3 Tor and Ext

3.1 Tor for Abelian Groups

The first question many people ask about $\text{Tor}_*(A, B)$ is "Why the name "Tor"?" The results of this section should answer that question. Historically, the first Tor groups to arise were the groups $\text{Tor}_1(\mathbb{Z}/p, B)$ associated to abelian groups. The following simple calculation describes these groups.

**Calculation 3.1.1**

$$\text{Tor}^\mathbb{Z}(\mathbb{Z}/p, B) = B/pB, \quad \text{Tor}^\mathbb{Z}_1(\mathbb{Z}/p, B) = pB = \{b \in B : pB = 0\} \quad \text{and} \quad \text{Tor}^\mathbb{Z}_n(\mathbb{Z}/p, B) = 0 \quad \text{for} \quad n \geq 2.$$ To see this, use the resolution

$$0 \rightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow \mathbb{Z}/p \rightarrow 0$$

to see that $\text{Tor}_*(\mathbb{Z}/p, B)$ is the homology of the complex $0 \rightarrow B \xrightarrow{p} B \rightarrow 0$.

**Proposition 3.1.2** For all abelian groups $A$ and $B$:

(a) $\text{Tor}^\mathbb{Z}_1(A, B)$ is a torsion abelian group.

(b) $\text{Tor}^\mathbb{Z}_n(A, B) = 0$ for $n \geq 2$.

**Proof** $A$ is the direct limit of its finitely generated subgroups $A_\alpha$, so by 2.6.17 $\text{Tor}_n(A, B)$ is the direct limit of the $\text{Tor}_n(A_\alpha, B)$. As the direct limit of torsion groups is a torsion group, we may assume that $A$ is finitely generated, that is, $A \cong \mathbb{Z}^m \oplus \mathbb{Z}/p_1 \oplus \mathbb{Z}/p_2 \oplus \cdots \oplus \mathbb{Z}/p_r$ for appropriate integers $m$, $p_1, \ldots, p_r$. As $\mathbb{Z}^m$ is projective, $\text{Tor}_n(\mathbb{Z}^m, -)$ vanishes for $n \neq 0$, and so we have

$$\text{Tor}_n(A, B) \cong \text{Tor}_n(\mathbb{Z}/p_1, B) \oplus \cdots \oplus \text{Tor}_n(\mathbb{Z}/p_r, B).$$

The proposition holds in this case by calculation 3.1.1 above. ☐
Proposition 3.1.3 \( \text{Tor}^\mathbb{Z}_{1}(\mathbb{Q}/\mathbb{Z}, B) \) is the torsion subgroup of \( B \) for every abelian group \( B \).

Proof As \( \mathbb{Q}/\mathbb{Z} \) is the direct limit of its finite subgroups, each of which is isomorphic to \( \mathbb{Z}/p \) for some integer \( p \), and \( \text{Tor} \) commutes with direct limits,

\[
\text{Tor}^\mathbb{Z}_{1}(\mathbb{Q}/\mathbb{Z}, B) \cong \lim_{\rightarrow} \text{Tor}^\mathbb{Z}_{1}(\mathbb{Z}/p, B) \cong \lim_{\rightarrow} \{ b \in B : p b = 0 \},
\]

which is the torsion subgroup of \( B \).

\[ \text{Proposition 3.1.4 If } A \text{ is a torsionfree abelian group, then } \text{Tor}^\mathbb{Z}_{n}(A, B) = 0 \text{ for } n \neq 0 \text{ and all abelian groups } B. \]

Proof \( A \) is the direct limit of its finitely generated subgroups, each of which is isomorphic to \( \mathbb{Z}^m \) for some \( m \). Therefore, \( \text{Tor}_n(A, B) \cong \lim_{\rightarrow} \text{Tor}_n(\mathbb{Z}^m, B) = 0. \)

Remark (Balancing Tor) If \( R \) is any commutative ring, then \( \text{Tor}^R_{n}(A, B) \cong \text{Tor}^R_{n}(B, A) \). In particular, this is true for \( R = \mathbb{Z} \), that is, for abelian groups. This is because for fixed \( B \), both are universal \( \delta \)-functors over \( F(A) = A \otimes B \cong B \otimes A \). Therefore \( \text{Tor}^\mathbb{Z}_{1}(A, \mathbb{Q}/\mathbb{Z}) \) is the torsion subgroup of \( A \). From this we obtain the following.

Corollary 3.1.5 For every abelian group \( A \),

\[
\text{Tor}^\mathbb{Z}_{1}(A, -) = 0 \iff A \text{ is torsionfree} \iff \text{Tor}^\mathbb{Z}_{1}(-, A) = 0.
\]

Calculation 3.1.6 All this fails if we replace \( \mathbb{Z} \) by another ring. For example, if we take \( R = \mathbb{Z}/m \) and \( A = \mathbb{Z}/d \) with \( d \mid m \), then we can use the periodic free resolution

\[
\cdots \xrightarrow{d} \mathbb{Z}/m \xrightarrow{m/d} \mathbb{Z}/m \xrightarrow{d} \mathbb{Z}/m \xrightarrow{\epsilon} \mathbb{Z}/d \to 0
\]

to see that for all \( \mathbb{Z}/m \)-modules \( B \) we have

\[
\text{Tor}^\mathbb{Z}_{n}(\mathbb{Z}/d, B) = \begin{cases} B/dB & \text{if } n = 0 \\ \{ b \in B : dB = 0 \}/(m/d)B & \text{if } n \text{ is odd, } n > 0 \\ \{ b \in B : (m/d)b = 0 \}/dB & \text{if } n \text{ is even, } n > 0. \end{cases}
\]
Example 3.1.7 Suppose that \( r \in R \) is a left nonzerodivisor on \( R \), that is, \( rR = \{ s \in R : rs = 0 \} \) is zero. For every \( R \)-module \( B \), set \( rB = \{ b \in B : rb = 0 \} \). We can repeat the above calculation with \( R/rR \) in place of \( \mathbb{Z}/p \) to see that \( \text{Tor}_0(R/rR, B) = B/rB \), \( \text{Tor}_1(R/rR, B) = rB \) and \( \text{Tor}_n(R/rR, B) = 0 \) for all \( B \) when \( n \geq 2 \).

Exercise 3.1.1 If \( r \neq 0 \), all we have is the non-projective resolution

\[
0 \to rR \to R \xrightarrow{r} R \to R/rR \to 0.
\]

Show that there is a short exact sequence

\[
0 \to \text{Tor}_2^R(R/rR, B) \to B \otimes R B \xrightarrow{\text{multiply}} B \to \text{Tor}_1^R(R/rR, B) \to 0
\]

and that \( \text{Tor}_n^R(R/rR, B) \cong \text{Tor}_{n-2}^R(rR, B) \) for \( n \geq 3 \).

Exercise 3.1.2 Suppose that \( R \) is a commutative domain with field of fractions \( F \). Show that \( \text{Tor}_1^R(F/R, B) \) is the torsion submodule \( \{ b \in B : (\exists r \neq 0) rb = 0 \} \) of \( B \) for every \( R \)-module \( B \).

Exercise 3.1.3 Show that \( \text{Tor}_1^R(R/I, R/J) \cong I \otimes J / JI \otimes I \) for every right ideal \( I \) and left ideal \( J \) of \( R \). In particular, \( \text{Tor}_1(R/I, R/I) \cong I/I^2 \) for every 2-sided ideal \( I \). Hint: Apply the Snake Lemma to

\[
\begin{array}{cccccc}
0 & \to & IJ & \to & I & \to & I \otimes R/J & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & J & \to & R & \to & R \otimes R/J & \to & 0.
\end{array}
\]

3.2 Tor and Flatness

In the last chapter, we saw that if \( A \) is a right \( R \)-module and \( B \) is a left \( R \)-module, then \( \text{Tor}_n^R(A, B) \) may be computed either as the left derived functors of \( A \otimes_R \) evaluated at \( B \) or as the left derived functors of \( \otimes_R B \) evaluated at \( A \). It follows that if either \( A \) or \( B \) is projective, then \( \text{Tor}_n(A, B) = 0 \) for \( n \neq 0 \).

Definition 3.2.1 A left \( R \)-module \( B \) is flat if the functor \( \otimes_R B \) is exact. Similarly, a right \( R \)-module \( A \) is flat if the functor \( A \otimes_R \) is exact. The above remarks show that projective modules are flat. The example \( R = \mathbb{Z}, B = \mathbb{Q} \) shows that flat modules need not be projective.
### Theorem 3.2.2
If $S$ is a central multiplicatively closed set in a ring $R$, then $S^{-1}R$ is a flat $R$-module.

**Proof** Form the filtered category $I$ whose objects are the elements of $S$ and whose morphisms are $\text{Hom}_I(s_1, s_2) = \{s \in S : s_1s = s_2\}$. Then $\text{colim} \ F(s) \cong S^{-1}R$ for the functor $F: I \rightarrow R\text{-mod}$ defined by $F(s) = R, \ F(s_1 \xrightarrow{s} s_2)$ being multiplication by $s$. (Exercise: Show that the maps $F(s) \rightarrow S^{-1}R$ sending 1 to $1/s$ induce an isomorphism $\text{colim} \ F(s) \cong S^{-1}R$.) Since $S^{-1}R$ is the filtered colimit of the free $R$-modules $F(s)$, it is flat by 2.6.17. \hfill \Box

### Exercise 3.2.1
Show that the following are equivalent for every left $R$-module $B$.

1. $B$ is flat.
2. $\text{Tor}^R_n(A, B) = 0$ for all $n \neq 0$ and all $A$.
3. $\text{Tor}^R_1(A, B) = 0$ for all $A$.

### Exercise 3.2.2
Show that if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is exact and both $B$ and $C$ are flat, then $A$ is also flat.

### Exercise 3.2.3
We saw in the last section that if $R = \mathbb{Z}$ (or more generally, if $R$ is a principal ideal domain), a module $B$ is flat iff $B$ is torsionfree. Here is an example of a torsionfree ideal $I$ that is not a flat $R$-module. Let $k$ be a field and set $R = k[x, y], I = (x, y)R$. Show that $k = R/I$ has the projective resolution

$$0 \rightarrow R \begin{bmatrix} -y & x \\ x & -y \end{bmatrix} R^2 \xrightarrow{(x, y)} R \rightarrow k \rightarrow 0.$$  

Then compute that $\text{Tor}^R_1(I, k) \cong R^2(k, k) \cong k$, showing that $I$ is not flat.

### Definition 3.2.3
The Pontrjagin dual $B^*$ of a left $R$-module $B$ is the right $R$-module $\text{Hom}_{\text{Ab}}(B, \mathbb{Q}/\mathbb{Z})$; an element $r$ of $R$ acts via $(fr)(b) = f(rb)$.

### Proposition 3.2.4
The following are equivalent for every left $R$-module $B$:

1. $B$ is a flat $R$-module.
2. $B^*$ is an injective right $R$-module.
3. $I \otimes_R B \cong IB = \{x_1b_1 + \cdots + x_nb_n \in B : x_i \in I, b_i \in B\} \subset B$ for every right ideal $I$ of $R$.
4. $\text{Tor}^R_1(R/I, B) = 0$ for every right ideal $I$ of $R$. 

---

The content is from the book "Commutative Algebra: with a View Toward Algebraic Geometry" by David Eisenbud. The page number is 69. The text is formatted in a natural reading style, without any additional annotations or modifications. The reference to the Cambridge Core terms of use and availability is included at the bottom of the snippet.
The equivalence of (3) and (4) follows from the exact sequence
\[ 0 \to \text{Tor}_1(R/I, B) \to I \otimes B \to B \to B/I B \to 0. \]

Now for every inclusion \( A' \subseteq A \) of right modules, the adjoint functors \( \otimes B \) and \( \text{Hom}(\cdot, B) \) give a commutative diagram
\[
\begin{array}{ccc}
\text{Hom}(A, B^*) & \to & \text{Hom}(A', B^*) \\
\downarrow \cong & & \downarrow \cong \\
(A \otimes B)^* = \text{Hom}(A \otimes B, \mathbb{Q}/\mathbb{Z}) & \to & \text{Hom}(A' \otimes B, \mathbb{Q}/\mathbb{Z}) = (A' \otimes B)^*.
\end{array}
\]

Using the lemma below and Baer's criterion 2.3.1, we see that
\( B^* \) is injective \( \iff (A \otimes B)^* \to (A' \otimes B)^* \) is surjective for all \( A' \subseteq A \).
\( \iff A' \otimes B \to A \otimes B \) is injective for all \( A' \subseteq A \iff B \) is flat.
\( B^* \) is injective \( \iff (R \otimes B)^* \to (I \otimes B)^* \) is surjective for all \( I \subseteq R \)
\( \iff I \otimes B \to R \otimes B \) is injective for all \( I \)
\( \iff I \otimes B \cong I B \) for all \( I \).

Lemma 3.2.5 A map \( f : B \to C \) is injective iff the dual map \( f^* : C^* \to B^* \) is surjective.

Proof If \( A \) is the kernel of \( f \), then \( A^* \) is the cokernel of \( f^* \), because \( \text{Hom}(\cdot, \mathbb{Q}/\mathbb{Z}) \) is contravariant exact. But we saw in exercise 2.3.3 that \( A = 0 \) iff \( A^* = 0 \).

Exercise 3.2.4 Show that a sequence \( A \to B \to C \) is exact iff its dual \( C^* \to B^* \to A^* \) is exact.

An \( R \)-module \( M \) is called finitely presented if it can be presented using finitely many generators \( (e_1, \ldots, e_n) \) and relations \( (\sum a_{ij} e_j = 0, \ j = 1, \ldots, m) \). That is, there is an \( m \times n \) matrix \( \alpha \) and an exact sequence \( R^n \to M \to 0 \). If \( M \) is finitely generated, the following exercise shows that the property of being finitely presented is independent of the choice of generators.

Exercise 3.2.5 Suppose that \( \varphi : F \to M \) is any surjection, where \( F \) is finitely generated and \( M \) is finitely presented. Use the Snake Lemma to show that \( \ker(\varphi) \) is finitely generated.

Still letting \( A^* \) denote the Pontrjagin dual 3.2.3 of \( A \), there is a natural map \( \sigma : A^* \otimes_R M \to \text{Hom}_R(M, A)^* \) defined by \( \sigma(f \otimes m) : h \mapsto f(h(m)) \) for \( f \in A^*, m \in M \) and \( h \in \text{Hom}(M, A) \). (Exercise: If \( M = \bigoplus_{i=1}^\infty R \), show that \( \sigma \) is not an isomorphism.)
3.2 Tor and Flatness

Lemma 3.2.6 The map σ is an isomorphism for every finitely presented M and all A.

Proof A simple calculation shows that σ is an isomorphism if M = R. By additivity, σ is an isomorphism if M = R^m or R^n. Now consider the diagram

\[
\begin{array}{ccc}
A^* \otimes R^m & \longrightarrow & A^* \otimes R^n & \longrightarrow & A^* \otimes M & \longrightarrow & 0 \\
\sigma \downarrow \cong & & \sigma \downarrow \cong & & \sigma \downarrow & \\
\hom(R^m, A)^* & \longrightarrow & \hom(R^n, A)^* & \longrightarrow & \hom(M, A)^* & \longrightarrow & 0.
\end{array}
\]

The rows are exact because \( \otimes \) is right exact, Hom is left exact, and Pontrjagin dual is exact by 2.3.3. The 5-lemma shows that σ is an isomorphism. ◊

Theorem 3.2.7 Every finitely presented flat R-module M is projective.

Proof In order to show that M is projective, we shall show that Hom_R(M, -) is exact. To this end, suppose that we are given a surjection B → C. Then C^* → B^* is an injection, so if M is flat, the top arrow of the square

\[
\begin{array}{ccc}
(C^*) \otimes_R M & \longrightarrow & (B^*) \otimes_R M \\
\cong \downarrow & & \cong \downarrow \\
\hom(M, C)^* & \longrightarrow & \hom(M, B)^*
\end{array}
\]

is an injection. Hence the bottom arrow is an injection. As we have seen, this implies that Hom(M, B) → Hom(M, C) is a surjection, as required. ◊

Flat Resolution Lemma 3.2.8 The groups Tor_n(A, B) may be computed using resolutions by flat modules. That is, if F → A is a resolution of A with the F_n being flat modules, then Tor_n(A, B) ≅ H_n(F \otimes B). Similarly, if F' → B is a resolution of B by flat modules, then Tor_n(A, B) ≅ H_n(A \otimes F').

Proof We use induction and dimension shifting (exercise 2.4.3) to prove that Tor_n(A, B) ≅ H_n(F \otimes B) for all n; the second part follows by arguing over R^{op}. The assertion is true for n = 0 because \( \otimes \)R is right exact. Let K be such that 0 → K → F_0 → A → 0 is exact; if E = (\cdots → F_2 → F_1 → 0), then E → K is a resolution of K by flat modules. For n = 1 we simply compute

\[
\text{Tor}_1(A, B) = \ker(K \otimes B \to F_0 \otimes B)
\]

\[
= \ker \left\{ \frac{F_1 \otimes B}{\text{im}(F_2 \otimes B)} \to F_0 \otimes B \right\} = H_1(F \otimes B).
\]
For \( n \geq 2 \) we use induction to see that
\[
\text{Tor}_n(A, B) \cong \text{Tor}_{n-1}(K, B) \cong H_{n-1}(E \otimes B) = H_n(F \otimes B).
\]

**Proposition 3.2.9** (Flat base change for \( \text{Tor} \)) Suppose \( R \to T \) is a ring map such that \( T \) is flat as an \( R \)-module. Then for all \( R \)-modules \( A \), all \( T \)-modules \( C \) and all \( n \)
\[
\text{Tor}_n^R(A, C) \cong \text{Tor}_n^T(A \otimes_R T, C).
\]

**Proof** Choose an \( R \)-module projective resolution \( P \to A \). Then \( \text{Tor}_n^R(A, C) \) is the homology of \( P \otimes_R C \). Since \( T \) is \( R \)-flat, and each \( P_n \otimes_R T \) is a projective \( T \)-module, \( P \otimes T \to A \otimes T \) is a \( T \)-module projective resolution. Thus \( \text{Tor}_n^T(A \otimes_R T, C) \) is the homology of the complex \( (P \otimes_R T) \otimes_T C \cong P \otimes_R C \) as well.

**Corollary 3.2.10** If \( R \) is commutative and \( T \) is a flat \( R \)-algebra, then for all \( R \)-modules \( A \) and \( B \), and for all \( n \)
\[
T \otimes_R \text{Tor}_n^R(A, B) \cong \text{Tor}_n^T(A \otimes_R T, T \otimes_R B).
\]

**Proof** Setting \( C = T \otimes_R B \), it is enough to show that \( \text{Tor}_n^R(A, T \otimes B) = T \otimes \text{Tor}_n^R(A, B) \). As \( T \otimes_R \) is an exact functor, \( T \otimes \text{Tor}_n^R(A, B) \) is the homology of \( T \otimes_R (P \otimes_R B) \cong P \otimes_R (T \otimes_R B) \), the complex whose homology is \( \text{Tor}_n^R(A, T \otimes B) \).

Now we shall suppose that \( R \) is a commutative ring, so that the \( \text{Tor}_n^R(A, B) \) are actually \( R \)-modules in order to show how \( \text{Tor}_n \) localizes.

**Lemma 3.2.11** If \( \mu : A \to A \) is multiplication by a central element \( r \in R \), so are the induced maps \( \mu_* : \text{Tor}_n^R(A, B) \to \text{Tor}_n^R(A, B) \) for all \( n \) and \( B \).

**Proof** Pick a projective resolution \( P \to A \). Multiplication by \( r \) is an \( R \)-module chain map \( \tilde{\mu} : P \to P \) over \( \mu \) (this uses the fact that \( r \) is central), and \( \tilde{\mu} \otimes B \) is multiplication by \( r \) on \( P \otimes B \). The induced map \( \mu_* \) on the quotient \( \text{Tor}_n(A, B) \) of \( P_n \otimes B \) is therefore also multiplication by \( r \).

**Corollary 3.2.12** If \( A \) is an \( R/r \)-module, then for every \( R \)-module \( B \) the \( R \)-modules \( \text{Tor}_n^R(A, B) \) are actually \( R/r \)-modules, that is, annihilated by the ideal \( rR \).
Corollary 3.2.13 (Localization for Tor) If \( R \) is commutative and \( A \) and \( B \) are \( R \)-modules, then the following are equivalent for each \( n \):

1. \( \text{Tor}^R_n(A, B) = 0 \).
2. For every prime ideal \( p \) of \( R \) \( \text{Tor}^R_p(A_p, B_p) = 0 \).
3. For every maximal ideal \( m \) of \( R \) \( \text{Tor}^R_m(A_m, B_m) = 0 \).

Proof For any \( R \)-module \( M, M = 0 \iff M_p = 0 \) for every prime \( p \iff M_m = 0 \) for every maximal ideal \( m \). In the case \( M = \text{Tor}^R(A, B) \) we have

\[
M_p = R_p \otimes_R M = \text{Tor}^R_p(A_p, B_p).
\]

\[\diamond\]

3.3 Ext for Nice Rings

We first turn to a calculation of \( \text{Ext}^* \) groups to get a calculational feel for what these derived functors do to abelian groups.

Lemma 3.3.1 \( \text{Ext}^n_Z(A, B) = 0 \) for \( n \geq 2 \) and all abelian groups \( A, B \).

Proof Embed \( B \) in an injective abelian group \( I^0 \); the quotient \( I^1 \) is divisible, hence injective. Therefore, \( \text{Ext}^*(A, B) \) is the cohomology of

\[
0 \to \text{Hom}(A, I^0) \to \text{Hom}(A, I^1) \to 0.
\]

\[\diamond\]

Calculation 3.3.2 \( (A = \mathbb{Z}/p) \) \( \text{Ext}^0_Z(\mathbb{Z}/p, B) = pB, \text{Ext}^1_Z(\mathbb{Z}/p, B) = B/pB \) and \( \text{Ext}^n_Z(\mathbb{Z}/p, B) = 0 \) for \( n \geq 2 \). To see this, use the resolution

\[
0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to 0
\]

to see that \( \text{Ext}^*(\mathbb{Z}/p, B) \) is the cohomology of \( 0 \leftarrow B \xleftarrow{p} B \leftarrow 0 \).

Since \( \mathbb{Z} \) is projective, \( \text{Ext}^1(\mathbb{Z}, B) = 0 \). Hence we can calculate \( \text{Ext}^*(A, B) \) for every finitely generated abelian group \( A \cong \mathbb{Z}^m \oplus \mathbb{Z}/p_1 \oplus \cdots \oplus \mathbb{Z}/p_n \) by taking a finite direct sum of \( \text{Ext}^*(\mathbb{Z}/p, B) \) groups. For infinitely generated groups, the calculation is much more complicated than it was for Tor.

Example 3.3.3 \( (B = \mathbb{Z}) \) Let \( A \) be a torsion group, and write \( A^* \) for its Pontrjagin dual \( \text{Hom}(A, \mathbb{Q}/\mathbb{Z}) \) as in 3.2.3. Using the injective resolution \( 0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \) to compute \( \text{Ext}^*(A, \mathbb{Z}) \), we see that \( \text{Ext}^0_Z(A, \mathbb{Z}) = 0 \) and
Ext$^1_\mathbb{Z}(A, \mathbb{Z}) = A^\ast$. To get a feel for this, note that because $\mathbb{Z}_{p^\infty}$ is the union (colimit) of its subgroups $\mathbb{Z}/p^n$, the group

$$\text{Ext}^1_\mathbb{Z}(\mathbb{Z}_{p^\infty}, \mathbb{Z}) = (\mathbb{Z}_{p^\infty})^\ast$$

is the torsionfree group of $p$-adic integers, $\hat{\mathbb{Z}}_p = \varprojlim(\mathbb{Z}/p^n)$. We will calculate $\text{Ext}^1_\mathbb{Z}(\mathbb{Z}_{p^\infty}, B)$ more generally in section 3.5, using $\text{lim}^1$.

**Exercise 3.3.1** Show that $\text{Ext}^1_\mathbb{Z}(\mathbb{Z}[1/p], \mathbb{Z}) \cong \hat{\mathbb{Z}}_p/\mathbb{Z} \cong \mathbb{Z}_{p^\infty}$. This shows that $\text{Ext}^1(\mathbb{Z}, \mathbb{Z})$ does not vanish on flat abelian groups.

**Exercise 3.3.2** When $R = \mathbb{Z}/m$ and $B = \mathbb{Z}/p$ with $p|m$, show that

$$0 \to \mathbb{Z}/p \hookrightarrow \mathbb{Z}/m \overset{p}{\to} \mathbb{Z}/m \overset{m/p}{\to} \mathbb{Z}/m \overset{p}{\to} \mathbb{Z}/m \overset{m/p}{\to} \cdots$$

is an infinite periodic injective resolution of $B$. Then compute the groups $\text{Ext}^n_{\mathbb{Z}/m}(A, \mathbb{Z}/p)$ in terms of $A^\ast = \text{Hom}(A, \mathbb{Z}/m)$. In particular, show that if $p^2|m$, then $\text{Ext}^n_{\mathbb{Z}/m}(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p$ for all $n$.

**Proposition 3.3.4** For all $n$ and all rings $R$

1. $\text{Ext}^n_R(\bigoplus A_\alpha, B) \cong \prod \text{Ext}^n_R(A_\alpha, B)$.
2. $\text{Ext}^n_R(A, \prod B_\beta) \cong \prod \text{Ext}^n_R(A, B_\beta)$.

**Proof** If $P_\alpha \to A_\alpha$ are projective resolutions, so is $\bigoplus P_\alpha \to \bigoplus A_\alpha$. If $B_\beta \to I_\beta$ are injective resolutions, so is $\prod B_\beta \to \prod I_\beta$. Since $\text{Hom}(\bigoplus P_\alpha, B) = \prod \text{Hom}(P_\alpha, B)$ and $\text{Hom}(A, \prod I_\beta) = \prod \text{Hom}(A, I_\beta)$, the result follows from the fact that for any family $C_\gamma$ of cochain complexes,

$$H^\ast(\prod C_\gamma) \cong \prod H^\ast(C_\gamma).$$

**Examples 3.3.5**

1. If $p^2|m$ and $A$ is a $\mathbb{Z}/p$-vector space of countably infinite dimension, then $\text{Ext}^n_{\mathbb{Z}/m}(A, \mathbb{Z}/p) \cong \prod_{i=1}^\infty \mathbb{Z}/p$ is a $\mathbb{Z}/p$-vector space of dimension $2^{\aleph_0}$.
2. If $B$ is the product $\mathbb{Z}/2 \times \mathbb{Z}/3 \times \mathbb{Z}/4 \times \mathbb{Z}/5 \times \cdots$ then $B$ is not a torsion group, and

$$\text{Ext}^1(A, B) = \prod_{p=2}^\infty A/pA = 0$$

vanishes if and only if $A$ is divisible.
Lemma 3.3.6 Suppose that $R$ is a commutative ring, so that $\Hom_R(A, B)$ and the $\Ext^n_R(A, B)$ are actually $R$-modules. If $\mu : A \to A$ and $\nu : B \to B$ are multiplication by $r \in R$, so are the induced endomorphisms $\mu^*$ and $\nu^*$ of $\Ext^n_R(A, B)$ for all $n$.

Proof Pick a projective resolution $P \to A$. Multiplication by $r$ is an $R$-module chain map $\tilde{\mu} : P \to P$ over $\mu$ (as $r$ is central); the map $\Hom(\tilde{\mu}, B)$ on $\Hom(P, B)$ is multiplication by $r$, because it sends $f \in \Hom(P_n, B)$ to $f \tilde{\mu}$, which takes $p \in P_n$ to $f(rp) = rf(p)$. Hence the map $\mu^*$ on the subquotient $\Ext^n(A, B)$ of $\Hom(P_n, B)$ is also multiplication by $r$. The argument for $\nu^*$ is similar, using an injective resolution $B \to I$.

Corollary 3.3.7 If $R$ is commutative and $A$ is actually an $R/r$-module, then for every $R$-module $B$ the $R$-modules $\Ext^n_R(A, B)$ are actually $R/r$-modules.

We would like to conclude, as we did for Tor, that Ext commutes with localization in some sense. Indeed, there is a natural map $\Phi$ from $S^{-1}\Hom_R(A, B)$ to $\Hom_{S^{-1}R}(S^{-1}A, S^{-1}B)$, but it need not be an isomorphism. A sufficient condition is that $A$ be finitely presented, that is, some $R^m \xrightarrow{\alpha} R^n \to A \to 0$ is exact.

Lemma 3.3.8 If $A$ is a finitely presented $R$-module, then for every central multiplicative set $S$ in $R$, $\Phi$ is an isomorphism:

$$\Phi : S^{-1}\Hom_R(A, B) \cong \Hom_{S^{-1}R}(S^{-1}A, S^{-1}B).$$

Proof $\Phi$ is trivially an isomorphism when $A = R$; as $\Hom$ is additive, $\Phi$ is also an isomorphism when $A = R^m$. The result now follows from the 5-lemma and the following diagram:

$$
\begin{array}{cccccc}
0 & \to & S^{-1}\Hom_R(A, B) & \to & S^{-1}\Hom_R(R^n, B) & \xrightarrow{\alpha} & S^{-1}\Hom_R(R^m, B) \\
\Phi \downarrow & & \cong \downarrow & & \cong \downarrow \\
0 & \to & \Hom(S^{-1}A, S^{-1}B) & \to & \Hom(S^{-1}R^n, S^{-1}B) & \xrightarrow{\alpha} & \Hom(S^{-1}R^m, S^{-1}B).
\end{array}
$$

Definition 3.3.9 A ring $R$ is (right) noetherian if every (right) ideal is finitely generated, that is, if every module $R/I$ is finitely presented. It is well known that if $R$ is noetherian, then every finitely generated (right) $R$-module is finitely presented. (See [BAI,§3.2].) It follows that every finitely generated module $A$ has a resolution $F \to A$ in which each $F_n$ is a finitely generated free $R$-module.
Proposition 3.3.10 Let $A$ be a finitely generated module over a commutative noetherian ring $R$. Then for every multiplicative set $S$, all modules $B$, and all $n$

$$\Phi: S^{-1}\mathbb{E}xt^n_R(A, B) \cong \mathbb{E}xt^n_{S^{-1}R}(S^{-1}A, S^{-1}B).$$

Proof Choose a resolution $F \rightarrow A$ by finitely generated free $R$-modules. Then $S^{-1}F \rightarrow S^{-1}A$ is a resolution by finitely generated free $S^{-1}R$-modules. Because $S^{-1}$ is an exact functor from $R$-modules to $S^{-1}R$-modules,

$$S^{-1}\mathbb{E}xt^*(A, B) = S^{-1}(H^* \mathbb{H}om_R(F, B)) \cong H^*(S^{-1} \mathbb{H}om_R(F, B))$$

$$\cong H^* \mathbb{H}om_{S^{-1}R}(S^{-1}F, S^{-1}B) = \mathbb{E}xt^*_{S^{-1}R}(S^{-1}A, S^{-1}B).$$

Corollary 3.3.11 (Localization for Ext) If $R$ is commutative noetherian and $A$ is a finitely generated $R$-module, then the following are equivalent for all modules $B$ and all $n$:

1. $\mathbb{E}xt^*_R(A, B) = 0.$
2. For every prime ideal $p$ of $R$, $\mathbb{E}xt^*_R(A_p, B_p) = 0.$
3. For every maximal ideal $m$ of $R$, $\mathbb{E}xt^*_R(A_m, B_m) = 0.$

3.4 Ext and Extensions

An extension $\xi$ of $A$ by $B$ is an exact sequence $0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0.$ Two extensions $\xi$ and $\xi'$ are equivalent if there is a commutative diagram

$$\begin{array}{cccccc}
\xi: & 0 & \rightarrow & B & \rightarrow & X & \rightarrow & A & \rightarrow & 0 \\
& & & \| & \downarrow \cong & \| \\
\xi': & 0 & \rightarrow & B & \rightarrow & X' & \rightarrow & A & \rightarrow & 0
\end{array}$$

An extension is split if it is equivalent to $0 \rightarrow B \xrightarrow{(0,1)} A \oplus B \rightarrow A \rightarrow 0.$

Exercise 3.4.1 Show that if $p$ is prime, there are exactly $p$ equivalence classes of extensions of $\mathbb{Z}/p$ by $\mathbb{Z}/p$ in $\mathbb{A}b$: the split extension and the extensions

$$0 \rightarrow \mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{i} \mathbb{Z}/p \rightarrow 0 \quad (i = 1, 2, \cdots, p - 1).$$
Lemma 3.4.1 If \( \text{Ext}^1(A, B) = 0 \), then every extension of \( A \) by \( B \) is split.

Proof Given an extension \( \xi \), applying \( \text{Ext}^*(A, -) \) yields the exact sequence
\[
\text{Hom}(A, X) \to \text{Hom}(A, A) \xrightarrow{\partial} \text{Ext}^1(A, B)
\]
so the identity map \( \text{id}_A \) lifts to a map \( \sigma: A \to X \) when \( \text{Ext}^1(A, B) = 0 \). As \( \sigma \) is a section of \( X \to A \), evidently \( X \cong A \oplus B \) and \( \xi \) is split. \( \Box \)

Porism 3.4.2 Taking the construction of this lemma to heart, we see that the class \( \Theta(\xi) = \partial(\text{id}_A) \) in \( \text{Ext}^1(A, B) \) is an obstruction to \( \xi \) being split: \( \xi \) is split iff \( \text{id}_A \) lifts to \( \text{Hom}(A, X) \) iff the class \( \Theta(\xi) \in \text{Ext}^1(A, B) \) vanishes. Equivalent extensions have the same obstruction by naturality of the map \( \partial \), so the obstruction \( \Theta(\xi) \) only depends on the equivalence class of \( \xi \).

Theorem 3.4.3 Given two \( R \)-modules \( A \) and \( B \), the mapping \( \Theta: \xi \mapsto \partial(\text{id}_A) \) establishes a 1–1 correspondence
\[
\left\{ \text{equivalence classes of extensions of } A \text{ by } B \right\} \leftrightarrow \text{Ext}^1(A, B)
\]
in which the split extension corresponds to the element \( 0 \in \text{Ext}^1(A, B) \).

Proof Fix an exact sequence \( 0 \to M \xrightarrow{j} P \to A \to 0 \) with \( P \) projective. Applying \( \text{Hom}(\_, B) \) yields an exact sequence
\[
\text{Hom}(P, B) \to \text{Hom}(M, B) \xrightarrow{\partial} \text{Ext}^1(A, B) \to 0.
\]
Given \( x \in \text{Ext}^1(A, B) \), choose \( \beta \in \text{Hom}(M, B) \) with \( \partial(\beta) = x \). Let \( X \) be the pushout of \( j \) and \( \beta \), i.e., the cokernel of \( M \to P \oplus B \) \((m) \mapsto (j(m), -\beta(m))\). There is a diagram
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{j} & P & \longrightarrow & A & \longrightarrow & 0 \\
& \downarrow{\beta} & \quad & \downarrow{\sigma} & & & & & \\
0 & \longrightarrow & B & \xrightarrow{i} & X & \longrightarrow & A & \longrightarrow & 0,
\end{array}
\]
where the map \( X \to A \) is induced by the maps \( B \xrightarrow{0} A \) and \( P \to A \). (Exercise: Show that the bottom sequence \( \xi \) is exact.) By naturality of the connecting map \( \partial \), we see that \( \Theta(\xi) = x \), that is, that \( \Theta \) is a surjection.
In fact, this construction gives a set map $\Psi$ from $\text{Ext}^1(A, B)$ to the set of equivalence classes of extensions. For if $\beta' \in \text{Hom}(M, B)$ is another lift of $x$, then there is an $f \in \text{Hom}(P, B)$ so that $\beta' = \beta + fj$. If $X'$ is the pushout of $j$ and $\beta'$, then the maps $i: B \to X$ and $\sigma + if: P \to X$ induce an isomorphism $X' \cong X$ and an equivalence between $\xi'$ and $\xi$. (Check this!)

Conversely, given an extension $\xi$ of $A$ by $B$, the lifting property of $P$ gives a map $\tau: P \to X$ and hence a commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & M & \rightarrow & P & \rightarrow & A & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & B & \rightarrow & X & \rightarrow & A & \rightarrow & 0.
\end{array}
$$

(\ast)

Now $X$ is the pushout of $j$ and $\gamma$. (Exercise: Check this!) Hence $\Psi(\Theta(\xi)) = \xi$, showing that $\Theta$ is injective.

**Definition 3.4.4 (Baer sum)** Let $\xi: 0 \to B \to X \to A \to 0$ and $\xi': 0 \to B \to X' \to A \to 0$ be two extensions of $A$ by $B$. Let $X''$ be the pullback $\{(x, x') \in X \times X' : \bar{x} = \bar{x}' \text{ in } A\}$.

$$
\begin{array}{ccc}
X'' & \rightarrow & X' \\
\downarrow & & \downarrow \\
X & \rightarrow & A
\end{array}
$$

$X''$ contains three copies of $B : B \times 0, 0 \times B$, and the skew diagonal $\{(b, b) : b \in B\}$. The copies $B \times 0$ and $0 \times B$ are identified in the quotient $Y$ of $X''$ by the skew diagonal. Since $X''/0 \times B \cong X$ and $X/B \cong A$, it is immediate that the sequence

$\varphi: \quad 0 \rightarrow B \rightarrow Y \rightarrow A \rightarrow 0$

is also an extension of $A$ by $B$. The class of $\varphi$ is called the *Baer sum* of the extensions $\xi$ and $\xi'$, since this construction was introduced by R. Baer in 1934.

**Corollary 3.4.5** The set of (equiv. classes of) extensions is an abelian group under Baer sum, with zero being the class of the split extension. The map $\Theta$ is an isomorphism of abelian groups.

**Proof** We will show that $\Theta(\varphi) = \Theta(\xi) + \Theta(\xi')$ in $\text{Ext}^1(A, B)$. This will prove that Baer sum is well defined up to equivalence, and the corollary will then follow. We shall adopt the notation used in (\ast) in the proof of the above
3.4 Ext and Extensions

Theorem. Let \( \tau'' : P \to X'' \) be the map induced by \( \tau : P \to X \) and \( \tau' : P \to X' \), and let \( \bar{\tau} : P \to Y \) be the induced map. The restriction of \( \bar{\tau} \) to \( M \) is induced by the map \( \gamma + \gamma' : M \to B \), so

\[
0 \to M \to P \to A \to 0
\]

\[
\gamma + \gamma' \downarrow \quad \bar{\tau} \downarrow \quad \text{commutes.}
\]

Hence, \( \Theta(\varphi) = \partial(\gamma + \gamma') \), where \( \partial \) is the map from \( \text{Hom}(M, B) \) to \( \text{Ext}^1(A, B) \). But \( \partial(\gamma + \gamma') = \partial(\gamma) + \partial(\gamma') = \Theta(\xi) + \Theta(\xi'). \)

**Vista 3.4.6 (Yoneda Ext groups)** We can define \( \text{Ext}^1(A, B) \) in any abelian category \( \mathcal{A} \), even if it has no projectives and no injectives, to be the set of equivalence classes of extensions under Baer sum (if indeed this is a set). The Freyd-Mitchell Embedding Theorem 1.6.1 shows that \( \text{Ext}^1(A, B) \) is an abelian group—but one could also prove this fact directly. Similarly, we can recapture the groups \( \text{Ext}^n(A, B) \) without mentioning projectives or injectives. This approach is due to Yoneda. An element of the Yoneda Ext\( n \)(A, B) is an equivalence class of exact sequences of the form

\[
\xi : 0 \to B \to X_n \to \cdots \to X_1 \to A \to 0.
\]

The equivalence relation is generated by the relation that \( \xi' \sim \xi'' \) if there is a diagram

\[
\begin{array}{ccc}
\xi' : & 0 \to B & \to X'_n \to \cdots \to X'_1 \to A \to 0 \\
& \downarrow \quad \downarrow \quad \downarrow \\
\xi'' : & 0 \to B & \to X''_n \to \cdots \to X''_1 \to A \to 0.
\end{array}
\]

To “add” \( \xi \) and \( \xi' \) when \( n \geq 2 \), let \( X'_n \) be the pullback of \( X_1 \) and \( X'_1 \) over \( A \), let \( X''_n \) be the pushout of \( X_n \) and \( X''_1 \) under \( B \), and let \( Y_n \) be the quotient of \( X''_n \) by the skew diagonal copy of \( B \). Then \( \xi + \xi' \) is the class of the extension

\[
0 \to B \to Y_n \to X_{n-1} \oplus X'_{n-1} \to \cdots \to X_2 \oplus X'_2 \to X'_1 \to A \to 0.
\]

Now suppose that \( \mathcal{A} \) has enough projectives. If \( P \to A \) is a projective resolution, the Comparison Theorem 2.2.6 yields a map from \( P \) to \( \xi \), hence a diagram.
By dimension shifting, there is an exact sequence

\[ \text{Hom}(P_{n-1}, B) \rightarrow \text{Hom}(M, B) \rightarrow \text{Ext}^n(A, B) \rightarrow 0. \]

The association \( \Theta(\xi) = \partial(\beta) \) gives the 1–1 correspondence between the Yoneda \( \text{Ext}^n \) and the derived functor \( \text{Ext}^n \). For more details we refer the reader to \([\text{BX}, \S 7.5]\) or \([\text{MacH}, \text{pp. 82–87}]\).

### 3.5 Derived Functors of the Inverse Limit

Let \( I \) be a small category and \( \mathcal{A} \) an abelian category. We saw in Chapter 2 that the functor category \( \mathcal{A}^I \) has enough injectives, at least when \( \mathcal{A} \) is complete and has enough injectives. (For example, \( \mathcal{A} \) could be \( \text{Ab}, R\text{-mod} \), or \( \text{Sheaves}(X) \).) Therefore we can define the right derived functors \( R^n \text{lim}_{i \in I} \) from \( \mathcal{A}^I \) to \( \mathcal{A} \).

We are most interested in the case in which \( \mathcal{A} \) is \( \text{Ab} \) and \( I \) is the poset \( \cdots \rightarrow 2 \rightarrow 1 \rightarrow 0 \) of whole numbers in reverse order. We shall call the objects of \( \text{Ab}^I \) (countable) towers of abelian groups; they have the form

\[ \{A_i\} : \cdots \rightarrow A_2 \rightarrow A_1 \rightarrow A_0. \]

In this section we shall give the alternative construction \( \text{lim}^1 \) of \( R^1 \text{lim} \) for countable towers due to Eilenberg and prove that \( R^n \text{lim} = 0 \) for \( n \neq 0, 1 \). This construction generalizes from \( \text{Ab} \) to other abelian categories that satisfy the following axiom, introduced by Grothendieck in \([\text{Tohoku}]\):

\((AB4^*)\). \( \mathcal{A} \) is complete, and the product of any set of surjections is a surjection.

**Explanation** If \( I \) is a discrete set, \( \mathcal{A}^I \) is the product category \( \prod_{i \in I} \mathcal{A} \) of indexed families of objects \( \{A_i\} \) in \( \mathcal{A} \). For \( \{A_i\} \) in \( \mathcal{A}^I \), \( \lim_{i \in I} A_i \) is the product \( \prod A_i \). Axiom \((AB4^*)\) states that the left exact functor \( \prod \) from \( \mathcal{A}^I \) to \( \mathcal{A} \) is exact for all discrete \( I \). Axiom \((AB4^*)\) fails \( (\prod_{i=1}^{\infty} \) is not exact) for some important abelian categories, such as \( \text{Sheaves}(X) \). On the other hand, axiom \((AB4^*)\) is satisfied by many abelian categories in which objects have underlying sets, such as \( \text{Ab}, \text{mod–R}, \text{and Ch(mod–R)} \).
3.5 Derived Functors of the Inverse Limit

**Definition 3.5.1** Given a tower \( \{A_i\} \) in \( \text{Ab} \), define the map

\[
\Delta: \prod_{i=0}^{\infty} A_i \rightarrow \prod_{i=0}^{\infty} A_i
\]

by the element-theoretic formula

\[
\Delta(\cdots, a_i, \cdots, a_0) = (\cdots, a_i - \bar{a}_{i+1}, \cdots, a_1 - \bar{a}_2, a_0 - \bar{a}_1),
\]

where \( \bar{a}_{i+1} \) denotes the image of \( a_{i+1} \in A_{i+1} \) in \( A_i \). The kernel of \( \Delta \) is \( \lim A_i \) (check this!). We define \( \lim^1 A_i \) to be the cokernel of \( \Delta \), so that \( \lim^1 \) is a functor from \( \text{Ab} \) to \( \text{Ab} \). We also set \( \lim^0 A_i = \lim A_i \) and \( \lim^n A_i = 0 \) for \( n \neq 0, 1 \).

**Lemma 3.5.2** The functors \( \{\lim^n\} \) form a cohomological \( \delta \)-functor.

**Proof** If \( 0 \rightarrow \{A_i\} \rightarrow \{B_i\} \rightarrow \{C_i\} \rightarrow 0 \) is a short exact sequence of towers, apply the Snake Lemma to

\[
\begin{array}{c}
0 \\
\downarrow \Delta
\end{array}
\rightarrow
\begin{array}{c}
\prod A_i \\
\downarrow \Delta
\end{array}
\rightarrow
\begin{array}{c}
\prod B_i \\
\downarrow \Delta
\end{array}
\rightarrow
\begin{array}{c}
\prod C_i \\
\downarrow \Delta
\end{array}
\rightarrow
0
\]

to get the requisite natural long exact sequence. \( \Box \)

**Lemma 3.5.3** If all the maps \( A_{i+1} \rightarrow A_i \) are onto, then \( \lim^1 A_i = 0 \). Moreover \( \lim A_i \neq 0 \) (unless every \( A_i = 0 \)), because each of the natural projections \( \lim A_i \rightarrow A_j \) are onto.

**Proof** Given elements \( b_i \in A_i \) \( (i = 0, 1, \cdots) \), and any \( a_0 \in A_0 \), inductively choose \( a_{i+1} \in A_{i+1} \) to be a lift of \( a_i - b_i \in A_i \). The map \( \Delta \) sends \( (\cdots, a_i, a_0) \) to \( (\cdots, b_1, b_0) \), so \( \Delta \) is onto and \( \text{coker}(\Delta) = 0 \). If all the \( b_i = 0 \), then \( (\cdots, a_i, a_0) \in \lim A_i \). \( \Box \)

**Corollary 3.5.4** \( \lim^1 A_i \cong (R^1 \lim)(A_i) \) and \( R^n \lim = 0 \) for \( n \neq 0, 1 \).

**Proof** In order to show that the \( \lim^n \) forms a universal \( \delta \)-functor, we only need to see that \( \lim^1 \) vanishes on enough injectives. In Chapter 2 we constructed
enough injectives by taking products of towers

\[ k_* E : \cdots = E = E \rightarrow 0 \rightarrow 0 \rightarrow 0 \cdots \rightarrow 0 \]

with \( E \) injective. All the maps in \( k_* E \) (and hence in the product towers) are onto, so \( \lim^1 \) vanishes on these injective towers.

**Remark** If we replace \( \textbf{Ab} \) by \( A = \textbf{mod}-R, \text{Ch(mod}-R) \) or any abelian category \( \mathcal{A} \) satisfying Grothendieck’s axiom \((AB4^*)\), the above proof goes through to show that \( \lim^1 = R^1(\lim) \) and \( R^n(\lim) = 0 \) for \( n \neq 0, 1 \) as functors on the category of towers in \( \mathcal{A} \). However, the proof breaks down for other abelian categories.

**Example 3.5.5** Set \( A_0 = \mathbb{Z} \) and let \( A_i = p^i \mathbb{Z} \) be the subgroup generated by \( p^i \). Applying \( \lim \) to the short exact sequence of towers

\[ 0 \rightarrow \{ p^i \mathbb{Z} \} \rightarrow \{ \mathbb{Z} \} \rightarrow \{ \mathbb{Z}/p^i \mathbb{Z} \} \rightarrow 0 \]

with \( p \) prime yields the uncountable group

\[ \lim^1 \{ p^i \mathbb{Z} \} \cong \hat{\mathbb{Z}}_p / \mathbb{Z}. \]

Here \( \hat{\mathbb{Z}}_p = \lim_{\leftarrow} \mathbb{Z}/p^i \mathbb{Z} \) is the group of \( p \)-adic integers.

**Exercise 3.5.1** Let \( \{ A_i \} \) be a tower in which the maps \( A_{i+1} \rightarrow A_i \) are inclusions. We may regard \( A = A_0 \) as a topological group in which the sets \( a + A_i (a \in A, i \geq 0) \) are the open sets. Show that \( \lim A_i = \bigcap A_i \) is zero iff \( A \) is Hausdorff. Then show that \( \lim^1 A_i = 0 \) iff \( A \) is complete in the sense that every Cauchy sequence has a limit, not necessarily unique. **Hint:** Show that \( A \) is complete iff \( A \cong \lim_{\leftarrow} (A/A_i) \).

**Definition 3.5.6** A tower \( \{ A_i \} \) of abelian groups satisfies the **Mittag-Leffler condition** if for each \( k \) there exists a \( j \geq k \) such that the image of \( A_j \rightarrow A_k \) equals the image of \( A_j \rightarrow A_k \) for all \( i \geq j \). (The images of the \( A_i \) in \( A_k \) satisfy the descending chain condition.) For example, the Mittag-Leffler condition is satisfied if all the maps \( A_{i+1} \rightarrow A_i \) in the tower \( \{ A_i \} \) are onto. We say that \( \{ A_i \} \) satisfies the **trivial Mittag-Leffler condition** if for each \( k \) there exists a \( j > k \) such that the map \( A_j \rightarrow A_k \) is zero.
Proposition 3.5.7 If \( \{A_i\} \) satisfies the Mittag-Leffler condition, then
\[
\lim^1 A_i = 0.
\]

Proof If \( \{A_i\} \) satisfies the trivial Mittag-Leffler condition, and \( b_i \in A_i \) are given, set \( a_k = b_k + \bar{b}_{k+1} + \cdots + \bar{b}_{j-1} \), where \( \bar{b}_i \) denotes the image of \( b_i \) in \( A_k \). (Note that \( \bar{b}_i = 0 \) for \( i \geq j \).) Then \( \Delta \) maps \( (\cdots, a_1, a_0) \) to \( (\cdots, b_1, b_0) \). Thus \( \Delta \) is onto and \( \lim^1 A_i = 0 \) when \( \{A_i\} \) satisfies the trivial Mittag-Leffler condition. In the general case, let \( B_k \subseteq A_k \) be the image of \( A_i \to A_k \) for large \( i \). The maps \( B_{k+1} \to B_k \) are all onto, so \( \lim^1 B_k = 0 \). The tower \( \{A_k/B_k\} \) satisfies the trivial Mittag-Leffler condition, so \( \lim^1 A_k/B_k = 0 \). From the short exact sequence
\[
0 \to \{B_i\} \to \{A_i\} \to \{A_i/B_i\} \to 0
\]
of towers, we see that \( \lim^1 A_i = 0 \) as claimed. \( \Diamond \)

Exercise 3.5.2 Show that \( \lim^1 A_i = 0 \) if \( \{A_i\} \) is a tower of finite abelian groups, or a tower of finite-dimensional vector spaces over a field.

The following formula presages the Universal Coefficient theorems of the next section, as well as the spectral sequences of Chapter 5.

Theorem 3.5.8 Let \( \cdots \to C_1 \to C_0 \) be a tower of chain complexes of abelian groups satisfying the Mittag-Leffler condition, and set \( C = \lim C_i \). Then there is an exact sequence for each \( q \):
\[
0 \to \lim^1 H_{q+1}(C_i) \to H_q(C) \to \lim H_q(C_i) \to 0.
\]

Proof Let \( B_i \subseteq Z_i \subseteq C_i \) be the subcomplexes of boundaries and cycles in the complex \( C_i \), so that \( Z_i/B_i \) is the chain complex \( H_*(C_i) \) with zero differentials. Applying the left exact functor \( \lim \) to \( 0 \to \{Z_i\} \to \{C_i\} \xrightarrow{d} \{C_i[-1]\} \) shows that in fact \( \lim Z_i \) is the subcomplex \( Z \) of cycles in \( C \). (The \([-1]\) refers to the supressed subscript on the chain complexes.) Let \( B \) denote the subcomplex \( d(C)[1] = (C/Z)[1] \) of boundaries in \( C \), so that \( Z/B \) is the chain complex \( H_*(C) \) with zero differentials. From the exact sequence of towers
\[
0 \to \{Z_i\} \to \{C_i\} \xrightarrow{d} \{B_i[-1]\} \to 0
\]
we see that \( \lim_{i} B_i = (\lim_{i} B_i[-1])[+1] = 0 \) and that
\[
0 \to B[-1] \to \lim_{i} B_i[-1] \to \lim_{i} Z_i \to 0
\]
is exact. From the exact sequence of towers
\[
0 \to \{B_i\} \to \{Z_i\} \to H_*(C_i) \to 0
\]
we see that \( \lim_{i} Z_i \cong \lim_{i} H_*(C_i) \) and that
\[
0 \to \lim_{i} B_i \to Z \to \lim_{i} H_*(C_i) \to 0
\]
is exact. Hence \( C \) has the filtration by subcomplexes
\[
0 \subset B \subset \lim_{i} B_i \subset Z \subset C
\]
whose filtration quotients are \( B, \lim_{i} H_*(C_i)[1], \lim_{i} H_*(C_i), \) and \( C/Z \) respectively. The theorem follows, since \( Z/B = H_*(C) \).

**Variant** If \( \cdots \to C_1 \to C_0 \) is a tower of cochain complexes satisfying the Mittag-Leffler condition, the sequences become
\[
0 \to \lim_{i} H^{q-1}(C_i) \to H^q(C) \to \lim_{i} H^q(C_i) \to 0.
\]

**Application 3.5.9** Let \( H^*(X) \) denote the integral cohomology of a topological CW complex \( X \). If \( \{X_i\} \) is an increasing sequence of subcomplexes with \( X = \bigcup X_i \), there is an exact sequence
\[
0 \to \lim_{i} H^{q-1}(X_i) \to H^q(X) \to \lim_{i} H^q(X_i) \to 0
\]
for each \( q \). This use of \( \lim_{i} \) to perform calculations in algebraic topology was discovered by Milnor in 1960 [Milnor] and thrust \( \lim_{i} \) into the limelight.

To derive this formula, let \( C_i \) denote the chain complex \( \text{Hom}(S(X_i), \mathbb{Z}) \) used to compute \( H^*(X_i) \). Since the inclusion \( S(X_i) \subset S(X_{i+1}) \) splits (because each \( S_n(X_{i+1})/S_n(X_i) \) is a free abelian group), the maps \( C_{i+1} \to C_i \) are onto, and the tower satisfies the Mittag-Leffler condition. Since \( X \) has the weak topology, \( S(X) \) is the union of the \( S(X_i) \), and therefore \( H^*(X) \) is the cohomology of the cochain complex
\[
\text{Hom}(\bigcup S(X_i), \mathbb{Z}) = \lim_{i} \text{Hom}(S(X_i), \mathbb{Z}) = \lim_{i} C_i.
\]
A historical remark: Milnor proved that the sequence (\star) is also valid if \( H^* \) is replaced by any generalized cohomology theory, such as topological \( K \)–theory.

**Application 3.5.10** Let \( A \) be an \( R \)-module that is the union of submodules 
\[ \cdots \subseteq A_i \subseteq A_{i+1} \subseteq \cdots \]
Then for every \( R \)-module \( B \) and every \( q \) the sequence
\[
0 \to \lim_{\rightarrow} \text{Ext}_{R}^{q-1}(A_i, B) \to \text{Ext}_{R}^{q}(A, B) \to \lim_{\leftarrow} \text{Ext}_{R}^{q}(A_i, B) \to 0
\]
is exact. For \( \mathbb{Z}_{p} = \bigcup \mathbb{Z}/p^i \), this gives a short exact sequence for every \( B \):
\[
0 \to \lim_{\rightarrow} \text{Hom}(\mathbb{Z}/p^i, B) \to \text{Ext}_{\mathbb{Z}}^{1}(\mathbb{Z}_{p}, B) \to \mathbb{B}_p \to 0,
\]
where the group \( \mathbb{B}_p = \lim_{\leftarrow} (B/p^i B) \) is the \( p \)-adic completion of \( B \). This generalizes the calculation \( \text{Ext}^1(\mathbb{Z}_{p}^{\infty}, \mathbb{Z}) \cong \mathbb{Z}_{p} \) of 3.3.3. To see this, let \( E \) be a fixed injective resolution of \( B \), and consider the tower of cochain complexes
\[
\text{Hom}(A_{i+1}, E) \to \text{Hom}(A_i, E) \to \cdots \to \text{Hom}(A_0, E).
\]
Each \( \text{Hom}(\cdot, E_n) \) is contravariant exact, so each map in the tower is a surjection. The cohomology of \( \text{Hom}(A_i, E) \) is \( \text{Ext}^*(A_i, B) \), and \( \text{Ext}^*(A, B) \) is the cohomology of
\[
\lim_{\leftarrow} \text{Hom}(A_i, E) = \text{Hom}(\bigcup A_i, E).
\]

**Exercise 3.5.3** Show that \( \text{Ext}_{1}^{1}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}) \cong \mathbb{Z}_p/\mathbb{Z} \) using \( \mathbb{Z}[\frac{1}{p}] = \cup p^{-i} \mathbb{Z} \); cf. exercise 3.3.1. Then show that \( \text{Ext}_{1}^{1}(\mathbb{Q}, B) = (\prod_{p} \mathbb{B}_p)/B \) for torsionfree \( B \).

**Application 3.5.11** Let \( C = C_{**} \) be a double chain complex, viewed as a lattice in the plane, and let \( T_n C \) be the quotient double complex obtained by brutally truncating \( C \) at the vertical line \( p = -n \):
\[
(T_n C)_{pq} = \begin{cases} 
C_{pq} & \text{if } p \geq -n \\
0 & \text{if } p < -n
\end{cases}
\]
Then \( \text{Tot}(C) \) is the inverse limit of the tower of surjections
\[
\cdots \to \text{Tot}(T_{i+1} C) \to \text{Tot}(T_i C) \to \cdots \to \text{Tot}(T_0 C).
\]
Therefore there is a short exact sequence for each \( q \):
\[
0 \to \lim_{\rightarrow} \text{H}_{q+1}(\text{Tot}(T_i C)) \to \text{H}_{q}(\text{Tot}(C)) \to \lim_{\leftarrow} \text{H}_{q}(\text{Tot}(T_i C)) \to 0.
\]
This is especially useful when $C$ is a second quadrant double complex, because the truncated complexes have only a finite number of nonzero rows.

**Exercise 3.5.4** Let $C$ be a second quadrant double complex with exact rows, and let $B_{pq}^h$ be the image of $d^h: C_{pq} \to C_{p-1,q}$. Show that $H_{p+q} \text{Tot}(T_{-p}C) \cong H_q(B_{p*}^h, d^v)$. Then let $b = d^h(a)$ be an element of $B_{pq}^h$ representing a cycle $\xi$ in $H_{p+q} \text{Tot}(T_{-p}C)$ and show that the image of $\xi$ in $H_{p+q} \text{Tot}(T_{-p-1}C)$ is represented by $d^v(a) \in B_{p+1,q-1}^h$. This provides an effective method for calculating $H_\ast \text{Tot}(C)$.

**Vista 3.5.12** Let $I$ be any poset and $\mathcal{A}$ any abelian category satisfying $(AB4^*)$. The following construction of the right derived functors of $\text{lim}$ is taken from [Roos] and generalizes the construction of $\text{lim}^1$ in this section.

Given $A: I \to \mathcal{A}$, we define $C_k$ to be the product over the set of all chains $i_k < \cdots < i_0$ in $I$ of the objects $A_{i_0}$. Letting $\text{pr}_{i_k \cdots i_1}$ denote the projection of $C_k$ onto the $(i_k < \cdots < i_1)^{st}$ factor and $f_0$ denote the map $A_{i_1} \to A_{i_0}$ associated to $i_1 < i_0$, we define $d^0 : C_{k-1} \to C_k$ to be the map whose $(i_k < \cdots < i_0)^{th}$ factor is $f_0(\text{pr}_{i_k \cdots i_1})$. For $1 \leq p \leq k$, we define $d^p : C_{k-1} \to C_k$ to be the map whose $(i_k < \cdots < i_0)^{th}$ factor is the projection onto the $(i_k < \cdots < \hat{i}_p < \cdots < i_0)^{th}$ factor. This data defines a cochain complex $C_\ast A$ whose differential $C_{k-1} \to C_k$ is the alternating sum $\sum_{p=0}^k (-1)^p d^p$, and we define $\text{lim}_i A$ to be $H^n(C_\ast A)$. (The data actually forms a cosimplicial object of $\mathcal{A}$; see Chapter 8.)

It is easy to see that $\lim_{i \in I}^0 A$ is the limit $\lim_{i \in I} A$. An exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}^I$ gives rise to a short exact sequence $0 \to C_\ast A \to C_\ast B \to C_\ast C \to 0$ in $\mathcal{A}$, whence an exact sequence

$$0 \to \lim_{i \in I} A \to \lim_{i \in I} B \to \lim_{i \in I} C \to \lim_{i \in I}^1 A \to \lim_{i \in I}^1 B \to \lim_{i \in I}^1 C \to \lim_{i \in I}^2 A \to \cdots.$$ 

Therefore the functors $\{\lim_{i \in I}^n\}$ form a cohomological $\delta$-functor. It turns out that they are universal when $\mathcal{A}$ has enough injectives, so in fact $R^n \lim_{i \in I} \cong \lim_{i \in I}^n$.

**Remark** Let $\aleph_d$ denote the $d^{th}$ infinite cardinal number, $\aleph_0$ being the cardinality of $\{1, 2, \cdots\}$. If $I$ is a directed poset of cardinality $\aleph_d$, or a filtered category with $\aleph_d$ morphisms, Mitchell proved in [Mitch] that $R^n \lim$ vanishes for $n \geq d + 2$.

**Exercise 3.5.5** (Pullback) Let $\to \leftarrow$ denote the poset $\{x, y, z\}$, $x < y$ and $y < z$, so that $\lim A_i$ is the pullback of $A_x$ and $A_y$ over $A_z$. Show that $\lim_{i \in I}^1 A_i$
3.6 Universal Coefficient Theorems

is the cokernel of the difference map $A_x \times A_y \to A_z$ and that $\lim_{n \to -\infty} = 0$ for $n \neq 0, 1$.

3.6 Universal Coefficient Theorems

There is a very useful formula for using the homology of a chain complex $P$ to compute the homology of the complex $P \otimes M$. Here is the most useful general formulation we can give:

**Theorem 3.6.1** (Künneth formula) *Let $P$ be a chain complex of flat right $R$-modules such that each submodule $d(P_n)$ of $P_{n-1}$ is also flat. Then for every $n$ and every left $R$-module $M$, there is an exact sequence*

$$0 \to H_n(P) \otimes_R M \to H_n(P \otimes_R M) \to \text{Tor}_1^R(H_{n-1}(P), M) \to 0.$$ 

**Proof** The long exact Tor sequence associated to $0 \to \mathbb{Z} \to P \to d(P) \to 0$ shows that each $\mathbb{Z}$ is also flat (exercise 3.2.2). Since $\text{Tor}_1^R(d(P_n), M) = 0$, 

$$0 \to \mathbb{Z} \otimes M \to P \otimes M \to d(P_n) \otimes M \to 0$$

is exact for every $n$. These assemble to give a short exact sequence of chain complexes $0 \to \mathbb{Z} \otimes M \to P \otimes M \to d(P) \otimes M \to 0$. Since the differentials in the $\mathbb{Z}$ and $d(P)$ complexes are zero, the homology sequence is

$$H_{n+1}(dP \otimes M) \overset{\partial}{\longrightarrow} H_n(Z \otimes M) \longrightarrow H_n(P \otimes M) \longrightarrow H_n(dP \otimes M) \overset{\partial}{\longrightarrow} H_{n-1}(Z \otimes M).$$

Using the definition of $\partial$, it is immediate that $\partial = i \otimes M$, where $i$ is the inclusion of $d(P_{n+1})$ in $\mathbb{Z}_n$. On the other hand,

$$0 \to d(P_{n+1}) \overset{i}{\longrightarrow} \mathbb{Z}_n \to H_n(P) \to 0$$

is a flat resolution of $H_n(P)$, so $\text{Tor}_*(H_n(P), M)$ is the homology of

$$0 \to d(P_{n+1}) \otimes M \overset{\partial}{\longrightarrow} \mathbb{Z}_n \otimes M \to 0.$$ 

\[\square\]

**Universal Coefficient Theorem for Homology** 3.6.2 *Let $P$ be a chain complex of free abelian groups. Then for every $n$ and every abelian group $M$ the*
K"unneth formula 3.6.1 splits noncanonically, yielding a direct sum decomposition

\[ H_n(P \otimes M) \cong H_n(P) \otimes M \oplus \text{Tor}_1^Z(H_{n-1}(P), M). \]

**Proof** We shall use the well-known fact that every subgroup of a free abelian group is free abelian [KapIAB, section 15]. Since \(d(P_n)\) is a subgroup of \(P_{n+1}\), it is free abelian. Hence the surjection \(P_n \to d(P_n)\) splits, giving a noncanonical decomposition

\[ P_n \cong Z_n \oplus d(P_n). \]

Applying \(\otimes M\), we see that \(Z_n \otimes M\) is a direct summand of \(P_n \otimes M\); a fortiori, \(Z_n \otimes M\) is a direct summand of the intermediate group \(\ker(d_n \otimes 1: P_n \otimes M \to P_{n-1} \otimes M)\).

Modding out \(Z_n \otimes M\) and \(\ker(d_n \otimes 1)\) by the common image of \(d_{n+1} \otimes 1\), we see that \(H_n(P) \otimes M\) is a direct summand of \(H_n(P \otimes M)\). Since \(P\) and \(d(P)\) are flat, the K"unneth formula tells us that the other summand is \(\text{Tor}_1^Z(H_{n-1}(P), M)\). \(\Box\)

**Theorem 3.6.3** (K"unneth formula for complexes) Let \(P\) and \(Q\) be right and left \(R\)-module complexes, respectively. Recall from 2.7.1 that the tensor product complex \(P \otimes_R Q\) is the complex whose degree \(n\) part is \(\bigoplus_{\rho+q=n} P_{\rho} \otimes Q_q\) and whose differential is given by \(d(a \otimes b) = (da) \otimes b + (-1)^\rho a \otimes (db)\) for \(a \in P_{\rho}, b \in Q_q\). If \(P_n\) and \(d(P_n)\) are flat for each \(n\), then there is an exact sequence

\[ 0 \to \bigoplus_{\rho+q=n} H_\rho(P) \otimes H_q(Q) \to H_n(P \otimes_R Q) \to \bigoplus_{\rho+q=n} \text{Tor}_1^R(H_\rho(P), H_q(Q)) \to 0 \]

for each \(n\). If \(R = \mathbb{Z}\) and \(P\) is a complex of free abelian groups, this sequence is noncanonically split.

**Proof** Modify the proof given in 3.6.1 for \(Q = M\). \(\Box\)

**Application 3.6.4** (Universal Coefficient Theorem in topology) Let \(S(X)\) denote the singular chain complex of a topological space \(X\); each \(S_n(X)\) is a free abelian group. If \(M\) is any abelian group, the homology of \(X\) with “coefficients” in \(M\) is

\[ H_*(X; M) = H_*(S(X) \otimes M). \]
Writing $H_\ast(X)$ for $H_\ast(X; \mathbb{Z})$, the formula in this case becomes

$$H_n(X; M) \cong H_n(X) \otimes M \oplus \text{Tor}_1^\mathbb{Z}(H_{n-1}(X), M).$$

This formula is often called the Universal Coefficient Theorem in topology.

If $Y$ is another topological space, the Eilenberg-Zilber theorem 8.5.1 (see [MacH, VIII.8]) states that $H_\ast(X \times Y)$ is the homology of the tensor product complex $S(X) \otimes S(Y)$. Therefore the Künneth formula yields the “Künneth formula for cohomology:"

$$H^n(X \times Y) \cong \left\{ \bigoplus_{p=0}^n H_p(X) \otimes H_{n-p}(Y) \right\} \otimes \left\{ \bigoplus_{p=1}^n \text{Tor}_1^\mathbb{Z}(H_{p-1}(X), H_{n-p}(Y)) \right\}.$$

We now turn to the analogue of the Künneth formula for Hom in place of $\otimes$.

**Universal Coefficient Theorem for Cohomology 3.6.5** Let $P$ be a chain complex of projective $R$-modules such that each $d(P_n)$ is also projective. Then for every $n$ and every $R$-module $M$, there is a (noncanonically) split exact sequence

$$0 \to \text{Ext}_R^1(H_{n-1}(P), M) \to H^n(\text{Hom}_R(P, M)) \to \text{Hom}_R(H_n(P), M) \to 0.$$

**Proof** Since $d(P_n)$ is projective, there is a (noncanonical) isomorphism $P_n \cong \mathbb{Z}_n \oplus d(P_n)$ for each $n$. Therefore each sequence

$$0 \to \text{Hom}(d(P_n), M) \to \text{Hom}(P_n, M) \to \text{Hom}(\mathbb{Z}_n, M) \to 0$$

is exact. We may now copy the proof of the Künneth formula 3.6.1 for $\otimes$, using $\text{Hom}(-, M)$ instead of $\otimes M$, to see that the sequence is indeed exact. We may copy the proof of the Universal Coefficient Theorem 3.6.2 for $\otimes$ in the same way to see that the sequence is split. \hfill \Box

**Application 3.6.6** (Universal Coefficient theorem in topology) The cohomology of a topological space $X$ with “coefficients” in $M$ is defined to be

$$H^\ast(X; M) = H^\ast(\text{Hom}(S(X), M)).$$

In this case, the Universal Coefficient theorem becomes

$$H^n(X; M) \cong \text{Hom}(H_n(X), M) \oplus \text{Ext}_\mathbb{Z}^1(H_{n-1}(X), M).$$
**Example 3.6.7** If $X$ is path-connected, then $H_0(X) = \mathbb{Z}$ and $H^1(X; \mathbb{Z}) \cong \text{Hom}(H_1(X), \mathbb{Z})$, which is a torsionfree abelian group.

**Exercise 3.6.1** Let $P$ be a chain complex and $Q$ a cochain complex of $R$-modules. As in 2.7.4, form the Hom double cochain complex $\text{Hom}(P, Q) = \{\text{Hom}_R(P_p, Q^q)\}$, and then write $H^* \text{Hom}(P, Q)$ for the cohomology of $\text{Tot}(\text{Hom}(P, Q))$. Show that if each $P_n$ and $d(P_n)$ is projective, there is an exact sequence

$$0 \to \prod_{p+q=n-1} \text{Ext}_R^1(H_p(P), H^q(Q)) \to H^n \text{Hom}(P, Q) \to \prod_{p+q=n} \text{Hom}_R(H_p(P), H^q(Q)) \to 0.$$

**Exercise 3.6.2** A ring $R$ is called *right hereditary* if every submodule of every (right) free module is a projective module. (See 4.2.10 and exercise 4.2.6 below.) Any principal ideal domain (for example, $R = \mathbb{Z}$) is hereditary, as is any commutative Dedekind domain. Show that the universal coefficient theorems of this section remain valid if $\mathbb{Z}$ is replaced by an arbitrary right hereditary ring $R$. 