4
Homological Dimension

4.1 Dimensions

Definitions 4.1.1 Let A be a right R-module.

1. The *projective dimension* $pd(A)$ is the minimum integer n (if it exists) such that there is a resolution of A by projective modules

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to A \to 0.$$

2. The *injective dimension* $id(A)$ is the minimum integer n (if it exists) such that there is a resolution of A by injective modules

$$0 \to A \to E^0 \to E^1 \to \cdots \to E^n \to 0.$$

3. The *flat dimension* $fd(A)$ is the minimum integer n (if it exists) such that there is a resolution of A by flat modules

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to A \to 0.$$

If no finite resolution exists, we set $pd(A)$, $id(A)$, or $fd(A)$ equal to ∞.

We are going to prove the following theorems in this section, which allow us to define the global and Tor dimensions of a ring R.

Global Dimension Theorem 4.1.2 The following numbers are the same for any ring R:

1. $\sup\{id(B) : B \in \mathbf{mod}\text{–}R\}$
2. $\sup\{pd(A) : A \in \mathbf{mod}\text{–}R\}$
3. $\sup\{pd(R/I) : I \text{ is a right ideal of } R\}$
4. $\sup\{d : \operatorname{Ext}_R^d(A, B) \neq 0 \text{ for some right modules } A, B\}$

This common number (possibly ∞) is called the (right) global dimension *of R, $r.gl.\dim(R)$. Bourbaki [BX] calls it the* homological dimension *of R.*

Remark One may define the left global dimension $\ell.gl.\dim(R)$ similarly. If R is commutative, we clearly have $\ell.gl.\dim(R) = r.gl.\dim(R)$. Equality also holds if R is left *and* right noetherian. Osofsky [Osof] proved that if every one-sided ideal can be generated by at most $\aleph_n$ elements, then $|\ell.gl.\dim(R) - r.gl.\dim(R)| \leq n+1$. The continuum hypothesis of set theory lurks at the fringe of this subject whenever we encounter non-constructible ideals over uncountable rings.

Tor-dimension Theorem 4.1.3 *The following numbers are the same for any ring R:*

1. $\sup\{fd(A) : A$ *is a right R-module*$\}$
2. $\sup\{fd(R/J) : J$ *is a right ideal of R*$\}$
3. $\sup\{fd(B) : B$ *is a left R-module*$\}$
4. $\sup\{fd(R/I) : I$ *is a left ideal of R*$\}$
5. $\sup\{d : \mathrm{Tor}_d^R(A, B) \neq 0$ *for some R-modules A, B*$\}$

This common number (possibly ∞) is called the Tor-dimension *of R. Due to the influence of [CE], the less descriptive name* weak dimension *of R is often used.*

Example 4.1.4 Obviously every field has both global and Tor-dimension zero. The Tor and Ext calculations for abelian groups show that $R = \mathbb{Z}$ has global dimension 1 and Tor-dimension 1. The calculations for $R = \mathbb{Z}/m$ show that if some $p^2|m$ (so R isn't a product of fields), then $\mathbb{Z}/m$ has global dimension ∞ and Tor-dimension ∞.

As projective modules are flat, $fd(A) \leq pd(A)$ for every R-module A. We need not have equality: over $\mathbb{Z}$, $fd(\mathbb{Q}) = 0$, but $pd(\mathbb{Q}) = 1$. Taking the supremum over all A shows that Tor–dim$(R) \leq r.gl.\dim(R)$. We will see examples in the next section where Tor–dim$(R) \neq r.gl.\dim(R)$. These examples are perforce non-noetherian, as we now prove, assuming the global and Tor-dimension theorems.

Proposition 4.1.5 *If R is right noetherian, then*

1. $fd(A) = pd(A)$ *for every finitely generated R-module A.*
2. Tor–dim$(R) = r.gl.\dim(R)$.

Proof Since we can compute Tor–dim(R) and $r.gl.\dim(R)$ using the modules R/I, it suffices to prove (1). Since $fd(A) \leq pd(A)$, it suffices to suppose that $fd(A) = n < \infty$ and prove that $pd(A) \le n$. As R is noetherian, there is a resolution

$$0 \to M \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$$

in which the P_i are finitely generated free modules and M is finitely presented. The fd lemma 4.1.10 below implies that the syzygy M is a flat R-module, so M must also be projective (3.2.7). This proves that $pd(A) \le n$, as required. $\diamond$

Exercise 4.1.1 Use the Tor-dimension theorem to prove that if R is both left and right noetherian, then $r.gl.\dim(R) = l.gl.\dim(R)$.

The pattern of proof for both theorems will be the same, so we begin with the characterization of projective dimension.

pd Lemma 4.1.6 *The following are equivalent for a right R-module A:*

1. $pd(A) \le d$.
2. $\mathrm{Ext}_R^n(A, B) = 0$ *for all $n > d$ and all R-modules B.*
3. $\mathrm{Ext}_R^{d+1}(A, B) = 0$ *for all R-modules B.*
4. *If* $0 \to M_d \to P_{d-1} \to P_{d-2} \to \cdots \to P_1 \to P_0 \to A \to 0$ *is* any *resolution with the P's projective, then the syzygy M_d is also projective.*

Proof Since $\mathrm{Ext}^*(A, B)$ may be computed using a projective resolution of A, it is clear that (4) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3). If we are given a resolution of A as in (4), then $\mathrm{Ext}^{d+1}(A, B) \cong \mathrm{Ext}^1(M_d, B)$ by dimension shifting. Now M_d is projective iff $\mathrm{Ext}^1(M_d, B) = 0$ for all B (exercise 2.5.2), so (3) implies (4). $\diamond$

Example 4.1.7 In 3.1.6 we produced an infinite projective resolution of $A = \mathbb{Z}/p$ over the ring $R = \mathbb{Z}/p^2$. Each syzygy was $\mathbb{Z}/p$, which is not a projective $\mathbb{Z}/p^2$-module. Therefore by (4) we see that $\mathbb{Z}/p$ has $pd = \infty$ over $R = \mathbb{Z}/p^2$. On the other hand, $\mathbb{Z}/p$ has $pd = 0$ over $R = \mathbb{Z}/p$ and $pd = 1$ over $R = \mathbb{Z}$.

The following two lemmas have the same proof as the preceding lemma.

id Lemma 4.1.8 *The following are equivalent for a right R-module B:*

1. $id(B) \le d$.
2. $\mathrm{Ext}_R^n(A, B) = 0$ *for all $n > d$ and all R-modules A.*
3. $\mathrm{Ext}_R^{d+1}(A, B) = 0$ *for all R-modules B.*
4. *If* $0 \to B \to E^0 \to \cdots \to E^{d-1} \to M^d \to 0$ *is a resolution with the E^i injective, then M^d is also injective.*

Example 4.1.9 In 3.1.6 we gave an infinite injective resolution of $B = \mathbb{Z}/p$ over $R = \mathbb{Z}/p^2$ and showed that $\mathrm{Ext}^n_R(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p$ for all n. Therefore $\mathbb{Z}/p$ has $id = \infty$ over $R = \mathbb{Z}/p^2$. On the other hand, it has $id = 0$ over $R = \mathbb{Z}/p$ and $id = 1$ over $\mathbb{Z}$.

fd Lemma 4.1.10 *The following are equivalent for a right R-module A:*

1. $fd(A) \le d$.
2. $\mathrm{Tor}^R_n(A, B) = 0$ *for all $n > d$ and all left R-modules B.*
3. $\mathrm{Tor}^R_{d+1}(A, B) = 0$ *for all left R-modules B.*
4. *If* $0 \to M_d \to F_{d-1} \to F_{d-2} \to \cdots \to F_0 \to A \to 0$ *is a resolution with the F_i all flat, then M_d is also a flat R-module.*

Lemma 4.1.11 *A left R-module B is injective iff* $\mathrm{Ext}^1(R/I, B) = 0$ *for all left ideals I.*

Proof Applying $\mathrm{Hom}(-, B)$ to $0 \to I \to R \to R/I \to 0$, we see that

$$\mathrm{Hom}(R, B) \to \mathrm{Hom}(I, B) \to \mathrm{Ext}^1(R/I, B) \to 0$$

is exact. By Baer's criterion 2.3.1, B is injective iff the first map is surjective, that is, iff $\mathrm{Ext}^1(R/I, B) = 0$. ◇

Proof of Global Dimension Theorem The lemmas characterizing $pd(A)$ and $id(A)$ show that $\sup(2) = \sup(4) = \sup(1)$. As $\sup(2) \ge \sup(3)$, we may assume that $d = \sup\{pd(R/I)\}$ is finite and that $id(B) > d$ for some R-module B. For this B, choose a resolution

$$0 \to B \to E^0 \to E^1 \to \cdots \to E^{d-1} \to M \to 0$$

with the E's injective. But then for all ideals I we have

$$0 = \mathrm{Ext}^{d+1}_R(R/I, B) \cong \mathrm{Ext}^1_R(R/I, M).$$

By the preceding lemma 4.1.11, M is injective, a contradiction to $id(B) > d$. ◇

Proof of Tor-dimension theorem The lemma 4.1.10 characterizing $fd(A)$ over R shows that $\sup(5) = \sup(1) \ge \sup(2)$. The same lemma over R^{op} shows that $\sup(5) = \sup(3) \ge \sup(4)$. We may assume that $\sup(2) \le \sup(4)$, that is, that $d = \sup\{fd(R/J) : J \text{ is a right ideal}\}$ is at most the supremum over left ideals.

We are done unless d is finite and $fd(B) > d$ for some left R-module B. For this B, choose a resolution $0 \to M \to F_{d-1} \to \cdots \to F_0 \to B \to 0$ with the F's flat. But then for all ideals J we have

$$0 = \mathrm{Tor}_{d+1}^R(R/J, B) = \mathrm{Tor}_1^R(R/J, M).$$

We saw in 3.2.4 that this implies that M is flat, contradicting $fd(B) > d$. $\diamond$

Exercise 4.1.2 If $0 \to A \to B \to C \to 0$ is an exact sequence, show that

1. $pd(B) \leq \max\{pd(A), pd(C)\}$ with equality except when $pd(C) = pd(A) + 1$.
2. $id(B) \leq \max\{id(A), id(C)\}$ with equality except when $id(A) = id(C) + 1$.
3. $fd(B) \leq \max\{fd(A), fd(C)\}$ with equality except when $fd(C) = fd(A) + 1$.

Exercise 4.1.3

1. Given a (possibly infinite) family $\{A_i\}$ of modules, show that

$$\mathrm{pd}\left(\bigoplus A_i\right) = \sup\{\mathrm{pd}(A_i)\}.$$

2. Conclude that if S is an R-algebra and P is a projective S-module considered as an R-module, the $pd_R(P) \leq pd_R(S)$.
3. Show that if $r.gl.\dim(R) = \infty$, there actually is an R-module A with $pd(A) = \infty$.

4.2 Rings of Small Dimension

Definition 4.2.1 A ring R is called *(right) semisimple* if every right ideal is a direct summand of R or, equivalently, if R is the direct sum of its minimal ideals. Wedderburn's theorem (see [Lang]) classifies semisimple rings: they are finite products $R = \prod_{i=1}^r R_i$ of matrix rings $R_i = M_{n_i}(D_i) = \mathrm{End}_{D_i}(V_i)$ ($n_i = \dim(V_i)$) over division rings D_i. It follows that right semisimple is the same as left semisimple, and that every semisimple ring is (both left and right) noetherian. By Maschke's theorem, the group ring $k[G]$ of a finite group G over a field k is semisimple if char(k) doesn't divide the order of G.

Theorem 4.2.2 *The following are equivalent for every ring R, where by "R-module" we mean either left R-module or right R-module.*

1. *R is semisimple.*
2. *R has (left and/or right) global dimension 0.*
3. *Every R-module is projective.*
4. *Every R-module is injective.*
5. *R is noetherian, and every R-module is flat.*
6. *R is noetherian and has Tor-dimension 0.*

Proof We showed in the last section that (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) for left R-modules and also for right R-modules. R is semisimple iff every short exact sequence $0 \to I \to R \to R/I \to 0$ splits, that is, iff $pd(R/I) = 0$ for every (right and/or left) ideal I. This proves that (1)$\Leftrightarrow$ (2). As (1) and (3) imply (5), and (5)$\Leftrightarrow$ (6) by definition, we only have to show that (5) implies (1). If I is an ideal of R, then (5) implies that R/I is finitely presented and flat, hence projective by 3.2.7. Since R/I is projective, $R \to R/I$ splits, and I is a direct summand of R, that is, (1) holds. $\diamond$

Definition 4.2.3 A ring R is *quasi-Frobenius* if it is (left and right) noetherian and R is an injective (left and right) R-module. Our interest in quasi-Frobenius rings stems from the following result of Faith and Faith-Walker, which we quote from [Faith].

Theorem 4.2.4 *The following are equivalent for every ring R:*

1. *R is quasi-Frobenius.*
2. *Every projective right R-module is injective.*
3. *Every injective right R-module is projective.*
4. *Every projective left R-module is injective.*
5. *Every injective left R-module is projective.*

Exercise 4.2.1 Show that $\mathbb{Z}/m$ is a quasi-Frobenius ring for every integer m.

Exercise 4.2.2 Show that if R is quasi-Frobenius, then either R is semisimple or R has global dimension ∞. *Hint:* Every finite projective resolution is split.

Definition 4.2.5 A *Frobenius algebra* over a field k is a finite-dimensional algebra R such that $R \cong \text{Hom}_k(R, k)$ as (right) R-modules. Frobenius algebras are quasi-Frobenius; more generally, $\text{Hom}_k(R, k)$ is an injective R-module for any algebra R over any field k, since k is an injective k-module and $\text{Hom}_k(R, -)$ preserves injectives (being right adjoint to the forgetful functor $\textbf{mod}\text{–}R \to \textbf{mod}\text{–}k$). Frobenius algebras were introduced in 1937 by Brauer and Nesbitt in order to generalize group algebras $k[G]$ of a finite group, especially when $\text{char}(k) = p$ divides the order of G so that $k[G]$ is not semisimple.

Proposition 4.2.6 *If G is a finite group, then $k[G]$ is a Frobenius algebra.*

Proof Set $R = k[G]$ and define $f: R \to k$ by letting $f(r)$ be the coefficient of $g = 1$ in the unique expression $r = \sum_{g \in G} r_g g$ of every element $r \in k[G]$. Let $\alpha: R \to \mathrm{Hom}_k(R, k)$ be the map $\alpha(r): x \mapsto f(rx)$. Since $\alpha(r) = fr$, α is a right R-module map; we claim that α is an isomorphism. If $\alpha(r) = 0$ for $r = \sum r_g g$, then $r = 0$ as each $r_g = f(rg^{-1}) = \alpha(r)(g^{-1}) = 0$. Hence α is an injection. As R and $\mathrm{Hom}_k(R, k)$ have the same finite dimension over k, α must be an isomorphism. ◇

Vista 4.2.7 Let R be a commutative noetherian ring. R is called a *Gorenstein ring* if $id(R)$ is finite; in this case $id(R)$ is the Krull dimension of R, defined in 4.4.1. Therefore a quasi-Frobenius ring is just a Gorenstein ring of Krull dimension zero, and in particular a finite product of 0-dimensional local rings. If R is a 0-dimensional local ring with maximal ideal $\mathfrak{m}$, then R is quasi-Frobenius $\Leftrightarrow \mathrm{ann}_R(\mathfrak{m}) = \{r \in R : r\mathfrak{m} = 0\} \cong R/\mathfrak{m}$. This recognition criterion is at the heart of current research into the Gorenstein rings that arise in algebraic geometry.

Now we shall characterize rings of Tor-dimension zero. A ring R is called *von Neumann regular* if for every $a \in R$ there is an $x \in R$ for which $axa = a$. These rings were introduced by J. von Neumann in 1936 in order to study continuous geometries such as the lattices of projections in "von Neumann algebras" of bounded operators on a Hilbert space. For more information about von Neumann regular rings, see [Good].

Remark A commutative ring R is von Neumann regular iff R has no nilpotent elements and has Krull dimension zero. On the other hand, a commutative ring R is semisimple iff it is a finite product of fields.

Exercise 4.2.3 Show that an infinite product of fields is von Neumann regular. This shows that not every von Neumann regular ring is semisimple.

Exercise 4.2.4 If V is a vector space over a field k (or a division ring k), show that $R = End_k(V)$ is von Neumann regular. Show that R is semisimple iff $\dim_k(V) < \infty$.

Lemma 4.2.8 *If R is von Neumann regular and I is a finitely generated right ideal of R, then there is an* idempotent *e (an element with $e^2 = e$) such that $I = eR$. In particular, I is a projective R-module, because $R \cong I \oplus (1 - e)R$.*

Proof Suppose first that $I = aR$ and that $axa = a$. It follows that $e = ax$ is idempotent and that $I = eR$. By induction on the number of generators of

I, we may suppose that $I = aR + bR$ with $a \in I$ idempotent. Since $bR = abR + (1-a)bR$, we have $I = aR + cR$ for $c = (1-a)b$. If $cyc = c$, then $f = cy$ is idempotent and $af = a(1-a)by = 0$. As fa may not vanish, we consider $e = f(1-a)$. Then $e \in I$, $ae = 0 = ea$, and e is idempotent:

$$e^2 = f(1-a)f(1-a) = f(f-af)(1-a) = f^2(1-a) = f(1-a) = e.$$

Moreover, $eR = cR$ because $c = fc = ffc = f(1-a)fc = efc$. Finally, we claim that I equals $J = (a+e)R$. Since $a + e \in I$, we have $J \subseteq I$; the reverse inclusion follows from the observation that $a = (a+e)a \in J$ and $e = (a+e)e \in J$. ◇

Exercise 4.2.5 Show that the converse holds: If every fin. gen. right ideal I of R is generated by an idempotent (*i.e.*, $R \cong I \oplus R/I$), then R is von Neumann regular.

Theorem 4.2.9 *The following are equivalent for every ring R:*

1. *R is von Neumann regular.*
2. *R has Tor-dimension 0.*
3. *Every R-module is flat.*
4. *R/I is projective for every finitely generated ideal I.*

Proof By definition, (2) ⇔ (3). If I is a fin. generated ideal, then R/I is finitely presented. Thus R/I is flat iff it is projective, hence iff $R \cong I \oplus R/I$ as a module. Therefore (3) ⇒ (4) ⇔ (1). Finally, any ideal I is the union of its finitely generated subideals I_α, and we have $R/I = \varinjlim(R/I_\alpha)$. Hence (4) implies that each R/I is flat, that is, that (2) holds. ◇

Remark Since the Tor-dimension of a ring is at most the global dimension, noetherian von Neumann regular rings must be semisimple (4.1.5). Von Neumann regular rings that are not semisimple show that we can have Tor–dim$(R) < gl.\dim(R)$. For example, the global dimension of $\prod_{i=1}^{\infty} \mathbb{C}$ is ≥ 2, with equality iff the Continuum Hypothesis holds.

Definition 4.2.10 A ring R is called (*right*) *hereditary* if every right ideal is projective. A commutative integral domain R is hereditary iff it is a *Dedekind domain* (noetherian, Krull dimension 0 or 1 and every local ring R_m is a discrete valuation ring). Principal ideal domains (*e.g*, $\mathbb{Z}$ or $k[t]$) are Dedekind, and of course every semisimple ring is hereditary.

Theorem 4.2.11 *A ring R is right hereditary iff $r.gl.\dim(R) \leq 1$.*

Proof The exact sequences $0 \to I \to R \to R/I \to 0$ show that R is hereditary iff $r.gl.\dim(R) \le 1$. $\diamondsuit$

Exercise 4.2.6 Show that R is right hereditary iff every submodule of every free module is projective. This was used in exercise 3.6.2.

4.3 Change of Rings Theorems

General Change of Rings Theorem 4.3.1 *Let $f: R \to S$ be a ring map, and let A be an S-module. Then as an R-module*

$$pd_R(A) \le pd_S(A) + pd_R(S).$$

Proof There is nothing to prove if $pd_S(A) = \infty$ or $pd_R(S) = \infty$, so assume that $pd_S(A) = n$ and $pd_R(S) = d$ are finite. Choose an S-module projective resolution $Q \to A$ of length n. Starting with R-module projective resolutions of A and of each syzygy in Q, the Horseshoe Lemma 2.2.8 gives us R-module projective resolutions $\tilde{P}_{*q} \to Q_q$ such that $\tilde{P}_{*q} \to \tilde{P}_{*,q-2}$ is zero. We saw in section 4.1 that $pd_R(Q_q) \le d$ for each q. The truncated resolutions $P_{*q} \to Q_q$ of length d ($P_{iq} = 0$ for $i > d$ and $P_{dq} = \tilde{P}_{dq}/\operatorname{im}(\tilde{P}_{d+1,q})$, as in 1.2.7) have the same property. By the sign trick, we have a double complex P_{**} and an augmentation $P_{0*} \to Q_*$.

$$
\begin{array}{c|ccccccccccc}
0 & 0 & & 0 & & & & & & 0 & & \\
\downarrow & \downarrow & & \downarrow & & & & & & \downarrow & & \\
Q_n & P_{0n} & \longleftarrow & P_{1n} & \longleftarrow & \cdots & \longleftarrow & \cdots & \longleftarrow & P_{dn} & \longleftarrow & 0 \\
\downarrow & \downarrow & & \downarrow & & & & & & \downarrow & & \\
\cdots & \cdots & & \cdots & & \cdots & & & & \cdots & & \\
\downarrow & \downarrow & & \downarrow & & & & & & \downarrow & & \\
Q_1 & P_{01} & \longleftarrow & P_{11} & \longleftarrow & P_{21} & \longleftarrow & \cdots & \longleftarrow & P_{d1} & \longleftarrow & 0 \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
Q_0 & P_{00} & \longleftarrow & P_{10} & \longleftarrow & P_{20} & \longleftarrow & \cdots & \longleftarrow & P_{d0} & \longleftarrow & 0 \\
\downarrow & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & 0 & & 0 & & 0 & & & & 0 & &
\end{array}
$$

The argument used in 2.7.2 to balance Tor shows that $\operatorname{Tot}(P) \to Q$ is a quasi-isomorphism, because the rows of the augmented double complex (add $Q[-1]$

in column -1) are exact. Hence $\text{Tot}(P) \to A$ is an R-module projective resolution of A. But then $pd_R(A)$ is at most the length of $\text{Tot}(P)$, that is, $d + n$. $\diamond$

Example 4.3.2 If R is a field and $pd_S(A) \neq 0$, we have strict inequality.

Remark The above argument presages the use of spectral sequences in getting more explicit information about $\text{Ext}_R^*(A, B)$. An important case in which we have equality is the case $S = R/xR$ when x is a nonzerodivisor, so $pd_R(R/xR) = 1$.

First Change of Rings Theorem 4.3.3 *Let x be a central nonzerodivisor in a ring R. If $A \neq 0$ is a R/x-module with $pd_{R/x}(A)$ finite, then*

$$pd_R(A) = 1 + pd_{R/x}(A).$$

Proof As $xA = 0$, A cannot be a projective R-module, so $pd_R(A) \geq 1$. On the other hand, if A is a projective R/x-module, then evidently $pd_R(A) = pd_R(R/x) = 1$. If $pd_{R/x}(A) \geq 1$, find an exact sequence

$$0 \to M \to P \to A \to 0$$

with P a projective R/x-module, so that $pd_{R/x}(A) = pd_{R/x}(M) + 1$. By induction, $pd_R(M) = 1 + pd_{R/x}(M) = pd_{R/x}(A) \geq 1$. Either $pd_R(A)$ equals $pd_R(M) + 1$ or $1 = pd_R(P) = \sup\{pd_R(M), pd_R(A)\}$. We shall conclude the proof by eliminating the possibility that $pd_R(A) = 1 = pd_{R/x}(A)$.

Map a free R-module F onto A with kernel K. If $pd_R(A) = 1$, then K is a projective R-module. Tensoring with R/xR yields the sequence of R/x-modules:

$$0 \to \text{Tor}_1^R(A, R/x) \to K/xK \to F/xF \to A \to 0.$$

If $pd_{R/x}(A) \leq 2$, then $\text{Tor}_1^R(A, R/x)$ is a projective R/x-module. But

$$\text{Tor}_1^R(A, R/x) \cong \{a \in A : xa = 0\} = A, \text{ so } pd_{R/x}(A) = 0. \qquad \diamond$$

Example 4.3.4 The conclusion of this theorem fails if $pd_{R/x}(A) = \infty$ but $pd_R(A) < \infty$. For example, $pd_{\mathbb{Z}/4}(\mathbb{Z}/2) = \infty$ but $pd_{\mathbb{Z}}(\mathbb{Z}/2) = 1$.

Exercise 4.3.1 Let R be the power series ring $k[[x_1, \cdots, x_n]]$ over a field k. R is a noetherian local ring with residue field k. Show that $gl.\dim(R) = pd_R(k) = n$.

Second Change of Rings Theorem 4.3.5 *Let x be a central nonzerodivisor in a ring R. If A is an R-module and x is a nonzerodivisor on A (i.e., $a \neq 0 \Rightarrow xa \neq 0$), then*

$$pd_R(A) \geq pd_{R/x}(A/xA).$$

Proof If $pd_R(A) = \infty$, there is nothing to prove, so we assume $pd_R(A) = n < \infty$ and proceed by induction on n. If A is a projective R-module, then A/xA is a projective R/x-module, so the result is true if $pd_R(A) = 0$. If $pd_R(A) \neq 0$, map a free R-module F onto A with kernel K. As $pd_R(K) = n - 1$, $pd_{R/x}(K/xK) \leq n - 1$ by induction. Tensoring with R/x yields the sequence

$$0 \to \operatorname{Tor}_1^R(A, R/x) \to K/xK \to F/xF \to A/xA \to 0.$$

As x is a nonzerodivisor on A, $\operatorname{Tor}_1(A, R/x) \cong \{a \in A : xa = 0\} = 0$. Hence either A/xA is projective or $pd_{R/x}(A/xA) = 1 + pd_{R/x}(K/xK) \leq 1 + (n - 1) = pd_R(A)$. ◇

Exercise 4.3.2 Use the first Change of Rings Theorem 4.3.3 to find another proof when $pd_{R/x}(A/xA)$ is finite.

Now let $R[x]$ be a polynomial ring in one variable over R. If A is an R-module, write $A[x]$ for the $R[x]$-module $R[x] \otimes_R A$.

Corollary 4.3.6 *$pd_{R[x]}(A[x]) = pd_R(A)$ for every R-module A.*

Proof Writing $T = R[x]$, we note that x is a nonzerodivisor on $A[x] = T \otimes_R A$. Hence $pd_T(A[x]) \geq pd_R(A)$ by the second Change of Rings theorem 4.3.5. On the other hand, if $P \to A$ is an R-module projective resolution, then $T \otimes_R P \to T \otimes_R A$ is a T-module projective resolution (T is flat over R), so $pd_R(A) \geq pd_T(T \otimes A)$. ◇

Theorem 4.3.7 *If $R[x_1, \cdots, x_n]$ denotes a polynomial ring in n variables, then $gl.\dim(R[x_1, \cdots, x_n]) = n + gl.\dim(R)$.*

Proof It suffices to treat the case $T = R[x]$. If $gl.\dim(R) = \infty$, then by the above corollary $gl.\dim(T) = \infty$, so we may assume $gl.\dim(R) = n < \infty$. By the first Change of Rings theorem 4.3.3, $gl.\dim(T) \geq 1 + gl.\dim(R)$. Given a T-module M, write $U(M)$ for the underlying R-module and consider the sequence of T-modules

$$(*) \qquad 0 \to T \otimes_R U(M) \xrightarrow{\beta} T \otimes_R U(M) \xrightarrow{\mu} M \to 0,$$

where μ is multiplication and β is defined by the bilinear map $\beta(t \otimes m) = t[x \otimes m - 1 \otimes (xm)]$ $(t \in T, m \in M)$. We claim that $(*)$ is exact, which yields the inequality $pd_T(M) \leq 1 + pd_T(T \otimes_R U(M)) = 1 + pd_R(U(M)) \leq 1 + n$. The supremum over all M gives the final inequality $gl.\dim(T) \leq 1 + n$.

To finish the proof, we must establish the claim that $(*)$ is exact. We first observe that, since T is a free R-module on basis $\{1, x, x^2, \cdots\}$, we can write every nonzero element f of $T \otimes U(M)$ as a polynomial with coefficients $m_i \in M$:

$$f = x^k \otimes m_k + \cdots + x^2 \otimes m_2 + x \otimes m_1 + 1 \otimes m_0 \quad (m_k \neq 0).$$

Since the leading term of $\beta(f)$ is $x^{k+1} \otimes m_k$, we see that β is injective. Clearly $\mu\beta = 0$. Finally, we prove by induction on k (the degree of f) that if $f \in \ker(\mu)$, then $f \in im(\beta)$. Since $\mu(1 \otimes m) = m$, the case $k = 0$ is trivial (if $\mu(f) = 0$, then $f = 0$). If $k \neq 0$, then $\mu(f) = \mu(g)$ for the polynomial $g = f - \beta(x^{k-1} \otimes m_k)$ of lower degree. By induction, if $f \in \ker(\mu)$, then $g = \beta(h)$ for some h, and hence $f = \beta(h + x^{k-1} \otimes m_k)$. ◇

Corollary 4.3.8 (Hilbert's theorem on syzygies) *If k is a field, then the polynomial ring $k[x_1, \cdots, x_n]$ has global dimension n. Thus the $(n-1)^{st}$ syzygy of every module is a projective module.* ◇

We now turn to the third Change of Rings theorem. For simplicity we deal with commutative local rings, that is, commutative rings with a unique maximal ideal. Here is the fundamental tool used to study local rings.

Nakayama's Lemma 4.3.9 *Let R be a commutative local ring with unique maximal ideal $\mathfrak{m}$ and let B be a nonzero finitely generated R-module. Then*

1. *$B \neq \mathfrak{m}B$.*
2. *If $A \subseteq B$ is a submodule such that $B = A + \mathfrak{m}B$, then $A = B$.*

Proof If we consider B/A then (2) is a special case of (1). Let m be the smallest integer such that B is generated $b_1, \cdots, b_m$; as $B \neq 0$, we have $m \neq 0$. If $B = \mathfrak{m}B$, then there are $r_i \in \mathfrak{m}$ such that $b_m = \sum r_i b_i$. This yields

$$(1 - r_m)b_m = r_1 b_1 + \cdots + r_{m-1} b_{m-1}.$$

Since $1 - r_m \notin \mathfrak{m}$, it is a unit of R. Multiplying by its inverse writes b_m as a linear combination of $\{b_1, \cdots, b_{m-1}\}$, so this set also generates B. This contradicts the choice of m. ◇

Remark If R is any ring, the set

$$J = \{r \in R : (\forall s \in R)\ 1 - rs \quad \text{is a unit of} \quad R\}$$

is a 2-sided ideal of R, called the *Jacobson radical* of R (see [BAII, 4.2]). The above proof actually proves the following:

General Version of Nakayama's Lemma 4.3.10 Let B be a nonzero finitely generated module over R and J the Jacobson radical of R. Then $B \neq JB$.

Proposition 4.3.11 *A finitely generated projective module P over a commutative local ring R is a free module.*

Proof Choose $u_1, \cdots, u_n \in P$ whose images form a basis of the k-vector space $P/\mathfrak{m}P$. By Nakayama's lemma the u's generate P, so the map $\epsilon: R^n \to P$ sending $(r_1, \cdots, r_n)$ to $\sum r_i u_i$ is onto. As P is projective, ϵ is split, that is, $R^n \cong P \oplus \ker(\epsilon)$. As $k^n = R^n/\mathfrak{m}R^n \cong P/\mathfrak{m}P$, we have $\ker(\epsilon) \subseteq \mathfrak{m}R^n$. But then considering P as a submodule of R^n we have $R^n = P + \mathfrak{m}R^n$, so Nakayama's lemma yields $R^n = P$. ◇

Third Change of Rings Theorem 4.3.12 *Let R be a commutative noetherian local ring with unique maximal ideal $\mathfrak{m}$, and let A be a finitely generated R-module. If $x \in \mathfrak{m}$ is a nonzerodivisor on both A and R, then*

$$pd_R(A) = pd_{R/x}(A/xA).$$

Proof We know $\geq$ holds by the second Change of Rings theorem 4.3.5, and we shall prove equality by induction on $n = pd_{R/x}(A/xA)$. If $n = 0$, then A/xA is projective, hence a free R/x-module because R/x is local.

Lemma 4.3.13 *If A/xA is a free R/x-module, A is a free R-module.*

Proof Pick elements $u_1, \cdots, u_n$ mapping onto a basis of A/xA; we claim they form a basis of A. Since $(u_1, \cdots, u_n)R + xA = A$, Nakayama's lemma states that $(u_1, \cdots, u_n)R = A$, that is, the u's span A. To show the u's are linearly independent, suppose $\sum r_i u_i = 0$ for $r_i \in R$. In A/xA, the images of the u's are linearly independent, so $r_i \in xR$ for all i. As x is a nonzerodivisor on R and A, we can divide to get $r_i/x \in R$ such that $\sum (r_i/x) u_i = 0$. Continuing this process, we get a sequence of elements $r_i, r_i/x, r_i/x^2, \cdots$ which generates a strictly ascending chain of ideals of R, unless $r_i = 0$. As R is noetherian, all the r_i must vanish. ◇

Resuming the proof of the theorem, we establish the inductive step $n \neq 0$. Map a free R-module F onto A with kernel K. As $\text{Tor}_1^R(A, R/x) = \{a \in A : xa = 0\} = 0$, tensoring with R/x yields the exact sequence

$$0 \to K/xK \to F/xF \to A/xA \to 0.$$

As F/xF is free, $pd_{R/x}(K/xK) = n - 1$ when $n \neq 0$. As R is noetherian, K is finitely generated, so by induction, $pd_R(K) = n - 1$. This implies that $pd_R(A) = n$, finishing the proof of the third Change of Rings theorem. ◇

Remark The third Change of Rings theorem holds in the generality that R is right noetherian, and $x \in R$ is a central element lying in the Jacobson radical of R. To prove this, reread the above proof, using the generalized version 4.3.10 of Nakayama's lemma.

Corollary 4.3.14 *Let R be a commutative noetherian local ring, and let A be a finitely generated R-module with $pd_R(A) < \infty$. If $x \in \mathfrak{m}$ is a nonzerodivisor on both A and R, then*

$$pd_R(A/xA) = 1 + pd_R(A).$$

Proof Combine the first and third Change of Rings theorems. ◇

Exercise 4.3.3 (Injective Change of Rings Theorems) Let x be a central nonzerodivisor in a ring R and let A be an R-module. Prove the following.

First Theorem. If $A \neq 0$ is an R/xR-module with $id_{R/xR}(A)$ finite, then

$$id_R(A) = 1 + id_{R/xR}(A).$$

Second Theorem. If x is a nonzerodivisor on both R and A, then either A is injective (in which case $A/xA = 0$) or else

$$id_R(A) \geq 1 + id_{R/xR}(A/xA).$$

Third Theorem. Suppose that R is a commutative noetherian local ring, A is finitely generated, and that $x \in \mathfrak{m}$ is a nonzerodivisor on both R and A. Then

$$id_R(A) = id_R(A/xA) = 1 + id_{R/xR}(A/xA).$$

4.4 Local Rings

In this section a *local ring* R will mean a commutative noetherian local ring R with a unique maximal ideal $\mathfrak{m}$. The residue field of R will be denoted $k = R/\mathfrak{m}$.

Definitions 4.4.1 The *Krull dimension* of a ring R, $\dim(R)$, is the length d of the longest chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_d$ of prime ideals in R; $\dim(R) < \infty$ for every local ring R. The *embedding dimension* of a local ring R is the finite number

$$emb.\dim(R) = \dim_k(\mathfrak{m}/\mathfrak{m}^2).$$

For any local ring we have $\dim(R) \leq emb.\dim(R)$; R is called a *regular local ring* if we have equality, that is, if $\dim(R) = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. Regular local rings have been long studied in algebraic geometry because the local coordinate rings of smooth algebraic varieties are regular local rings.

Examples 4.4.2 A regular local ring of dimension 0 must be a field. Every 1-dimensional regular local ring is a discrete valuation ring. The power series ring $k[[x_1, \cdots, x_n]]$ over a field k is regular local of dimension n, as is the local ring $k[x_1, \cdots, x_n]_{\mathfrak{m}}$, $\mathfrak{m} = (x_1, \cdots, x_n)$.

Let R be the local ring of a complex algebraic variety X at a point P. The embedding dimension of R is the smallest integer n such that some analytic neighborhood of P in X embeds in $\mathbf{C}^n$. If the variety X is smooth as a manifold, R is a regular local ring and $\dim(R) = \dim(X)$.

More Definitions 4.4.3 If A is a finitely generated R-module, a *regular sequence on* A, or A*-sequence*, is a sequence $(x_1, \cdots, x_n)$ of elements in $\mathfrak{m}$ such that x_1 is a nonzerodivisor on A (*i.e.*, if $a \neq 0$, then $x_1 a \neq 0$) and such that each x_i $(i > 1)$ is a nonzerodivisor on $A/(x_1, \cdots, x_{i-1})A$. The *grade* of A, $G(A)$, is the length of the longest regular sequence on A. For any local ring R we have $G(R) \leq \dim(R)$.

R is called *Cohen-Macaulay* if $G(R) = \dim(R)$. We will see below that regular local rings are Cohen-Macaulay; in fact, any $x_1, \cdots, x_d \in \mathfrak{m}$ mapping to a basis of $\mathfrak{m}/\mathfrak{m}^2$ will be an R-sequence; by Nakayama's lemma they will also generate $\mathfrak{m}$ as an ideal. For more details, see [KapCR].

Examples 4.4.4 Every 0-dimensional local ring R is Cohen-Macaulay (since $G(R) = 0$), but cannot be a regular local ring unless R is a field. The 1-dimensional local ring $k[[x, \epsilon]]/(x\epsilon = \epsilon^2 = 0)$ is not Cohen-Macaulay; every element of $\mathfrak{m} = (x, \epsilon)R$ kills $\epsilon \in R$. Unless the maximal ideal consists entirely of zerodivisors, a 1-dimensional local ring R is always Cohen-Macaulay; R is regular only when it is a discrete valuation ring. For example, the local ring $k[[x]]$ is a discrete valuation ring, and the subring $k[[x^2, x^3]]$ is Cohen-Macaulay of dimension 1 but is not a regular local ring.

Exercise 4.4.1 If R is a regular local ring and $x_1, \cdots, x_d \in \mathfrak{m}$ map to a basis of $\mathfrak{m}/\mathfrak{m}^2$, show that each quotient ring $R/(x_1, \cdots, x_i)R$ is regular local of dimension $d - i$.

Proposition 4.4.5 *A regular local ring is an integral domain.*

Proof We use induction on $\dim(R)$. Pick $x \in \mathfrak{m} - \mathfrak{m}^2$; by the above exercise, R/xR is regular local of dimension $\dim(R) - 1$. Inductively, R/xR is a domain, so xR is a prime ideal. If there is a prime ideal Q properly contained in xR, then $Q \subset x^n R$ for all n (inductively, if $q = rx^n \in Q$, then $r \in Q \subset xR$, so $q \in x^{n+1}R$). In this case $Q \subseteq \cap x^n R = 0$, whence $Q = 0$ and R is a domain. If R were not a domain, this would imply that xR is a minimal prime ideal of R for all $x \in \mathfrak{m} - \mathfrak{m}^2$. Hence $\mathfrak{m}$ would be contained in the union of $\mathfrak{m}^2$ and the finitely many minimal prime ideals $P_1, \cdots, P_t$ of R. This would imply that $\mathfrak{m} \subseteq P_i$ for some i. But then $\dim(R) = 0$, a contradiction. ◇

Corollary 4.4.6 *If R is a regular local ring, then $G(R) = \dim(R)$, and any $x_1, \cdots, x_d \in \mathfrak{m}$ mapping to a basis of $\mathfrak{m}/\mathfrak{m}^2$ is an R−sequence.*

Proof As $G(R) \leq \dim(R)$, and $x_1 \in R$ is a nonzerodivisor on R, it suffices to prove that $x_2, \cdots, x_d$ form a regular sequence on R/x_1R. This follows by induction on d. ◇

Exercise 4.4.2 Let R be a regular local ring and I an ideal such that R/I is also regular local. Prove that $I = (x_1, \cdots, x_i)R$, where $(x_1, \cdots, x_i)$ form a regular sequence in R.

Standard Facts 4.4.7 Part of the standard theory of associated prime ideals in commutative noetherian rings implies that if every element of $\mathfrak{m}$ is a zerodivisor on a finitely generated R-module A, then $\mathfrak{m}$ equals $\{r \in R : ra = 0\}$ for some nonzero $a \in A$ and therefore $aR \cong R/\mathfrak{m} = k$. Hence if $G(A) = 0$, then $\operatorname{Hom}_R(k, A) \neq 0$.

If $G(A) \neq 0$ and $G(R) \neq 0$, then some element of $\mathfrak{m} - \mathfrak{m}^2$ must also be a nonzerodivisor on both R and A. Again, this follows from the standard theory of associated prime ideals. Another standard fact is that if $x \in \mathfrak{m}$ is a nonzerodivisor on R, then the Krull dimension of R/xR is $\dim(R) - 1$.

Theorem 4.4.8 *If R is a local ring and $A \neq 0$ is a finitely generated R-module, then every maximal A-sequence has the same length, $G(A)$. Moreover, $G(A)$ is characterized as the smallest n such that $\operatorname{Ext}^n_R(k, A) \neq 0$.*

Proof We saw above that if $G(A) = 0$, then $\mathrm{Hom}_R(k, A) \neq 0$. Conversely, if $\mathrm{Hom}_R(k, A) \neq 0$, then some nonzero $a \in A$ has $aR \cong k$, that is, $ax = 0$ for all $x \in \mathfrak{m}$. In this case $G(A) = 0$ is clear. We now proceed by induction on the length n of a maximal regular A-sequence $x_1, \cdots, x_n$ on A. If $n \geq 1$, $x = x_1$ is a nonzerodivisor on A, so the sequence $0 \to A \xrightarrow{x} A \to A/xA \to 0$ is exact, and $x_2, \cdots, x_n$ is a maximal regular sequence on A/xA. This yields the exact sequence

$$\mathrm{Ext}^{i-1}(k, A) \xrightarrow{x} \mathrm{Ext}^{i-1}(k, A) \to \mathrm{Ext}^{i-1}(k, A/xA) \to \mathrm{Ext}^{i}(k, A) \xrightarrow{x} \mathrm{Ext}^{i}(k, A).$$

Now $xk = 0$, so $\mathrm{Ext}^i(k, A)$ is an R/xR-module. Hence the maps "x" in this sequence are zero. By induction, this proves that $\mathrm{Ext}^i(k, A) = 0$ for $0 \leq i < n$ and that $\mathrm{Ext}^n(k, A) \neq 0$. This finishes the inductive step, proving the theorem. ◇

Remark The injective dimension $id(A)$ is the largest integer n such that $\mathrm{Ext}^n_R(k, A) \neq 0$. This follows from the next result, which we cite without proof from [KapCR, section 4.5] because the proof involves more ring theory than we want to use.

Theorem 4.4.9 *If R is a local ring and A is a finitely generated R-module, then*

$$id(A) \leq d \Leftrightarrow \mathrm{Ext}^n_R(k, A) = 0 \text{ for all } n > d.$$

Corollary 4.4.10 *If R is a Gorenstein local ring (i.e., $id_R(R) < \infty$), then R is also Cohen-Macaulay. In this case $G(R) = id_R(R) = \dim(R)$ and*

$$\mathrm{Ext}^q_R(k, R) \neq 0 \Leftrightarrow q = \dim(R).$$

Proof The last two theorems imply that $G(R) \leq id(R)$. Now suppose that $G(R) = 0$ but that $id(R) \neq 0$. For each $s \in R$ and $n \geq 0$ we have an exact sequence

$$\mathrm{Ext}^n_R(R, R) \to \mathrm{Ext}^n_R(sR, R) \to \mathrm{Ext}^{n+1}_R(R/sR, R).$$

For $n = id(R) > 0$, the outside terms vanish, so $\mathrm{Ext}^n_R(sR, R) = 0$ as well. Choosing $s \in R$ so that $sR \cong k$ contradicts the previous theorem so if $G(R) = 0$ then $id(R) = 0$. If $G(R) = d > 0$, choose a nonzerodivisor $x \in \mathfrak{m}$ and set $S = R/xR$. By the third Injective Change of Rings theorem (exercise

4.3.3), $id_S(S) = id_R(R) - 1$, so S is also a Gorenstein ring. Inductively, S is Cohen-Macaulay, and $G(S) = id_S(S) = \dim(S) = \dim(R) - 1$. Hence $id_R(R) = \dim(R)$. If $x_2, \cdots, x_d$ are elements of $\mathfrak{m}$ mapping onto a maximal S-sequence in $\mathfrak{m}S$, then $x_1, x_2, \cdots, x_d$ forms a maximal R-sequence, that is, $G(R) = 1 + G(S) = \dim(R)$. ◇

Proposition 4.4.11 *If R is a local ring with residue field k, then for every finitely generated R-module A and every integer d*

$$pd(A) \leq d \Leftrightarrow \mathrm{Tor}_{d+1}^R(A, k) = 0.$$

In particular, $pd(A)$ is the largest d such that $\mathrm{Tor}_d^R(A, k) \neq 0$.

Proof As $fd(A) \leq pd(A)$, the $\Rightarrow$ direction is clear. We prove the converse by induction on d. Nakayama's lemma 4.3.9 states that the finitely generated R-module A can be generated by $m = \dim_k(A/\mathfrak{m}A)$ elements. Let $\{u_1, \cdots, u_m\}$ be a minimal set of generators for A, and let K be the kernel of the surjection $\epsilon: R^m \to A$ defined by $\epsilon(r_1, \cdots, r_m) = \sum r_i u_i$. The inductive step is clear, since if $d \neq 0$, then

$$\mathrm{Tor}_{d+1}(A, k) = \mathrm{Tor}_d(K, k) \text{ and } pd(A) \leq 1 + pd(K).$$

If $d = 0$, then the assumption that $\mathrm{Tor}_1(A, k) = 0$ gives exactness of

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K \otimes k & \longrightarrow & R^m \otimes k & \longrightarrow & A \otimes k & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & K/\mathfrak{m}K & \longrightarrow & k^m & \xrightarrow{\epsilon \otimes k} & A/\mathfrak{m}A & \longrightarrow & 0. \end{array}$$

By construction, the map $\epsilon \otimes k$ is an isomorphism. Hence $K/\mathfrak{m}K = 0$, so the finitely generated R-module K must be zero by Nakayama's lemma. This forces $R^m \cong A$, so $pd(A) = 0$ as asserted. ◇

Corollary 4.4.12 *If R is a local ring, then $gl.\dim(R) = pd_R(R/\mathfrak{m})$.*

Proof $pd(R/\mathfrak{m}) \leq gl.\dim(R) = \sup\{pd(R/I)\} \leq fd(R/\mathfrak{m}) \leq pd(R/\mathfrak{m})$. ◇

Corollary 4.4.13 *If R is local and $x \in \mathfrak{m}$ is a nonzerodivisor on R, then either $gl.\dim(R/xR) = \infty$ or $gl.\dim(R) = 1 + gl.\dim(R/xR)$.*

Proof Set $S = R/xR$ and suppose that $gl.\dim(S) = d$ is finite. By the First Change of Rings Theorem, the residue field $k = R/\mathfrak{m} = S/\mathfrak{m}S$ has

$$pd_R(k) = 1 + pd_S(k) = 1 + d.$$ ◇

Grade 0 Lemma 4.4.14 *If R is local and $G(R) = 0$ (i.e., every element of the maximal ideal $\mathfrak{m}$ is a zerodivisor on R), then for any finitely generated R-module A,*

$$\text{either } pd(A) = 0 \quad \text{or} \quad pd(A) = \infty.$$

Proof If $0 < pd(A) < \infty$ for some A then an appropriate syzygy M of A is finitely generated and has $pd(M) = 1$. Nakayama's lemma states that M can be generated by $m = \dim_k(M/\mathfrak{m}M)$ elements. If $u_1, \cdots, u_m$ generate M, there is a projective resolution $0 \to P \to R^m \xrightarrow{\epsilon} M \to 0$ with $\epsilon(r_1, \ldots, r_m) = \sum r_i u_i$; visibly $R^m/\mathfrak{m}R^m \cong k^m \cong M/\mathfrak{m}M$. But then $P \subseteq \mathfrak{m}R^m$, so $sP = 0$, where $s \in R$ is any element such that $\mathfrak{m} = \{r \in R : sr = 0\}$. On the other hand, P is projective, hence a free R-module (4.3.11), so $sP = 0$ implies that $s = 0$, a contradiction. ◇

Theorem 4.4.15 (Auslander-Buchsbaum Equality) *Let R be a local ring, and A a finitely generated R-module. If $pd(A) < \infty$, then $G(R) = G(A) + pd(A)$.*

Proof If $G(R) = 0$ and $pd(A) < \infty$, then A is projective (hence free) by the Grade 0 lemma 4.4.14. In this case $G(R) = G(A)$, and $pd(A) = 0$. If $G(R) \neq 0$, we shall perform a double induction on $G(R)$ and on $G(A)$.

Suppose first that $G(R) \neq 0$ and $G(A) = 0$. Choose $x \in m$ and $0 \neq a \in A$ so that x is a nonzerodivisor on R and $\mathfrak{m}a = 0$. Resolve A:

$$0 \to K \to R^m \xrightarrow{\epsilon} A \to 0$$

and choose $u \in R^m$ with $\epsilon(u) = a$. Now $\mathfrak{m}u \subseteq K$ so $xu \in K$ and $\mathfrak{m}(xu) \subseteq xK$, yet $xu \notin xK$ as $u \notin K$ and x is a nonzerodivisor on R^m. Hence $G(K/xK) = 0$. Since K is a submodule of a free module, x is a nonzerodivisor on K. By the third Change of Rings theorem, and the fact that A is not free (as $G(R) \neq G(A)$),

$$pd_{R/xR}(K/xK) = pd_R(K) = pd_R(A) - 1.$$

Since $G(R/xR) = G(R) - 1$, induction gives us the required identity:

$$G(R) = 1 + G(R/xR) = 1 + G(K/xK) + pd_{R/xR}(K/xK) = pd_R(A).$$

Finally, we consider the case $G(R) \neq 0$, $G(A) \neq 0$. We can pick $x \in \mathfrak{m}$, which is a nonzerodivisor on both R and A (see the *Standard Facts* 4.4.7

cited above). Since we may begin a maximal A-sequence with x, $G(A/xA) = G(A) - 1$. Induction and the corollary 4.3.14 to the third Change of Rings theorem now give us the required identity:

$$\begin{aligned} G(R) &= G(A/xA) + pd_R(A/xA) \\ &= (G(A) - 1) + (1 + pd_R(A)) \\ &= G(A) + pd_R(A). \end{aligned} \qquad \diamond$$

Main Theorem 4.4.16 *A local ring R is regular iff* $gl.\dim(R) < \infty$. *In this case*

$$G(R) = \dim(R) = emb.\dim(R) = gl.\dim(R) = pd_R(k).$$

Proof First, suppose R is regular. If $\dim(R) = 0$, R is a field, and the result is clear. If $d = \dim(R) > 0$, choose an R-sequence $x_1, \cdots, x_d$ generating $\mathfrak{m}$ and set $S = R/x_1R$. Then $x_2, \cdots, x_d$ is an S-sequence generating the maximal ideal of S, so S is regular of dimension $d - 1$. By induction on d, we have

$$gl.\dim(R) = 1 + gl.\dim(S) = 1 + (d - 1) = d.$$

If $gl.\dim(R) = 0$, R must be semisimple and local (a field). If $gl.\dim(R) \neq 0, \infty$ then $\mathfrak{m}$ contains a nonzerodivisor x by the Grade 0 lemma 4.4.14; we may even find an $x = x_1$ not in $\mathfrak{m}^2$ (see the *Standard Facts* 4.4.7 cited above). To prove that R is regular, we will prove that $S = R/xR$ is regular; as $\dim(S) = \dim(R) - 1$, this will prove that the maximal ideal $\mathfrak{m}S$ of S is generated by an S-sequence $y_2, \cdots, y_d$. Lift the $y_i \in \mathfrak{m}S$ to elements $x_i \in \mathfrak{m}$ $(i = 2, \cdots, d)$. By definition $x_1, \cdots, x_d$ is an R-sequence generating $\mathfrak{m}$, so this will prove that R is regular.

By the third Change of Rings theorem 4.3.12 with $A = \mathfrak{m}$,

$$pd_S(\mathfrak{m}/x\mathfrak{m}) = pd_R(\mathfrak{m}) = pd_R(k) - 1 = gl.\dim(R) - 1.$$

Now the image of $\mathfrak{m}/x\mathfrak{m}$ in $S = R/xR$ is $\mathfrak{m}/xR = \mathfrak{m}S$, so we get exact sequences

$$0 \to xR/x\mathfrak{m} \to \mathfrak{m}/x\mathfrak{m} \to \mathfrak{m}S \to 0 \quad \text{and} \quad 0 \to \mathfrak{m}S \to S \to k \to 0.$$

Moreover, $xR/x\mathfrak{m} \cong \operatorname{Tor}_1^R(R/xR, k) \cong \{a \in k : xa = 0\} = k$, and the image of x in $xR/x\mathfrak{m}$ is nonzero. We claim that $\mathfrak{m}/x\mathfrak{m} \cong \mathfrak{m}S \oplus k$ as S-modules. This will imply that

$$gl.\dim(S) = pd_S(k) \leq pd_S(\mathfrak{m}/x\mathfrak{m}) = gl.\dim(R) - 1.$$

By induction on global dimension, this will prove that S is regular.

To see the claim, set $r = emb.\dim(R)$ and find elements $x_2, \cdots, x_r$ in $\mathfrak{m}$ such that the image of $\{x_1, \cdots, x_r\}$ in $\mathfrak{m}/\mathfrak{m}^2$ forms a basis. Set $I = (x_2, \cdots, x_r)R + x\mathfrak{m}$ and observe that $I/x\mathfrak{m} \subseteq \mathfrak{m}/x\mathfrak{m}$ maps onto $\mathfrak{m}S$. As the kernel $xR/x\mathfrak{m}$ of $\mathfrak{m}/x\mathfrak{m} \to \mathfrak{m}S$ is isomorphic to k and contains $x \notin I$, it follows that $(xR/x\mathfrak{m}) \cap (I/x\mathfrak{m}) = 0$. Hence $I/x\mathfrak{m} \cong \mathfrak{m}S$ and $k \oplus \mathfrak{m}S \cong \mathfrak{m}/x\mathfrak{m}$, as claimed. ◇

Corollary 4.4.17 *A regular ring is both Gorenstein and Cohen-Macaulay.*

Corollary 4.4.18 *If R is a regular local ring and $\mathfrak{p}$ is any prime ideal of R, then the localization $R_\mathfrak{p}$ is also a regular local ring.*

Proof We shall show that if S is any multiplicative set in R, then the localization $S^{-1}R$ has finite global dimension. As $R_\mathfrak{p} = S^{-1}R$ for $S = R - \mathfrak{p}$, this will suffice. Considering an $S^{-1}R$-module A as an R-module, there is a projective resolution $P \to A$ of length at most $gl.\dim(R)$. Since $S^{-1}R$ is a flat R-module and $S^{-1}A = A$, $S^{-1}P \to A$ is a projective $S^{-1}R$-module resolution of length at most $gl.\dim(R)$. ◇

Remark The only non-homological proof of this result, due to Nagata, is very long and hard. This ability of homological algebra to give easy proofs of results outside the scope of homological algebra justifies its importance. Here is another result, quoted without proof from [KapCR], which uses homological algebra (projective resolutions) in the proof but not in the statement.

Theorem 4.4.19 *Every regular local ring is a Unique Factorization Domain.*

4.5 Koszul Complexes

An efficient way to perform calculations is to use Koszul complexes. If $x \in R$ is central, we let $K(x)$ denote the chain complex

$$0 \to R \xrightarrow{x} R \to 0$$

concentrated in degrees 1 and 0. It is convenient to identify the generator of the degree 1 part of $K(x)$ as the element e_x, so that $d(e_x) = x$. If $\boldsymbol{x} = (x_1, \cdots, x_n)$ is a finite sequence of central elements in R, we define the *Koszul complex* $K(\boldsymbol{x})$ to be the total tensor product complex (see 2.7.1):

$$K(x_1) \otimes_R K(x_2) \otimes_R \cdots \otimes_R K(x_n).$$

Notation 4.5.1 If A is an R-module, we define

$$H_q(\boldsymbol{x}, A) = H_q(K(\boldsymbol{x}) \otimes_R A);$$
$$H^q(\boldsymbol{x}, A) = H^q(\mathrm{Hom}(K(\boldsymbol{x}), A)).$$

The degree p part of $K(\boldsymbol{x})$ is a free R-module generated by the symbols

$$e_{i_1} \wedge \cdots \wedge e_{i_p} = 1 \otimes \cdots \otimes 1 \otimes e_{x_{i_1}} \otimes \cdots \otimes e_{x_{i_p}} \otimes \cdots \otimes 1 \quad (i_1 < \cdots < i_p).$$

In particular, $K_p(\boldsymbol{x})$ is isomorphic to the p^{th} exterior product $\Lambda^p R^n$ of R^n and has rank $\binom{n}{p}$, so $K(\boldsymbol{x})$ is often called the *exterior algebra complex*. The derivative $K_p(\boldsymbol{x}) \to K_{p-1}(\boldsymbol{x})$ sends $e_{i_1} \wedge \cdots \wedge e_{i_p}$ to $\sum (-1)^{k+1} x_{i_k} e_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_p}$. As an example, $K(x, y)$ is the complex

$$\begin{array}{cccccccc} & 0 \longrightarrow & R & \xrightarrow{(x,-y)} & R^2 & \xrightarrow{\binom{x}{y}} & R & \longrightarrow 0. \\ \text{basis:} & & \{e_x \wedge e_y\} & & \{e_y, e_x\} & & \{1\} & \end{array}$$

DG-Algebras 4.5.2 A *graded R-algebra* K_* is a family $\{K_p, p \geq 0\}$ of R-modules, equipped with a bilinear product $K_p \otimes_R K_q \to K_{p+q}$ and an element $1 \in K_0$ making K_0 and $\oplus K_p$ into associative R-algebras with unit. K_* is *graded-commutative* if for every $a \in K_p$, $b \in K_q$ we have $a \cdot b = (-1)^{pq} b \cdot a$. A *differential graded algebra*, or *DG-algebra*, is a graded R-algebra K_* equipped with a map $d\colon K_p \to K_{p-1}$, satisfying $d^2 = 0$ and satisfying the *Leibnitz rule:*

$$d(a \cdot b) = d(a) \cdot b + (-1)^p a \cdot d(b) \qquad \text{for } a \in K_p.$$

Exercise 4.5.1

1. Let K be a DG-algebra. Show that the homology $H_*(K) = \{H_p(K)\}$ forms a graded R-algebra, and that $H_*(K)$ is graded-commutative whenever K_* is.
2. Show that the Koszul complex $K(\boldsymbol{x}) \cong \Lambda^*(R^n)$ is a graded-commutative DG-algebra. If R is commutative, use this to obtain an external product $H_p(\boldsymbol{x}, A) \otimes_R H_q(\boldsymbol{x}, B) \to H_{p+q}(\boldsymbol{x}, A \otimes_R B)$. Conclude that if A is a commutative R-algebra then the Koszul homology $H_*(\boldsymbol{x}, A)$ is a graded-commutative R-algebra.
3. If $x_1, \cdots \in I$ and $A = R/I$, show that $H_*(\boldsymbol{x}, A)$ is the exterior algebra $\Lambda^*(A^n)$.

Exercise 4.5.2 Show that $\{H_q(\boldsymbol{x}, -)\}$ is a homological δ-functor, and that $\{H^q(\boldsymbol{x}, -)\}$ is a cohomological δ-functor with

$$H_0(\boldsymbol{x}, A) = A/(x_1, \cdots, x_n)A$$

$$H^0(\boldsymbol{x}, A) = \mathrm{Hom}(R/\boldsymbol{x}R, A) = \{a \in A : x_i a = 0 \text{ for all } i\}.$$

Then show that there are isomorphisms $H_p(\boldsymbol{x}, A) \cong H^{n-p}(\boldsymbol{x}, A)$ for all p.

Lemma 4.5.3 (Künneth formula for Koszul complexes) *If $C = C_*$ is a chain complex of R-modules and $x \in R$, there are exact sequences*

$$0 \to H_0(x, H_q(C)) \to H_q(K(x) \otimes_R C) \to H_1(x, H_{q-1}(C)) \to 0.$$

Proof Considering R as a complex concentrated in degree zero, there is a short exact sequence of complexes $0 \to R \to K(x) \to R[-1] \to 0$. Tensoring with C yields a short exact sequence of complexes whose homology long exact sequence is

$$H_{q+1}(C[-1]) \xrightarrow{\partial} H_q(C) \to H_q(K(x) \otimes C) \to H_q(C[-1]) \xrightarrow{\partial} H_q(C).$$

Identifying $H_{q+1}(C[-1])$ with $H_q(C)$, the map ∂ is multiplication by x (check this!), whence the result. ◇

Exercise 4.5.3 If x is a nonzerodivisor on R, that is, $H_1(K(x)) = 0$, use the Künneth formula for complexes 3.6.3 to give another proof of this result.

Exercise 4.5.4 Show that if one of the x_i is a unit of R, then the complex $K(\boldsymbol{x})$ is split exact. Deduce that in this case $H_*(\boldsymbol{x}, A) = H^*(\boldsymbol{x}, A) = 0$ for all modules A.

Corollary 4.5.4 (Acyclicity) *If $\boldsymbol{x}$ is a regular sequence on an R-module A, then $H_q(\boldsymbol{x}, A) = 0$ for $q \neq 0$ and $H_0(\boldsymbol{x}, A) = A/\boldsymbol{x}A$, where $\boldsymbol{x}A = (x_1, \cdots, x_n)A$.*

Proof Since x is a nonzerodivisor on A, the result is true for $n = 1$. Inductively, letting $x = x_n$, $\boldsymbol{y} = (x_1, \cdots, x_{n-1})$, and $C = K(\boldsymbol{y}) \otimes A$, $H_q(C) = 0$ for $q \neq 0$ and $K(x) \otimes H_0(C)$ is the complex

$$0 \to A/\boldsymbol{y}A \xrightarrow{x} A/\boldsymbol{y}A \to 0.$$

The result follows from 4.5.3, since x is a nonzerodivisor on $A/\boldsymbol{y}A$. ◇

Corollary 4.5.5 (Koszul resolution) *If $\mathbf{x}$ is a regular sequence in R, then $K(\mathbf{x})$ is a free resolution of R/I, $I = (x_1, \cdots, x_n)R$. That is, the following sequence is exact:*

$$0 \to \Lambda^n(R^n) \to \cdots \to \Lambda^2(R^n) \to R^n \xrightarrow{\mathbf{x}} R \to R/I \to 0.$$

In this case we have

$$\mathrm{Tor}_p^R(R/I, A) = H_p(\mathbf{x}, A);$$

$$\mathrm{Ext}_R^p(R/I, A) = H^p(\mathbf{x}, A).$$

Exercise 4.5.5 If $\mathbf{x}$ is a regular sequence in R, show that the external and internal products for Tor (2.7.8 and exercise 2.7.5(4)) agree with the external and internal products for $H_*(\mathbf{x}, A)$ constructed in this section.

Exercise 4.5.6 Let R be a regular local ring with residue field k. Show that

$$\mathrm{Tor}_p^R(k, k) \cong \mathrm{Ext}_R^p(k, k) \cong \Lambda^p k^n \cong k^{\binom{n}{p}}, \quad \text{where } n = \dim(R).$$

Conclude that $id_R(k) = \dim(R)$ and that as rings $\mathrm{Tor}_*^R(k, k) \cong \wedge^*(k^n)$.

Application 4.5.6 (Scheja-Storch) Here is a computational proof of Hilbert's Syzygy Theorem 4.3.8. Let F be a field, and set $R = F[x_1, \cdots, x_n]$, $S = R[y_1, \cdots, y_n]$. Let $\mathbf{t}$ be the sequence $(t_1, \cdots, t_n)$ of elements $t_i = y_i - x_i$ of S. Since $S = R[t_1, \cdots, t_n]$, $\mathbf{t}$ is a regular sequence, and $H_0(\mathbf{t}, S) \cong R$, so the augmented Koszul complex of $K(\mathbf{t})$ is exact:

$$0 \to \Lambda^n S^n \to \Lambda^{n-1} S^n \to \cdots \to \Lambda^2 S^n \to S^n \xrightarrow{\mathbf{t}} S \to R \to 0.$$

Since each $\Lambda^p S^n$ is a free R-module, this is in fact a split exact sequence of R-modules. Hence applying $\otimes_R A$ yields an exact sequence for every R-module A. That is, each $K(\mathbf{t}) \otimes_R A$ is an S-module resolution of A. Set $R' = F[y_1, \cdots, y_n]$, a subring of S. Since $t_i = 0$ on A, we may identify the R-module structure on A with the R'-module structure on A. But $S \otimes_R A \cong R' \otimes_F A$ is a free R'-module because F is a field. Therefore each $\Lambda^p S^n \otimes_R A$ is a free R'-module, and $K(\mathbf{t}) \otimes_R A$ is a canonical, natural resolution of A by free R'-modules. Since $K(\mathbf{t}) \otimes_R A$ has length n, this proves that

$$pd_R(A) = pd_{R'}(A) \le n$$

for every R-module A. On the other hand, since $\mathrm{Tor}_n^R(F, F) \cong F$, we see that $pd_R(F) = n$. Hence the ring $R = F[x_1, \cdots, x_n]$ has global dimension n.

4.6 Local Cohomology

Definition 4.6.1 If I is a finitely generated ideal in a commutative ring R and A is an R-module, we define

$$H_I^0(A) = \{a \in A : (\exists i) I^i a = 0\} = \varinjlim \operatorname{Hom}(R/I^i, A).$$

Since each $\operatorname{Hom}(R/I^i, -)$ is left exact and $\varinjlim$ is exact, we see that H_I^0 is an additive left exact functor from R–**mod** to itself. We set

$$H_I^q(A) = (R^q H_I^0)(A).$$

Since the direct limit is exact, we also have

$$H_I^q(A) = \varinjlim \operatorname{Ext}_R^q(R/I^i, A).$$

Exercise 4.6.1 Show that if $J \subseteq I$ are finitely generated ideals such that $I^i \subseteq J$ for some i, then $H_J^q(A) \cong H_I^q(A)$ for all R-modules A and all q.

Exercise 4.6.2 (Mayer-Vietoris sequence) Let I and J be ideals in a noetherian ring R. Show that there is a long exact sequence for every R-module A:

$$\cdots \xrightarrow{\delta} H_{I+J}^q(A) \to H_I^q(A) \oplus H_J^q(A) \to H_{I \cap J}^q(A) \to H_{I+J}^{q+1}(A) \xrightarrow{\delta} \cdots.$$

Hint: Apply $\operatorname{Ext}^*(-, A)$ to the family of sequences

$$0 \to R/I^i \cap J^i \to R/I^i \oplus R/J^i \to R/(I^i + J^i) \to 0.$$

Then pass to the limit, observing that $(I+J)^{2i} \subseteq (I^i + J^i) \subseteq (I+J)^i$ and that, by the Artin-Rees lemma ([BA II, 7.13]), for every i there is an $N \geq i$ so that $I^N \cap J^N \subseteq (I \cap J)^i \subseteq I^i \cap J^i$.

Generalization 4.6.2 (Cohomology with supports; See [GLC]) Let Z be a closed subspace of a topological space X. If F is a sheaf on X, let $H_Z^0(X, F)$ be the kernel of $H^0(X, F) \to H^0(X - Z, F)$, that is, all global sections of F with support in Z. H_Z^0 is a left exact functor on Sheaves(X), and we write $H_Z^n(X, F)$ for its right derived functors.

If I is any ideal of R, then $H_I^n(A)$ is defined to be $H_Z^n(X, \tilde{A})$, where $X = \operatorname{Spec}(R)$ is the topological space of prime ideals of R, $Z = \{\mathfrak{p} : I \subseteq \mathfrak{p}\}$, and $\tilde{A}$ is the sheaf on $\operatorname{Spec}(R)$ associated to A. If I is a finitely generated ideal, this agrees with our earlier definition. For more details see [GLC], including the construction of the long exact sequence

$$0 \to H_Z^0(X,F) \to H^0(X,F) \to H^0(X-Z,F) \to H_Z^1(X,F) \to \cdots.$$

A standard result in algebraic geometry states that $H^n(\mathrm{Spec}(R), \tilde{A}) = 0$ for $n \neq 0$, so for the *punctured spectrum* $U = \mathrm{Spec}(R) - Z$ the sequence

$$0 \to H_I^0(A) \to A \to H^0(U, \tilde{A}) \to H_I^1(A) \to 0$$

is exact, and for $n \neq 0$ we can calculate the cohomology of $\tilde{A}$ on U via

$$H^n(U, \tilde{A}) \cong H_I^{n+1}(A).$$

Exercise 4.6.3 Let $\mathcal{A}$ be the full subcategory of R–**mod** consisting of the modules with $H_I^0(A) = A$.

1. Show that $\mathcal{A}$ is an abelian category, that $H_I^0\colon R$–**mod** $\to \mathcal{A}$ is right adjoint to the inclusion $\iota\colon \mathcal{A} \hookrightarrow R$–**mod**, and that ι is an exact functor.
2. Conclude that H_I^0 preserves injectives (2.3.10), and that $\mathcal{A}$ has enough injectives.
3. Conclude that each $H_I^n(A)$ belongs to the subcategory $\mathcal{A}$ of R–**mod**.

Theorem 4.6.3 *Let R be a commutative noetherian local ring with maximal ideal* $\mathfrak{m}$. *Then the grade $G(A)$ of any finitely generated R-module A is the smallest integer n such that $H_{\mathfrak{m}}^n(A) \neq 0$.*

Proof For each i we have the exact sequence

$$\mathrm{Ext}^{n-1}(\mathfrak{m}^i/\mathfrak{m}^{i+1}, A) \to \mathrm{Ext}^n(R/\mathfrak{m}^i, A) \to \mathrm{Ext}^n(R/\mathfrak{m}^{i+1}, A) \to \mathrm{Ext}^n(\mathfrak{m}^i/\mathfrak{m}^{i+1}, A).$$

We saw in 4.4.8 that $\mathrm{Ext}^n(R/\mathfrak{m}, A)$ is zero if $n < G(A)$ and nonzero if $n = G(A)$; as $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a finite direct sum of copies of $R/\mathfrak{m}$, the same is true for $\mathrm{Ext}^n(\mathfrak{m}^i/\mathfrak{m}^{i+1}, A)$. By induction on i, this proves that $\mathrm{Ext}^n(R/\mathfrak{m}^{i+1}, A)$ is zero if $n < G(A)$ and that it contains the nonzero module $\mathrm{Ext}^n(R/\mathfrak{m}^i, A)$ if $n = G(A)$. Now take the direct limit as $i \to \infty$. ◇

Application 4.6.4 Let R be a 2-dimensional local domain. Since $G(R) \neq 0$, $H_{\mathfrak{m}}^0(R) = 0$. From the exact sequence

$$0 \to \mathfrak{m}^i \to R \to R/\mathfrak{m}^i \to 0$$

we obtain the exact sequence

$$0 \to R \to \mathrm{Hom}_R(\mathfrak{m}^i, R) \to \mathrm{Ext}^1_R(R/\mathfrak{m}^i, R) \to 0.$$

As R is a domain, there is a natural inclusion of $\mathrm{Hom}_R(\mathfrak{m}^i, R)$ in the field F of fractions of R as the submodule

$$\mathfrak{m}^{-i} \equiv \{x \in F : x\mathfrak{m}^i \subseteq R\}.$$

Set $C = \cup \mathfrak{m}^{-i}$. (*Exercise:* Show that C is a subring of F.) Evidently

$$H^1_{\mathfrak{m}}(R) = \varinjlim \mathrm{Ext}^1(R/\mathfrak{m}^i, R) \cong C/R.$$

If R is Cohen-Macaulay, that is, $G(R) = 2$, then $H^1_{\mathfrak{m}}(R) = 0$, so $R = C$ and $\mathrm{Hom}_R(\mathfrak{m}^i, R) = R$ for all i. Otherwise $R \neq C$ and $G(R) = 1$. When the integral closure of R is finitely generated as an R-module, C is actually a Cohen-Macaulay ring—the smallest Cohen-Macaulay ring containing R [EGA, IV.5.10.17].

Here is an alternative construction of local cohomology due to Serre [EGA, III.1.1]. If $x \in R$ there is a natural map from $K(x^{i+1})$ to $K(x^i)$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \xrightarrow{x^{i+1}} & R & \longrightarrow & 0 \\ & & {\scriptstyle x}\downarrow & & \| & & \\ 0 & \longrightarrow & R & \xrightarrow{x^{i}} & R & \longrightarrow & 0. \end{array}$$

By tensoring these maps together, and writing $\boldsymbol{x}^i$ for $(x_1^i, \cdots, x_n^i)$, this gives a map from $K(\boldsymbol{x}^{i+1})$ to $K(\boldsymbol{x}^i)$, hence a tower $\{H_q(K(\boldsymbol{x}^i)\}$ of R-modules. Applying $\mathrm{Hom}_R(-, A)$ and taking cohomology yields a map from $H^q(\boldsymbol{x}^i, A)$ to $H^q(\boldsymbol{x}^{i+1}, A)$.

Definition 4.6.5 $H^q_{\boldsymbol{x}}(A) = \varinjlim H^q(\boldsymbol{x}^i, A)$.

For our next result, recall from 3.5.6 that a tower $\{A_i\}$ satisfies the *trivial Mittag-Leffler condition* if for every i there is a $j > i$ so that $A_j \to A_i$ is zero.

Exercise 4.6.4 If $\{A_i\} \to \{B_i\} \to \{C_i\}$ is an exact sequence of towers of R-modules and both $\{A_i\}$ and $\{C_i\}$ satisfy the trivial Mittag-Leffler condition, then $\{B_i\}$ also satisfies the trivial Mittag-Leffler condition (3.5.6).

Proposition 4.6.6 *Let R be a commutative noetherian ring and A a finitely generated R-module. Then the tower $\{H_q(\boldsymbol{x}^i, A)\}$ satisfies the trivial Mittag-Leffler condition for every $q \neq 0$.*

Proof We proceed by induction on the length n of $\boldsymbol{x}$. If $n = 1$, one sees immediately that $H_1(x^i, A)$ is the submodule $A_i = \{a \in A : x^i a = 0\}$. The submodules A_i of A form an ascending chain, which must be stationary since R is noetherian and A is finitely generated. This means that there is an integer k such that $A_k = A_{k+1} = \cdots$, that is, $x^k A_i = 0$ for all i. Since the map $A_{i+j} \to A_i$ is multiplication by x^j, it is zero whenever $j \geq k$. Thus the lemma holds if $n = 1$.

Inductively, set $\boldsymbol{y} = (x_1, \ldots, x_{n-1})$ and write x for x_n. Since $K(\boldsymbol{x}^i) \otimes K(\boldsymbol{y}^i) = K(\boldsymbol{x}^i)$, the Künneth formula for Koszul complexes 4.5.3 (and its proof) yields the following exact sequences of towers:

$$\{H_q(\boldsymbol{y}^i, A)\} \to \{H_q(\boldsymbol{x}^i, A)\} \to \{H_{q-1}(\boldsymbol{y}^i, A)\};$$

$$\{H_1(\boldsymbol{y}^i, A)\} \to \{H_1(\boldsymbol{x}^i, A)\} \to \{H_1(x^i, A/\boldsymbol{y}^i A)\} \to 0.$$

If $q \geq 2$, the outside towers satisfy the trivial Mittag-Leffler condition by induction, so $\{H_q(\boldsymbol{x}^i, A)\}$ does too. If $q = 1$ and we set $A_{ij} = \{a \in A/\boldsymbol{y}^i A : x^j a = 0\} = H_1(x^j, A/\boldsymbol{y}^i A)$, it is enough to show that the diagonal tower $\{A_{ii}\}$ satisfies the trivial Mittag-Leffler condition. For fixed i, we saw above that there is a k such that every map $A_{ij} \to A_{i,j+k}$ is zero. Hence the map $A_{ii} \to A_{i,i+k} \to A_{i+k,i+k}$ is zero, as desired. ◇

Corollary 4.6.7 *Let R be commutative noetherian, and let E be an injective R-module. Then $H^q_{\boldsymbol{x}}(E) = 0$ for all $q \neq 0$.*

Proof Because E is injective, $\operatorname{Hom}_R(-, E)$ is exact. Therefore

$$H^q(\boldsymbol{x}^i, E) = H^q \operatorname{Hom}_R(K(\boldsymbol{x}^i, R), E) \cong \operatorname{Hom}_R(H_q(\boldsymbol{x}^i, R), E).$$

Because the tower $\{H_q(\boldsymbol{x}^i, R)\}$ satisfies the trivial Mittag-Leffler condition,

$$H^q_{\boldsymbol{x}}(E) \cong \varinjlim \operatorname{Hom}_R(H_q(\boldsymbol{x}^i, R), E) = 0. \qquad \diamond$$

Theorem 4.6.8 *If R is commutative noetherian, $\boldsymbol{x} = (x_1, \cdots, x_n)$ is any sequence of elements of R, and $I = (x_1, \cdots, x_n)R$, then for every R-module A*

$$H^q_I(A) \cong H^q_{\boldsymbol{x}}(A).$$

Proof Both H^q_I and $H^q_{\boldsymbol{x}}$ are universal δ-functors, and

$$H^0_I(A) = \varinjlim \operatorname{Hom}(R/\boldsymbol{x}^i R, A) = \varinjlim H^0(\boldsymbol{x}^i, A) = H^0_{\boldsymbol{x}}(A). \qquad \diamond$$

Corollary 4.6.9 *If R is a noetherian local ring, then $H^q_{\mathfrak{m}}(A) \neq 0$ only when $G(A) \leq q \leq \dim(R)$. In particular, if R is a Cohen-Macauley local ring, then*

$$H^q_m(R) \neq 0 \Leftrightarrow q = \dim(R).$$

Proof Set $d = \dim(R)$. By standard commutative ring theory ([KapCR, Thm.153]), there is a sequence $x = (x_1, \cdots, x_d)$ of elements of $\mathfrak{m}$ such that $\mathfrak{m}^j \subseteq I \subseteq \mathfrak{m}$ for some j, where $I = (x_1, \cdots, x_d)R$. But then $H^q_{\mathfrak{m}}(A) = H^q_I(A) = H^q_x(A)$, and this vanishes for $q > d$ because the Koszul complexes $K(\mathbf{x}^i)$ have length d. Now use (4.6.3). ◇

Exercise 4.6.5 If I is a finitely generated ideal of R and $R \to S$ is a ring map, show that $H^q_I(A) \cong H^q_{IS}(A)$ for every S-module A. This result is rather surprising, because there isn't any nice relationship between the groups $\mathrm{Ext}^*_R(R/I^i, A)$ and $\mathrm{Ext}^*_S(S/I^i, A)$. Consequently, if $\mathrm{ann}_R(A)$ denotes $\{r \in R : rA = 0\}$, then $H^q_I(A) = 0$ for $q > \dim(R/\mathrm{ann}_R(A))$.

Application 4.6.10 (Hartshorne) Let $R = \mathbb{C}[x_1, x_2, y_1, y_2]$, $P = (x_1, x_2)R$, $Q = (y_1, y_2)R$, and $I = P \cap Q$. As P, Q, and $\mathfrak{m} = P + Q = (x_1, x_2, y_1, y_2)R$ are generated by regular sequences, the outside terms in the Mayer-Vietoris sequence (exercise 4.6.2)

$$H^3_P(R) \oplus H^3_Q(R) \to H^3_I(R) \to H^4_{\mathfrak{m}}(R) \to H^4_P(R) \oplus H^4_Q(R)$$

vanish, yielding $H^3_I(R) \cong H^4_{\mathfrak{m}}(R) \neq 0$. This implies that the union of two planes in $\mathbb{C}^4$ that meet in a point cannot be described as the solutions of only two equations $f_1 = f_2 = 0$. Indeed, if this were the case, then we would have $I^i \subseteq (f_1, f_2)R \subseteq I$ for some i, so that $H^3_I(R)$ would equal $H^3_f(R)$, which is zero.