3

Tor and Ext

3.1 Tor for Abelian Groups

The first question many people ask about $\mathrm{Tor}_*(A, B)$ is "Why the name 'Tor'?" The results of this section should answer that question. Historically, the first Tor groups to arise were the groups $\mathrm{Tor}_1(\mathbb{Z}/p, B)$ associated to abelian groups. The following simple calculation describes these groups.

Calculation 3.1.1 $\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/p, B) = B/pB$, $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p, B) = {}_pB = \{b \in B : pB = 0\}$ and $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/p, B) = 0$ for $n \geq 2$. To see this, use the resolution

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to 0$$

to see that $\mathrm{Tor}_*(\mathbb{Z}/p, B)$ is the homology of the complex $0 \to B \xrightarrow{p} B \to 0$.

Proposition 3.1.2 *For all abelian groups A and B:*

(a) $\mathrm{Tor}_1^{\mathbb{Z}}(A, B)$ *is a torsion abelian group.*
(b) $\mathrm{Tor}_n^{\mathbb{Z}}(A, B) = 0$ *for* $n \geq 2$.

Proof A is the direct limit of its finitely generated subgroups A_α, so by 2.6.17 $\mathrm{Tor}_n(A, B)$ is the direct limit of the $\mathrm{Tor}_n(A_\alpha, B)$. As the direct limit of torsion groups is a torsion group, we may assume that A is finitely generated, that is, $A \cong \mathbb{Z}^m \oplus \mathbb{Z}/p_1 \oplus \mathbb{Z}/p_2 \oplus \cdots \oplus \mathbb{Z}/p_r$ for appropriate integers $m, p_1, \ldots, p_r$. As $\mathbb{Z}^m$ is projective, $\mathrm{Tor}_n(\mathbb{Z}^m, -)$ vanishes for $n \neq 0$, and so we have

$$\mathrm{Tor}_n(A, B) \cong \mathrm{Tor}_n(\mathbb{Z}/p_1, B) \oplus \cdots \oplus \mathrm{Tor}_n(\mathbb{Z}/p_r, B).$$

The proposition holds in this case by calculation 3.1.1 above. $\diamond$

Proposition 3.1.3 $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B)$ *is the torsion subgroup of B for every abelian group B.*

Proof As $\mathbb{Q}/\mathbb{Z}$ is the direct limit of its finite subgroups, each of which is isomorphic to $\mathbb{Z}/p$ for some integer p, and Tor commutes with direct limits,

$$\mathrm{Tor}_*^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B) \cong \varinjlim \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p, B) \cong \varinjlim({}_pB) = \cup_p\{b \in B : pb = 0\},$$

which is the torsion subgroup of B. ◇

Proposition 3.1.4 *If A is a torsionfree abelian group, then* $\mathrm{Tor}_n^{\mathbb{Z}}(A, B) = 0$ *for $n \neq 0$ and all abelian groups B.*

Proof A is the direct limit of its finitely generated subgroups, each of which is isomorphic to $\mathbb{Z}^m$ for some m. Therefore, $\mathrm{Tor}_n(A, B) \cong \varinjlim \mathrm{Tor}_n(\mathbb{Z}^m, B) = 0$. ◇

Remark (Balancing Tor) If R is any commutative ring, then $\mathrm{Tor}_*^R(A, B) \cong \mathrm{Tor}_*^R(B, A)$. In particular, this is true for $R = \mathbb{Z}$, that is, for abelian groups. This is because for fixed B, both are universal δ-functors over $F(A) = A \otimes B \cong B \otimes A$. Therefore $\mathrm{Tor}_1^{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ is the torsion subgroup of A. From this we obtain the following.

Corollary 3.1.5 *For every abelian group A,*

$$\mathrm{Tor}_1^{\mathbb{Z}}(A, -) = 0 \Leftrightarrow A \textit{ is torsionfree} \Leftrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(-, A) = 0.$$

Calculation 3.1.6 All this fails if we replace $\mathbb{Z}$ by another ring. For example, if we take $R = \mathbb{Z}/m$ and $A = \mathbb{Z}/d$ with $d|m$, then we can use the periodic free resolution

$$\cdots \xrightarrow{d} \mathbb{Z}/m \xrightarrow{m/d} \mathbb{Z}/m \xrightarrow{d} \mathbb{Z}/m \xrightarrow{\epsilon} \mathbb{Z}/d \to 0$$

to see that for all $\mathbb{Z}/m$-modules B we have

$$\mathrm{Tor}_n^{\mathbb{Z}/m}(\mathbb{Z}/d, B) = \begin{cases} B/dB & \text{if } n = 0 \\ \{b \in B : db = 0\}/(m/d)B & \text{if } n \text{ is odd, } n > 0 \\ \{b \in B : (m/d)b = 0\}/dB & \text{if } n \text{ is even, } n > 0. \end{cases}$$

Example 3.1.7 Suppose that $r \in R$ is a left nonzerodivisor on R, that is, ${}_rR = \{s \in R : rs = 0\}$ is zero. For every R-module B, set ${}_rB = \{b \in B : rb = 0\}$. We can repeat the above calculation with R/rR in place of $\mathbb{Z}/p$ to see that $\mathrm{Tor}_0(R/rR, B) = B/rB$, $\mathrm{Tor}_1^R(R/rR, B) = {}_rB$ and $\mathrm{Tor}_n^R(R/rR, B) = 0$ for all B when $n \geq 2$.

Exercise 3.1.1 If ${}_rR \neq 0$, all we have is the non-projective resolution

$$0 \to {}_rR \to R \xrightarrow{r} R \to R/rR \to 0.$$

Show that there is a short exact sequence

$$0 \longrightarrow \mathrm{Tor}_2^R(R/rR, B) \longrightarrow {}_rR \otimes_R B \xrightarrow{\text{multiply}} {}_rB \longrightarrow \mathrm{Tor}_1^R(R/rR, B) \longrightarrow 0$$

and that $\mathrm{Tor}_n^R(R/rR, B) \cong \mathrm{Tor}_{n-2}^R({}_rR, B)$ for $n \geq 3$.

Exercise 3.1.2 Suppose that R is a commutative domain with field of fractions F. Show that $\mathrm{Tor}_1^R(F/R, B)$ is the torsion submodule $\{b \in B : (\exists r \neq 0)\, rb = 0\}$ of B for every R-module B.

Exercise 3.1.3 Show that $\mathrm{Tor}_1^R(R/I, R/J) \cong \frac{I \cap J}{IJ}$ for every right ideal I and left ideal J of R. In particular, $\mathrm{Tor}_1(R/I, R/I) \cong I/I^2$ for every 2-sided ideal I. *Hint:* Apply the Snake Lemma to

$$\begin{array}{ccccccccc}
0 & \longrightarrow & IJ & \longrightarrow & I & \longrightarrow & I \otimes R/J & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J & \longrightarrow & R & \longrightarrow & R \otimes R/J & \longrightarrow & 0.
\end{array}$$

3.2 Tor and Flatness

In the last chapter, we saw that if A is a right R-module and B is a left R-module, then $\mathrm{Tor}_*^R(A, B)$ may be computed either as the left derived functors of $A \otimes_R$ evaluated at B or as the left derived functors of $\otimes_R B$ evaluated at A. It follows that if either A or B is projective, then $\mathrm{Tor}_n(A, B) = 0$ for $n \neq 0$.

Definition 3.2.1 A left R-module B is *flat* if the functor $\otimes_R B$ is exact. Similarly, a right R-module A is *flat* if the functor $A \otimes_R$ is exact. The above remarks show that projective modules are flat. The example $R = \mathbb{Z}$, $B = \mathbb{Q}$ shows that flat modules need not be projective.

Theorem 3.2.2 *If S is a central multiplicatively closed set in a ring R, then $S^{-1}R$ is a flat R-module.*

Proof Form the filtered category I whose objects are the elements of S and whose morphisms are $\mathrm{Hom}_I(s_1, s_2) = \{s \in S : s_1 s = s_2\}$. Then $\underrightarrow{\mathrm{colim}}\, F(s) \cong S^{-1}R$ for the functor $F \colon I \to R$–**mod** defined by $F(s) = R$, $F(s_1 \xrightarrow{s} s_2)$ being multiplication by s. (*Exercise:* Show that the maps $F(s) \to S^{-1}R$ sending 1 to $1/s$ induce an isomorphism $\underrightarrow{\mathrm{colim}}\, F(s) \cong S^{-1}R$.) Since $S^{-1}R$ is the filtered colimit of the free R-modules $F(s)$, it is flat by 2.6.17. $\diamond$

Exercise 3.2.1 Show that the following are equivalent for every left R-module B.

1. B is flat.
2. $\mathrm{Tor}_n^R(A, B) = 0$ for all $n \neq 0$ and all A.
3. $\mathrm{Tor}_1^R(A, B) = 0$ for all A.

Exercise 3.2.2 Show that if $0 \to A \to B \to C \to 0$ is exact and both B and C are flat, then A is also flat.

Exercise 3.2.3 We saw in the last section that if $R = \mathbb{Z}$ (or more generally, if R is a principal ideal domain), a module B is flat iff B is torsionfree. Here is an example of a torsionfree ideal I that is not a flat R-module. Let k be a field and set $R = k[x, y]$, $I = (x, y)R$. Show that $k = R/I$ has the projective resolution

$$0 \to R \xrightarrow{\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]} R^2 \xrightarrow{(x\ y)} R \to k \to 0.$$

Then compute that $\mathrm{Tor}_1^R(I, k) \cong \mathrm{Tor}_2^R(k, k) \cong k$, showing that I is not flat.

Definition 3.2.3 The *Pontrjagin dual* B^* of a left R-module B is the right R-module $\mathrm{Hom}_{\mathbf{Ab}}(B, \mathbb{Q}/\mathbb{Z})$; an element r of R acts via $(fr)(b) = f(rb)$.

Proposition 3.2.4 *The following are equivalent for every left R-module B :*

1. *B is a flat R-module.*
2. *B^* is an injective right R-module.*
3. *$I \otimes_R B \cong IB = \{x_1 b_1 + \cdots + x_n b_n \in B : x_i \in I, b_i \in B\} \subset B$ for every right ideal I of R.*
4. *$\mathrm{Tor}_1^R(R/I, B) = 0$ for every right ideal I of R.*

Proof The equivalence of (3) and (4) follows from the exact sequence

$$0 \to \mathrm{Tor}_1(R/I, B) \to I \otimes B \to B \to B/IB \to 0.$$

Now for every inclusion $A' \subset A$ of right modules, the adjoint functors $\otimes B$ and $\mathrm{Hom}(-, B)$ give a commutative diagram

$$\begin{array}{ccccc} & \mathrm{Hom}(A, B^*) & \longrightarrow & \mathrm{Hom}(A', B^*) & \\ & \downarrow\cong & & \cong\downarrow & \\ (A \otimes B)^* = & \mathrm{Hom}(A \otimes B, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(A' \otimes B, \mathbb{Q}/\mathbb{Z}) & = (A' \otimes B)^*. \end{array}$$

Using the lemma below and Baer's criterion 2.3.1, we see that

$$\begin{aligned} B^* \text{ is injective} &\Leftrightarrow (A \otimes B)^* \to (A' \otimes B)^* \text{ is surjective for all } A' \subset A. \\ &\Leftrightarrow A' \otimes B \to A \otimes B \text{ is injective for all } A' \subset A \Leftrightarrow B \text{ is flat.} \\ B^* \text{ is injective} &\Leftrightarrow (R \otimes B)^* \to (I \otimes B)^* \text{ is surjective for all } I \subset R \\ &\Leftrightarrow I \otimes B \to R \otimes B \text{ is injective for all } I \\ &\Leftrightarrow I \otimes B \cong IB \text{ for all } I. \end{aligned}$$

◇

Lemma 3.2.5 *A map $f: B \to C$ is injective iff the dual map $f^*: C^* \to B^*$ is surjective.*

Proof If A is the kernel of f, then A^* is the cokernel of f^*, because $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ is contravariant exact. But we saw in exercise 2.3.3 that $A = 0$ iff $A^* = 0$. ◇

Exercise 3.2.4 Show that a sequence $A \to B \to C$ is exact iff its dual $C^* \to B^* \to A^*$ is exact.

An R-module M is called *finitely presented* if it can be presented using finitely many generators $(e_1, \dots, e_n)$ and relations $(\sum \alpha_{ij} e_j = 0,\ j = 1, \dots, m)$. That is, there is an $m \times n$ matrix α and an exact sequence $R^m \xrightarrow{\alpha} R^n \to M \to 0$. If M is finitely generated, the following exercise shows that the property of being finitely presented is independent of the choice of generators.

Exercise 3.2.5 Suppose that $\varphi: F \to M$ is *any* surjection, where F is finitely generated and M is finitely presented. Use the Snake Lemma to show that $\ker(\varphi)$ is finitely generated.

Still letting A^* denote the Pontrjagin dual 3.2.3 of A, there is a natural map $\sigma: A^* \otimes_R M \to \mathrm{Hom}_R(M, A)^*$ defined by $\sigma(f \otimes m): h \mapsto f(h(m))$ for $f \in A^*$, $m \in M$ and $h \in \mathrm{Hom}(M, A)$. (*Exercise:* If $M = \oplus_{i=1}^{\infty} R$, show that σ is not an isomorphism.)

Lemma 3.2.6 *The map σ is an isomorphism for every finitely presented M and all A.*

Proof A simple calculation shows that σ is an isomorphism if $M = R$. By additivity, σ is an isomorphism if $M = R^m$ or R^n. Now consider the diagram

$$\begin{array}{ccccccc} A^* \otimes R^m & \longrightarrow & A^* \otimes R^n & \longrightarrow & A^* \otimes M & \longrightarrow & 0 \\ \sigma \downarrow \cong & & \sigma \downarrow \cong & & \sigma \downarrow & & \\ \mathrm{Hom}(R^m, A)^* & \xrightarrow{\alpha^*} & \mathrm{Hom}(R^n, A)^* & \longrightarrow & \mathrm{Hom}(M, A)^* & \longrightarrow & 0. \end{array}$$

The rows are exact because $\otimes$ is right exact, Hom is left exact, and Pontrjagin dual is exact by 2.3.3. The 5-lemma shows that σ is an isomorphism. $\diamond$

Theorem 3.2.7 *Every finitely presented flat R-module M is projective.*

Proof In order to show that M is projective, we shall show that $\mathrm{Hom}_R(M, -)$ is exact. To this end, suppose that we are given a surjection $B \to C$. Then $C^* \to B^*$ is an injection, so if M is flat, the top arrow of the square

$$\begin{array}{ccc} (C^*) \otimes_R M & \longrightarrow & (B^*) \otimes_R M \\ \cong \downarrow & & \cong \downarrow \\ \mathrm{Hom}(M, C)^* & \longrightarrow & \mathrm{Hom}(M, B)^* \end{array}$$

is an injection. Hence the bottom arrow is an injection. As we have seen, this implies that $\mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C)$ is a surjection, as required. $\diamond$

Flat Resolution Lemma 3.2.8 *The groups* $\mathrm{Tor}_*(A, B)$ *may be computed using resolutions by flat modules. That is, if $F \to A$ is a resolution of A with the F_n being flat modules, then* $\mathrm{Tor}_*(A, B) \cong H_*(F \otimes B)$. *Similarly, if $F' \to B$ is a resolution of B by flat modules, then* $\mathrm{Tor}_*(A, B) \cong H_*(A \otimes F')$.

Proof We use induction and dimension shifting (exercise 2.4.3) to prove that $\mathrm{Tor}_n(A, B) \cong H_n(F \otimes B)$ for all n; the second part follows by arguing over R^{op}. The assertion is true for $n = 0$ because $\otimes B$ is right exact. Let K be such that $0 \to K \to F_0 \to A \to 0$ is exact; if $E = (\cdots \to F_2 \to F_1 \to 0)$, then $E \to K$ is a resolution of K by flat modules. For $n = 1$ we simply compute

$$\begin{aligned} \mathrm{Tor}_1(A, B) &= \ker(K \otimes B \to F_0 \otimes B) \\ &= \ker\left\{\frac{F_1 \otimes B}{\mathrm{im}(F_2 \otimes B)} \to F_0 \otimes B\right\} = H_1(F \otimes B). \end{aligned}$$

For $n \geq 2$ we use induction to see that

$$\mathrm{Tor}_n(A, B) \cong \mathrm{Tor}_{n-1}(K, B) \cong H_{n-1}(E \otimes B) = H_n(F \otimes B). \qquad \diamond$$

Proposition 3.2.9 (Flat base change for Tor) *Suppose $R \to T$ is a ring map such that T is flat as an R-module. Then for all R-modules A, all T-modules C and all n*

$$\mathrm{Tor}_n^R(A, C) \cong \mathrm{Tor}_n^T(A \otimes_R T, C).$$

Proof Choose an R-module projective resolution $P \to A$. Then $\mathrm{Tor}_*^R(A, C)$ is the homology of $P \otimes_R C$. Since T is R-flat, and each $P_n \otimes_R T$ is a projective T-module, $P \otimes T \to A \otimes T$ is a T-module projective *resolution*. Thus $\mathrm{Tor}_*^T(A \otimes_R T, C)$ is the homology of the complex $(P \otimes_R T) \otimes_T C \cong P \otimes_R C$ as well. $\diamond$

Corollary 3.2.10 *If R is commutative and T is a flat R-algebra, then for all R-modules A and B, and for all n*

$$T \otimes_R \mathrm{Tor}_n^R(A, B) \cong \mathrm{Tor}_n^T(A \otimes_R T, T \otimes_R B).$$

Proof Setting $C = T \otimes_R B$, it is enough to show that $\mathrm{Tor}_*^R(A, T \otimes B) = T \otimes \mathrm{Tor}_*^R(A, B)$. As $T\otimes_R$ is an exact functor, $T \otimes \mathrm{Tor}_*^R(A, B)$ is the homology of $T \otimes_R (P \otimes_R B) \cong P \otimes_R (T \otimes_R B)$, the complex whose homology is $\mathrm{Tor}_*^R(A, T \otimes B)$. $\diamond$

Now we shall suppose that R is a commutative ring, so that the $\mathrm{Tor}_*^R(A, B)$ are actually R-modules in order to show how Tor_* localizes.

Lemma 3.2.11 *If $\mu: A \to A$ is multiplication by a central element $r \in R$, so are the induced maps $\mu_*: \mathrm{Tor}_n^R(A, B) \to \mathrm{Tor}_n^R(A, B)$ for all n and B.*

Proof Pick a projective resolution $P \to A$. Multiplication by r is an R-module chain map $\tilde{\mu}: P \to P$ over μ (this uses the fact that r is central), and $\tilde{\mu} \otimes B$ is multiplication by r on $P \otimes B$. The induced map μ_* on the subquotient $\mathrm{Tor}_n(A, B)$ of $P_n \otimes B$ is therefore also multiplication by r. $\diamond$

Corollary 3.2.12 *If A is an R/r-module, then for every R-module B the R-modules $\mathrm{Tor}_*^R(A, B)$ are actually R/r-modules, that is, annihilated by the ideal rR.*

Corollary 3.2.13 (Localization for Tor) *If R is commutative and A and B are R-modules, then the following are equivalent for each n:*

1. $\mathrm{Tor}_n^R(A, B) = 0$.
2. *For every prime ideal p of R* $\mathrm{Tor}_n^{R_p}(A_p, B_p) = 0$.
3. *For every maximal ideal m of R* $\mathrm{Tor}_n^{R_m}(A_m, B_m) = 0$.

Proof For any R-module M, $M = 0 \Leftrightarrow M_p = 0$ for every prime $p \Leftrightarrow M_m = 0$ for every maximal ideal m. In the case $M = \mathrm{Tor}_m^R(A, B)$ we have

$$M_p = R_p \otimes_R M = \mathrm{Tor}_n^{R_p}(A_p, B_p). \qquad \diamond$$

3.3 Ext for Nice Rings

We first turn to a calculation of $\mathrm{Ext}_{\mathbb{Z}}^*$ groups to get a calculational feel for what these derived functors do to abelian groups.

Lemma 3.3.1 $\mathrm{Ext}_{\mathbb{Z}}^n(A, B) = 0$ *for $n \geq 2$ and all abelian groups A, B.*

Proof Embed B in an injective abelian group I^0; the quotient I^1 is divisible, hence injective. Therefore, $\mathrm{Ext}^*(A, B)$ is the cohomology of

$$0 \to \mathrm{Hom}(A, I^0) \to \mathrm{Hom}(A, I^1) \to 0. \qquad \diamond$$

Calculation 3.3.2 $(A = \mathbb{Z}/p)$ $\mathrm{Ext}_{\mathbb{Z}}^0(\mathbb{Z}/p, B) = {}_pB$, $\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/p, B) = B/pB$ and $\mathrm{Ext}_{\mathbb{Z}}^n(\mathbb{Z}/p, B) = 0$ for $n \geq 2$. To see this, use the resolution

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to 0 \text{ and the fact that } \mathrm{Hom}(\mathbb{Z}, B) \cong B$$

to see that $\mathrm{Ext}^*(\mathbb{Z}/p, B)$ is the cohomology of $0 \leftarrow B \xleftarrow{p} B \leftarrow 0$.

Since $\mathbb{Z}$ is projective, $\mathrm{Ext}^1(\mathbb{Z}, B) = 0$. Hence we can calculate $\mathrm{Ext}^*(A, B)$ for every finitely generated abelian group $A \cong \mathbb{Z}^m \oplus \mathbb{Z}/p_1 \oplus \cdots \oplus \mathbb{Z}/p_n$ by taking a finite direct sum of $\mathrm{Ext}^*(\mathbb{Z}/p, B)$ groups. For infinitely generated groups, the calculation is much more complicated than it was for Tor.

Example 3.3.3 $(B = \mathbb{Z})$ Let A be a torsion group, and write A^* for its Pontrjagin dual $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ as in 3.2.3. Using the injective resolution $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ to compute $\mathrm{Ext}^*(A, \mathbb{Z})$, we see that $\mathrm{Ext}_{\mathbb{Z}}^0(A, \mathbb{Z}) = 0$ and

$\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) = A^*$. To get a feel for this, note that because $\mathbb{Z}_{p^\infty}$ is the union (colimit) of its subgroups $\mathbb{Z}/p^n$, the group

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, \mathbb{Z}) = (\mathbb{Z}_{p^\infty})^*$$

is the torsionfree group of p-adic integers, $\hat{\mathbb{Z}}_p = \varprojlim(\mathbb{Z}/p^n)$. We will calculate $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, B)$ more generally in section 3.5, using $\varprojlim^1$.

Exercise 3.3.1 Show that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}) \cong \hat{\mathbb{Z}}_p/\mathbb{Z} \cong \mathbb{Z}_{p^\infty}$. This shows that $\mathrm{Ext}^1(-, \mathbb{Z})$ does not vanish on flat abelian groups.

Exercise 3.3.2 When $R = \mathbb{Z}/m$ and $B = \mathbb{Z}/p$ with $p|m$, show that

$$0 \to \mathbb{Z}/p \overset{\iota}{\hookrightarrow} \mathbb{Z}/m \xrightarrow{p} \mathbb{Z}/m \xrightarrow{m/p} \mathbb{Z}/m \xrightarrow{p} \mathbb{Z}/m \xrightarrow{m/p} \cdots$$

is an infinite periodic injective resolution of B. Then compute the groups $\mathrm{Ext}^n_{\mathbb{Z}/m}(A, \mathbb{Z}/p)$ in terms of $A^* = \mathrm{Hom}(A, \mathbb{Z}/m)$. In particular, show that if $p^2|m$, then $\mathrm{Ext}^n_{\mathbb{Z}/m}(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p$ for all n.

Proposition 3.3.4 *For all n and all rings R*

1. $\mathrm{Ext}^n_R(\oplus_\alpha A_\alpha, B) \cong \prod_\alpha \mathrm{Ext}^n_R(A_\alpha, B)$.
2. $\mathrm{Ext}^n_R(A, \prod_\beta B_\beta) \cong \prod_\beta \mathrm{Ext}^n_R(A, B_\beta)$.

Proof If $P_\alpha \to A_\alpha$ are projective resolutions, so is $\oplus P_\alpha \to \oplus A_\alpha$. If $B_\beta \to I_\beta$ are injective resolutions, so is $\prod B_\beta \to \prod I_\beta$. Since $\mathrm{Hom}(\oplus P_\alpha, B) = \prod \mathrm{Hom}(P_\alpha, B)$ and $\mathrm{Hom}(A, \prod I_\beta) = \prod \mathrm{Hom}(A, I_\beta)$, the result follows from the fact that for any family C_γ of cochain complexes,

$$H^*(\prod C_\gamma) \cong \prod H^*(C_\gamma). \qquad \diamond$$

Examples 3.3.5

1. If $p^2|m$ and A is a $\mathbb{Z}/p$-vector space of countably infinite dimension, then $\mathrm{Ext}^n_{\mathbb{Z}/m}(A, \mathbb{Z}/p) \cong \prod_{i=1}^\infty \mathbb{Z}/p$ is a $\mathbb{Z}/p$-vector space of dimension $2^{\aleph_0}$.
2. If B is the product $\mathbb{Z}/2 \times \mathbb{Z}/3 \times \mathbb{Z}/4 \times \mathbb{Z}/5 \times \cdots$ then B is *not* a torsion group, and

$$\mathrm{Ext}^1(A, B) = \prod_{p=2}^{\infty} A/pA = 0$$

vanishes if and only if A is divisible.

Lemma 3.3.6 *Suppose that R is a commutative ring, so that* $\mathrm{Hom}_R(A, B)$ *and the* $\mathrm{Ext}^*_R(A, B)$ *are actually R-modules. If $\mu: A \to A$ and $\nu: B \to B$ are multiplication by $r \in R$, so are the induced endomorphisms μ^* and ν_* of* $\mathrm{Ext}^n_R(A, B)$ *for all n.*

Proof Pick a projective resolution $P \to A$. Multiplication by r is an R-module chain map $\tilde{\mu}: P \to P$ over μ (as r is central); the map $\mathrm{Hom}(\tilde{\mu}, B)$ on $\mathrm{Hom}(P, B)$ is multiplication by r, because it sends $f \in \mathrm{Hom}(P_n, B)$ to $f\tilde{\mu}$, which takes $p \in P_n$ to $f(rp) = rf(p)$. Hence the map μ^* on the subquotient $\mathrm{Ext}^n(A, B)$ of $\mathrm{Hom}(P_n, B)$ is also multiplication by r. The argument for ν_* is similar, using an injective resolution $B \to I$. ◇

Corollary 3.3.7 *If R is commutative and A is actually an R/r-module, then for every R-module B the R-modules* $\mathrm{Ext}^*_R(A, B)$ *are actually R/r-modules.*

We would like to conclude, as we did for Tor, that Ext commutes with localization in some sense. Indeed, there is a natural map Φ from $S^{-1}\mathrm{Hom}_R(A, B)$ to $\mathrm{Hom}_{S^{-1}R}(S^{-1}A, S^{-1}B)$, but it need not be an isomorphism. A sufficient condition is that A be finitely presented, that is, some $R^m \xrightarrow{\alpha} R^n \to A \to 0$ is exact.

Lemma 3.3.8 *If A is a finitely presented R-module, then for every central multiplicative set S in R, Φ is an isomorphism:*

$$\Phi : S^{-1}\mathrm{Hom}_R(A, B) \cong \mathrm{Hom}_{S^{-1}R}(S^{-1}A, S^{-1}B).$$

Proof Φ is trivially an isomorphism when $A = R$; as Hom is additive, Φ is also an isomorphism when $A = R^m$. The result now follows from the 5-lemma and the following diagram:

$$\begin{array}{ccccccc}
0 \to & S^{-1}\mathrm{Hom}_R(A, B) & \to & S^{-1}\mathrm{Hom}_R(R^n, B) & \xrightarrow{\alpha} & S^{-1}\mathrm{Hom}_R(R^m, B) \\
& \Phi\downarrow & & \cong\downarrow & & \cong\downarrow \\
0 \to & \mathrm{Hom}(S^{-1}A, S^{-1}B) & \to & \mathrm{Hom}(S^{-1}R^n, S^{-1}B) & \xrightarrow{\alpha} & \mathrm{Hom}(S^{-1}R^m, S^{-1}B).
\end{array}$$ ◇

Definition 3.3.9 A ring R is *(right) noetherian* if every (right) ideal is finitely generated, that is, if every module R/I is finitely presented. It is well known that if R is noetherian, then every finitely generated (right) R-module is finitely presented. (See [BAII,§3.2].) It follows that every finitely generated module A has a resolution $F \to A$ in which each F_n is a finitely generated free R-module.

Proposition 3.3.10 *Let A be a finitely generated module over a commutative noetherian ring R. Then for every multiplicative set S, all modules B, and all n*

$$\Phi\colon S^{-1}\operatorname{Ext}^n_R(A,B)\cong\operatorname{Ext}^n_{S^{-1}R}(S^{-1}A,S^{-1}B).$$

Proof Choose a resolution $F\to A$ by finitely generated free R-modules. Then $S^{-1}F\to S^{-1}A$ is a resolution by finitely generated free $S^{-1}R$-modules. Because S^{-1} is an exact functor from R-modules to $S^{-1}R$-modules,

$$\begin{aligned} S^{-1}\operatorname{Ext}^*_R(A,B) &= S^{-1}(H^*\operatorname{Hom}_R(F,B))\cong H^*(S^{-1}\operatorname{Hom}_R(F,B))\\ &\cong H^*\operatorname{Hom}_{S^{-1}R}(S^{-1}F,S^{-1}B)=\operatorname{Ext}^*_{S^{-1}R}(S^{-1}A,S^{-1}B).\diamond\end{aligned}$$

Corollary 3.3.11 (Localization for Ext) *If R is commutative noetherian and A is a finitely generated R-module, then the following are equivalent for all modules B and all n:*

1. $\operatorname{Ext}^n_R(A,B)=0$.
2. *For every prime ideal p of R,* $\operatorname{Ext}^n_{R_p}(A_p,B_p)=0$.
3. *For every maximal ideal m of R,* $\operatorname{Ext}^n_{R_m}(A_m,B_m)=0$.

3.4 Ext and Extensions

An *extension* ξ of A by B is an exact sequence $0\to B\to X\to A\to 0$. Two extensions ξ and ξ' are *equivalent* if there is a commutative diagram

$$\begin{array}{rccccccccc} \xi\colon & 0 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0\\ & & & \| & & \downarrow\cong & & \| & & \\ \xi'\colon & 0 & \longrightarrow & B & \longrightarrow & X' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

An extension is *split* if it is equivalent to $0\to B\xrightarrow{(0,1)}A\oplus B\to A\to 0$.

Exercise 3.4.1 Show that if p is prime, there are exactly p equivalence classes of extensions of $\mathbb{Z}/p$ by $\mathbb{Z}/p$ in **Ab**: the split extension and the extensions

$$0\to\mathbb{Z}/p\xrightarrow{p}\mathbb{Z}/p^2\xrightarrow{i}\mathbb{Z}/p\to 0\qquad(i=1,2,\cdots,p-1).$$

Lemma 3.4.1 *If* $\mathrm{Ext}^1(A, B) = 0$*, then every extension of* A *by* B *is split.*

Proof Given an extension ξ, applying $\mathrm{Ext}^*(A, -)$ yields the exact sequence

$$\mathrm{Hom}(A, X) \to \mathrm{Hom}(A, A) \xrightarrow{\partial} \mathrm{Ext}^1(A, B)$$

so the identity map id_A lifts to a map $\sigma\colon A \to X$ when $\mathrm{Ext}^1(A, B) = 0$. As σ is a section of $X \to A$, evidently $X \cong A \oplus B$ and ξ is split. ◇

Porism 3.4.2 Taking the construction of this lemma to heart, we see that the class $\Theta(\xi) = \partial(\mathrm{id}_A)$ in $\mathrm{Ext}^1(A, B)$ is an *obstruction* to ξ being split: ξ is split iff id_A lifts to $\mathrm{Hom}(A, X)$ iff the class $\Theta(\xi) \in \mathrm{Ext}^1(A, B)$ vanishes. Equivalent extensions have the same obstruction by naturality of the map ∂, so the obstruction $\Theta(\xi)$ only depends on the equivalence class of ξ.

Theorem 3.4.3 *Given two* R*-modules* A *and* B*, the mapping* $\Theta\colon \xi \mapsto \partial(\mathrm{id}_A)$ *establishes a 1–1 correspondence*

$$\left\{ \begin{matrix} \text{equivalence classes of} \\ \text{extensions of } A \text{ by } B \end{matrix} \right\} \overset{1-1}{\longleftrightarrow} \mathrm{Ext}^1(A, B)$$

in which the split extension corresponds to the element $0 \in \mathrm{Ext}^1(A, B)$.

Proof Fix an exact sequence $0 \to M \xrightarrow{j} P \to A \to 0$ with P projective. Applying $\mathrm{Hom}(-, B)$ yields an exact sequence

$$\mathrm{Hom}(P, B) \to \mathrm{Hom}(M, B) \xrightarrow{\partial} \mathrm{Ext}^1(A, B) \to 0.$$

Given $x \in \mathrm{Ext}^1(A, B)$, choose $\beta \in \mathrm{Hom}(M, B)$ with $\partial(\beta) = x$. Let X be the pushout of j and β, i.e., the cokernel of $M \to P \oplus B$ $(m \mapsto (j(m), -\beta(m)))$. There is a diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & M \arrow[r, "j"] \arrow[d, "\beta"'] & P \arrow[r] \arrow[d, "\sigma"] \arrow[dl, phantom, "\lrcorner"] & A \arrow[r] \arrow[d, equal] & 0 \\
\xi\colon & 0 \arrow[r] & B \arrow[r, "i"] & X \arrow[r] & A \arrow[r] & 0,
\end{tikzcd}$$

where the map $X \to A$ is induced by the maps $B \xrightarrow{0} A$ and $P \to A$. (*Exercise:* Show that the bottom sequence ξ is exact.) By naturality of the connecting map ∂, we see that $\Theta(\xi) = x$, that is, that Θ is a surjection.

In fact, this construction gives a set map Ψ from $\mathrm{Ext}^1(A, B)$ to the set of equivalence classes of extensions. For if $\beta' \in \mathrm{Hom}(M, B)$ is another lift of x, then there is an $f \in \mathrm{Hom}(P, B)$ so that $\beta' = \beta + fj$. If X' is the pushout of j and β', then the maps $i: B \to X$ and $\sigma + if: P \to X$ induce an isomorphism $X' \cong X$ and an equivalence between ξ' and ξ. (Check this!)

Conversely, given an extension ξ of A by B, the lifting property of P gives a map $\tau: P \to X$ and hence a commutative diagram

$$(*) \qquad \begin{array}{ccccccccc} & 0 & \longrightarrow & M & \xrightarrow{j} & P & \longrightarrow & A & \longrightarrow & 0 \\ & & & \downarrow\gamma & & \downarrow\tau & & \| & & \\ \xi: & 0 & \longrightarrow & B & \xrightarrow{i} & X & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

Now X is the pushout of j and γ. (*Exercise:* Check this!) Hence $\Psi(\Theta(\xi)) = \xi$, showing that Θ is injective. $\diamond$

Definition 3.4.4 (Baer sum) Let $\xi: 0 \to B \to X \to A \to 0$ and $\xi': 0 \to B \to X' \to A \to 0$ be two extensions of A by B. Let X'' be the pullback $\{(x, x') \in X \times X' : \bar{x} = \bar{x}' \text{ in } A\}$.

$$\begin{array}{ccc} X'' & \longrightarrow & X' \\ \downarrow\ulcorner & & \downarrow \\ X & \longrightarrow & A \end{array}$$

X'' contains three copies of B : $B \times 0$, $0 \times B$, and the skew diagonal $\{(-b, b) : b \in B\}$. The copies $B \times 0$ and $0 \times B$ are identified in the quotient Y of X'' by the skew diagonal. Since $X''/0 \times B \cong X$ and $X/B \cong A$, it is immediate that the sequence

$$\varphi: \quad 0 \to B \to Y \to A \to 0$$

is also an extension of A by B. The class of φ is called the *Baer sum* of the extensions ξ and ξ', since this construction was introduced by R. Baer in 1934.

Corollary 3.4.5 *The set of (equiv. classes of) extensions is an abelian group under Baer sum, with zero being the class of the split extension. The map Θ is an isomorphism of abelian groups.*

Proof We will show that $\Theta(\varphi) = \Theta(\xi) + \Theta(\xi')$ in $\mathrm{Ext}^1(A, B)$. This will prove that Baer sum is well defined up to equivalence, and the corollary will then follow. We shall adopt the notation used in $(*)$ in the proof of the above theorem. Let $\tau'': P \to X''$ be the map induced by $\tau: P \to X$ and $\tau': P \to X'$, and let $\bar{\tau}: P \to Y$ be the induced map. The restriction of $\bar{\tau}$ to M is induced by the map $\gamma + \gamma': M \to B$, so

$$\begin{array}{ccccccccc} & 0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & A & \longrightarrow & 0 \\ & & & {\scriptstyle \gamma+\gamma'}\downarrow & & {\scriptstyle \bar{\tau}}\downarrow & & \| & & \\ \varphi: & 0 & \longrightarrow & B & \longrightarrow & Y & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

commutes. Hence, $\Theta(\varphi) = \partial(\gamma + \gamma')$, where ∂ is the map from $\mathrm{Hom}(M, B)$ to $\mathrm{Ext}^1(A, B)$. But $\partial(\gamma + \gamma') = \partial(\gamma) + \partial(\gamma') = \Theta(\xi) + \Theta(\xi')$. $\diamond$

Vista 3.4.6 (Yoneda Ext groups) We can define $\mathrm{Ext}^1(A, B)$ in *any* abelian category $\mathcal{A}$, even if it has no projectives and no injectives, to be the set of equivalence classes of extensions under Baer sum (if indeed this is a set). The Freyd-Mitchell Embedding Theorem 1.6.1 shows that $\mathrm{Ext}^1(A, B)$ is an abelian group—but one could also prove this fact directly. Similarly, we can recapture the groups $\mathrm{Ext}^n(A, B)$ without mentioning projectives or injectives. This approach is due to Yoneda. An element of the Yoneda $\mathrm{Ext}^n(A, B)$ is an equivalence class of exact sequences of the form

$$\xi: \quad 0 \to B \to X_n \to \cdots \to X_1 \to A \to 0.$$

The equivalence relation is generated by the relation that $\xi' \sim \xi''$ if there is a diagram

$$\begin{array}{cccccccccccccc} \xi': & 0 & \longrightarrow & B & \longrightarrow & X'_n & \longrightarrow & \cdots & \longrightarrow & X'_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & & \| & & \downarrow & & & & \downarrow & & \| & & \\ \xi'': & 0 & \longrightarrow & B & \longrightarrow & X''_n & \longrightarrow & \cdots & \longrightarrow & X''_1 & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

To "add" ξ and ξ' when $n \geq 2$, let X''_1 be the pullback of X_1 and X'_1 over A, let X''_n be the pushout of X_n and X''_n under B, and let Y_n be the quotient of X''_n by the skew diagonal copy of B. Then $\xi + \xi'$ is the class of the extension

$$0 \to B \to Y_n \to X_{n-1} \oplus X'_{n-1} \to \cdots \to X_2 \oplus X'_2 \to X''_1 \to A \to 0.$$

Now suppose that $\mathcal{A}$ has enough projectives. If $P \to A$ is a projective resolution, the Comparison Theorem 2.2.6 yields a map from P to ξ, hence a diagram

$$\begin{array}{ccccccccccccc} & 0 & \longrightarrow & M & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow 0 \\ & & & \beta\downarrow & & \downarrow\gamma_n & & & & \downarrow & & \| & \\ \xi\colon & 0 & \longrightarrow & B & \longrightarrow & X_n & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & A & \longrightarrow 0. \end{array}$$

By dimension shifting, there is an exact sequence

$$\operatorname{Hom}(P_{n-1}, B) \to \operatorname{Hom}(M, B) \xrightarrow{\partial} \operatorname{Ext}^n(A, B) \to 0.$$

The association $\Theta(\xi) = \partial(\beta)$ gives the 1–1 correspondence between the Yoneda Ext^n and the derived functor Ext^n. For more details we refer the reader to [BX, §7.5] or [MacH, pp. 82–87].

3.5 Derived Functors of the Inverse Limit

Let I be a small category and $\mathcal{A}$ an abelian category. We saw in Chapter 2 that the functor category $\mathcal{A}^I$ has enough injectives, at least when $\mathcal{A}$ is complete and has enough injectives. (For example, $\mathcal{A}$ could be **Ab**, R–**mod**, or Sheaves(X).) Therefore we can define the right derived functors $R^n \lim_{i\in I}$ from $\mathcal{A}^I$ to $\mathcal{A}$.

We are most interested in the case in which $\mathcal{A}$ is **Ab** and I is the poset $\cdots \to 2 \to 1 \to 0$ of whole numbers in reverse order. We shall call the objects of $\mathbf{Ab}^I$ (countable) *towers* of abelian groups; they have the form

$$\{A_i\}\colon \quad \cdots \to A_2 \to A_1 \to A_0.$$

In this section we shall give the alternative construction $\varprojlim^1$ of $R^1 \varprojlim$ for countable towers due to Eilenberg and prove that $R^n \varprojlim = 0$ for $n \neq 0, 1$. This construction generalizes from **Ab** to other abelian categories that satisfy the following axiom, introduced by Grothendieck in [Tohoku]:

$(AB4^*)$ $\mathcal{A}$ is complete, and the product of any set of surjections is a surjection.

Explanation If I is a discrete set, $\mathcal{A}^I$ is the product category $\Pi_{i\in I}\mathcal{A}$ of indexed families of objects $\{A_i\}$ in $\mathcal{A}$. For $\{A_i\}$ in $\mathcal{A}^I$, $\lim_{i\in I} A_i$ is the product $\prod A_i$. Axiom $(AB4^*)$ states that the left exact functor $\prod$ from $\mathcal{A}^I$ to $\mathcal{A}$ is exact for all discrete I. Axiom $(AB4^*)$ fails ($\prod_{i=1}^{\infty}$ is not exact) for some important abelian categories, such as Sheaves(X). On the other hand, axiom $(AB4^*)$ is satisfied by many abelian categories in which objects have underlying sets, such as **Ab**, **mod**–R, and **Ch**(**mod**–R).

Definition 3.5.1 Given a tower $\{A_i\}$ in **Ab**, define the map

$$\Delta\colon \prod_{i=0}^{\infty} A_i \to \prod_{i=0}^{\infty} A_i$$

by the element-theoretic formula

$$\Delta(\cdots, a_i, \cdots, a_0) = (\cdots, a_i - \bar{a}_{i+1}, \cdots, a_1 - \bar{a}_2, a_0 - \bar{a}_1),$$

where $\bar{a}_{i+1}$ denotes the image of $a_{i+1} \in A_{i+1}$ in A_i. The kernel of Δ is $\varprojlim A_i$ (check this!). We define $\varprojlim^1 A_i$ to be the cokernel of Δ, so that $\varprojlim^1$ is a functor from $\mathbf{Ab}^I$ to **Ab**. We also set $\varprojlim^0 A_i = \varprojlim A_i$ and $\varprojlim^n A_i = 0$ for $n \neq 0, 1$.

Lemma 3.5.2 *The functors $\{\varprojlim^n\}$ form a cohomological δ-functor.*

Proof If $0 \to \{A_i\} \to \{B_i\} \to \{C_i\} \to 0$ is a short exact sequence of towers, apply the Snake Lemma to

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod A_i & \longrightarrow & \prod B_i & \longrightarrow & \prod C_i & \longrightarrow & 0 \\
 & & \downarrow\Delta & & \downarrow\Delta & & \downarrow\Delta & & \\
0 & \longrightarrow & \prod A_i & \longrightarrow & \prod B_i & \longrightarrow & \prod C_i & \longrightarrow & 0
\end{array}$$

to get the requisite natural long exact sequence. ◇

Lemma 3.5.3 *If all the maps $A_{i+1} \to A_i$ are onto, then $\varprojlim^1 A_i = 0$. Moreover $\varprojlim A_i \neq 0$ (unless every $A_i = 0$), because each of the natural projections $\varprojlim A_i \to A_j$ are onto.*

Proof Given elements $b_i \in A_i$ $(i = 0, 1, \cdots)$, and any $a_0 \in A_0$, inductively choose $a_{i+1} \in A_{i+1}$ to be a lift of $a_i - b_i \in A_i$. The map Δ sends $(\cdots, a_1, a_0)$ to $(\cdots, b_1, b_0)$, so Δ is onto and $\operatorname{coker}(\Delta) = 0$. If all the $b_i = 0$, then $(\cdots, a_1, a_0) \in \varprojlim A_i$. ◇

Corollary 3.5.4 $\varprojlim^1 A_i \cong (R^1 \varprojlim)(A_i)$ *and* $R^n \varprojlim = 0$ *for* $n \neq 0, 1$.

Proof In order to show that the $\varprojlim^n$ forms a universal δ-functor, we only need to see that $\varprojlim^1$ vanishes on enough injectives. In Chapter 2 we constructed enough injectives by taking products of towers

$$k_*E: \quad \cdots = E = E \to 0 \to 0 \cdots \to 0$$

with E injective. All the maps in k_*E (and hence in the product towers) are onto, so $\varprojlim^1$ vanishes on these injective towers. $\diamond$

Remark If we replace **Ab** by $\mathcal{A} = \textbf{mod}\text{–}R$, $\textbf{Ch}(\textbf{mod}\text{–}R)$ or any abelian category $\mathcal{A}$ satisfying Grothendieck's axiom $(AB4^*)$, the above proof goes through to show that $\varprojlim^1 = R^1(\varprojlim)$ and $R^n(\varprojlim) = 0$ for $n \neq 0, 1$ as functors on the category of towers in $\mathcal{A}$. However, the proof breaks down for other abelian categories.

Example 3.5.5 Set $A_0 = \mathbb{Z}$ and let $A_i = p^i\mathbb{Z}$ be the subgroup generated by p^i. Applying $\varprojlim$ to the short exact sequence of towers

$$0 \to \{p^i\mathbb{Z}\} \to \{\mathbb{Z}\} \to \{\mathbb{Z}/p^i\mathbb{Z}\} \to 0$$

with p prime yields the uncountable group

$$\varprojlim{}^1\{p^i\mathbb{Z}\} \cong \hat{\mathbb{Z}}_p/\mathbb{Z}.$$

Here $\hat{\mathbb{Z}}_p = \varprojlim \mathbb{Z}/p^i\mathbb{Z}$ is the group of p-adic integers.

Exercise 3.5.1 Let $\{A_i\}$ be a tower in which the maps $A_{i+1} \to A_i$ are inclusions. We may regard $A = A_0$ as a topological group in which the sets $a + A_i (a \in A, i \geq 0)$ are the open sets. Show that $\varprojlim A_i = \cap A_i$ is zero iff A is *Hausdorff*. Then show that $\varprojlim^1 A_i = 0$ iff A is *complete* in the sense that every Cauchy sequence has a limit, not necessarily unique. *Hint:* Show that A is complete iff $A \cong \varprojlim(A/A_i)$.

Definition 3.5.6 A tower $\{A_i\}$ of abelian groups satisfies the *Mittag-Leffler condition* if for each k there exists a $j \geq k$ such that the image of $A_i \to A_k$ equals the image of $A_j \to A_k$ for all $i \geq j$. (The images of the A_i in A_k satisfy the *descending chain condition*.) For example, the Mittag-Leffler condition is satisfied if all the maps $A_{i+1} \to A_i$ in the tower $\{A_i\}$ are onto. We say that $\{A_i\}$ satisfies the *trivial* Mittag-Leffler condition if for each k there exists a $j > k$ such that the map $A_j \to A_k$ is zero.

Proposition 3.5.7 *If $\{A_i\}$ satisfies the Mittag-Leffler condition, then*

$$\varprojlim{}^1 A_i = 0.$$

Proof If $\{A_i\}$ satisfies the trivial Mittag-Leffler condition, and $b_i \in A_i$ are given, set $a_k = b_k + \bar{b}_{k+1} + \cdots + \bar{b}_{j-1}$, where $\bar{b}_i$ denotes the image of b_i in A_k. (Note that $\bar{b}_i = 0$ for $i \geq j$.) Then Δ maps $(\cdots, a_1, a_0)$ to $(\cdots, b_1, b_0)$. Thus Δ is onto and $\varprojlim^1 A_i = 0$ when $\{A_i\}$ satisfies the trivial Mittag-Leffler condition. In the general case, let $B_k \subseteq A_k$ be the image of $A_i \to A_k$ for large i. The maps $B_{k+1} \to B_k$ are all onto, so $\varprojlim^1 B_k = 0$. The tower $\{A_k/B_k\}$ satisfies the trivial Mittag-Leffler condition, so $\varprojlim^1 A_k/B_k = 0$. From the short exact sequence

$$0 \to \{B_i\} \to \{A_i\} \to \{A_i/B_i\} \to 0$$

of towers, we see that $\varprojlim^1 A_i = 0$ as claimed. $\diamond$

Exercise 3.5.2 Show that $\varprojlim^1 A_i = 0$ if $\{A_i\}$ is a tower of finite abelian groups, or a tower of finite-dimensional vector spaces over a field.

The following formula presages the Universal Coefficient theorems of the next section, as well as the spectral sequences of Chapter 5.

Theorem 3.5.8 *Let $\cdots \to C_1 \to C_0$ be a tower of chain complexes of abelian groups satisfying the Mittag-Leffler condition, and set $C = \varprojlim C_i$. Then there is an exact sequence for each q:*

$$0 \to \varprojlim{}^1 H_{q+1}(C_i) \to H_q(C) \to \varprojlim H_q(C_i) \to 0.$$

Proof Let $B_i \subseteq Z_i \subseteq C_i$ be the subcomplexes of boundaries and cycles in the complex C_i, so that Z_i/B_i is the chain complex $H_*(C_i)$ with zero differentials. Applying the left exact functor $\varprojlim$ to $0 \to \{Z_i\} \to \{C_i\} \xrightarrow{d} \{C_i[-1]\}$ shows that in fact $\varprojlim Z_i$ is the subcomplex Z of cycles in C. (The $[-1]$ refers to the surpressed subscript on the chain complexes.) Let B denote the subcomplex $d(C)[1] = (C/Z)[1]$ of boundaries in C, so that Z/B is the chain complex $H_*(C)$ with zero differentials. From the exact sequence of towers

$$0 \to \{Z_i\} \to \{C_i\} \xrightarrow{d} \{B_i[-1]\} \to 0$$

we see that $\varprojlim^1 B_i = (\varprojlim{}^1 B_i[-1])[+1] = 0$ and that

$$0 \to B[-1] \to \varprojlim B_i[-1] \to \varprojlim{}^1 Z_i \to 0$$

is exact. From the exact sequence of towers

$$0 \to \{B_i\} \to \{Z_i\} \to H_*(C_i) \to 0$$

we see that $\varprojlim^1 Z_i \cong \varprojlim^1 H_*(C_i)$ and that

$$0 \to \varprojlim B_i \to Z \to \varprojlim H_*(C_i) \to 0$$

is exact. Hence C has the filtration by subcomplexes

$$0 \subseteq B \subseteq \varprojlim B_i \subseteq Z \subseteq C$$

whose filtration quotients are B, $\varprojlim^1 H_*(C_i)[1]$, $\varprojlim H_*(C_i)$, and C/Z respectively. The theorem follows, since $Z/B = H_*(C)$. $\diamond$

Variant If $\cdots \to C_1 \to C_0$ is a tower of cochain complexes satisfying the Mittag-Leffler condition, the sequences become

$$0 \to \varprojlim{}^1 H^{q-1}(C_i) \to H^q(C) \to \varprojlim H^q(C_i) \to 0.$$

Application 3.5.9 Let $H^*(X)$ denote the integral cohomology of a topological CW complex X. If $\{X_i\}$ is an increasing sequence of subcomplexes with $X = \cup X_i$, there is an exact sequence

$$(*) \qquad 0 \to \varprojlim{}^1 H^{q-1}(X_i) \to H^q(X) \to \varprojlim H^q(X_i) \to 0$$

for each q. This use of $\varprojlim^1$ to perform calculations in algebraic topology was discovered by Milnor in 1960 [Milnor] and thrust $\varprojlim^1$ into the limelight.

To derive this formula, let C_i denote the chain complex $\mathrm{Hom}(S(X_i), \mathbb{Z})$ used to compute $H^*(X_i)$. Since the inclusion $S(X_i) \subseteq S(X_{i+1})$ splits (because each $S_n(X_{i+1})/S_n(X_i)$ is a free abelian group), the maps $C_{i+1} \to C_i$ are onto, and the tower satisfies the Mittag-Leffler condition. Since X has the weak topology, $S(X)$ is the union of the $S(X_i)$, and therefore $H^*(X)$ is the cohomology of the cochain complex

$$\mathrm{Hom}(\cup S(X_i), \mathbb{Z}) = \varprojlim \mathrm{Hom}(S(X_i), \mathbb{Z}) = \varprojlim C_i.$$

A historical remark: Milnor proved that the sequence $(*)$ is also valid if H^* is replaced by any generalized cohomology theory, such as topological K–theory.

Application 3.5.10 Let A be an R-module that is the union of submodules $\cdots \subseteq A_i \subseteq A_{i+1} \subseteq \cdots$. Then for every R-module B and every q the sequence

$$0 \to \underleftarrow{\lim}{}^1 \operatorname{Ext}_R^{q-1}(A_i, B) \to \operatorname{Ext}_R^q(A, B) \to \underleftarrow{\lim} \operatorname{Ext}_R^q(A_i, B) \to 0$$

is exact. For $\mathbb{Z}_{p^\infty} = \cup \mathbb{Z}/p^i$, this gives a short exact sequence for every B:

$$0 \to \underleftarrow{\lim}{}^1 \operatorname{Hom}(\mathbb{Z}/p^i, B) \to \operatorname{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_{p^\infty}, B) \to \hat{B}_p \to 0,$$

where the group $\hat{B}_p = \underleftarrow{\lim}(B/p^i B)$ is the p-adic completion of B. This generalizes the calculation $\operatorname{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_{p^\infty}, \mathbb{Z}) \cong \hat{\mathbb{Z}}_p$ of 3.3.3. To see this, let E be a fixed injective resolution of B, and consider the tower of cochain complexes

$$\operatorname{Hom}(A_{i+1}, E) \to \operatorname{Hom}(A_i, E) \to \cdots \to \operatorname{Hom}(A_0, E).$$

Each $\operatorname{Hom}(-, E_n)$ is contravariant exact, so each map in the tower is a surjection. The cohomology of $\operatorname{Hom}(A_i, E)$ is $\operatorname{Ext}^*(A_i, B)$, and $\operatorname{Ext}^*(A, B)$ is the cohomology of

$$\operatorname{Hom}(\cup A_i, E) = \underleftarrow{\lim} \operatorname{Hom}(A_i, E).$$

Exercise 3.5.3 Show that $\operatorname{Ext}_{\mathbb{Z}}^1(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}) \cong \hat{\mathbb{Z}}_p/\mathbb{Z}$ using $\mathbb{Z}[\frac{1}{p}] = \cup p^{-i}\mathbb{Z}$; cf. exercise 3.3.1. Then show that $\operatorname{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, B) = (\prod_p \hat{B}_p)/B$ for torsionfree B.

Application 3.5.11 Let $C = C_{**}$ be a double chain complex, viewed as a lattice in the plane, and let $T_n C$ be the quotient double complex obtained by brutally truncating C at the vertical line $p = -n$:

$$(T_n C)_{pq} = \begin{cases} C_{pq} & \text{if } p \geq -n \\ 0 & \text{if } p < -n \end{cases}.$$

Then $\operatorname{Tot}(C)$ is the inverse limit of the tower of surjections

$$\cdots \to \operatorname{Tot}(T_{i+1}C) \to \operatorname{Tot}(T_i C) \to \cdots \to \operatorname{Tot}(T_0 C).$$

Therefore there is a short exact sequence for each q:

$$0 \to \underleftarrow{\lim}{}^1 H_{q+1}(\operatorname{Tot}(T_i C)) \to H_q(\operatorname{Tot}(C)) \to \underleftarrow{\lim} H_q(\operatorname{Tot}(T_i C)) \to 0.$$

This is especially useful when C is a second quadrant double complex, because the truncated complexes have only a finite number of nonzero rows.

Exercise 3.5.4 Let C be a second quadrant double complex with exact rows, and let B^h_{pq} be the image of $d^h\colon C_{pq} \to C_{p-1,q}$. Show that $H_{p+q}\operatorname{Tot}(T_{-p}C) \cong H_q(B^h_{p*}, d^v)$. Then let $b = d^h(a)$ be an element of B^h_{pq} representing a cycle ξ in $H_{p+q}\operatorname{Tot}(T_{-p}C)$ and show that the image of ξ in $H_{p+q}\operatorname{Tot}(T_{-p-1}C)$ is represented by $d^v(a) \in B^h_{p+1,q-1}$. This provides an effective method for calculating $H_*\operatorname{Tot}(C)$.

Vista 3.5.12 Let I be any poset and $\mathcal{A}$ any abelian category satisfying $(AB4^*)$. The following construction of the right derived functors of lim is taken from [Roos] and generalizes the construction of $\varprojlim^1$ in this section.

Given $A\colon I \to \mathcal{A}$, we define C_k to be the product over the set of all chains $i_k < \cdots < i_0$ in I of the objects A_{i_0}. Letting $pr_{i_k\cdots i_1}$ denote the projection of C_k onto the $(i_k < \cdots < i_1)^{st}$ factor and f_0 denote the map $A_{i_1} \to A_{i_0}$ associated to $i_1 < i_0$, we define $d^0\colon C_{k-1} \to C_k$ to be the map whose $(i_k < \cdots < i_0)^{th}$ factor is $f_0(pr_{i_k\cdots i_1})$. For $1 \le p \le k$, we define $d^p\colon C_{k-1} \to C_k$ to be the map whose $(i_k < \cdots < i_0)^{th}$ factor is the projection onto the $(i_k < \cdots < \hat{i}_p < \cdots < i_0)^{th}$ factor. This data defines a cochain complex C_*A whose differential $C_{k-1} \to C_k$ is the alternating sum $\sum_{p=0}^{k}(-1)^p d^p$, and we define $\lim^n_{i\in I} A$ to be $H^n(C_*A)$. (The data actually forms a *cosimplicial object* of $\mathcal{A}$; see Chapter 8.)

It is easy to see that $\lim^0_{i\in I} A$ is the limit $\lim_{i\in I} A$. An exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}^I$ gives rise to a short exact sequence $0 \to C_*A \to C_*B \to C_*C \to 0$ in $\mathcal{A}$, whence an exact sequence

$$0 \to \lim_{i\in I} A \to \lim_{i\in I} B \to \lim_{i\in I} C \to \lim_{i\in I}{}^1 A \to \lim_{i\in I}{}^1 B \to \lim_{i\in I}{}^1 C \to \lim_{i\in I}{}^2 A \to \cdots.$$

Therefore the functors $\{\lim^n_{i\in I}\}$ form a cohomological δ-functor. It turns out that they are universal when $\mathcal{A}$ has enough injectives, so in fact $R^n \lim_{i\in I} \cong \lim^n_{i\in I}$.

Remark Let $\aleph_d$ denote the d^{th} infinite cardinal number, $\aleph_0$ being the cardinality of $\{1, 2, \cdots\}$. If I is a directed poset of cardinality $\aleph_d$, or a filtered category with $\aleph_d$ morphisms, Mitchell proved in [Mitch] that $R^n \varprojlim$ vanishes for $n \ge d + 2$.

Exercise 3.5.5 (Pullback) Let $\to\leftarrow$ denote the poset $\{x, y, z\}$, $x < z$ and $y < z$, so that $\lim\limits_{\to\leftarrow} A_i$ is the pullback of A_x and A_y over A_z. Show that $\lim\limits_{\to\leftarrow}{}^1 A_i$

is the cokernel of the difference map $A_x \times A_y \to A_z$ and that $\varinjlim\varprojlim^n = 0$ for $n \neq 0, 1$.

3.6 Universal Coefficient Theorems

There is a very useful formula for using the homology of a chain complex P to compute the homology of the complex $P \otimes M$. Here is the most useful general formulation we can give:

Theorem 3.6.1 (Künneth formula) *Let P be a chain complex of flat right R-modules such that each submodule $d(P_n)$ of P_{n-1} is also flat. Then for every n and every left R-module M, there is an exact sequence*

$$0 \to H_n(P) \otimes_R M \to H_n(P \otimes_R M) \to \mathrm{Tor}_1^R(H_{n-1}(P), M) \to 0.$$

Proof The long exact Tor sequence associated to $0 \to Z_n \to P_n \to d(P_n) \to 0$ shows that each Z_n is also flat (exercise 3.2.2). Since $\mathrm{Tor}_1^R(d(P_n), M) = 0$,

$$0 \to Z_n \otimes M \to P_n \otimes M \to d(P_n) \otimes M \to 0$$

is exact for every n. These assemble to give a short exact sequence of chain complexes $0 \to Z \otimes M \to P \otimes M \to d(P) \otimes M \to 0$. Since the differentials in the Z and $d(P)$ complexes are zero, the homology sequence is

$$\begin{array}{ccccccccc}
H_{n+1}(dP \otimes M) & \xrightarrow{\partial} & H_n(Z \otimes M) & \to & H_n(P \otimes M) & \to & H_n(dP \otimes M) & \xrightarrow{\partial} & H_{n-1}(Z \otimes M) \\
\| \wr & & \| \wr & & & & \| \wr & & \| \wr \\
d(P_{n+1}) \otimes M & & Z_n \otimes M & & & & d(P_n) \otimes M & & Z_{n-1} \otimes M.
\end{array}$$

Using the definition of ∂, it is immediate that $\partial = i \otimes M$, where i is the inclusion of $d(P_{n+1})$ in Z_n. On the other hand,

$$0 \to d(P_{n+1}) \xrightarrow{i} Z_n \to H_n(P) \to 0$$

is a flat resolution of $H_n(P)$, so $\mathrm{Tor}_*(H_n(P), M)$ is the homology of

$$0 \to d(P_{n+1}) \otimes M \xrightarrow{\partial} Z_n \otimes M \to 0.$$

◇

Universal Coefficient Theorem for Homology 3.6.2 *Let P be a chain complex of free abelian groups. Then for every n and every abelian group M the*

Künneth formula 3.6.1 splits noncanonically, yielding a direct sum decomposition

$$H_n(P \otimes M) \cong H_n(P) \otimes M \,\oplus\, \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(P), M).$$

Proof We shall use the well-known fact that every subgroup of a free abelian group is free abelian [KapIAB, section 15]. Since $d(P_n)$ is a subgroup of P_{n+1}, it is free abelian. Hence the surjection $P_n \to d(P_n)$ splits, giving a noncanonical decomposition

$$P_n \cong Z_n \oplus d(P_n).$$

Applying $\otimes M$, we see that $Z_n \otimes M$ is a direct summand of $P_n \otimes M$; a fortiori, $Z_n \otimes M$ is a direct summand of the intermediate group

$$\ker(d_n \otimes 1\colon P_n \otimes M \to P_{n-1} \otimes M).$$

Modding out $Z_n \otimes M$ and $\ker(d_n \otimes 1)$ by the common image of $d_{n+1} \otimes 1$, we see that $H_n(P) \otimes M$ is a direct summand of $H_n(P \otimes M)$. Since P and $d(P)$ are flat, the Künneth formula tells us that the other summand is $\mathrm{Tor}_1(H_{n-1}(P), M)$. ◇

Theorem 3.6.3 (Künneth formula for complexes) *Let P and Q be right and left R-module complexes, respectively. Recall from 2.7.1 that the* tensor product complex *$P \otimes_R Q$ is the complex whose degree n part is $\bigoplus_{p+q=n} P_p \otimes Q_q$ and whose differential is given by $d(a \otimes b) = (da) \otimes b + (-1)^p a \otimes (db)$ for $a \in P_p$, $b \in Q_q$. If P_n and $d(P_n)$ are flat for each n, then there is an exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(P) \otimes H_q(Q) \to H_n(P \otimes_R Q) \to \bigoplus_{\substack{p+q=\\ n-1}} \mathrm{Tor}_1^R(H_p(P), H_q(Q)) \to 0$$

for each n. If $R = \mathbb{Z}$ and P is a complex of free abelian groups, this sequence is noncanonically split.

Proof Modify the proof given in 3.6.1 for $Q = M$. ◇

Application 3.6.4 (Universal Coefficient Theorem in topology) Let $S(X)$ denote the singular chain complex of a topological space X; each $S_n(X)$ is a free abelian group. If M is any abelian group, the homology of X with "coefficients" in M is

$$H_*(X; M) = H_*(S(X) \otimes M).$$

Writing $H_*(X)$ for $H_*(X; \mathbb{Z})$, the formula in this case becomes

$$H_n(X; M) \cong H_n(X) \otimes M \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X), M).$$

This formula is often called the Universal Coefficient Theorem in topology.

If Y is another topological space, the Eilenberg-Zilber theorem 8.5.1 (see [MacH, VIII.8]) states that $H_*(X \times Y)$ is the homology of the tensor product complex $S(X) \otimes S(Y)$. Therefore the Künneth formula yields the "Künneth formula for cohomology:"

$$H_n(X \times Y) \cong \left\{ \bigoplus_{p=0}^{n} H_p(X) \otimes H_{n-p}(Y) \right\} \otimes \left\{ \bigoplus_{p=1}^{n} \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(X), H_{n-p}(Y)) \right\}.$$

We now turn to the analogue of the Künneth formula for Hom in place of $\otimes$.

Universal Coefficient Theorem for Cohomology 3.6.5 *Let P be a chain complex of projective R-modules such that each $d(P_n)$ is also projective. Then for every n and every R-module M, there is a (noncanonically) split exact sequence*

$$0 \to \mathrm{Ext}_R^1(H_{n-1}(P), M) \to H^n(\mathrm{Hom}_R(P, M)) \to \mathrm{Hom}_R(H_n(P), M) \to 0.$$

Proof Since $d(P_n)$ is projective, there is a (noncanonical) isomorphism $P_n \cong Z_n \oplus d(P_n)$ for each n. Therefore each sequence

$$0 \to \mathrm{Hom}(d(P_n), M) \to \mathrm{Hom}(P_n, M) \to \mathrm{Hom}(Z_n, M) \to 0$$

is exact. We may now copy the proof of the Künneth formula 3.6.1 for $\otimes$, using $\mathrm{Hom}(-, M)$ instead of $\otimes M$, to see that the sequence is indeed exact. We may copy the proof of the Universal Coefficient Theorem 3.6.2 for $\otimes$ in the same way to see that the sequence is split. ◇

Application 3.6.6 (Universal Coefficient theorem in topology) The cohomology of a topological space X with "coefficients" in M is defined to be

$$H^*(X; M) = H^*(\mathrm{Hom}(S(X), M)).$$

In this case, the Universal Coefficient theorem becomes

$$H^n(X; M) \cong \mathrm{Hom}(H_n(X), M) \oplus \mathrm{Ext}_{\mathbb{Z}}^1(H_{n-1}(X), M).$$

Example 3.6.7 If X is path-connected, then $H_0(X) = \mathbb{Z}$ and $H^1(X; \mathbb{Z}) \cong \text{Hom}(H_1(X), \mathbb{Z})$, which is a torsionfree abelian group.

Exercise 3.6.1 Let P be a chain complex and Q a cochain complex of R-modules. As in 2.7.4, form the Hom double cochain complex $\text{Hom}(P, Q) = \{\text{Hom}_R(P_p, Q^q)\}$, and then write $H^* \text{Hom}(P, Q)$ for the cohomology of $\text{Tot}(\text{Hom}(P, Q))$. Show that if each P_n and $d(P_n)$ is projective, there is an exact sequence

$$0 \to \prod_{\substack{p+q \\ n-1}} \text{Ext}^1_R(H_p(P), H^q(Q)) \to H^n \text{Hom}(P, Q) \to \prod_{\substack{p+q= \\ n}} \text{Hom}_R(H_p(P), H^q(Q)) \to 0.$$

Exercise 3.6.2 A ring R is called *right hereditary* if every submodule of every (right) free module is a projective module. (See 4.2.10 and exercise 4.2.6 below.) Any principal ideal domain (for example, $R = \mathbb{Z}$) is hereditary, as is any commutative Dedekind domain. Show that the universal coefficient theorems of this section remain valid if $\mathbb{Z}$ is replaced by an arbitrary right hereditary ring R.