mathbb{Z}$ and the exact sequence

$$0 \to H_2(SO(3)) \to \mathbb{Z} \xrightarrow{d^2} \mathbb{Z} \to H_1(SO(3)) \to 0.$$

Classically, we know that $\pi_1 SO(3) = \mathbb{Z}/2$, so that $H_1(SO(3)) = \mathbb{Z}/2$. Therefore $H_2(SO(3)) \cong \mathbb{Z}$, although $H_2(SO(3)) \to H_2(S^2)$ is not an isomorphism.

Application 5.3.7 (Gysin sequence) If $S^n \to E \xrightarrow{\pi} B$ is a fibration with B simply connected and $n \neq 0$, there is an exact sequence

$$\cdots \to H_{p-n}(B) \to H_pE \xrightarrow{\pi} H_p(B) \xrightarrow{d^{n+1}} H_{p-n-1}(B) \to H_{p-1}(E) \xrightarrow{\pi} \cdots.$$

In particular, $H_p(E) \cong H_p(B)$ for $0 \leq p < n$.

Proof This is similar to the Wang sequence 5.3.5, except that now the nonzero terms E^2_{pq} all lie on the two rows $q = 0, n$. The only nontrivial differentials are d^{n+1}_{p0} from $H_p(B) = E^{n+1}_{p0}$ to $E^{n+1}_{p-n-1,n} \cong H_{p-n-1}(B)$. ◇

Exercise 5.3.2 If $n \neq 0$, the complex projective n-space $\mathbb{CP}^n$ is a simply connected manifold of dimension $2n$. As such $H_p(\mathbb{CP}^n) = 0$ for $p > 2n$. Given that there is a fibration $S^1 \to S^{2n+1} \to \mathbb{CP}^n$, show that for $0 \leq p \leq 2n$

$$H_p(\mathbb{CP}^n) \cong \left\{ \begin{matrix} \mathbb{Z} & p \text{ even} \\ 0 & p \text{ odd} \end{matrix} \right\}.$$

5.4 Spectral Sequence of a Filtration

A *filtration* F on a chain complex C is an ordered family of chain subcomplexes $\cdots \subseteq F_{p-1}C \subseteq F_pC \subseteq \cdots$ of C. In this section, we construct a spectral sequence associated to every such filtration; we will discuss convergence of the spectral sequence in the next section.

We say that a filtration is *exhaustive* if $C = \cup F_pC$. It will be clear from the construction that both $\cup F_pC$ and C give rise to the same spectral sequence. In practice, therefore, we always insist that filtrations be exhaustive.

Construction Theorem 5.4.1 *A filtration F of a chain complex C naturally determines a spectral sequence starting with $E^0_{pq} = F_pC_{p+q}/F_{p-1}C_{p+q}$ and $E^1_{pq} = H_{p+q}(E^0_{p*})$.*

Before constructing the spectral sequence, let us make some elementary remarks about the "shape" of the spectral sequence.

Definition 5.4.2 A filtration on a chain complex C is called *bounded* if for each n there are integers $s < t$ such that $F_sC_n = 0$ and $F_tC_n = C_n$. In this case, there are only finitely many nonzero terms of total degree n in E^0_{**}, so the spectral sequence is bounded. We will see in 5.5.1 that the spectral sequence always converges to $H_*(C)$.

A filtration on a chain complex C is called *bounded below* if for each n there is an integer s so that $F_sC_n = 0$, and it is called *bounded above* if for each n there is a t so that $F_tC_n = C_n$. Bounded filtrations are bounded above and below. Being bounded above is merely an easy way to ensure that a filtration is exhaustive. Bounded below filtrations give rise to bounded below spectral sequences. The Classical Convergence Theorem 5.5.1 of the next section says that the spectral sequence always converges to $H_*(C)$ when the filtration is bounded below and exhaustive.

Example 5.4.3 (First quadrant spectral sequences) We call the filtration *canonically bounded* if $F_{-1}C = 0$ and $F_nC_n = C_n$ for each n. As $E^0_{pq} = F_pC_{p+q}/F_{p-1}C_{p+q}$, every canonically bounded filtration gives rise to a first quadrant spectral sequence (converging to $H_*(C)$). For example, the Leray-Serre spectral sequence 5.3.2 arises from a canonically bounded filtration of the singular chain complex $S_*(E)$.

Here are some related notions, which we introduce now in order to give a better perspective on the construction of the spectral sequence.

Definition 5.4.4 A filtration on a chain complex C is called *Hausdorff* if $\cap F_pC = 0$. It will be clear from the construction that both C and its Hausdorff quotient $C^h = C/\cap F_pC$ give rise to the same spectral sequence.

A filtration on C is called *complete* if $C = \varprojlim C/F_pC$. Complete filtrations are Hausdorff because $\cap F_pC$ is the kernel of the map from C to its completion $\widehat{C} = \varprojlim C/F_pC$ (which is also a filtered complex: $F_n\widehat{C} = \varprojlim F_nC/F_pC$). Bounded below filtrations are complete, and hence Hausdorff, because $F_sH_n(C) = 0$ for each n. The following addendum to the Construction Theorem 5.4.1 explains why the most interesting applications of spectral sequences arise from complete filtrations. It will follow from exercise 5.4.1.

Addendum 5.4.5 The two spectral sequences arising from C and $\widehat{C}$ are the same.

The Construction 5.4.6 For legibility, we drop the bookkeeping subscript q and write η_p for the surjection $F_pC \to F_pC/F_{p-1}C = E_p^0$. Next we introduce

$$A_p^r = \{c \in F_pC : d(c) \in F_{p-r}C\},$$

the elements of F_pC that are cycles modulo $F_{p-r}C$ ("approximately cycles") and their images $Z_p^r = \eta_p(A_p^r)$ in E_p^0 and $B_{p-r}^{r+1} = \eta_{p-r}(d(A_p^r))$ in E_{p-r}^0. The indexing is chosen so that Z_p^r and $B_p^r = \eta_p(d(A_{p+r-1}^{r-1}))$ are subobjects of E_p^0.

Set $Z_p^\infty = \cap_{r=1}^\infty Z_p^r$ and $B_p^\infty = \cup_{r=1}^\infty B_p^r$. Assembling the above definitions, we see that we have defined a tower of subobjects of each E_p^0:

$$0 = B_p^0 \subseteq B_p^1 \subseteq \cdots \subseteq B_p^r \subseteq \cdots \subseteq B_p^\infty \subseteq Z_p^\infty \subseteq \cdots \subseteq Z_p^r \subseteq \cdots \subseteq Z_p^1 \subseteq Z_p^0 = E_p^0.$$

Note that $A_p^r \cap F_{p-1}C = A_{p-1}^{r-1}$, so that $Z_p^r \cong A_p^r/A_{p-1}^{r-1}$. Hence

$$E_p^r = \frac{Z_p^r}{B_p^r} \cong \frac{A_p^r + F_{p-1}(C)}{d(A_{p+r-1}^{r-1}) + F_{p-1}(C)} \cong \frac{A_p^r}{d(A_{p+r-1}^{r-1}) + A_{p-1}^{r-1}}.$$

Let $d_p^r\colon E_p^r \to E_{p-r}^r$ be the map induced by the differential of C. To define the spectral sequence, we only need to give the isomorphism between E^{r+1} and $H_*(E^r)$.

Lemma 5.4.7 *The map d determines isomorphisms*

$$Z_p^r/Z_p^{r+1} \xrightarrow{\cong} B_{p-r}^{r+1}/B_{p-r}^r.$$

Proof This is largely an exercise in decoding notation. First, note that $d(A_p^r) \cap F_{p-r-1}C = d(A_p^{r+1})$, so that $B_{p-r}^{r+1} \cong d(A_p^r)/d(A_p^{r+1})$ and hence B_{p-r}^{r+1}/B_{p-r}^r is isomorphic to $d(A_p^r)/d(A_p^{r+1} + A_{p-1}^{r-1})$. The other term Z_p^r/Z_p^{r+1} is isomorphic to $A_p^r/(A_p^{r+1} + A_{p-1}^{r-1})$. As the kernel of $d\colon A_p^r \to F_{p-r}C$ is contained in A_p^{r+1}, the two sides are isomorphic. $\diamond$

Resuming the construction of the spectral sequence, the kernel of d_p^r is

$$\frac{\left\{z \in A_p^r : d(z) \in d(A_{p-1}^{r-1}) + A_{p-r-1}^{r-1}\right\}}{d(A_{p+r-1}^{r-1}) + A_{p-1}^{r-1}} = \frac{A_{p-1}^{r-1} + A_p^{r+1}}{d(A_{p+r-1}^{r-1}) + A_{p-1}^{r-1}} \cong \frac{Z_p^{r+1}}{B_p^r}.$$

By lemma 5.4.7, the map d^r_p factors as

$$E^r_p = Z^r_p/B^r_p \to Z^r_p/Z^{r+1}_p \xrightarrow{\cong} B^{r+1}_{p-r}/B^r_{p-r} \hookrightarrow Z^r_{p-r}/B^r_{p-r} = E^r_{p-r}.$$

From this we see that the image of d^r_p is B^{r+1}_{p-r}/B^r_{p-r}; replacing p with $p+r$, the image of d^r_{p+r} is B^{r+1}_p/B^r_p. This provides the isomorphism

$$E^{r+1}_p = Z^{r+1}_p/B^{r+1}_p \cong \ker(d^r_p)/\operatorname{im}(d^r_{p+r})$$

needed to complete the construction of the spectral sequence. ◇

Observation Fix p and $k \geq 1$, and set $C' = C/F_{p-k}C$, $C'' = F_{p+k}C/F_{p-k}C$. The complex C' is bounded below, C'' is bounded, and there are maps $C \to C' \leftarrow C''$. For $0 \leq r \leq k$ these maps induce isomorphisms on the associated groups $A^r_p/F_{p-k}C$ and $\{d(A^{r-1}_{p+r-1}) + F_{p-k}C\}/F_{p-k}C$. (Check this!) Hence the associated groups Z^r_p, B^r_p and E^r_p are isomorphic. That is, the associated spectral sequences for C, C', and C'' agree in the (p, q) spots through the E^k terms.

Exercise 5.4.1 Recall that the completion $\widehat{C}$ is also a filtered complex. Show that $C/F_{p-k}C$ and $\widehat{C}/F_{p-k}\widehat{C}$ are naturally isomorphic.

We can now establish the addendum 5.4.5. For each p, q, and k, we have shown that the maps $C \to \widehat{C} \to C'$ induce isomorphisms between the corresponding E^k_{pq} terms. Letting k go to infinity, we see that the map $\{f^r_{pq}\colon E^r_{pq}(C) \to E^r_{pq}(\widehat{C})\}$ of spectral sequences is an isomorphism, because each f^r_{pq} is an isomorphism.

Exercise 5.4.2 Show that the spectral sequences for C, $\cup F_pC$, and $C/\cap F_pC$ are all isomorphic.

Multiplicative Structure 5.4.8 Suppose that C is a differential graded algebra (4.5.2) and that the filtration is multiplicative in the sense that for every s and t, $(F_sC)(F_tC) \subseteq F_{s+t}C$. Since $E^0_{p,n-p}$ is $F_pC_n/F_{p-1}C_n$, it is clear that we have a product

$$E^0_{p_1q_1} \times E^0_{p_2q_2} \to E^0_{p_1+p_2,q_1+q_2}$$

satisfying the Leibnitz relation. Hence the spectral sequence has a multiplicative structure in the sense of 5.2.13. Moreover, we saw in exercise 4.5.1 that $H_*(C)$ is an algebra and that the images $F_pH_*(C)$ of the $H_*(F_pC)$ form a multiplicative system of ideals in $H_*(C)$. Therefore whenever the spectral sequence (weakly) converges to $H_*(C)$ it follows that E^∞ is the associated graded algebra of $H_*(C)$. This convergence is the topic of the next section.

Exercise 5.4.3 (Shifting or Décalage) Given a filtration F on a chain complex C, define two new filtrations $\tilde{F}$ and $\mathrm{Dec}F$ on C by $\tilde{F}_pC_n = F_{p-n}C_n$ and $(\mathrm{Dec}F)_pC_n = \{x \in F_{p+n}C_n : dx \in F_{p+n-1}C_{n-1}\}$. Show that the spectral sequences for these three filtrations are isomorphic after reindexing: $E^r_{pq}(F) \cong E^{r+1}_{p+n,q-n}(\tilde{F})$ for $r \geq 0$, and $E^r_{pq}(F) \cong E^{r-1}_{p-n,q+n}(\mathrm{Dec}F)$ for $r \geq 2$.

Exercise 5.4.4 (Eilenberg-Moore) Let $f : B \to C$ be a map of filtered chain complexes. For each $r \geq 0$, define a filtration on the mapping cone cone(f) 1.5.1 by

$$F_p\mathrm{cone}(f)_n = F_{p-r}B_{n-1} \oplus F_pC_n.$$

Show that $E^r_p(\mathrm{cone} f)$ is the mapping cone of $f^r : E^r_p(B) \to E^r_p(C)$. By 1.5.2 this gives a long exact sequence

$$\cdots E^r_{p+r}(\mathrm{cone} f) \to E^r_p(B) \to E^r_p(C) \to E^r_p(\mathrm{cone} f) \cdots.$$

5.5 Convergence

A filtration on a chain complex C induces a filtration on the homology of $C : F_pH_n(C)$ is the image of the map $H_n(F_pC) \to H_n(C)$. If the filtration on C is exhaustive, then the filtration on H_n is also exhaustive ($H_n = \cup F_pH_n$), because every element of H_n is represented by an element c of some F_pC_n such that $d(c) = 0$. If the filtration on C is bounded below then the filtration on each $H_n(C)$ is also bounded below, since $F_pC = 0$ implies that $F_pH_n(C) = 0$.

Exercise 5.5.1 Give an example of a complete Hausdorff filtered complex C such that the filtration on $H_0(C)$ is not Hausdorff, that is, such that $\cap F_pH_0(C) \neq 0$.

Here are the two classical criteria used to establish convergence; we will discuss convergence for complete filtrations later on.

Classical Convergence Theorem 5.5.1

1. *Suppose that the filtration on C is* bounded. *Then the spectral sequence is bounded and converges to $H_*(C)$:*

$$E^1_{pq} = H_{p+q}(F_pC/F_{p-1}C) \Rightarrow H_{p+q}(C).$$

2. *Suppose that the filtration on C is* bounded below *and* exhaustive. *Then the spectral sequence is bounded below and also converges to $H_*(C)$.*

   Moreover, the convergence is natural in the sense that if $f: C \to C'$ is a map of filtered complexes, then the map $f_*: H_*(C) \to H_*(C')$ is compatible with the corresponding map of spectral sequences.

Example 5.5.2 (First quadrant spectral sequences) Suppose that the filtration is *canonically bounded* ($F_{-1}C = 0$ and $F_nC_n = C_n$ for each n), so that the spectral sequence lies in the first quadrant. Then it converges to $H_*(C)$. Along the y-axis of E^1 we have $E^1_{0q} = H_q(F_0C)$, and E^∞_{0q} is a quotient of this (see 5.2.6). Along the x-axis, E^1_{p0} is the homology $H_p(\bar{C})$ of C's top quotient chain complex $\bar{C}$, $\bar{C}_n = C_n/F_{n-1}C_n$; E^∞_{p0} is therefore a subobject of $H_p(\bar{C})$.

Corollary 5.5.3 *If the filtration is canonically bounded, then E^∞_{0q} is the image of $H_q(F_0C)$ in $H_q(C)$ and E^∞_{p0} is the image of $H_p(C)$ in $H_p(\bar{C})$.*

Proof By definition, $E^\infty_{0q} = F_0H_q(C)$ is the image of $H_q(F_0C)$ in $H_q(C)$. Now consider the exact sequence of chain complexes $0 \to F_{p-1}C \to C_p \to \bar{C}_p \to 0$. From the associated homology exact sequence we see that the image of $H_p(C)$ in $H_p(\bar{C})$ is the cokernel of the map from $H_p(F_{p-1}C)$ to $H_p(C)$, which by definition is $E^\infty_{p0} = H_p(C)/F_{p-1}H_p(C)$. ◇

Proof of Classical Convergence Theorem Suppose that the filtration is exhaustive and bounded below (resp. bounded). Then the filtration on H_* is exhaustive and bounded below (resp. bounded), and the spectral sequence is bounded below (resp. bounded). By Definition 5.2.11, the spectral sequence will converge to H_* whenever it weakly converges. For this, we observe that since the filtration is bounded below and p and n are fixed, the groups $A^r_p = \{c \in F_pC_n : d(c) \in F_{p-r}C_{n-1}\}$ stabilize for large r; write A^∞_p for this stable value, and observe that since $Z^r_p = \eta_p(A^r_p)$ we have $Z^\infty_p = \eta_p(A^\infty_p)$. Now A^∞_p is the kernel of $d\colon F_pC_n \to F_pC_{n-1}$, $(dC) \cap F_pC$ is the union of the $d(A^r_{p+r})$, and A^∞_{p-1} is the kernel of the map $\eta_p\colon A^\infty_p \to E^0_{pq}$. Thus

$$\begin{aligned} F_pH_n(C)/F_{p-1}H_n(C) &\cong A^\infty_p/\{A^\infty_{p-1} + d(\cup A^r_{p+r})\} \\ &\cong \eta_p(A^\infty_p)/\ \eta_p d(\cup A^r_{p+r}) \\ &= Z^\infty_p/B^\infty_p = E^\infty_p. \end{aligned}$$ ◇

When the filtration is not bounded below, convergence is more delicate. Of course we have to work within an abelian category such as R-mod, because we need axiom ($AB4^*$) in order to even talk about E^∞ (see 5.2.8). But there are more basic problems. For example, the filtration on $H_*(C)$ need not be Hausdorff. This is not surprising, since by 5.4.5 the completion $\widehat{C}$ has the same spectral sequence but different homology. (And see exercise 5.5.1.)

Example 5.5.4 Let C be the chain complex $0 \to \mathbb{Z} \xrightarrow{3} \mathbb{Z} \to 0$, and let F_pC be 2^pC. Then the Hausdorff quotient of $H_*(C)$ is zero, because $F_pH_*(C) = H_*(C)$ for all p, even though $H_0(C) = \mathbb{Z}/3$. Each row of E^0 is $\mathbb{Z}/2 \xleftarrow{3} \mathbb{Z}/2$ and the spectral sequence collapses to zero at E^1, so the spectral sequence is weakly converging (but not converging) to $H_*(C)$. It converges to $H_*(\widehat{C}) = 0$.

Theorem 5.5.5 (Eilenberg-Moore Filtration Sequence for complete complexes) *Suppose that C is complete with respect to a filtration by subcomplexes. Associated to the tower $\{C/F_pC\}$ is the sequence of 3.5.8:*

$$0 \to \varprojlim{}^1 H_{n+1}(C/F_pC) \to H_n(C) \xrightarrow{\pi} \varprojlim H_n(C/F_pC) \to 0.$$

This sequence is associated to the filtration on $H_*(C)$ as follows. The left-hand term $\varprojlim^1 H_{n+1}(C/F_pC)$ is $\cap F_pH_n(C)$, and the right-hand term is the Hausdorff quotient of $H_*(C)$:

$$H_*(C)/\cap F_pH_n(C) \cong \varprojlim H_n(C)/F_pH_n(C) \cong \varprojlim H_n(C/F_pC).$$

Proof Taking the inverse limit of the exact sequences of towers

$$0 \to \{F_pH_*(C)\} \to H_*(C) \to \{H_*(C)/F_pH_*(C)\} \to 0;$$

$$0 \to \{H_*(C)/F_pH_*(C)\} \to \{H_*(C/F_pC)\}$$

shows that $H_*(C)/\cap F_pH_*(C)$ is a subobject of $\varprojlim H_*(C)/F_pH_*(C)$, which is in turn a subobject of $\varprojlim H_n(C/F_pC)$. Now combine this with the $\varprojlim^1$ sequence of 3.5.8. $\diamond$

Corollary 5.5.6 *If the spectral sequence weakly converges, then $H_*(C) \cong H_*(\widehat{C})$.*

A careful reading of the proof of the Classical Convergence Theorem 5.5.1 yields the following lemma for all Hausdorff, exhaustive filtrations. To avoid confusion, we reintroduce the fixed subscripts q and $n = p + q$. Write $A^\infty_{pq} = \cap_{r=1}^\infty A^r_{pq}$, recalling that in our notation $A^r_{pq} = \{c \in F_pC_n : d(c) \in F_{p-r}C_{n-1}\}$. In $E^0_{pq} = F_pC_n/F_{p-1}C_n$, $\eta_p(A^\infty_{pq})$ is contained in Z^∞_{pq} and contains $B^\infty_{pq} = \eta_p(F_pC \cap d(C))$. (Check this!) Hence $e^\infty_{pq} = \eta_p(A^\infty_{pq})/B^\infty_{pq}$ is contained in E^∞_{pq}.

Lemma 5.5.7 *Assume that the filtration on C is Hausdorff and exhaustive. Then*

1. *A^∞_{pq} is the kernel of $d\colon F_pC_n \to F_pC_{n-1}$;*
2. *$F_pH_n(C) \cong A^\infty_{pq}/\cup_{r=1}^\infty d(A^r_{p+r,q-r+1})$;*
3. *The subgroup e^∞_{pq} of E^∞_{pq} is related to $H_*(C)$ by*

$$e^\infty_{pq} \cong F_pH_n(C)/F_{p-1}H_n(C).$$

Proof Recall that $F_pH_n(C)$ is the image of the map $H_n(F_pC) \to H_n(C)$. Since $\cap F_pC = 0$, the kernel of $d\colon F_pC_n \to F_pC_{n-1}$ is A^∞_{pq}, so $H_n(F_pC) \cong$

$A_{pq}^{\infty}/d(F_pC_{n+1})$. As $\cup F_pC = C$, the kernel of $A_{pq}^{\infty} \to H_n(C)$ is the union $\cup d(A_{p+r,q-r+1}^r)$. For part (3) observe that $A_{pq}^{\infty} \cap F_{p-1}C_n = A_{p-1,q+1}^{\infty}$ by definition, so that $\eta_p A_{pq}^{\infty} = A_{pq}^{\infty}/A_{p-1,q+1}^{\infty}$. Hence we may calculate in E_{pq}^0

$$\begin{aligned} F_pH_n(C)/F_{p-1}H_n(C) &\cong A_{pq}^{\infty}/A_{p-1,q+1}^{\infty} + \cup d(A_{p+r,q-r+1}^r) \\ &\cong \eta_p(A_{pq}^{\infty})/\cup \eta_p d(A_{p+r,q-r+1}^r) \\ &= \eta_p(A_{pq}^{\infty})/B_{pq}^{\infty} = e_{pq}^{\infty}. \end{aligned}$$ ◇

Corollary 5.5.8 (Boardman's Criterion) *Let Q_p denote $\varprojlim^1\{A_{pq}^r\}$ for fixed p and q. The inclusions $A_{p-1,q+1}^{r-1} \subset A_{pq}^r$ induce a map $a\colon Q_{p-1} \to Q_p$, and there is an exact sequence*

$$0 \to e_{pq}^{\infty} \to E_{pq}^{\infty} \to Q_{p-1} \xrightarrow{a} Q_p \to \varprojlim{}^1\{Z_{pq}^r\} \to 0.$$

In particular, if the filtration is Hausdorff and exhaustive, then the spectral sequence weakly converges to $H_*(C)$ if and only if the maps $a\colon Q_{p-1} \to Q_p$ are all injections.

Proof The short exact sequence of towers from 5.4.6

$$0 \to \{A_{p-1}^{r-1}\} \to \{A_p^r\} \xrightarrow{\eta} \{Z_p^r\} \to 0$$

yields

$$0 \to A_{p-1}^{\infty} \to A_p^{\infty} \xrightarrow{\eta} Z_p^{\infty} \to Q_{p-1} \xrightarrow{a} Q_p \to \varprojlim{}^1\{Z_p^r\} \to 0.$$

Now mod out by B_p^{∞}, recalling that e_{pq}^{∞} is $\eta(A_{pq}^{\infty})/B_{pq}^{\infty}$. ◇

Exercise 5.5.2 Set $R_p = \cap_r \operatorname{image}\{H(F_rC) \to H(F_pC)\}$. Show that the spectral sequence is weakly convergent iff the maps $R_{p-1} \to R_p$ are injections for all p. *Hint:* $R_p \subset Q_p$.

Exercise 5.5.3 Suppose that the filtration on C is Hausdorff and exhaustive. If for any $p+q=n$ we have $E_{pq}^r = 0$, show that $F_pH_n(C) = F_{p-1}H_n(C)$. Conclude that $H_n(C) = \cap F_pH_n(C)$, provided that every $E_{p,q}^r$ with $p+q$ equalling n vanishes.

Proposition 5.5.9 (Boardman) *Suppose that the filtration on C_n is complete, and form the tower of groups $Q_p = \varprojlim^1\{A_{p,n-p}^r\}$ as in 5.5.8 along the maps $a\colon Q_{p-1} \to Q_p$. Then $\varprojlim Q_p = 0$.* ◇

Proof Let I denote the poset of negative numbers $\cdots < p-1 < p < p+1 < \cdots < 0$. For each negative p and t, the subgroups $A(p,t) = A_p^{t-p} = \{c \in F_pC_n : d(c) \in F_tC_{n-1}\}$ of C_n form a functor $A\colon I \times I \to \mathbf{Ab}$, that is, a "double tower" of subgroups. If we fix t and vary p, then for $p \le t$ we have $A(p,t) = F_pC_n$. Hence we have $\varprojlim_p A(p,t) = \varprojlim_p F_pC_n = 0$ and $\varprojlim^1_p A(p,t) = \varprojlim^1_p F_pC_n = 0$ (see 3.5.7). We assert that the derived functor $R^1 \lim_{I\times I}$ from double towers to abelian groups fits into two short exact sequences:

$$(\dagger)\quad \begin{aligned} 0 \to \varprojlim_t{}^1(\varprojlim_p A(p,t)) \to R^1 \lim_{I\times I} A(p,t) \to \varprojlim_t(\varprojlim_p{}^1 A(p,t)) \to 0,\\ 0 \to \varprojlim_p{}^1(\varprojlim_t A(p,t)) \to R^1 \lim_{I\times I} A(p,t) \to \varprojlim_p(\varprojlim_t{}^1 A(p,t)) \to 0. \end{aligned}$$

We will postpone the proof of this assertion until 5.8.7 below, even though it follows from the Classical Convergence Theorem 5.5.1, as it is an easy application of the Grothendieck spectral sequence 5.8.3. The first of the sequences in ($\dagger$) implies that $R^1 \lim_{I\times I} A(p,t) = 0$, so from the second sequence in ($\dagger$) we deduce that $\varprojlim_p (\varprojlim^1_t A(p,t)) = 0$.

To finish, it suffices to prove that $\varprojlim^1_t A(p,t)$ is isomorphic to Q_p for each $p < 0$. Fix p, so that there is a short exact sequence of towers in t:

$$(*)\qquad 0 \to \{A(p,p+t)\} \to \{A(p,t)\} \to \{A(p,t)/A(p,p+t)\} \to 0.$$

If $t' < p+t$ the map $A(p,t')/A(p,p+t') \to A(p,t)/A(p,p+t)$ is obviously zero. Therefore the third tower of $(*)$ satisfies the trivial Mittag-Leffler condition (3.5.6), which means that

$$\varprojlim_t A(p,t)/A(p,p+t) = \varprojlim_t{}^1 A(p,t)/A(p,p+t) = 0.$$

From the $\varprojlim$ exact sequence of $(*)$ we obtain the described isomorphism

$$Q_p = \varprojlim_t A_p^t = \varprojlim_t A(p,p+t) \cong \varprojlim_t A(p,t). \qquad \diamond$$

Complete Convergence Theorem 5.5.10 *Suppose that the filtration on C is complete and exhaustive and the spectral sequence is* regular *(5.2.10). Then*

1. *The spectral sequence weakly converges to $H_*(C)$.*
2. *If the spectral sequence is bounded above, it converges to $H_*(C)$.*

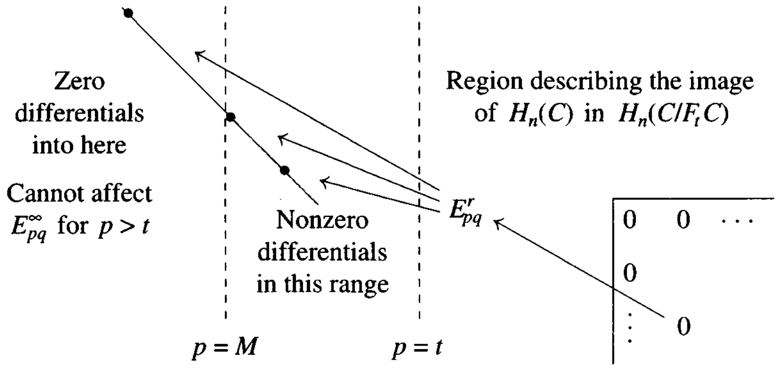


Figure 5.2. Complete convergence for regular, bounded above spectral sequences.

Proof When the spectral sequence is regular, Z^∞_{pq} equals $Z^r_{pq} = \eta_p A^r_{pq}$ for large r. By Boardman's criterion 5.5.8, all the maps $Q_{p-1} \to Q_p$ are onto, and the spectral sequence weakly converges if and only if $Q_p = 0$ for all p. This is indeed the case since the group $\varprojlim Q_p$ maps onto each Q_p (3.5.3), and we have just seen in 5.5.9 that $\varprojlim Q_p = 0$. This proves (1).

To see that the spectral sequence converges to $H_*(C)$, it suffices to show that the filtration on $H_*(C)$ is Hausdorff. By the Eilenberg-Moore Filtration Sequence 5.5.5, it suffices to show that the tower $\{H_n(C/F_tC)\}$ is Mittag-Leffler for every n, since then its $\lim^1$ groups vanish by 3.5.7. Each C/F_tC has a bounded below filtration, so it has a convergent spectral sequence whose associated graded groups $E^\infty_{pq}(C/F_tC)$ are subquotients of $E^0_{pq}(C)$ for $p > t$. For $m < t$, the images of the maps $E^\infty_{pq}(C/F_mC) \to E^\infty_{pq}(C/F_tC)$ are the associated graded groups of the image of $H_*(C/F_mC) \to H_*(C/F_tC)$, so it suffices to show that these images are independent of m as $m \to -\infty$.

Now assume that the spectral sequence for C is regular and bounded above. Then for each n and t there is an M such that the differentials $E^r_{pq}(C) \to E^r_{p-r,q+1-r}(C)$ are zero whenever $p + q = n$, $p > t$, and $p - r \leq M$. By inspection, this implies that $E^\infty_{pq}(C) = E^\infty_{pq}(C/F_mC)$ for every $p + q = n$ with $p > t$ and every $m \leq M$. Thus the image of $E^\infty_{pq}(C/F_mC) \to E^\infty_{pq}(C/F_tC)$ is independent of $m \leq M$ for $p + q = n$ and $p > t$, as was to be shown. $\diamond$

Exercise 5.5.4 (Complete nonconverging spectral sequences) Let $\mathbb{Z} < x >$ denote an infinite cyclic group with generator x, and let C be the chain complex with

$$C_1 = \bigoplus_{i=1}^{\infty} \mathbb{Z} < x_i >, \quad C_0 = \prod_{i=-\infty}^{i=0} \mathbb{Z} < y_i >, \quad C_n = 0 \qquad \text{for } n \neq 0, 1$$

and $d\colon C_1 \to C_0$ defined by $d(x_i) = y_{1-i} - y_{-i}$. For $p \leq 0$ define $F_pC_1 = 0$ and $F_pC_0 = \prod_{i \leq p} \mathbb{Z} < y_i >$; this is a complete filtration on C.

1. Show that $F_pH_0(C) = H_0(C)$ for every $p \leq 0$, so that the filtration on $H_0(C)$ is not Hausdorff. (Since C_1 is countable and C_0 is not, we have $H_0(C) \neq 0$.) Hence no spectral sequence constructed with this filtration can approach $H_*(C)$, let alone converge to it; such a spectral sequence will weakly converge to $H_*(C)$ if and only if it converges to zero.
2. Here is an example of an (essentially) second quadrant spectral sequence that weakly converges but does not converge to $H_*(C)$. For $p \geq 1$ define $F_pC_1 = C_1$ and $F_pC_0 = C_0$. The resulting spectral sequence has $E^0_{10} = C_1$, $E^0_{p,-p} = \mathbb{Z} < y_p >$ for $p \leq 0$ and $E^0_{pq} = 0$ otherwise. Show that $d^r(x_r)$ is $[y_{1-r}]$ and $d^r(x_i) = 0$ for $i \neq r$, and conclude that $E^\infty_{pq} = 0$ for every p and q.
3. Here is a regular spectral sequence that does not converge to $H_*(C)$. For $p \geq 1$ let F_pC_1 be the subgroup of C_1 spanned by $x_1, \cdots, x_p$ and set $F_pC_0 = C_0$. The resulting spectral sequence has $E^0_{p,1-p} = \mathbb{Z} < x_p >$ for $p \geq 1$, $E^0_{p,-p} = \mathbb{Z} < y_p >$ for $p \leq 0$ and $E^0_{pq} = 0$ otherwise. Show that this spectral sequence is regular and converges to zero.

The following result generalizes the Comparison Theorem 5.2.12 to non-convergent spectral sequences.

Eilenberg-Moore Comparison Theorem 5.5.11 *Let $f : B \to C$ be a map of filtered complexes of modules, where both B and C are complete and exhaustive. Fix $r \geq 0$. Suppose that $f^r : E^r_{pq}(B) \cong E^r_{pq}(C)$ is an isomorphism for all p and q. Then $f : H_*(B) \to H_*(C)$ is an isomorphism.*

Proof Consider the filtration on the mapping cone complex given by the formula $F_p\text{cone}(f) = F_{p-r}B[-1] \oplus F_pC$. This filtration is complete and exhaustive. Since f^r is an isomorphism, the long exact sequence of Exercise 5.4.4 shows that $E^r_{pq}(\text{cone} f) = 0$ for all p and q. By 5.5.10, this spectral sequence converges to $H_*\text{cone}(f)$. Hence cone(f) is an exact complex, and 1.5.4 applies. ◇

5.6 Spectral Sequences of a Double Complex

There are two filtrations associated to every double complex C, resulting in two spectral sequences related to the homology of Tot(C). Playing these spectral sequences off against each other is an easy way to calculate homology.

Definition 5.6.1 (Filtration by columns) If $C = C_{**}$ is a double complex, we may filter the (product or direct sum) total complex Tot(C) by the columns of C, letting ${}^IF_n\,\text{Tot}(C)$ be the total complex of the double subcomplex

$$\left({}^I\tau_{\leq n}C\right)_{pq} = \begin{cases} C_{pq} & \text{if } p \leq n \\ 0 & \text{if } p > n \end{cases} \qquad \begin{array}{ccc|cc} \dots & * & * & 0 & 0 \\ \dots & * & * & 0 & 0 \\ \dots & * & * & 0 & 0 \\ & * & * & 0 & 0 \end{array}$$

of C. This gives rise to a spectral sequence $\{{}^I E^r_{pq}\}$, starting with ${}^I E^0_{pq} = C_{pq}$. The maps d^0 are just the vertical differentials d^v of C, so

$$ {}^I E^1_{pq} = H^v_q(C_{p*}). $$

The maps $d^1\colon H^v_q(C_{p*}) \to H^v_q(C_{p-1,*})$ are induced on homology from the horizontal differentials d^h of C, so we may use the suggestive notation:

$$ {}^I E^2_{pq} = H^h_p H^v_q(C). $$

If C is a first quadrant double complex, the filtration is canonically bounded, and we have the convergent spectral sequence discussed in section 5.1:

$$ {}^I E^2_{pq} = H^h_p H^v_q(C) \Rightarrow H_{p+q}(\mathrm{Tot}(C)). $$

If C is a fourth quadrant double complex (or more generally if $C_{pq} = 0$ in the second quadrant), the filtration on $\mathrm{Tot}^\Pi(C)$ is bounded below but is not exhaustive. The filtration on the direct sum total complex $\mathrm{Tot}^\oplus(C)$ is both bounded below and exhaustive, so by the Classical Convergence Theorem 5.5.1 the spectral sequence ${}^I E^r_{**}$ converges to $H_*(\mathrm{Tot}^\oplus C)$ and not to $H_*(\mathrm{Tot}^\Pi C)$.

If C is a second quadrant double complex (or more generally if $C_{pq} = 0$ in the fourth quadrant), the filtration on the product total complex $\mathrm{Tot}^\Pi(C)$ is complete and exhaustive. By the Complete Convergence Theorem 5.5.10, the spectral sequence ${}^I E^r_{**}$ weakly converges to $H_*(\mathrm{Tot}^\Pi C)$, and we have the Eilenberg-Moore filtration sequence (5.5.5)

$$ 0 \to \varprojlim{}^1 H_{n+1}(C/\tau_{\le n}C) \to H_n(\mathrm{Tot}^\Pi C) \to \varprojlim H_n(C/\tau_{\le n}C) \to 0. $$

We will encounter a spectral sequence of this type in Chapter 9, 9.6.17.

Definition 5.6.2 (Filtration by rows) If C is a double complex, we may also filter $\mathrm{Tot}(C)$ by the rows of C, letting ${}^{II}F_n \,\mathrm{Tot}(C)$ be the total complex of

$$ \left({}^{II}\tau_{\le n} C\right)_{pq} = \begin{cases} C_{pq} & \text{if } q \le n \\ 0 & \text{if } q > n \end{cases} \qquad \begin{array}{cccccc} & \cdots & & & \cdots & \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline * & * & * & * & * & * \\ * & * & * & * & * & * \\ & \cdots & & & \cdots & \end{array}. $$

Since $F_p \,\mathrm{Tot}(C)/F_{p-1}\,\mathrm{Tot}(C)$ is the *row* C_{*p}, ${}^{II}E^0_{pq} = C_{qp}$ and ${}^{II}E^1_{pq} = H^h_q(C_{*p})$. (Beware the interchange of p and q in the notation!) The maps d^1 are induced from the vertical differentials d^v of C, so we may use the suggestive notation

$$ {}^{II}E^2_{pq} = H^v_p H^h_q(C). $$

Of course, this should not be surprising, since interchanging the roles of p and q converts the filtration by rows into the filtration by columns, and interchanges the spectral sequences ${}^{I}E$ and ${}^{II}E$.

As before, if C is a first quadrant double complex, this filtration is canonically bounded, and the spectral sequence converges to $H_*\operatorname{Tot}(C)$. If C is a second quadrant double complex (or more generally if $C_{pq} = 0$ in the fourth quadrant), the spectral sequence ${}^{II}E^r_{**}$ converges to $H_*\operatorname{Tot}^{\oplus}(C)$. If C is a fourth quadrant double complex (or if $C_{pq} = 0$ in the second quadrant), then the spectral sequence ${}^{II}E^r_{**}$ weakly converges to $H_*\operatorname{Tot}^{\Pi}(C)$. $\diamond$

Application 5.6.3 (Balancing Tor) In Chapter 2, 2.7.1, we used a disguised spectral sequence argument to prove that $L_n(A\otimes)(B) \cong L_n(\otimes B)(A)$, that is, that $\operatorname{Tor}_*(A, B)$ could be computed by taking either a projective resolution $P \to A$ or a projective resolution $Q \to B$. In our new vocabulary, there are two spectral sequences converging to the homology of $\operatorname{Tot}(P \otimes Q)$. Since $H^v_q(P_p \otimes Q) = P_p \otimes H_q(A)$, the first has

$$ {}^{I}E^2_{pq} = \left\{ \begin{array}{ll} H^h_p(P \otimes B) = L_p(\otimes B)(A) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{array} \right\}. $$

This spectral sequence collapses to yield $H_p(P \otimes Q) = L_p(\otimes B)(A)$. Therefore the second spectral sequence converges to $L_p(\otimes B)(A)$. Since $H^h_q(P \otimes Q_n) = H_q(P) \otimes Q_n$,

$$ {}^{II}E^2_{pq} = \left\{ \begin{array}{ll} H^v_p(A \otimes Q) = L_q(A\otimes)(B) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{array} \right\}. $$

This spectral sequence collapses to yield $H_p(P \otimes Q) = L_p(A\otimes)(B)$, whence the result.

Theorem 5.6.4 (Künneth spectral sequence) *Let P be a bounded below complex of flat R-modules and M an R-module. Then there is a boundedly converging right half-plane spectral sequence*

$$ E^2_{pq} = \operatorname{Tor}^R_p(H_q(P), M) \Rightarrow H_{p+q}(P \otimes_R M). $$

Proof Let $Q \to M$ be a projective resolution and consider the upper half-plane double complex $P \otimes Q$. Since P_p is flat, $H^v_q(P \otimes Q) = P_p \otimes H_q(Q)$, so the first spectral sequence has

$$ {}^{I}E^2_{pq} = \left\{ \begin{array}{ll} H_p(P \otimes M) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{array} \right\}. $$

This spectral sequence collapses to yield $H_p(P \otimes Q) = H_p(P \otimes M)$. Since Q_q is flat, $H_q(P \otimes Q_n) = H_q(P) \otimes Q_n$, so the second spectral sequence has the desired E^2 term

$$^{II}E^2_{pq} = H_p(H_q(P) \otimes Q) = \mathrm{Tor}^R_p(H_q(P), M). \qquad \diamond$$

Künneth Formula 5.6.5 In Chapter 3, 3.6.1, we could have given the following spectral sequence argument to compute $H_*(P \otimes M)$, assuming that $d(P)$ (and hence Z) is flat. The flat dimension of $H_q(P)$ is at most 1, since

$$0 \to d(P_{q+1}) \to Z_q \to H_q(P) \to 0$$

is a flat resolution. In this case only the columns $p = 0, 1$ are nonzero, so all the differentials vanish and $E^2_{pq} = E^\infty_{pq}$. The 2-stage filtration of $H_p(P \otimes Q)$ yields the Künneth formula.

$$\begin{array}{cc|cc|cc}
0 & 0 & \cdots & \cdots & 0 & 0 \\
0 & 0 & H_q(P) \otimes M & \mathrm{Tor}_1(H_q(P), M) & 0 & 0 \\
0 & 0 & H_{q-1}(P) \otimes M & \mathrm{Tor}_1(H_{q-1}(P), M) & 0 & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0
\end{array}$$

Exercise 5.6.1 Give a spectral sequence proof of the Universal Coefficient Theorem 3.6.5 for cohomology.

Theorem 5.6.6 (Base-change for Tor) *Let $f\colon R \to S$ be a ring map. Then there is a first quadrant homology spectral sequence*

$$E^2_{pq} = \mathrm{Tor}^S_p(\mathrm{Tor}^R_q(A, S), B) \Rightarrow \mathrm{Tor}^R_{p+q}(A, B)$$

for every $A \in$ **mod**–*R and $B \in S$*–**mod**.

Proof Let $P \to A$ be an R-module projective resolution, and $Q \to B$ an S-module projective resolution. As in 2.7.1, form the first quadrant double complex $P \otimes Q$ and write $H_*(P \otimes Q)$ for $H_*(\mathrm{Tot}(P \otimes_R Q))$. Since $P_p \otimes_R$ is an exact functor, the p^{th} column of $P \otimes Q$ is a resolution of $P_p \otimes B$. Therefore the first spectral sequence 5.6.1 collapses at $^IE^1 = H^v_q(P \otimes Q)$ to yield $H_*(P \otimes Q) \cong H_*(P \otimes B) = \mathrm{Tor}^R_*(A, B)$. Therefore the second spectral sequence 5.6.2 converges to $\mathrm{Tor}^R_*(A, B)$ and has

$$\begin{aligned} ^{II}E^1_{pq} &= H_q(P \otimes_R Q_p) = H_q((P \otimes_R S) \otimes_S Q_p) \\ &= H_q(P \otimes_R S) \otimes_S Q_p = \mathrm{Tor}^R_q(A, S) \otimes_S Q_p \end{aligned}$$

and hence the prescribed E^2_{pq} term: $H_p(^{II}E^1_{pq}) = \mathrm{Tor}^S_p(\mathrm{Tor}^R_q(A, S), B)$. $\diamond$

Exercise 5.6.2 (Bourbaki) Given rings R and S, let L be a right R-module, M an R–S bimodule, and N a left S-module, so that the tensor product $L \otimes_R M \otimes_S N$ makes sense.

1. Show that there are two spectral sequences, such that

$$^{I}E^2_{pq} = \mathrm{Tor}^R_p(L, \mathrm{Tor}^S_q(M, N)) \quad {}^{II}E^2_{pq} = \mathrm{Tor}^S_p(\mathrm{Tor}^R_q(L, M), N)$$

converging to the same graded abelian group H_*. *Hint:* Consider a double complex $P \otimes M \otimes Q$, where $P \to L$ and $Q \to N$.

2. If M is a flat S-module, show that the spectral sequence ^{II}E converges to $\mathrm{Tor}^R_*(L, M \otimes_S N)$. If M is a flat R-module, show that the spectral sequence ^{I}E converges to $\mathrm{Tor}^S_*(L \otimes_R M, N)$.

Exercise 5.6.3 (Base-change for Ext) Let $f: R \to S$ be a ring map. Show that there is a first quadrant cohomology spectral sequence

$$E_2^{pq} = \mathrm{Ext}^p_S(A, \mathrm{Ext}^q_R(S, B)) \Rightarrow \mathrm{Ext}^{p+q}_R(A, B)$$

for every S-module A and every R-module B.

Exercise 5.6.4 Use spectral sequences to prove the Acyclic Assembly Lemma 2.7.3.

5.7 Hyperhomology

Definition 5.7.1 Let $\mathcal{A}$ be an abelian category that has enough projectives. A (left) *Cartan-Eilenberg* resolution P_{**} of a chain complex A_* in $\mathcal{A}$ is an upper half-plane double complex ($P_{pq} = 0$ if $q < 0$), consisting of projective objects of $\mathcal{A}$, together with a chain map ("augmentation") $P_{*0} \xrightarrow{\epsilon} A_*$ such that for every p

1. If $A_p = 0$, the column P_{p*} is zero.
2. The maps on boundaries and homology

$$B_p(\epsilon): B_p(P, d^h) \to B_p(A)$$

$$H_p(\epsilon): H_p(P, d^h) \to H_p(A)$$

are projective resolutions in $\mathcal{A}$. Here $B_p(P, d^h)$ denotes the horizontal boundaries in the (p, q) spot, that is, the chain complex whose q^{th} term is $d^h(P_{p+1,q})$. The chain complexes $Z_p(P, d^h)$ and $H_p(P, d^h) = Z_p(P, d^h)/B_p(P, d^h)$ are defined similarly.

Exercise 5.7.1 In a Cartan-Eilenberg resolution show that the induced maps

$$\epsilon^p\colon P_{p*} \to A_p$$
$$Z^p(\epsilon)\colon Z_p(P, d^h) \to Z_p(A)$$

are projective resolutions in $\mathcal{A}$. Then show that the augmentation $\mathrm{Tot}^{\oplus}(P) \to A$ is a quasi-isomorphism in $\mathcal{A}$; when A isn't bounded below, you will need to assume axiom (AB4) holds.

Lemma 5.7.2 *Every chain complex A_* has a Cartan-Eilenberg resolution $P_{**} \to A$.*

Proof For each p select projective resolutions P^B_{p*} of $B_p(A)$ and P^H_{p*} of $H_p(A)$. By the Horseshoe Lemma 2.2.8 there is a projective resolution P^Z_{p*} of $Z_p(A)$ so that

$$0 \to P^B_{p*} \to P^Z_{p*} \to P^H_{p*} \to 0$$

is an exact sequence of chain complexes lying over

$$0 \to B_p(A) \to Z_p(A) \to H_p(A) \to 0.$$

Applying the Horseshoe Lemma again, we find a projective resolution P^A_{p*} of A_p fitting into an exact sequence

$$0 \to P^Z_{p*} \to P^A_{p*} \to P^B_{p-1,*} \to 0.$$

We now define P_{**} to be the double complex whose p^{th} column is P^A_{p*} except that (using the Sign Trick 1.2.5) the vertical differential is multiplied by $(-1)^p$; the horizontal differential of P_{**} is the composite

$$P^A_{p+1,*} \to P^B_{p*} \hookrightarrow P^Z_{p*} \hookrightarrow P^A_{p*}.$$

The construction guarantees that the maps $\epsilon_p\colon P_{p0} \to A_p$ assemble to give a chain map ϵ, and that each $B_p(\epsilon)$ and $H_p(\epsilon)$ give projective resolutions (check this!). $\diamond$

Exercise 5.7.2 If $f\colon A \to B$ is a chain map and $P \to A$, $Q \to B$ are Cartan-Eilenberg resolutions, show that there is a double complex map $\tilde{f}\colon P \to Q$ over f. *Hint:* Modify the proof of 2.4.6 that $L_* f$ is a homological δ-functor.

Definition 5.7.3 Let $f, g\colon D \to E$ be two maps of double complexes. A *chain homotopy* from f to g consists of maps $s^h_{pq}\colon D_{pq} \to E_{p+1,q}$ and $s^v_{pq}\colon D_{pq} \to E_{p,q+1}$ so that

$$g - f = (d^h s^h + s^h d^h) + (d^v s^v + s^v d^v)$$

$$s^v d^h + d^h s^v = s^h d^v + d^v s^h = 0.$$

This definition is set up so that $\{s^h + s^v : \mathrm{Tot}(D)_n \to \mathrm{Tot}(E)_{n+1}\}$ forms an ordinary chain homotopy between the maps $\mathrm{Tot}(f)$ and $\mathrm{Tot}(g)$ from $\mathrm{Tot}^{\oplus}(D)$ to $\mathrm{Tot}^{\oplus}(E)$.

Exercise 5.7.3

1. If $f, g: A \to B$ are homotopic maps of chain complexes, and $\tilde{f}, \tilde{g}: P \to Q$ are maps of Cartan-Eilenberg resolutions lying over them, show that $\tilde{f}$ is chain homotopic to $\tilde{g}$.
2. Show that any two Cartan-Eilenberg resolutions P, Q of A are chain homotopy equivalent. Conclude that for any additive functor F the chain complexes $\mathrm{Tot}^{\oplus}(F(P))$ and $\mathrm{Tot}^{\oplus}(F(Q))$ are chain homotopy equivalent.

Definition 5.7.4 ($\mathbb{L}_*F$) Let $F: \mathcal{A} \to \mathcal{B}$ be a right exact functor, and assume that $\mathcal{A}$ has enough projectives. If A is a chain complex in $\mathcal{A}$ and $P \to A$ is a Cartan-Eilenberg resolution, define $\mathbb{L}_i F(A)$ to be $H_i\,\mathrm{Tot}^{\oplus}(F(P))$. Exercise 5.7.3 shows that $\mathbb{L}_i F(A)$ is independent of the choice of P.

If $f: A \to B$ is a chain map and $\tilde{f}: P \to Q$ is a map of Cartan-Eilenberg resolutions over f, define $\mathbb{L}_i F(f)$ to be the map $H_i(\mathrm{Tot}(\tilde{f}))$ from $\mathbb{L}_i F(A)$ to $\mathbb{L}_i F(B)$. The exercise above implies that $\mathbb{L}_i F$ is a functor from $\mathbf{Ch}(\mathcal{A})$ to $\mathcal{B}$, at least when $\mathcal{B}$ is cocomplete. The $\mathbb{L}_i F$ are called the *left hyper-derived functors* of F.

Warning: If $\mathcal{B}$ is not cocomplete, $\mathrm{Tot}^{\oplus}(F(P))$ and $\mathbb{L}_i F(A)$ may not exist for all chain complexes A. In this case we restrict to the category $\mathbf{Ch}_+(\mathcal{A})$ of all chain complexes A which are bounded below in the sense that there is a p_0 such that $A_p = 0$ for $p < p_0$. Since $P_{pq} = 0$ if $p < p_0$ or $q < 0$, $\mathrm{Tot}^{\oplus}(F(P))$ exists in $\mathbf{Ch}(\mathcal{B})$ and we may consider $\mathbb{L}_i F$ to be a functor from $\mathbf{Ch}_+(\mathcal{A})$ to $\mathcal{B}$. ◇

Exercises 5.7.4

1. If A is an object of $\mathcal{A}$, considered as a chain complex concentrated in degree zero, show that $\mathbb{L}_i F(A)$ is the ordinary derived functor $L_i F(A)$.
2. Let $\mathbf{Ch}_{\geq 0}(\mathcal{A})$ be the subcategory of complexes A with $A_p = 0$ for $p < 0$. Show that the functors $\mathbb{L}_i F$ restricted to $\mathbf{Ch}_{\geq 0}(\mathcal{A})$ are the left derived functors of the right exact functor $H_0 F$.
3. (Dimension shifting) Show that $\mathbb{L}_i F(A[n]) = \mathbb{L}_{n+i} F(A)$ for all n. Here $A[n]$ is the translate of A with $A[n]_i = A_{n+i}$.

Lemma 5.7.5 *If* $0 \to A \to B \to C \to 0$ *is a short exact sequence of bounded below complexes, there is a long exact sequence*

$$\cdots \mathbb{L}_{i+1}F(C) \xrightarrow{\delta} \mathbb{L}_i F(A) \to \mathbb{L}_i F(B) \to \mathbb{L}_i F(C) \xrightarrow{\delta} \cdots .$$

Proof By dimension shifting, we may assume that A, B, and C belong to $\mathbf{Ch}_{\geq 0}(\mathcal{A})$. The sequence in question is just the long exact sequence for the derived functors of the right exact functor $H_0 F$. ◇

Proposition 5.7.6 *There is always a convergent spectral sequence*

$$^{II}E^2_{pq} = (L_p F)(H_q(A)) \Rightarrow \mathbb{L}_{p+q}F(A).$$

If A is bounded below, there is a convergent spectral sequence

$$^{I}E^2_{pq} = H_p(L_q F(A)) \Rightarrow \mathbb{L}_{p+q}F(A).$$

Proof We have merely written out the two spectral sequences arising from the upper half-plane double chain complex $F(P)$. ◇

Corollary 5.7.7

1. *If A is exact,* $\mathbb{L}_i F(A) = 0$ *for all* i.
2. *Any quasi-isomorphism* $f\colon A \to B$ *induces isomorphisms*

$$\mathbb{L}_* F(A) \cong \mathbb{L}_* F(B).$$

3. *If each* A_p *is* F-acyclic *(2.4.3), that is,* $L_q F(A_p) = 0$ *for* $q \neq 0$, *and* A *is bounded below, then*

$$\mathbb{L}_p F(A) = H_p(F(A)) \text{ for all } p.$$

Application 5.7.8 (Hypertor) Let R be a ring and B a left R-module. The *hypertor* groups $\mathbf{Tor}_i^R(A_*, B)$ of a chain complex A_* of right R-modules are defined to be the hyper-derived functors $\mathbb{L}_i F(A_*)$ for $F = \otimes_R B$. This extends the usual Tor to chain complexes, and if A is a bounded below complex of flat modules, then $\mathbf{Tor}_i^R(A_*, B) = H_i(A_* \otimes B)$ for all i. The hypertor spectral sequences coming from 5.7.6 are

$$^{II}E^2_{pq} = \mathrm{Tor}_p(H_q(A), B) \Rightarrow \mathbf{Tor}^R_{p+q}(A_*, B)$$

and (when A is bounded below)

$$^{I}E^1_{pq} = \mathrm{Tor}_q(A_p, B),\ ^{I}E^2_{pq} = H_p \mathrm{Tor}_q(A_*, B) \Rightarrow \mathbf{Tor}^R_{p+q}(A_*, B).$$

Even more generally, if B_* is also a chain complex, we can define the hypertor of the bifunctor $A \otimes_R B$ to be

$$\mathbf{Tor}^R_i(A_*, B_*) = H_i \mathrm{Tot}^{\oplus}(P \otimes_R Q),$$

where $P \to A$ and $Q \to B$ are Cartan-Eilenberg resolutions. Since $\mathrm{Tot}(P \otimes Q)$ is unique up to chain homotopy equivalence, the hypertor is independent of the choice of P and Q. If B is a module, considered as a chain complex, this agrees with the above definition (exercise!); by symmetry the same is true for A. By definition, hypertor is a balanced functor in the sense of 2.7.7. A lengthy discussion of hypertor may be found in [EGA, III.6].

Exercise 5.7.5 Show that there is a convergent spectral sequence

$$^{II}E^2_{pq} = \bigoplus_{q'+q''=q} \mathrm{Tor}^R_p\left(H_{q'}(A_*), H_{q''}(B_*)\right) \Rightarrow \mathbf{Tor}^R_{p+q}(A_*, B_*).$$

If A_* and B_* are bounded below, show that there is a spectral sequence

$$^{I}E^2_{pq} = H_p \mathrm{Tot}^{\oplus} \mathrm{Tor}_q(A_*, B_*) \Rightarrow \mathbf{Tor}^R_{p+q}(A_*, B_*).$$

Exercise 5.7.6 Let A be the mapping cone complex $0 \to A_1 \xrightarrow{f} A_0 \to 0$ with only two nonzero rows. Show that there is a long exact sequence:

$$\cdots \mathbb{L}_{i+1}F(A) \to L_iF(A_1) \xrightarrow{f} L_iF(A_0) \to \mathbb{L}_iF(A) \to L_{i-1}F(A_1) \cdots.$$

Cohomology Variant 5.7.9 Let $\mathcal{A}$ be an abelian category that has enough injectives. A (right) *Cartan-Eilenberg resolution* of a cochain complex A^* in $\mathcal{A}$ is an upper half-plane complex I^{**} of injective objects of $\mathcal{A}$, together with an augmentation $A^* \to I^{*0}$ such that the maps on coboundaries and cohomology are injective resolutions of $B^p(A)$ and $H^p(A)$. Every cochain complex has a Cartan-Eilenberg resolution $A \to I$. If $F: \mathcal{A} \to \mathcal{B}$ is a left exact functor, we define $\mathbb{R}^iF(A)$ to be $H^i \mathrm{Tot}^{\Pi}(F(I))$, at least when $\mathrm{Tot}^{\Pi}(F(I))$ exists in $\mathcal{B}$. By appealing to the functor $F^{op}: \mathcal{A}^{op} \to \mathcal{B}^{op}$, we see that $\mathbb{R}^iF$ is a functor from $\mathbf{Ch}^+(\mathcal{A})$ (the complexes A^* with $A^p = 0$ for $p << 0$) to $\mathcal{B}$, and even from

$\mathbf{Ch}(\mathcal{A})$ to $\mathcal{B}$ when $\mathcal{B}$ is complete. The $\mathbb{R}^i F$ are called the *right hyper-derived functors* of F.

If A is in $\mathbf{Ch}(\mathcal{A})$, the two spectral sequences arising from the upper half-plane double cochain complex $F(I)$ become

$$^{II}E_2^{pq} = (R^pF)(H^q(A)) \Rightarrow \mathbb{R}^{p+q}F(A), \text{ weakly convergent; and}$$

$$^{I}E_2^{pq} = H^p(R^qF(A)) \Rightarrow \mathbb{R}^{p+q}F(A), \text{ if } A \text{ is bounded below.}$$

Hence $\mathbb{R}^*F$ vanishes on exact complexes and sends quasi-isomorphisms of (bounded below) complexes to isomorphisms.

Application 5.7.10 (Hypercohomology) Let X be a topological space and $\mathcal{F}^*$ a cochain complex of sheaves on X. The *hypercohomology* $\mathbb{H}^i(X, \mathcal{F}^*)$ is $\mathbb{R}^i\Gamma(\mathcal{F}^*)$, where Γ is the global sections functor 2.5.4. This generalizes sheaf cohomology to complexes of sheaves, and if $\mathcal{F}^*$ is a bounded below complex of injective sheaves, then $\mathbb{H}^i(X, \mathcal{F}^*) = H^i(\Gamma(\mathcal{F}^*))$. The hypercohomology spectral sequence is $^{II}E_2^{pq} = H^p(X, H^q(\mathcal{F}^*)) \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{F}^*)$.

5.8 Grothendieck Spectral Sequences

In his classic paper [Tohoku], Grothendieck introduced a spectral sequence associated to the composition of two functors. Today it is one of the organizational principles of Homological Algebra.

Cohomological Setup 5.8.1 Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be abelian categories such that both $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. We are given left exact functors $G: \mathcal{A} \to \mathcal{B}$ and $F: \mathcal{B} \to \mathcal{C}$.

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "G"] \arrow[dr, "FG"'] & & \mathcal{B} \arrow[dl, "F"] \\ & \mathcal{C} & \end{tikzcd}$$

Definition 5.8.2 Let $F: \mathcal{B} \to \mathcal{C}$ be a left exact functor. An object B of $\mathcal{B}$ is called *F-acyclic* if the derived functors of F vanish on B, that is, if $R^iF(B) = 0$ for $i \neq 0$. (Compare with 2.4.3.)

Grothendieck Spectral Sequence Theorem 5.8.3 *Given the above cohomological setup, suppose that G sends injective objects of $\mathcal{A}$ to F-acyclic objects* *of $\mathcal{B}$. Then there exists a convergent first quadrant cohomological spectral sequence for each A in $\mathcal{A}$:*

$$^{I}E_2^{pq} = (R^pF)(R^qG)(A) \Rightarrow R^{p+q}(FG)(A).$$

The edge maps in this spectral sequence are the natural maps

$$(R^pF)(GA) \to R^p(FG)(A) \quad \text{and} \quad R^q(FG)(A) \to F(R^qG(A)).$$

The exact sequence of low degree terms is

$$0 \to (R^1F)(GA) \to R^1(FG)A \to F(R^1G(A)) \to (R^2F)(GA) \to R^2(FG)A.$$

Proof Choose an injective resolution $A \to I$ of A in $\mathcal{A}$, and apply G to get a cochain complex $G(I)$ in $\mathcal{B}$. Using a first quadrant Cartan-Eilenberg resolution of $G(I)$, form the hyper-derived functors $\mathbb{R}^nF(G(I))$ as in 5.7.9. There are two spectral sequences converging to these hyper-derived functors. The first spectral sequence is

$$^{I}E_2^{pq} = H^p((R^qF)(GI)) \Rightarrow (\mathbb{R}^{p+q}F)(GI).$$

By hypothesis, each $G(I^p)$ is F-acyclic, so $(R^qF)(G(I^p)) = 0$ for $q \neq 0$. Therefore this spectral sequence collapses to yield

$$(\mathbb{R}^pF)(GI) \cong H^p(FG(I)) = R^p(FG)(A).$$

The second spectral sequences is therefore

$$^{II}E_2^{pq} = (R^pF)H^q(G(I)) \Rightarrow R^p(FG)(A).$$

Since $H^q(G(I)) = R^pG(A)$, it is Grothendieck's spectral sequence. $\diamond$

Corollary 5.8.4 (Homology spectral sequence) *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be abelian categories such that both $\mathcal{A}$ and $\mathcal{B}$ have enough projectives. Suppose given right exact functors $G\colon \mathcal{A} \to \mathcal{B}$ and $F\colon \mathcal{B} \to \mathcal{C}$ such that G sends projective objects of $\mathcal{A}$ to F-acyclic objects of $\mathcal{B}$. Then there is a convergent first quadrant homology spectral sequence for each A in $\mathcal{A}$:*

$$E^2_{pq} = (L_pF)(L_qG)(A) \Rightarrow L_{p+q}(FG)(A).$$

The exact sequence of low degree terms is

$$L_2(FG)A \to (L_2F)(GA) \to F(L_1G(A)) \to L_1(FG)A \to (L_1F)(GA) \to 0.$$

Proof Dualizing allows us to consider $G^{op}\colon \mathcal{A}^{op} \to \mathcal{B}^{op}$ and $F^{op}\colon \mathcal{B}^{op} \to \mathcal{C}^{op}$, and the corollary is just translation of Grothendieck's spectral sequence using the dictionary $L_pF = R^pF^{op}$, and so on. $\diamond$

Applications 5.8.5 The base-change spectral sequences for Tor and Ext of section 5.7 are actually special instances of the Grothendieck spectral sequence: Given a ring map $R \to S$ and an S-module B, one considers the composites

$$R\text{–}\mathbf{mod} \xrightarrow{\otimes_S R} S\text{–}\mathbf{mod} \xrightarrow{\otimes_S B} \mathbf{Ab}$$

and

$$R\text{–}\mathbf{mod} \xrightarrow{\mathrm{Hom}_R(S,-)} S\text{–}\mathbf{mod} \xrightarrow{\mathrm{Hom}_S(B,-)} \mathbf{Ab}.$$

Leray Spectral Sequence 5.8.6 Let $f\colon X \to Y$ be a continuous map of topological spaces. The direct image sheaf functor f_* (2.6.6) has the exact functor f^{-1} as its left adjoint (exercise 2.6.2), so f_* is left exact and preserves injectives by 2.3.10. If $\mathcal{F}$ is a sheaf of abelian groups on X, the global sections of $f_*\mathcal{F}$ is the group $(f_*\mathcal{F})(Y) = \mathcal{F}(f^{-1}Y) = \mathcal{F}(X)$. Thus we are in the situation

$$\begin{tikzcd} \mathrm{Sheaves}(X) \arrow[rr, "f_*"] \arrow[dr, "\Gamma"'] & & \mathrm{Sheaves}(Y) \arrow[dl, "\Gamma"] \\ & \mathbf{Ab} & \end{tikzcd}$$

The Grothendieck spectral sequence in this case is called the *Leray spectral sequence:* Since $R^p\Gamma$ is sheaf cohomology (2.5.4), it is usually written as

$$E_2^{pq} = H^p(Y; R^qf_*\mathcal{F}) \Rightarrow H^{p+q}(X; \mathcal{F}).$$

This spectral sequence is a central tool to much of modern algebraic geometry.

We will see other applications of the Grothendieck spectral sequence in 6.8.2 and 7.5.2. Here is one we needed in section 5.5.9.

Recall from Chapter 3, section 5 that a tower $\cdots A_1 \to A_0$ of abelian groups is a functor $I \to \mathbf{Ab}$, where I is the poset of whole numbers in reverse order. A *double tower* is a functor $A\colon I \times I \to \mathbf{Ab}$; it may be helpful to think of the groups A_{ij} as forming a lattice in the first quadrant of the plane.

Proposition 5.8.7 ($\lim^1$ of a double tower) *For each double tower $A: I \times I \to \mathbf{Ab}$ we have $\lim_{I\times I} A_{ij} = \varprojlim_i \varprojlim_j A_{ij}$, a short exact sequence*

$$0 \to \varprojlim{}_i^1(\varprojlim{}_j A_{ij}) \to \left(R^1 \lim_{I\times I}\right) A_{ij} \to \varprojlim{}_i(\varprojlim{}_j^1 A_{ij}) \to 0,$$

$$\left(R^2 \lim_{I\times I}\right) A_{ij} = \varprojlim{}_i^1(\varprojlim{}_j^1 A_{ij}), \quad \text{and} \quad \left(R^n \lim_{I\times I}\right) A_{ij} = 0 \quad \text{for } n \geq 3.$$

Proof We may form the inverse limit as $\lim A_{ij} = \varprojlim_i \varprojlim_j A_{ij}$, that is, as the composition of $\varprojlim_j : (\mathbf{Ab}^I)^I \to \mathbf{Ab}^I$ and $\varprojlim_i : \mathbf{Ab}^I \to \mathbf{Ab}$. From 2.3.10 and 2.6.9 we see that $\varprojlim$ preserves injectives; it is right adjoint to the "constant tower" functor. Therefore we have a Grothendieck spectral sequence

$$E_2^{pq} = \varprojlim{}_i^p \varprojlim{}_j^q A_{ij} \Rightarrow (R^{p+q} \lim) A_{ij}.$$

Since both $\mathbf{Ab}$ and $\mathbf{Ab}^I$ satisfy $(AB4^*)$, $\varprojlim^p = \varprojlim^q = 0$ for $p, q \neq 0, 1$. Thus the spectral sequence degenerates as described. ◇

5.9 Exact Couples

An alternative construction of spectral sequences can be given via "exact couples" and is due to Massey [Massey]. It is often encountered in algebraic topology but rarely in commutative algebra.

It is convenient to forget all subscripts for a while and to work in the category of modules over some ring (or more generally in any abelian category satisfying axiom $AB5$). An *exact couple* $\mathcal{E}$ is a pair (D, E) of modules, together with three morphisms i, j, k

$$\mathcal{E}: \begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

which form an *exact triangle* in the sense that kernel = image at each vertex.

Definition 5.9.1 (Derived couple) The composition jk from E to itself satisfies $(jk)(jk) = j(kj)k = 0$, so we may form the homology module $H(E) =$

$\ker(jk)/\operatorname{image}(jk)$. Construct the triangle

$$\begin{array}{ccccc} & i(D) & \xrightarrow{i'} & i(D) & \\ \mathcal{E}': & {\scriptstyle k'}\nwarrow & & \swarrow{\scriptstyle j'} & \\ & & H(E) & & \end{array}$$

where i' is the restriction of i to $i(D)$, while j' and k' are given by

$$j'(i(d)) = [j(d)], \quad k'([e]) = k(e).$$

The map j' is well defined since $i(d) = 0$ implies that for some $e \in E$ $d = k(e)$ and $j(d) = jk(e)$ is a boundary. Similarly, $k(jk(e)) = 0$ implies that the map k' is well defined. We call $\mathcal{E}'$ the *derived couple* of $\mathcal{E}$. A diagram chase (left to the reader) shows that $\mathcal{E}'$ is also an exact couple.

If we iterate the process of taking exact couples r times, the result is called the r^{th} *derived couple* $\mathcal{E}^r$ of $\mathcal{E}$.

$$\begin{array}{ccccc} & D^r & \xrightarrow{i} & D^r & \\ \mathcal{E}^r: & {\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j^{(r)}} & \\ & & E^r & & \end{array}$$

Here $D^r = i^r(D)$ is a submodule of D, and $E^r = H(E^{r-1})$ is a subquotient of E. The maps i and k are induced from the i and k of $\mathcal{E}$, while $j^{(r)}$ sends $[i^r(d)]$ to $[j(d)]$.

Exercise 5.9.1 Show that $H(E) = k^{-1}(iD)/j(\ker(i))$ and more generally, that $E^r = Z^r/B^r$, with $Z^r = k^{-1}(i^r D)$ and $B^r = j(\ker(i^r))$.

With this generic background established, we now introduce subscripts (for D_{pq} and E_{pq}) in such a way that i has bidegree $(1, -1)$, k has bidegree $(-1, 0)$, and

$$\operatorname{bidegree}(j) = (-a, a).$$

Thus i and j preserve total degree $(p + q)$, while k drops the total degree by 1. Setting $D'_{pq} = i(D_{p-1,q+1}) \subseteq D_{pq}$ and letting E'_{pq} be the corresponding subquotient of E_{pq}, it is easy to see that in $\mathcal{E}'$ the maps i and k still have bidegrees $(1, -1)$ and $(-1, 0)$, while j' now has bidegree $(-1 - a, 1 + a)$. It is convenient to reindex so that $\mathcal{E} = \mathcal{E}^a$ and $\mathcal{E}^r$ denotes the $(r - a)^{th}$ derived couple of $\mathcal{E}$, so that $j^{(r)}$ has bidegree $(-r, r)$ and the E^r-differential has bidegree

$(-r, r-1)$.

$$E^r_{pq} \xrightarrow{k} D^r_{p-1,q} \xrightarrow{i} D^r_{p,q-1} \xrightarrow{j^{(r)}} E^r_{p-r,q+r-1}.$$

In summary, we have established the following result.

Proposition 5.9.2 *An exact couple $\mathcal{E}$ in which $i, k,$ and j have bidegrees $(1,-1)$, $(-1,0)$, and $(-a,a)$ determines a homology spectral sequence $\{E^r_{pq}\}$ starting with E^a. A morphism of exact couples induces a morphism of the corresponding spectral sequences.*

Example 5.9.3 (Exact couple of a filtration) Let C_* be a filtered chain complex of modules, and consider the bigraded homology modules

$$D^1_{pq} = H_n(F_pC),\ E^1_{pq} = H_n(F_pC/F_{p-1}C), \quad n = p+q.$$

Then the short exact sequences $0 \to F_{p-1} \to F_p \to F_p/F_{p-1} \to 0$ may be rolled up into an exact triangle of complexes (see Chapter 10 or 1.3.6)

$$\begin{tikzcd}
\oplus F_pC \arrow[rr, "i"] & & \oplus F_pC \arrow[dl, "\oplus\eta_p"] \\
& \oplus F_pC/F_{p-1}C \arrow[ul, "0"] &
\end{tikzcd}$$

whose homology forms an exact couple

$$\mathcal{E}^1: \quad \begin{tikzcd}
\oplus H_{p+q}(F_pC) \arrow[rr, "i"] & & \oplus H_{p+q}(F_pC) \arrow[dl, "j"] \\
& \oplus H_{p+q}(F_pC/F_{p-1}C) \arrow[ul, "k"] &
\end{tikzcd}$$

Theorem 5.9.4 *Let C_* be a filtered chain complex. The spectral sequence arising from the exact couple $\mathcal{E}^1$ (which starts at E^1) is naturally isomorphic to the spectral sequence constructed in section 5.4 (which starts at E^0).*

Proof In both spectral sequences, the groups E^r_{pq} are subquotients of $E^0_p = F_pC_{p+q}/F_{p-1}C_{p+q}$; we shall show they are the same subquotients. Since the differentials in both are induced from $d: C \to C$, this will establish the result.

In the exact couple spectral sequence, we see from exercise 5.9.1 that the numerator of E^r in E^1 is $k^{-1}(i^{r-1}D^1)$ and the denominator is $j(\ker\ i^{r-1})$.

If $c \in F_pC_n$ represents $[c] \in H_n(F_pC/F_{p-1}C)$, then $d(c) \in F_{p-1}C$ and $k([c])$ is the class of $d(c)$. Therefore the numerator in F_p/F_{p-1} for E^r is $Z^r_p = \{c \in F_pC : d(c) = a + d(b)$ for some $a \in F_{p-r}C, b \in F_pC\}/F_{p-1}C$. Similarly, the kernel of $i^{r-1}\colon H_n(F_pC) \to H_n(F_{p+r-1}C)$ is represented by those cycles $c \in F_pC$ with $c = d(b)$ for some $b \in F_{p+r-1}C$. That is, $\ker(i^{r-1})$ is the image of A^{r-1}_{p+r-1} in $H_n(F_pC)$. Since j is induced on homology by η_p, we see that the denominator is $B^r_p = \eta_p d(A^{r-1}_{p+r-1})$. Since the spectral sequence of section 5.4 had $E^r_p = Z^r_p/B^r_p$, we have finished the proof. ◇

Convergence 5.9.5 Let $\mathcal{E}$ be an exact couple in which i, j, and k have bidegrees $(-1, 1)$, $(-a, a)$ and, $(-1, 0)$, respectively. The associated spectral sequence is related to the direct limits $H_n = \varinjlim D_{p,n-p}$ of the D_{pq} along the maps $i\colon D_{pq} \to D_{p+1,q-1}$. Let F_pH_n denote the image of $D_{p+a,q-a}$ in H_n $(p + q = n)$; the system $\dots F_{p-1}H_n \subseteq F_pH_n \subseteq \dots$ forms an exhaustive filtration of H_n.

Proposition 5.9.6 *There is a natural inclusion of $F_pH_n/F_{p-1}H_n$ in $E^\infty_{p,n-p}$. The spectral sequence E^r_{pq} weakly converges to H_* if and only if:*

$$Z^\infty = \cap_r k^{-1}(i^r D) \text{ equals } k^{-1}(0) = j(D).$$

Proof Fix p, q, and $n = p + q$. The kernel $K_{p+a,q-a}$ of $D_{p+a,q-a} \to H_n$ is the union of the $\ker(i^r)$, so $j(K_{p+a,q-a}) = \cup j(\ker(i^r)) = \cup B^r_{pq} = B^\infty_{pq}$. (This is where axiom $AB5$ is used.) Applying the Snake Lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{p-1+a} & \longrightarrow & D_{p+a-1} & \longrightarrow & F_{p-1}H_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle i} & & \downarrow & & \\ 0 & \longrightarrow & K_{p+a} & \longrightarrow & D_{p+a} & \longrightarrow & F_pH_n & \longrightarrow & 0 \end{array}$$

yields the exact sequence

$$0 \to B^\infty_{pq} \to j(D_{p+a,q-a}) \to F_pH_n/F_{p-1}H_n \to 0.$$

But $j(D_{p+a,q-a}) = k^{-1}(0)$, so it is contained in $Z^r_{pq} = k^{-1}(i^r D_{p-r-1,q+r})$ for all r. The result now follows. ◇

We say that an exact couple is *bounded below* if for each n there is an integer $f(n)$ such that $D_{p,q} = 0$ whenever $p < f(p+q)$. In this case, for each p and q there is an r such that $i^r(D_{p-r-1,q+r}) = i^r(0) = 0$, i.e., $Z^r_{pq} = k^{-1}(0)$. As an immediate corollary, we obtain the following convergence result.

Classical Convergence Theorem 5.9.7 *If an exact couple is bounded below, then the spectral sequence is bounded below and converges to* $H_* = \varinjlim D$.

$$E^a_{pq} \Rightarrow H_{p+q}$$

The spectral sequence is bounded and converges to H_* *if for each* n *there is a* p *such that* $D_{p,n-p} \xrightarrow{\cong} H_n$.

Exercise 5.9.2 (Complete convergence) Let $\mathcal{E}$ be an exact couple that is bounded above ($D_{p,q} = 0$ whenever $p > f(p+q)$). Suppose that the spectral sequence is regular (5.2.10). Show that the spectral sequence converges to $\widehat{D}_n = \varprojlim D_{p,n-p}$.

Application 5.9.8 Here is an exact couple that does not arise from a filtered chain complex. Let C_* be an exact sequence of left R-modules and M a right R-module. Let $Z_p \subset C_p$ be the kernel of $d\colon C_p \to C_p$; associated to the short exact sequences $0 \to Z_p \to C_p \to Z_{p-1} \to 0$ are the long exact sequences

$$\cdots \mathrm{Tor}_q(M, Z_p) \xrightarrow{j} \mathrm{Tor}_q(M, C_p) \xrightarrow{k} \mathrm{Tor}_q(M, Z_{p-1}) \xrightarrow{i} \mathrm{Tor}_{q-1}(M, Z_p) \cdots$$

which we can assemble into an exact couple $\mathcal{E} = \mathcal{E}^0$ with

$$D^0_{pq} = \mathrm{Tor}_q(M, Z_p) \quad \text{and} \quad E^0_{pq} = \mathrm{Tor}_q(M, C_p).$$

By inspection, the map $d = jk\colon \mathrm{Tor}_q(M, C_p) \to \mathrm{Tor}_q(M, C_{p-1})$ is induced via $\mathrm{Tor}_q(M, -)$ by the differential $d\colon C_p \to C_{p-1}$, so we may write

$$E^1_{pq} = H_p(\mathrm{Tor}_q(M, C_*)).$$

More generally, if we replace $\mathrm{Tor}_*(M, -)$ by the derived functors L_*F of any right exact functor, the exact couple yields a spectral sequence with $E^0_{pq} = L_qF(C_p)$ and $E^1_{pq} = H_p(L_qF(C))$. These are essentially the hyperhomology sequences of section 5.7 related to the hyperhomology modules $\mathbb{L}_*F(C)$, which are zero. Therefore this spectral sequence converges to zero whenever C_* is bounded below.

Bockstein Spectral Sequence 5.9.9 Fix a prime ℓ and let H_* be a (graded) abelian group. Suppose that multiplication by ℓ fits into a long exact sequence

$$\cdots E_{n+1} \xrightarrow{\partial} H_n \xrightarrow{\ell} H_n \xrightarrow{j} E_n \xrightarrow{\partial} H_{n-1} \xrightarrow{\ell} \cdots.$$

If we roll this up into the exact couple

$$\begin{tikzcd} H_* \arrow[rr, "\ell"] & & H_* \arrow[dl, "j"] \\ & E_* \arrow[ul, "\partial"] & \end{tikzcd}$$

then we obtain a spectral sequence with $E^0_* = E_*$, called the *Bockstein spectral sequence* associated to H_*. This spectral sequence was first studied by W. Browder in [Br], who noted the following applications:

1. $H_* = H_*(X; \mathbb{Z})$ and $E_* = H_*(X; \mathbb{Z}/\ell)$ for a topological space X
2. $H_* = \pi_*(X)$ and $E_* = \pi_*(X; \mathbb{Z}/\ell)$ for a topological space X
3. $H_* = H_*(G; \mathbb{Z})$ and $E_* = H_*(G; \mathbb{Z}/\ell)$ for a group G
4. $H_* = H_*(C)$ for a torsionfree chain complex C, and $E_* = H_*(C/\ell C)$

We note that the differential $d = j\partial$ sends E^r_n to E^r_{n-1}, so that the bigrading subscripts we formally require for a spectral sequence are completely artificial. The next result completely describes the convergence of the Bockstein spectral sequence. To state it, it is convenient to adapt the notation that for $q \in \mathbb{Z}$

$${}_qH_* = \{x \in H_* : qx = 0\}.$$

Proposition 5.9.10 *For every $r \geq 0$, there is an exact sequence*

$$0 \to \frac{H_n}{\ell H_n + {}_{\ell^r}H_n} \xrightarrow{j} E^r_n \xrightarrow{\partial} (\ell^r H_{n-1}) \cap ({}_\ell H_{n-1}) \to 0.$$

In particular, if T_n denotes the ℓ-primary torsion subgroup of H_n and Q_n denotes the infinitely ℓ-divisible part of ${}_\ell H_n$, then there is an exact sequence

$$0 \to \frac{H_n}{{}_\ell H_n + T_n} \xrightarrow{j} E^\infty_n \xrightarrow{\partial} Q_{n-1} \to 0.$$

Proof For $r = 0$ we are given an extension

$$0 \to H_n/{}_\ell H_n \xrightarrow{j} E^0_n \xrightarrow{\partial} {}_\ell H_{n-1} \to 0.$$

Now E^r is the subquotient of E^0 with numerator $\partial^{-1}(\ell^r H)$ and denominator $j({}_{\ell^r}H)$ by the above exercise, so from the extension

$$0 \to H/\ell H \xrightarrow{j} \partial^{-1}(\ell^r H) \xrightarrow{\partial} (\ell^r H \cap {}_\ell H) \to 0$$

the result is immediate. ◇

Corollary 5.9.11 *If each H_n is finitely generated and* $\dim(H_n \otimes \mathbb{Q}) = d_n$, *then the Bockstein spectral sequence converges to $E_n^\infty = (\mathbb{Z}/p)^{d_n}$ and is bounded in the sense that $E_n^\infty = E_n^r$ for large r.*

Actually, it turns out that the Bockstein spectral sequence can be used to completely describe H_* when each H_n is finitely generated. For example, if X is a simply connected H-space whose homology is finitely generated (such as a Lie group), Browder used the Bockstein spectral sequence in [Br] to prove that $\pi_2(X) = 0$.

For this, note that j induces a map $H_n \to E_n^r$ for each r. If $X \in E_n^r$ has $\alpha(x) = p^r y$, then $d(x) = j^{(r)}\alpha(x) = \overline{j(y)}$ in the notation of the proposition. In particular, x is a cycle if and only if $\alpha(x)$ is divisible by p^{r+1}. We can summarize these observations as follows.

Corollary 5.9.12 *In the Bockstein spectral sequence*

1. *Elements of E_n that survive to E^r but not to E^{r+1} (because they are not cycles) correspond to elements of exponent p in H_{n-1}, which are divisible by p^r but not by p^{r+1}.*
2. *An element $y \in H_n$ yields an element $j(y)$ of E^r for all r; if $j(y) \neq 0$ in E^{r-1}, but $j(y) = 0$ in E^r, then y generates a direct summand of H_n isomorphic to $\mathbb{Z}/p^r$.*

Exercise 5.9.3 Study the exact couple for $H = \mathbb{Z}/p^3$, and show directly that $E^2 \neq 0$ but $E^3 = 0$.