2

Derived Functors

2.1 δ-Functors

The right context in which to view derived functors, according to Grothendieck [Tohoku], is that of δ-functors between two abelian categories $\mathcal{A}$ and $\mathcal{B}$.

Definition 2.1.1 A (covariant) *homological* (resp. *cohomological*) *δ-functor* between $\mathcal{A}$ and $\mathcal{B}$ is a collection of additive functors $T_n: \mathcal{A} \to \mathcal{B}$ (resp. $T^n: \mathcal{A} \to \mathcal{B}$) for $n \geq 0$, together with morphisms

$$\delta_n: T_n(C) \to T_{n-1}(A)$$

$$(\text{resp. } \delta^n: T^n(C) \to T^{n+1}(A))$$

defined for each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$. Here we make the convention that $T^n = T_n = 0$ for $n < 0$. These two conditions are imposed:

1. For each short exact sequence as above, there is a long exact sequence

$$\cdots\ T_{n+1}(C) \xrightarrow{\delta} T_n(A) \to T_n(B) \to T_n(C) \xrightarrow{\delta} T_{n-1}(A)\ \cdots$$

(resp.

$$\cdots\ T^{n-1}(C) \xrightarrow{\delta} T^n(A) \to T^n(B) \to T^n(C) \xrightarrow{\delta} T^{n+1}(A)\ \cdots).$$

In particular, T_0 is right exact, and T^0 is left exact.

2. For each morphism of short exact sequences from $0 \to A' \to B' \to C' \to 0$ to $0 \to A \to B \to C \to 0$, the δ's give a commutative diagram

$$\begin{array}{ccc} T_n(C') & \xrightarrow{\delta} & T_{n-1}(A') \\ \downarrow & & \downarrow \\ T_n(C) & \xrightarrow{\delta} & T_{n-1}(A) \end{array} \quad \text{resp.} \quad \begin{array}{ccc} T^n(C') & \xrightarrow{\delta} & T^{n+1}(A') \\ \downarrow & & \downarrow \\ T^n(C) & \xrightarrow{\delta} & T^{n+1}(A). \end{array}$$

Example 2.1.2 Homology gives a homological δ-functor H_* from $\mathbf{Ch}_{\geq 0}(\mathcal{A})$ to $\mathcal{A}$; cohomology gives a cohomological δ-functor H^* from $\mathbf{Ch}^{\geq 0}(\mathcal{A})$ to $\mathcal{A}$.

Exercise 2.1.1 Let $\mathcal{S}$ be the category of short exact sequences

$$(*) \qquad 0 \to A \to B \to C \to 0$$

in $\mathcal{A}$. Show that δ_i is a natural transformation from the functor sending $(*)$ to $T_i(C)$ to the functor sending $(*)$ to $T_{i-1}(A)$.

Example 2.1.3 (p-torsion) If p is an integer, the functors $T_0(A) = A/pA$ and

$$T_1(A) = {}_pA \equiv \{a \in A : pa = 0\}$$

fit together to form a homological δ-functor, or a cohomological δ-functor (with $T^0 = T_1$ and $T^1 = T_0$) from **Ab** to **Ab**. To see this, apply the Snake Lemma to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & p\downarrow & & p\downarrow & & p\downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

to get the exact sequence

$$0 \to {}_pA \to {}_pB \to {}_pC \xrightarrow{\delta} A/pA \to B/pB \to C/pC \to 0.$$

Generalization The same proof shows that if r is any element in a ring R, then $T_0(M) = M/rM$ and $T_1(M) = {}_rM$ fit together to form a homological δ-functor (or cohomological δ-functor, if that is one's taste) from R–**mod** to **Ab**.

Vista We will see in 2.6.3 that $T_n(M) = \mathrm{Tor}_n^R(R/r, M)$ is also a homological δ-functor with $T_0(M) = M/rM$. If r is a left nonzerodivisor (meaning that ${}_rR = \{s \in R : rs = 0\}$ is zero), then in fact $\mathrm{Tor}_1^R(R/r, M) = {}_rM$ and $\mathrm{Tor}_n^R(R/r, M) = 0$ for $n \geq 2$; see 3.1.7. However, in general ${}_rR \neq 0$, while $\mathrm{Tor}_1^R(R/r, R) = 0$, so they aren't the same; $\mathrm{Tor}_1^R(M, R/r)$ is the quotient of ${}_rM$ by the submodule $({}_rR)M$ generated by $\{sm : rs = 0, s \in R, m \in M\}$. The Tor_n will be *universal* δ-functors in a sense that we shall now make precise.

Definition 2.1.4 A *morphism* $S \to T$ of δ-functors is a system of natural transformations $S_n \to T_n$ (resp. $S^n \to T^n$) that commute with δ. This is fancy language for the assertion that there is a commutative ladder diagram connecting the long exact sequences for S and T associated to any short exact sequence in $\mathcal{A}$.

A homological δ-functor T is *universal* if, given any other δ-functor S and a natural transformation $f_0: S_0 \to T_0$, there exists a unique morphism $\{f_n: S_n \to T_n\}$ of δ-functors that extends f_0.

A cohomological δ-functor T is *universal* if, given S and $f^0: T^0 \to S^0$, there exists a unique morphism $T \to S$ of δ-functors extending f^0.

Example 2.1.5 We will see in section 2.4 that homology $H_*: \mathbf{Ch}_{\geq 0}(\mathcal{A}) \to \mathcal{A}$ and cohomology $H^*: \mathbf{Ch}^{\geq 0}(\mathcal{A}) \to \mathcal{A}$ are universal δ-functors.

Exercise 2.1.2 If $F: \mathcal{A} \to \mathcal{B}$ is an exact functor, show that $T_0 = F$ and $T_n = 0$ for $n \neq 0$ defines a universal δ-functor (of both homological and cohomological type).

Remark If $F: \mathcal{A} \to \mathcal{B}$ is an additive functor, then we can ask if there is *any* δ-functor T (universal or not) such that $T_0 = F$ (resp. $T^0 = F$). One obvious obstruction is that T_0 must be right exact (resp. T^0 must be left exact). By definition, however, we see that there is at most one (up to isomorphism) *universal* δ-functor T with $T_0 = F$ (resp. $T^0 = F$). If a universal T exists, the T_n are sometimes called the *left satellite functors* of F (resp. the T^n are called the *right satellite functors* of F). This terminology is due to the pervasive influence of the book [CE].

We will see that derived functors, when they exist, are indeed universal δ-functors. For this we need the concept of projective and injective resolutions.

2.2 Projective Resolutions

An object P in an abelian category $\mathcal{A}$ is *projective* if it satisfies the following universal lifting property: Given a surjection $g\colon B \to C$ and a map $\gamma\colon P \to C$, there is at least one map $\beta\colon P \to B$ such that $\gamma = g \circ \beta$.

$$\begin{array}{ccccc} & & P & & \\ & {}^{\exists\beta}\swarrow & \downarrow\gamma & & \\ B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

We shall be mostly concerned with the special case of projective modules ($\mathcal{A}$ being the category **mod**–R). The notion of projective module first appeared in the book [CE]. It is easy to see that free R-modules are projective (lift a basis). Clearly, direct summands of free modules are also projective modules.

Proposition 2.2.1 *An R-module is projective iff it is a direct summand of a free R-module.*

Proof Letting $F(A)$ be the free R-module on the set underlying an R-module A, we see that for every R-module A there is a surjection $\pi\colon F(A) \to A$. If A is a projective R-module, the universal lifting property yields a map $i\colon A \to F(A)$ so that $\pi i = 1_A$, that is, A is a direct summand of the free module $F(A)$. ◇

Example 2.2.2 Over many nice rings ($\mathbb{Z}$, fields, division rings, $\cdots$) every projective module is in fact a free module. Here are two examples to show that this is not always the case:

1. If $R = R_1 \times R_2$, then $P = R_1 \times 0$ and $0 \times R_2$ are projective because their sum is R. P is not free because $(0, 1)P = 0$. This is true, for example, when R is the ring $\mathbb{Z}/6 = \mathbb{Z}/2 \times \mathbb{Z}/3$.
2. Consider the ring $R = M_n(F)$ of $n \times n$ matrices over a field F, acting on the left on the column vector space $V = F^n$. As a left R-module, R is the direct sum of its columns, each of which is the left R-module V. Hence $R \cong V \oplus \cdots \oplus V$, and V is a projective R-module. Since any free R-module would have dimension dn^2 over F for some cardinal number d, and $\dim_F(V) = n$, V cannot possibly be free over R.

Remark The category $\mathcal{A}$ of finite abelian groups is an example of an abelian category that has *no* projective objects. We say that $\mathcal{A}$ *has enough projectives* if for every object A of $\mathcal{A}$ there is a surjection $P \to A$ with P projective.

Here is another characterization of projective objects in $\mathcal{A}$:

Lemma 2.2.3 *M is projective iff $\mathrm{Hom}_{\mathcal{A}}(M, -)$ is an exact functor. That is, iff the sequence of groups*

$$0 \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C) \to 0$$

is exact for every exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$.

Proof Suppose that $\mathrm{Hom}(M, -)$ is exact and that we are given a surjection $g: B \to C$ and a map $\gamma: M \to C$. We can lift $\gamma \in \mathrm{Hom}(M, C)$ to $\beta \in \mathrm{Hom}(M, B)$ such that $\gamma = g_*\beta = g \circ \beta$ because g_* is onto. Thus M has the universal lifting property, that is, it is projective. Conversely, suppose M is projective. In order to show that $\mathrm{Hom}(M, -)$ is exact, it suffices to show that g_* is onto for every short exact sequence as above. Given $\gamma \in \mathrm{Hom}(M, C)$, the universal lifting property of M gives $\beta \in \mathrm{Hom}(M, B)$ so that $\gamma = g \circ \beta = g_*(\beta)$, that is, g_* is onto. $\diamond$

A chain complex P in which each P_n is projective in $\mathcal{A}$ is called a *chain complex of projectives*. It need not be a projective object in **Ch**.

Exercise 2.2.1 Show that a chain complex P is a projective object in **Ch** if and only if it is a split exact complex of projectives. *Hint:* To see that P must be split exact, consider the surjection from $\mathrm{cone}(\mathrm{id}_P)$ to $P[-1]$. To see that split exact complexes are projective objects, consider the special case $0 \to P_1 \cong P_0 \to 0$.

Exercise 2.2.2 Use the previous exercise 2.2.1 to show that if $\mathcal{A}$ has enough projectives, then so does the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes over $\mathcal{A}$.

Definition 2.2.4 Let M be an object of $\mathcal{A}$. A *left resolution* of M is a complex $P_.$ with $P_i = 0$ for $i < 0$, together with a map $\epsilon: P_0 \to M$ so that the augmented complex

$$\cdots \xrightarrow{d} P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\epsilon} M \to 0$$

is exact. It is a *projective resolution* if each P_i is projective.

Lemma 2.2.5 *Every R-module M has a projective resolution. More generally, if an abelian category $\mathcal{A}$ has enough projectives, then every object M in $\mathcal{A}$ has a projective resolution.*

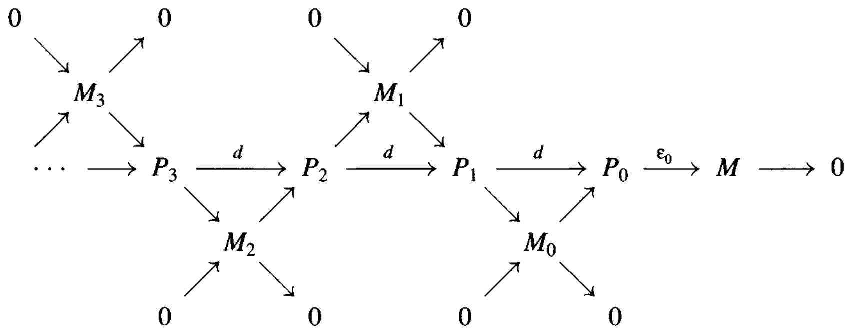

Figure 2.1. Forming a resolution by splicing.

Proof Choose a projective P_0 and a surjection $\epsilon_0 \colon P_0 \to M$, and set $M_0 = \ker(\epsilon_0)$. Inductively, given a module M_{n-1}, we choose a projective P_n and a surjection $\epsilon_n \colon P_n \to M_{n-1}$. Set $M_n = \ker(\epsilon_n)$, and let d_n be the composite $P_n \to M_{n-1} \to P_{n-1}$. Since $d_n(P_n) = M_{n-1} = \ker(d_{n-1})$, the chain complex $P_.$ is a resolution of M. (See Figure 2.1.) ◇

Exercise 2.2.3 Show that if $P_.$ is a complex of projectives with $P_i = 0$ for $i < 0$, then a map $\epsilon \colon P_0 \to M$ giving a resolution for M is the same thing as a chain map $\epsilon \colon P_. \to M$, where M is considered as a complex concentrated in degree zero.

Comparison Theorem 2.2.6 *Let $P_. \xrightarrow{\epsilon} M$ be a projective resolution of M and $f' \colon M \to N$ a map in $\mathcal{A}$. Then for every resolution $Q_. \xrightarrow{\eta} N$ of N there is a chain map $f \colon P_. \to Q_.$ lifting f' in the sense that $\eta \circ f_0 = f' \circ \epsilon$. The chain map f is unique up to chain homotopy equivalence.*

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow & 0 \\
 & & \exists\downarrow & & \exists\downarrow & & \exists\downarrow & & \downarrow f' & & \\
\cdots & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \xrightarrow{\eta} & N & \longrightarrow & 0
\end{array}$$

Porism 2.2.7 The proof will make it clear that the hypothesis that $P \to M$ be a projective resolution is too strong. It suffices to be given a chain complex

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

with the P_i projective. Then for every resolution $Q \to N$ of N, every map $M \to N$ lifts to a map $P \to Q$, which is unique up to chain homotopy. This stronger version of the Comparison Theorem will be used in section 2.7 to construct the external product for Tor.

Proof We will construct the f_n and show their uniqueness by induction on n, thinking of f_{-1} as f'. Inductively, suppose f_i has been constructed for $i \le n$ so that $f_{i-1}d = df_i$. In order to construct f_{n+1} we consider the n-cycles of P and Q. If $n = -1$, we set $Z_{-1}(P) = M$ and $Z_{-1}(Q) = N$; if $n \ge 0$, the fact that $f_{n-1}d = df_n$ means that f_n induces a map f'_n from $Z_n(P)$ to $Z_n(Q)$. Therefore we have two diagrams with exact rows

$$\begin{array}{ccccccc} \cdots \xrightarrow{d} & P_{n+1} & \xrightarrow{d} & Z_n(P) & \longrightarrow 0 \\ & \exists\downarrow & & \downarrow f'_n & \\ \cdots \longrightarrow & Q_{n+1} & \xrightarrow{d} & Z_n(Q) & \longrightarrow 0 \end{array} \quad \text{and} \quad \begin{array}{ccccccc} 0 \longrightarrow & Z_n(P) & \longrightarrow & P_n & \longrightarrow & P_{n-1} \\ & \downarrow f'_n & & \downarrow f_n & & \downarrow f_{n-1} \\ 0 \longrightarrow & Z_n(Q) & \longrightarrow & Q_n & \longrightarrow & Q_{n-1} \end{array}$$

The universal lifting property of the projective P_{n+1} yields a map f_{n+1} from P_{n+1} to Q_{n+1}, so that $df_{n+1} = f'_n d = f_n d$. This finishes the inductive step and proves that the chain map $f: P \to Q$ exists.

To see uniqueness of f up to chain homotopy, suppose that $g: P \to Q$ is another lift of f' and set $h = f - g$; we will construct a chain contraction $\{s_n: P_n \to Q_{n+1}\}$ of h by induction on n. If $n < 0$, then $P_n = 0$, so we set $s_n = 0$. If $n = 0$, note that since $\eta h_0 = \epsilon(f' - f') = 0$, the map h_0 sends P_0 to $Z_0(Q) = d(Q_1)$. We use the lifting property of P_0 to get a map $s_0: P_0 \to Q_1$ so that $h_0 = ds_0 = ds_0 + s_{-1}d$. Inductively, we suppose given maps $s_i (i < n)$ so that $ds_{n-1} = h_{n-1} - s_{n-2}d$ and consider the map $h_n - s_{n-1}d$ from P_n to Q_n. We compute that

$$d(h_n - s_{n-1}d) = dh_n - (h_{n-1} - s_{n-2}d)d = (dh - hd) + s_{n-2}dd = 0.$$

Therefore $h_n - s_{n-1}d$ lands in $Z_n(Q)$, a quotient of Q_{n+1}. The lifting property of P_n yields the desired map $s_n: P_n \to Q_{n+1}$ such that $ds_n = h_n - s_{n-1}d$. $\diamond$

$$\begin{array}{ccccc} & & P_n & & \\ & \exists\swarrow & \downarrow h-sd & & \\ Q_{n+1} & \xrightarrow{d} & Z_n(Q) & \longrightarrow & 0 \end{array} \quad \text{and} \quad \begin{array}{ccccc} P_n & \xrightarrow{d} & P_{n-1} & \xrightarrow{d} & P_{n-2} \\ \downarrow h & \swarrow s & \downarrow h & \swarrow s & \\ Q_n & \longrightarrow & Q_{n-1} & & \end{array}$$

Here is another way to construct projective resolutions. It is called the Horseshoe Lemma because we are required to fill in the horseshoe-shaped diagram.

Horseshoe Lemma 2.2.8 *Suppose given a commutative diagram*

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
\cdots & P_2' & \longrightarrow & P_1' & \longrightarrow & P_0' & \xrightarrow{\epsilon'} & A' & \longrightarrow 0 \\
 & & & & & & & \downarrow{i_A} & \\
 & & & & & & & A & \\
 & & & & & & & \downarrow{\pi_A} & \\
\cdots & P_2'' & \longrightarrow & P_1'' & \longrightarrow & P_0'' & \xrightarrow{\epsilon''} & A'' & \longrightarrow 0 \\
 & & & & & & & \downarrow & \\
 & & & & & & & 0 &
\end{array}$$

where the column is exact and the rows are projective resolutions. Set $P_n = P_n' \oplus P_n''$. Then the P_n assemble to form a projective resolution P of A, and the right-hand column lifts to an exact sequence of complexes

$$0 \to P' \xrightarrow{i} P \xrightarrow{\pi} P'' \to 0,$$

where $i_n \colon P_n' \to P_n$ and $\pi_n \colon P_n \to P_n''$ are the natural inclusion and projection, respectively.

Proof Lift ϵ'' to a map $P_0'' \to A$; the direct sum of this with the map $i_A\epsilon' \colon P_0' \to A$ gives a map $\epsilon \colon P_0 \to A$. The diagram $(*)$ below commutes.

$$(*) \qquad \begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(\epsilon') & \longrightarrow & P_0' & \xrightarrow{\epsilon'} & A' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(\epsilon) & \longrightarrow & P_0 & \xrightarrow{\epsilon} & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(\epsilon'') & \longrightarrow & P_0'' & \xrightarrow{\epsilon''} & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The right two columns of $(*)$ are short exact sequences. The Snake Lemma 1.3.2 shows that the left column is exact and that $\text{coker}(\epsilon) = 0$, so that P_0 maps onto A. This finishes the initial step and brings us to the situation

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ \cdots & \longrightarrow & P'_1 & \xrightarrow{d'} & \ker(\epsilon') & \longrightarrow & 0 \\ & & & & \downarrow & & \\ & & & & \ker(\epsilon) & & \\ & & & & \downarrow & & \\ \cdots & \longrightarrow & P''_1 & \xrightarrow{d''} & \ker(\epsilon'') & \longrightarrow & 0 \\ & & & & \downarrow & & \\ & & & & 0. & & \end{array}$$

The filling in of the "horseshoe" now proceeds by induction. ◇

Exercise 2.2.4 Show that there are maps $\lambda_n \colon P''_n \to P'_{n-1}$ so that

$$d = \begin{bmatrix} d' & \lambda \\ 0 & d'' \end{bmatrix}, \quad \text{i.e.,} \quad d' \begin{bmatrix} p' \\ p'' \end{bmatrix} = \begin{bmatrix} d'(p') + \lambda(p'') \\ d''(p'') \end{bmatrix}.$$

2.3 Injective Resolutions

An object I in an abelian category $\mathcal{A}$ is *injective* if it satisfies the following universal lifting property: Given an injection $f \colon A \to B$ and a map $\alpha \colon A \to I$, there exists at least one map $\beta \colon B \to I$ such that $\alpha = \beta \circ f$.

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B \\ & & \alpha \downarrow & \swarrow \exists\beta & \\ & & I & & \end{array}$$

We say that $\mathcal{A}$ *has enough injectives* if for every object A in $\mathcal{A}$ there is an injection $A \to I$ with I injective. Note that if $\{I_\alpha\}$ is a family of injectives, then the product $\prod I_\alpha$ is also injective. The notion of injective module was invented by R. Baer in 1940, long before projective modules were thought of.

Baer's Criterion 2.3.1 *A right R-module E is injective if and only if for every right ideal J of R, every map $J \to E$ can be extended to a map $R \to E$.*

Proof The "only if" direction is a special case of the definition of injective. Conversely, suppose given an R-module B, a submodule A and a map $\alpha: A \to E$. Let $\mathcal{E}$ be the poset of all extensions $\alpha': A' \to E$ of α to an intermediate submodule $A \subseteq A' \subseteq B$; the partial order is that $\alpha' \leq \alpha''$ if α'' extends α'. By Zorn's lemma there is a maximal extension $\alpha': A' \to E$ in $\mathcal{E}$; we have to show that $A' = B$. Suppose there is some $b \in B$ not in A'. The set $J = \{r \in R : br \in A'\}$ is a right ideal of R. By assumption, the map $J \xrightarrow{b} A' \xrightarrow{\alpha'} E$ extends to a map $f: R \to E$. Let A'' be the submodule $A' + bR$ of B and define $\alpha'': A'' \to E$ by

$$\alpha''(a + br) = \alpha'(a) + f(r), \quad a \in A' \text{ and } r \in R.$$

This is well defined because $\alpha'(br) = f(r)$ for br in $A' \cap bR$, and α'' extends α', contradicting the existence of b. Hence $A' = B$. ◇

Exercise 2.3.1 Let $R = \mathbb{Z}/m$. Use Baer's criterion to show that R is an injective R-module. Then show that $\mathbb{Z}/d$ is *not* an injective R-module when $d|m$ and some prime p divides both d and m/d. (The hypothesis ensures that $\mathbb{Z}/m \neq \mathbb{Z}/d \oplus \mathbb{Z}/e$.)

Corollary 2.3.2 *Suppose that $R = \mathbb{Z}$, or more generally that R is a principal ideal domain. An R-module A is injective iff it is* divisible, *that is, for every $r \neq 0$ in R and every $a \in A$, $a = br$ for some $b \in A$.*

Example 2.3.3 The divisible abelian groups $\mathbb{Q}$ and $\mathbb{Z}_{p^\infty} = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ are injective ($\mathbb{Z}[\frac{1}{p}]$ is the group of rational numbers of the form a/p^n, $n \geq 1$). Every injective abelian group is a direct sum of these [KapIAB,section 5]. In particular, the injective abelian group $\mathbb{Q}/\mathbb{Z}$ is isomorphic to $\oplus\mathbb{Z}_{p^\infty}$.

We will now show that **Ab** has enough injectives. If A is an abelian group, let $I(A)$ be the product of copies of the injective group $\mathbb{Q}/\mathbb{Z}$, indexed by the set $\text{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z})$. Then $I(A)$ is injective, being a product of injectives, and there is a canonical map $e_A: A \to I(A)$. This is our desired injection of A into an injective by the following exercise.

Exercise 2.3.2 Show that e_A is an injection. *Hint:* If $a \in A$, find a map $f: a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ with $f(a) \neq 0$ and extend f to a map $f': A \to \mathbb{Q}/\mathbb{Z}$.

Exercise 2.3.3 Show that an abelian group A is zero iff $\mathrm{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z}) = 0$.

Now it is a fact, easily verified, that if $\mathcal{A}$ is an abelian category, then the opposite category $\mathcal{A}^{op}$ is also abelian. The definition of injective is dual to that of projective, so we immediately can deduce the following results (2.3.4–2.3.7) by arguing in $\mathcal{A}^{op}$.

Lemma 2.3.4 *The following are equivalent for an object I in an abelian category $\mathcal{A}$:*

1. *I is injective in $\mathcal{A}$.*
2. *I is projective in $\mathcal{A}^{op}$.*
3. *The contravariant functor $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact, that is, it takes short exact sequences in $\mathcal{A}$ to short exact sequences in $\mathbf{Ab}$.*

Definition 2.3.5 Let M be an object of $\mathcal{A}$. A *right resolution* of M is a cochain complex $I^{\cdot}$ with $I^i = 0$ for $i < 0$ and a map $M \to I^0$ such that the augmented complex

$$0 \to M \to I^0 \xrightarrow{d} I^1 \xrightarrow{d} I^2 \xrightarrow{d} \cdots$$

is exact. This is the same as a cochain map $M \to I^{\cdot}$, where M is considered as a complex concentrated in degree 0. It is called an *injective resolution* if each I^i is injective.

Lemma 2.3.6 *If the abelian category $\mathcal{A}$ has enough injectives, then every object in $\mathcal{A}$ has an injective resolution.*

Comparison Theorem 2.3.7 *Let $N \to I^{\cdot}$ be an injective resolution of N and $f'\colon M \to N$ a map in $\mathcal{A}$. Then for every resolution $M \to E^{\cdot}$ there is a cochain map $F\colon E^{\cdot} \to I^{\cdot}$ lifting f'. The map f is unique up to cochain homotopy equivalence.*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M & \longrightarrow & E^0 & \longrightarrow & E^1 & \longrightarrow & E^2 & \longrightarrow & \cdots \\
 & & f'\downarrow & & \exists\downarrow & & \exists\downarrow & & \exists\downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & I^0 & \longrightarrow & I^1 & \xrightarrow{\eta} & I^2 & \longrightarrow & \cdots
\end{array}$$

Exercise 2.3.4 Show that I is an injective object in the category of chain complexes iff I is a split exact complex of injectives. Then show that if $\mathcal{A}$ has enough injectives, so does the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes over $\mathcal{A}$. *Hint:* $\mathbf{Ch}(\mathcal{A})^{op} \approx \mathbf{Ch}(\mathcal{A}^{op})$.

We now show that there are enough injective R-modules for every ring R. Recall that if A is an abelian group and B is a left R-module, then $\mathrm{Hom}_{\mathbf{Ab}}(B, A)$ is a right R-module via the rule $fr: b \mapsto f(rb)$.

Lemma 2.3.8 *For every right R-module M, the natural map*

$$\tau: \mathrm{Hom}_{\mathbf{Ab}}(M, A) \to \mathrm{Hom}_{\mathbf{mod}-R}(M, \mathrm{Hom}_{\mathbf{Ab}}(R, A))$$

is an isomorphism, where $(\tau f)(m)$ is the map $r \mapsto f(mr)$.

Proof We define a map μ backwards as follows: If $g: M \to \mathrm{Hom}(R, A)$ is an R-module map, μg is the abelian group map sending m to $g(m)(1)$. Since $\tau(\mu g) = g$ and $\mu\tau(f) = f$ (check this!), τ is an isomorphism. ◇

Definition 2.3.9 A pair of functors $L: \mathcal{A} \to \mathcal{B}$ and $R: \mathcal{B} \to \mathcal{A}$ are *adjoint* if there is a natural bijection for all A in $\mathcal{A}$ and B in $\mathcal{B}$:

$$\tau = \tau_{AB} : \mathrm{Hom}_{\mathcal{B}}(L(A), B) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(A, R(B)).$$

Here "natural" means that for all $f: A \to A'$ in $\mathcal{A}$ and $g: B \to B'$ in $\mathcal{B}$ the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{B}}(L(A'), B) & \xrightarrow{Lf^*} & \mathrm{Hom}_{\mathcal{B}}(L(A), B) & \xrightarrow{g_*} & \mathrm{Hom}_{\mathcal{B}}(L(A), B') \\
\downarrow \tau & & \downarrow \tau & & \downarrow \tau \\
\mathrm{Hom}_{\mathcal{A}}(A', R(B)) & \xrightarrow{f^*} & \mathrm{Hom}_{\mathcal{A}}(A, R(B)) & \xrightarrow{Rg_*} & \mathrm{Hom}_{\mathcal{A}}(A, R(B')).
\end{array}$$

We call L the *left adjoint* and R the *right adjoint* of this pair. The above lemma states that the forgetful functor from **mod**–R to **Ab** has $\mathrm{Hom}_{\mathbf{Ab}}(R, -)$ as its right adjoint.

Proposition 2.3.10 *If an additive functor $R: \mathcal{B} \to \mathcal{A}$ is right adjoint to an exact functor $L: \mathcal{A} \to \mathcal{B}$ and I is an injective object of $\mathcal{B}$, then $R(I)$ is an injective object of $\mathcal{A}$. (We say that R* preserves injectives.*)*

Dually, if an additive functor $L: \mathcal{A} \to \mathcal{B}$ is left adjoint to an exact functor $R: \mathcal{B} \to \mathcal{A}$ and P is a projective object of $\mathcal{A}$, then $L(P)$ is a projective object of $\mathcal{B}$. (We say that L preserves projectives.*)*

Proof We must show that $\mathrm{Hom}_{\mathcal{A}}(-, R(I))$ is exact. Given an injection $f: A \to A'$ in $\mathcal{A}$ the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{B}}(L(A'), I) & \xrightarrow{Lf^*} & \mathrm{Hom}_{\mathcal{B}}(L(A), I) \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{Hom}_{\mathcal{A}}(A', R(I)) & \xrightarrow{f^*} & \mathrm{Hom}_{\mathcal{A}}(A, R(I)) \end{array}$$

commutes by naturality of τ. Since L is exact and I is injective, the top map Lf^* is onto. Hence the bottom map f^* is onto, proving that $R(I)$ is an injective object in $\mathcal{A}$. $\diamond$

Corollary 2.3.11 *If I is an injective abelian group, then* $\mathrm{Hom}_{\mathbf{Ab}}(R, I)$ *is an injective R-module.*

Exercise 2.3.5 If M is an R-module, let $I(M)$ be the product of copies of $I_0 = \mathrm{Hom}_{\mathbf{Ab}}(R, \mathbb{Q}/\mathbb{Z})$, indexed by the set $\mathrm{Hom}_R(M, I_0)$. There is a canonical map $e_M: M \to I(M)$; show that e_M is an injection. Being a product of injectives, $I(M)$ is injective, so this will prove that R–**mod** has enough injectives. An important consequence of this is that every R-module has an injective resolution.

Example 2.3.12 The category Sheaves(X) of abelian group sheaves (1.6.5) on a topological space X has enough injectives. To see this, we need two constructions. The *stalk* of a sheaf $\mathcal{F}$ at a point $x \in X$ is the abelian group $\mathcal{F}_x = \varinjlim\{\mathcal{F}(U) : x \in U\}$. "Stalk at x" is an exact functor from Sheaves(X) to **Ab**. If A is any abelian group, the *skyscraper sheaf* x_*A at the point $x \in X$ is defined to be the presheaf

$$(x_*A)(U) = \begin{cases} A & \text{if } x \in U \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 2.3.6 Show that x_*A is a sheaf and that

$$\mathrm{Hom}_{\mathbf{Ab}}(\mathcal{F}_x, A) \cong \mathrm{Hom}_{\text{Sheaves(X)}}(\mathcal{F}, x_*A)$$

for every sheaf $\mathcal{F}$. Use 2.3.10 to conclude that if A_x is an injective abelian group, then $x_*(A_x)$ is an injective object in Sheaves(X) for each x, and that $\prod_{x\in X} x_*(A_x)$ is also injective.

Given a fixed sheaf $\mathcal{F}$, choose an injection $\mathcal{F}_x \to I_x$ with I_x injective in **Ab** for each $x \in X$. Combining the natural maps $\mathcal{F} \to x_*\mathcal{F}_x$ with $x_*\mathcal{F}_x \to x_*I_x$ yields a map from $\mathcal{F}$ to the injective sheaf $\mathcal{I} = \prod_{x\in X} x_*(I_x)$. The map $\mathcal{F} \to \mathcal{I}$ is an injection (see [Gode], for example) showing that Sheaves(X) has enough injectives.

Example 2.3.13 Let I be a small category and $\mathcal{A}$ an abelian category. If the product of any set of objects exists in $\mathcal{A}$ ($\mathcal{A}$ is *complete*) and $\mathcal{A}$ has enough injectives, we will show that the functor category $\mathcal{A}^I$ has enough injectives. For each k in I, the k^{th} coordinate $A \mapsto A(k)$ is an exact functor from $\mathcal{A}^I$ to $\mathcal{A}$. Given A in $\mathcal{A}$, define the functor $k_*A\colon I \to \mathcal{A}$ by sending $i \in I$ to

$$k_*A(i) = \prod_{\mathrm{Hom}_I(i,k)} A.$$

If $\eta\colon i \to j$ is a map in I, the map $k_*A(i) \to k_*A(j)$ is determined by the index map $\eta^*\colon \mathrm{Hom}(j,k) \to \mathrm{Hom}(i,k)$. That is, the coordinate $k_*A(i) \to A$ of this map corresponding to $\varphi \in \mathrm{Hom}(j,k)$ is the projection of $k_*A(i)$ onto the factor corresponding to $\eta^*\varphi = \varphi\eta \in \mathrm{Hom}(i,k)$. If $f\colon A \to B$ is a map in $\mathcal{A}$, there is a corresponding map $k_*A \to k_*B$ defined slotwise. In this way, k_* becomes an additive functor from $\mathcal{A}$ to $\mathcal{A}^I$, assuming that $\mathcal{A}$ has enough products for k_*A to be defined.

Exercise 2.3.7 Assume that $\mathcal{A}$ is complete and has enough injectives. Show that k_* is right adjoint to the k^{th} coordinate functor, so that k_* preserves injectives by 2.3.10. Given $F \in \mathcal{A}^I$, embed each $F(k)$ in an injective object A_k of $\mathcal{A}$, and let $F \to k_*A_k$ be the corresponding adjoint map. Show that the product $E = \prod_{k\in I} k_*A_k$ exists in $\mathcal{A}^I$, that E is an injective object, and that $F \to E$ is an injection. Conclude that $\mathcal{A}^I$ has enough injectives.

Exercise 2.3.8 Use the isomorphism $(\mathcal{A}^I)^{op} \cong \mathcal{A}^{(I^{op})}$ to dualize the previous exercise. That is, assuming that $\mathcal{A}$ is cocomplete and has enough projectives, show that $\mathcal{A}^I$ has enough projectives.

2.4 Left Derived Functors

Let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor between two abelian categories. If $\mathcal{A}$ has enough projectives, we can construct the *left derived functors* L_iF $(i \geq 0)$ of F as follows. If A is an object of $\mathcal{A}$, choose (once and for all) a projective resolution $P \to A$ and define

$$L_iF(A) = H_i(F(P)).$$

Note that since $F(P_1) \to F(P_0) \to F(A) \to 0$ is exact, we always have $L_0F(A) \cong F(A)$. The aim of this section is to show that the L_*F form a universal homological δ-functor.

Lemma 2.4.1 *The objects $L_iF(A)$ of $\mathcal{B}$ are well defined up to natural isomorphism. That is, if $Q \to A$ is a second projective resolution, then there is a canonical isomorphism:*

$$L_iF(A) = H_i(F(P)) \xrightarrow{\cong} H_i(F(Q)).$$

In particular, a different choice of the projective resolutions would yield new functors $\hat{L}_iF$, which are naturally isomorphic to the functors L_iF.

Proof By the Comparison Theorem (2.2.6), there is a chain map $f\colon P \to Q$ lifting the identity map id_A, yielding a map f_* from $H_iF(P)$ to $H_iF(Q)$. Any other such chain map $f'\colon P \to Q$ is a chain homotopic to f, so $f_* = f'_*$. Therefore, the map f_* is canonical. Similarly, there is a chain map $g\colon Q \to P$ lifting id_A and a map g_*. Since gf and id_P are both chain maps $P \to P$ lifting id_A, we have

$$g_*f_* = (gf)_* = (\mathrm{id}_P)_* = \text{ identity map on } H_iF(P).$$

Similarly, fg and id_Q both lift id_A, so f_*g_* is the identity. This proves that f_* and g_* are isomorphisms. ◇

Corollary 2.4.2 *If A is projective, then $L_iF(A) = 0$ for $i \neq 0$.*

F-Acyclic Objects 2.4.3 An object Q is called *F-acyclic* if $L_iF(Q) = 0$ for all $i \neq 0$, that is, if the higher derived functors of F vanish on Q. Clearly, projectives are F-acyclic for every right exact functor F, but there are others; flat modules are acyclic for tensor products, for example. An *F-acyclic resolution* of A is a left resolution $Q \to A$ for which each Q_i is F-acyclic. We will see later (using dimension shifting, exercise 2.4.3 and 3.2.8) that we can also compute left derived functors from F-acyclic resolutions, that is, that $L_i(A) \cong H_i(F(Q))$ for any F-acyclic resolution Q of A.

Lemma 2.4.4 *If $f\colon A' \to A$ is any map in $\mathcal{A}$, there is a natural map $L_iF(f)$: $L_iF(A') \to L_iF(A)$ for each i.*

Proof Let $P' \to A'$ and $P \to A$ be the chosen projective resolutions. The comparison theorem yields a lift of f to a chain map $\tilde{f}$ from P' to P, hence a map $\tilde{f}_*$ from $H_iF(P')$ to $H_iF(P)$. Any other lift is chain homotopic to $\tilde{f}$, so the map $\tilde{f}_*$ is independent of the choice of $\tilde{f}$. The map $L_iF(f)$ is $\tilde{f}_*$. ◇

Exercise 2.4.1 Show that $L_0F(f) = F(f)$ under the identification $L_0F(A) \cong F(A)$.

Theorem 2.4.5 *Each L_iF is an additive functor from $\mathcal{A}$ to $\mathcal{B}$.*

Proof The identity map on P lifts the identity on A, so $L_iF(id_A)$ is the identity map. Given maps $A' \xrightarrow{f} A \xrightarrow{g} A''$ and chain maps $\tilde{f}, \tilde{g}$ lifting f and g, the composite $\tilde{g}\tilde{f}$ lifts gf. Therefore $g_* f_* = (gf)_*$, proving that L_iF is a functor. If $f_i: A' \to A$ are two maps with lifts $\tilde{f}_i$, the sum $\tilde{f}_1 + \tilde{f}_2$ lifts $f_1 + f_2$. Therefore $f_{1*} + f_{2*} = (f_1 + f_2)_*$, proving that L_iF is additive. ◇

Exercise 2.4.2 (Preserving derived functors) If $U: \mathcal{B} \to \mathcal{C}$ is an exact functor, show that

$$U(L_iF) \cong L_i(UF).$$

Forgetful functors such as **mod**–$R \to$ **Ab** are often exact, and it is often easier to compute the derived functors of UF due to the absence of cluttering restrictions.

Theorem 2.4.6 *The derived functors L_*F form a homological δ-functor.*

Proof Given a short exact sequence

$$0 \to A' \to A \to A'' \to 0,$$

choose projective resolutions $P' \to A'$ and $P'' \to A''$. By the Horseshoe Lemma 2.2.8, there is a projective resolution $P \to A$ fitting into a short exact sequence $0 \to P' \to P \to P'' \to 0$ of projective complexes in $\mathcal{A}$. Since the P''_n are projective, each sequence $0 \to P'_n \to P_n \to P''_n \to 0$ is split exact. As F is additive, each sequence

$$0 \to F(P'_n) \to F(P_n) \overset{\longleftarrow}{\longrightarrow} F(P''_n) \to 0$$

is split exact in $\mathcal{B}$. Therefore

$$0 \to F(P') \to F(P) \to F(P'') \to 0$$

is a short exact sequence of chain complexes. Writing out the corresponding long exact homology sequence, we get

$$\cdots \xrightarrow{\partial} L_iF(A') \to L_iF(A) \to L_iF(A'') \xrightarrow{\partial} L_{i-1}F(A') \to L_{i-1}F(A) \to L_{i-1}F(A'') \xrightarrow{\partial} \cdots$$

To see the naturality of the ∂_i, assume we are given a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & f'\downarrow & & f\downarrow & & \downarrow f'' & & \\ 0 & \longrightarrow & B' & \underset{i_B}{\longrightarrow} & B & \underset{\pi_B}{\longrightarrow} & B'' & \longrightarrow & 0 \end{array}$$

in $\mathcal{A}$, and projective resolutions of the corners: $\epsilon'\colon P' \to A'$, $\epsilon''\colon P'' \to A''$, $\eta'\colon Q' \to B'$ and $\eta''\colon Q'' \to B''$. Use the Horseshoe Lemma 2.2.8 to get projective resolutions $\epsilon\colon P \to A$ and $\eta\colon Q \to B$. Use the Comparison Theorem 2.2.6 to obtain chain maps $F'\colon P' \to Q'$ and $F''\colon P'' \to Q''$ lifting the maps f' and f'', respectively. We shall show that there is also a chain map $F\colon P \to Q$ lifting f, and giving a commutative diagram of chain complexes with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & P'' & \longrightarrow & 0 \\ & & F'\downarrow & & F\downarrow & & \downarrow F'' & & \\ 0 & \longrightarrow & Q' & \longrightarrow & Q & \longrightarrow & Q'' & \longrightarrow & 0. \end{array}$$

The naturality of the connecting homomorphism in the long exact homology sequence now translates into the naturality of the ∂_i. In order to produce F, we will construct maps (*not* chain maps) $\gamma_n\colon P''_n \to Q'_n$ such that F_n is

$$F_n = \begin{bmatrix} F'_n & \gamma_n \\ 0 & F''_n \end{bmatrix}\colon \begin{array}{c} P'_n \\ \oplus \\ P''_n \end{array} \longrightarrow \begin{array}{c} Q'_n \\ \oplus \\ Q''_n \end{array}$$

$$F_n(p', p'') = (F'(p') + \gamma(p''), F''(p'')).$$

Assuming that F is a chain map over f, this choice of F will yield our commutative diagram of chain complexes. In order for F to be a lifting of f, the map $(\eta F_0 - f\epsilon)$ from $P_0 = P'_0 \oplus P''_0$ to B must vanish. On P'_0 this is no problem, so this just requires that

$$i_B\eta'\gamma_0 = f\lambda_P - \lambda_Q F''_0$$

as maps from P''_0 to B, where λ_P and λ_Q are the restrictions of ϵ and η to P''_0 and Q''_0, and i_B is the inclusion of B' in B. There is some map $\beta\colon P''_0 \to B'$ so that $i_B\beta = f\lambda - \lambda F''_0$ because in B'' we have

$$\pi_B(f\lambda - \lambda F''_0) = f''\pi_A\lambda_P - \pi_B\lambda F''_0 = f''\epsilon'' - \eta'' F''_0 = 0.$$

We may therefore define γ_0 to be any lift of β to Q'_0.

$$\begin{array}{ccccc} & & P''_0 & & \\ & \gamma_0 \swarrow & \downarrow \beta & & \\ Q'_0 & \xrightarrow{\eta'} & B' & \longrightarrow & 0 \end{array}$$

In order for F to be a chain map, we must have

$$dF - Fd = \left[\begin{pmatrix} d' & \lambda \\ 0 & d'' \end{pmatrix}, \begin{pmatrix} F' & \gamma \\ 0 & F'' \end{pmatrix}\right]$$
$$= \begin{pmatrix} d'F' - F'd' & (d'\gamma - \gamma d'' + \lambda F'' - F'\lambda') \\ 0 & d''F'' - F''d'' \end{pmatrix}$$

vanishing. That is, the map $d'\gamma_n \colon P''_n \to Q'_{n-1}$ must equal

$$g_n = \gamma_{n-1} d'' - \lambda_n F'_n + F''_{n-1}\lambda_n.$$

Inductively, we may suppose γ_i defined for $i < n$, so that g_n exists. A short calculation, using the inductive formula for $d'\gamma_{n-1}$, shows that $d'g_n = 0$. As the complex Q' is exact, the map g_n factors through a map $\beta \colon P''_n \to d(Q'_n)$. We may therefore define γ_n to be any lift of β to Q'_n. This finishes the construction of the chain map F and the proof. ◇

Exercise 2.4.3 (Dimension shifting) If $0 \to M \to P \to A \to 0$ is exact with P projective (or F-acyclic 2.4.3), show that $L_iF(A) \cong L_{i-1}F(M)$ for $i \geq 2$ and that $L_1F(A)$ is the kernel of $F(M) \to F(P)$. More generally, show that if

$$0 \to M_m \to P_m \to P_{m-1} \to \cdots \to P_0 \to A \to 0$$

is exact with the P_i projective (or F-acyclic), then $L_iF(A) \cong L_{i-m-1}F(M_m)$ for $i \geq m+2$ and $L_{m+1}F(A)$ is the kernel of $F(M_m) \to F(P_m)$. Conclude that if $P \to A$ is an F-acyclic resolution of A, then $L_iF(A) = H_i(F(P))$.

The object M_m, which obviously depends on the choices made, is called the m^{th} *syzgy* of A. The word "syzygy" comes from astronomy, where it was originally used to describe the alignment of the Sun, Earth, and Moon.

Theorem 2.4.7 *Assume that $\mathcal{A}$ has enough projectives. Then for any right exact functor $F \colon \mathcal{A} \to \mathcal{B}$, the derived functors L_nF form a universal δ-functor.*

Remark This result was first proven in [CE, III.5], but is commonly attributed to [Tohoku], where the term "universal δ-functor" first appeared.

Proof Suppose that T_* is a homological δ-functor and that $\varphi_0: T_0 \to F$ is given. We need to show that φ_0 admits a unique extension to a morphism $\varphi: T_* \to L_*F$ of δ-functors. Suppose inductively that $\varphi_i: T_i \to L_iF$ are already defined for $0 \le i < n$, and that they commute with all the appropriate δ_i's. Given A in $\mathcal{A}$, select an exact sequence $0 \to K \to P \to A \to 0$ with P projective. Since $L_nF(P) = 0$, this yields a commutative diagram with exact rows:

$$\begin{array}{ccccccc} & & T_n(A) & \xrightarrow{\delta_n} & T_{n-1}(K) & \longrightarrow & T_{n-1}(P) \\ & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_{n-1}} \\ 0 & \longrightarrow & L_nF(A) & \xrightarrow{\delta_n} & L_{n-1}F(K) & \longrightarrow & L_{n-1}F(P). \end{array}$$

A diagram chase reveals that there exists a *unique* map $\varphi_n(A)$ from $T_n(A)$ to $L_nF(A)$ commuting with the given δ_n's. We need to show that φ_n is a natural transformation commuting with all δ_n's for all short exact sequences.

To see that φ_n is a natural transformation, suppose given $f: A' \to A$ and an exact sequence $0 \to K' \to P' \to A' \to 0$ with P' projective. As P' is projective we can lift f to $g: P' \to P$, which induces a map $h: K' \to K$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K' & \longrightarrow & P' & \longrightarrow & A' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

To see that φ_n commutes with f, we note that in the following diagram that each small quadrilateral commutes.

$$\begin{array}{ccccccc} T_n(A') & & & \xrightarrow{T_n(f)} & & & T_n(A) \\ & \searrow{\scriptstyle \delta} & & & & \swarrow{\scriptstyle \delta} & \\ & & T_{n-1}(K') & \xrightarrow{T_{n-1}(h)} & T_{n-1}(K) & & \\ \downarrow{\scriptstyle \varphi_n(A')} & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_n(A)} \\ & & L_{n-1}F(K') & \xrightarrow{L_{n-1}F(h)} & L_{n-1}F(K) & & \\ & \nearrow{\scriptstyle \delta} & & & & \nwarrow{\scriptstyle \delta} & \\ L_nF(A') & & & \xrightarrow{L_nF(f)} & & & L_nF(A) \end{array}$$

A chase reveals that

$$\delta \circ L_n(f) \circ \varphi_n(A') = \delta \circ \varphi_n(A) \circ T_n(f).$$

Because $\delta: L_nF(A) \to L_{n-1}F(K)$ is monic, we can cancel it from the equation to see that the outer square commutes, that is, that φ_n is a natural transformation. Incidentally, this argument (with $A = A'$ and $f = id_A$) also shows that $\varphi_n(A)$ doesn't depend on the choice of P.

Finally, we need to verify that φ_n commutes with δ_n. Given a short exact sequence $0 \to A' \to A \to A'' \to 0$ and a chosen exact sequence $0 \to K'' \to P'' \to A'' \to 0$ with P'' projective, we can construct maps f and g making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K'' & \longrightarrow & P'' & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow g & & \downarrow f & & \| & & \\ 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \end{array}$$

commute. This yields a commutative diagram

$$\begin{array}{ccccc} T_n(A'') & \xrightarrow{\delta} & T_{n-1}(K'') & \xrightarrow{T(g)} & T_{n-1}(A') \\ \varphi_n \downarrow & & \downarrow \varphi_{n-1} & & \downarrow \varphi_{n-1} \\ L_nF(A'') & \xrightarrow{\delta} & L_{n-1}F(K'') & \xrightarrow{LF(g)} & L_{n-1}F(A'). \end{array}$$

Since the horizontal composites are the δ_n maps of the bottom row, this implies the desired commutativity relation. ◇

Exercise 2.4.4 Show that homology $H_*: \mathbf{Ch}_{\geq 0}(\mathcal{A}) \to \mathcal{A}$ and cohomology $H^*: \mathbf{Ch}^{\geq 0}(\mathcal{A}) \to \mathcal{A}$ are universal δ-functors. *Hint:* Copy the proof above, replacing P by the mapping cone cone(A) of exercise 1.5.1.

Exercise 2.4.5 ([Tohoku]) An additive functor $F: \mathcal{A} \to \mathcal{B}$ is called *effaceable* if for each object A of $\mathcal{A}$ there is a monomorphism $u: A \to I$ such that $F(u) = 0$. We call F *coeffaceable* if for every A there is a surjection $u: P \to A$ such that $F(u) = 0$. Modify the above proof to show that if T_* is a homological δ-functor such that each T_n is coeffaceable (except T_0), then T_* is universal. Dually, show that if T^* is a cohomological δ-functor such that each T^n is effaceable (except T^0), then T^* is universal.

2.5 Right Derived Functors

2.5.1 Let $F: \mathcal{A} \to \mathcal{B}$ be a left exact functor between two abelian categories. If $\mathcal{A}$ has enough injectives, we can construct the *right derived functors*

$R^iF (i \geq 0)$ of F as follows. If A is an object of $\mathcal{A}$, choose an injective resolution $A \to I^\cdot$ and define

$$R^iF(A) = H^i(F(I)).$$

Note that since $0 \to F(A) \to F(I^0) \to F(I^1)$ is exact, we always have $R^0F(A) \cong F(A)$.

Since F also defines a right exact functor $F^{op}: \mathcal{A}^{op} \to \mathcal{B}^{op}$, and $\mathcal{A}^{op}$ has enough projectives, we can construct the left derived functors L_iF^{op} as well. Since $I^\cdot$ becomes a projective resolution of A in $\mathcal{A}^{op}$, we see that

$$R^iF(A) = (L_iF^{op})^{op}(A).$$

Therefore all the results about right exact functors apply to left exact functors. In particular, the objects $R^iF(A)$ are independent of the choice of injective resolutions, R^*F is a universal cohomological δ-functor, and $R^iF(I) = 0$ for $i \neq 0$ whenever I is injective. Calling an object Q *F-acyclic* if $R^iF(Q) = 0\ (i \neq 0)$, as in 2.4.3, we see that the right derived functors of F can also be computed from F-acyclic resolutions.

Definition 2.5.2 (Ext functors) For each R-module A, the functor $F(B) = \text{Hom}_R(A, B)$ is left exact. Its right derived functors are called the *Ext* groups:

$$\text{Ext}^i_R(A, B) = R^i\,\text{Hom}_R(A, -)(B).$$

In particular, $\text{Ext}^0(A, B)$ is $\text{Hom}(A, B)$, and injectives are characterized by Ext via the following exercise.

Exercise 2.5.1 Show that the following are equivalent.

1. B is an *injective* R-module.
2. $\text{Hom}_R(-, B)$ is an exact functor.
3. $\text{Ext}^i_R(A, B)$ vanishes for all $i \neq 0$ and all A (B is $\text{Hom}_R(-, B)$-acyclic for all A).
4. $\text{Ext}^1_R(A, B)$ vanishes for all A.

The behavior of Ext with respect to the variable A characterizes projectives.

Exercise 2.5.2 Show that the following are equivalent.

1. A is a *projective* R-module.
2. $\text{Hom}_R(A, -)$ is an exact functor.
3. $\text{Ext}^i_R(A, B)$ vanishes for all $i \neq 0$ and all B (A is $\text{Hom}_R(-, B)$-acyclic for all B).
4. $\text{Ext}^1_R(A, B)$ vanishes for all B.

The notion of derived functor has obvious variations for contravariant functors. For example, let F be a contravariant left exact functor from $\mathcal{A}$ to $\mathcal{B}$. This is the same as a covariant left exact functor from $\mathcal{A}^{op}$ to $\mathcal{B}$, so if $\mathcal{A}$ has enough projectives (i.e., $\mathcal{A}^{op}$ has enough injectives), we can define the right derived functors $R^*F(A)$ to be the cohomology of $F(P_.)$, $P_. \to A$ being a projective resolution in $\mathcal{A}$. This too is a universal δ-functor with $R^0F(A) = F(A)$, and $R^iF(P) = 0$ for $i \neq 0$ whenever P is projective.

Example 2.5.3 For each R-module B, the functor $G(A) = \mathrm{Hom}_R(A, B)$ is contravariant and left exact. It is therefore entitled to right derived functors $R^*G(A)$. However, we will see in 2.7.6 that these are just the functors $\mathrm{Ext}^*(A, B)$. That is,

$$R^* \mathrm{Hom}(-, B)(A) \cong R^* \mathrm{Hom}(A, -)(B) = \mathrm{Ext}^*(A, B).$$

Application 2.5.4 Let X be a topological space. The *global sections* functor Γ from Sheaves(X) to **Ab** is the functor $\Gamma(\mathcal{F}) = \mathcal{F}(X)$. It turns out (see 2.6.1 and exercise 2.6.3 below) that Γ is right adjoint to the constant sheaves functor, so Γ is left exact. The right derived functors of Γ are the *cohomology functors* on X:

$$H^i(X, \mathcal{F}) = R^i\Gamma(\mathcal{F}).$$

The cohomology of a sheaf is arguably the central notion in modern algebraic geometry. For more details about sheaf cohomology, we refer the reader to [Hart].

Exercise 2.5.3 Let X be a topological space and $\{A_x\}$ any family of abelian groups, parametrized by the points $x \in X$. Show that the skyscraper sheaves $x_*(A_x)$ of 2.3.12 as well as their product $\mathcal{F} = \Pi x_*(A_x)$ are Γ-acyclic, that is, that $H^i(X, \mathcal{F}) = 0$ for $i \neq 0$. This shows that sheaf cohomology can also be computed from resolutions by products of skyscraper sheaves.

2.6 Adjoint Functors and Left/Right Exactness

We begin with a useful trick for constructing left and right exact functors.

Theorem 2.6.1 *Let $L: \mathcal{A} \to \mathcal{B}$ and $R: \mathcal{B} \to \mathcal{A}$ be an adjoint pair of additive functors. That is, there is a natural isomorphism*

$$\tau: \mathrm{Hom}_{\mathcal{B}}(L(A), B) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(A, R(B)).$$

Then L is right exact, and R is left exact.

Proof Suppose that $0 \to B' \to B \to B'' \to 0$ is exact in $\mathcal{B}$. By naturality of τ there is a commutative diagram for every A in $\mathcal{A}$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(L(A), B') & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(L(A), B) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(L(A), B'') \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A, R(B')) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A, R(B)) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A, R(B'')) \end{array}$$

The top row is exact because $\mathrm{Hom}(LA, -)$ is left exact, so the bottom row is exact for all A. By the Yoneda Lemma 1.6.11,

$$0 \to R(B') \to R(B) \to R(B'')$$

must be exact. This proves that every right adjoint R is left exact. In particular $L^{op}: \mathcal{A}^{op} \to \mathcal{B}^{op}$ (which is a right adjoint) is left exact, that is, L is right exact. $\diamond$

Remark Left adjoints have left derived functors, and right adjoints have right derived functors. This of course assumes that $\mathcal{A}$ has enough projectives, and that $\mathcal{B}$ has enough injectives for the derived functors to be defined.

Application 2.6.2 Let R be a ring and B a left R-module. The following standard proposition shows that $\otimes_R B$: **mod**–$R \to$ **Ab** is left adjoint to $\mathrm{Hom}_{\mathbf{Ab}}(B, -)$, so $\otimes_R B$ is right exact. More generally, if S is another ring, and B is an R–S bimodule, then $\otimes_R B$ takes **mod**–R to **mod**–S and is a left adjoint, so it is right exact.

Proposition 2.6.3 *If B is an R–S bimodule and C a right S-module, then* $\mathrm{Hom}_S(B, C)$ *is naturally a right R-module by the rule $(fr)(b) = f(rb)$ for $f \in \mathrm{Hom}(B, C)$, $r \in R$ and $b \in B$. The functor* $\mathrm{Hom}_S(B, -)$ *from* **mod**–S *to* **mod**–R *is right adjoint to $\otimes_R B$. That is, for every R-module A and S-module C there is a natural isomorphism*

$$\tau: \mathrm{Hom}_S(A \otimes_R B, C) \xrightarrow{\cong} \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)).$$

Proof Given $f: A \otimes_R B \to C$, we define $(\tau f)(a)$ as the map $b \mapsto f(a \otimes b)$ for each $a \in A$. Given $g: A \to \mathrm{Hom}_S(B, C)$, we define $\tau^{-1}(g)$ to be the map defined by the bilinear form $a \otimes b \mapsto g(a)(b)$. We leave the verification that

$\tau(f)(a)$ is an S-module map, that $\tau(f)$ is an R-module map, $\tau^{-1}(g)$ is an R-module map, τ is an isomorphism with inverse τ^{-1}, and that τ is natural as an exercise for the reader. ◇

Definition 2.6.4 Let B be a left R-module, so that $T(A) = A \otimes_R B$ is a right exact functor from **mod**–R to **Ab**. We define the abelian groups

$$\mathrm{Tor}_n^R(A, B) = (L_n T)(A).$$

In particular, $\mathrm{Tor}_0^R(A, B) \cong A \otimes_R B$. Recall that these groups are computed by finding a projective resolution $P \to A$ and taking the homology of $P \otimes_R B$. In particular, if A is a projective R-module, then $\mathrm{Tor}_n(A, B) = 0$ for $n \neq 0$.

More generally, if B is an R–S bimodule, we can think of $T(A) = A \otimes_R B$ as a right exact functor landing in **mod**–S, so we can think of the $\mathrm{Tor}_n^R(A, B)$ as S-modules. Since the forgetful functor U from **mod**–S to **Ab** is exact, this generalization does not change the underlying abelian groups, it merely adds an S-module structure, because $U(L_* \otimes B) \cong L_* U(\otimes B)$ as derived functors.

The reader may notice that the functor $A \otimes_R$ is also right exact, so we could also form the derived functors $L_*(A \otimes_R)$. We will see in section 2.7 that this yields nothing new in the sense that $L_*(A \otimes_R)(B) \cong L_*(\otimes_R B)(A)$.

Application 2.6.5 Now we see why the inclusion "incl" of Sheaves(X) into Presheaves(X) is a left exact functor, as claimed in 1.6.7; it is the right adjoint to the sheafification functor. The fact that sheafification is right exact is automatic; it is a theorem that sheafification is exact.

Exercise 2.6.1 Show that the derived functor $R^i(\mathrm{incl})$ sends a sheaf $\mathcal{F}$ to the presheaf $U \mapsto H^i(U, \mathcal{F}|U)$, where $\mathcal{F}|U$ is the restriction of $\mathcal{F}$ to U and H^i is the sheaf cohomology of 2.5.4. *Hint:* Compose $R^i(\mathrm{incl})$ with the exact functors Presheaves$(X) \to$ **Ab** sending $\mathcal{F}$ to $\mathcal{F}(U)$.

Application 2.6.6 Let $f: X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathcal{F}$ on X, we define the *direct image sheaf* $f_*\mathcal{F}$ on Y by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}V)$ for every open V in Y. (*Exercise:* Show that $f_*\mathcal{F}$ is a sheaf!) For any sheaf $\mathcal{G}$ on Y, we define the *inverse image sheaf* $f^{-1}\mathcal{G}$ to be the sheafification of the presheaf sending an open set U in X to the direct limit $\varinjlim \mathcal{G}(V)$ over the poset of all open sets V in Y containing $f(U)$. The following exercise shows that f^{-1} is right exact and that f_* is left exact because they are adjoint. The derived functors $R^i f_*$ are called the *higher direct image sheaf functors* and also play a key role in algebraic geometry. (See [Hart] for more details.)

Exercise 2.6.2 Show that for any sheaf $\mathcal{F}$ on X there is a natural map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$, and that for any sheaf $\mathcal{G}$ on Y there is a natural map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$. Conclude that f^{-1} and f_* are adjoint to each other, that is, that there is a natural isomorphism

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

Exercise 2.6.3 Let $*$ denote the one-point space, so that Sheaves$(*) \cong$ **Ab**.

1. If $f: X \to *$ is the collapse map, show that f_* and f^{-1} are the global sections functor Γ and the constant sheaves functor, respectively. This proves that Γ is right adjoint to the constant sheaves functor. By 2.6.1, Γ is left exact, as asserted in 2.5.4.
2. If $x: * \to X$ is the inclusion of a point in X, show that x_* and x^{-1} are the skyscraper sheaf and stalk functors of 2.3.12.

Application 2.6.7 (Colimits) Let I be a fixed category. There is a diagonal functor Δ from every category $\mathcal{A}$ to the functor category $\mathcal{A}^I$; if $A \in \mathcal{A}$, then ΔA is the constant functor: $(\Delta A)_i = A$ for all i. Recall that the *colimit* of a functor $F: I \to \mathcal{A}$ is an object of $\mathcal{A}$, written $\mathrm{colim}_{i \in I} F_i$, together with a natural transformation from F to $\Delta(\mathrm{colim}\ F_i)$, which is universal among natural transformations $F \to \Delta A$ with $A \in \mathcal{A}$. (See the appendix or [MacCW, III.3].) This universal property implies that colim is a functor from $\mathcal{A}^I$ to $\mathcal{A}$, at least when the colimit exists for all $F: I \to \mathcal{A}$.

Exercise 2.6.4 Show that colim is left adjoint to Δ. Conclude that colim is a right exact functor when $\mathcal{A}$ is abelian (and colim exists). Show that pushout (the colimit when I is $\cdot \leftarrow \cdot \to \cdot$) is not an exact functor in **Ab**.

Proposition 2.6.8 *The following are equivalent for an abelian category $\mathcal{A}$:*

1. *The direct sum $\oplus A_i$ exists in $\mathcal{A}$ for every set $\{A_i\}$ of objects in $\mathcal{A}$.*
2. *$\mathcal{A}$ is* cocomplete, *that is,* $\mathrm{colim}_{i \in I} A_i$ *exists in $\mathcal{A}$ for each functor $A: I \to \mathcal{A}$ whose indexing category I has only a set of objects.*

Proof As (1) is a special case of (2), we assume (1) and prove (2). Given $A: I \to \mathcal{A}$, the cokernel C of

$$\bigoplus_{\varphi: i \to j} A_i \longrightarrow \bigoplus_{i \in I} A_i$$

$$a_i[\varphi] \mapsto \varphi(a_i) - a_i$$

solves the universal problem defining the colimit, so $C = \underset{i \in I}{\mathrm{colim}}\ A_i$. ◇

Remark **Ab**, **mod**–R, Presheaves(X), and Sheaves(X) are cocomplete because (1) holds. (If I is infinite, the direct sum in Sheaves(X) is the sheafification of the direct sum in Presheaves(X)). The category of finite abelian groups has only *finite* direct sums, so it is not cocomplete.

Variation 2.6.9 (Limits) The limit of a functor $A: I \to \mathcal{A}$ is the colimit of the corresponding functor $A^{op}: I^{op} \to \mathcal{A}^{op}$, so all the above remarks apply in dual form to limits. In particular, lim: $\mathcal{A}^I \to \mathcal{A}$ is right adjoint to the diagonal functor Δ, so lim is a left exact functor when it exists. If the product ΠA_i of every set $\{A_i\}$ of objects exists in $\mathcal{A}$, then $\mathcal{A}$ is *complete*, that is, $\lim_{i \in I} A_i$ exists for every $A: I \to \mathcal{A}$ with I having only a set of objects. **Ab**, **mod**–R, Presheaves(X), and Sheaves(X) are complete because such products exist.

One of the most useful properties of adjoint functors is the following result, which we quote without proof from [MacCW, V.5].

Adjoints and Limits Theorem 2.6.10 *Let $L: \mathcal{A} \to \mathcal{B}$ be left adjoint to a functor $R: \mathcal{B} \to \mathcal{A}$, where $\mathcal{A}$ and $\mathcal{B}$ are arbitrary categories. Then*

1. *L preserves all colimits (coproducts, direct limits, cokernels, etc.). That is, if $A: I \to \mathcal{A}$ has a colimit, then so does $LA: I \to \mathcal{B}$, and*

$$L(\operatorname*{colim}_{i \in I} A_i) = \operatorname*{colim}_{i \in I} L(A_i).$$

2. *R preserves all limits (products, inverse limits, kernels, etc.). That is, if $B: I \to \mathcal{B}$ has a limit, then so does $RB: I \to \mathcal{A}$, and*

$$R(\lim_{i \in I} B_i) = \lim_{i \in I} R(B_i).$$

Here are two consequences that use the fact that homology commutes with arbitrary direct sums of chain complexes. (Homology does not commute with arbitrary colimits; the derived functors of colim intervene via a spectral sequence.)

Corollary 2.6.11 *If a cocomplete abelian category $\mathcal{A}$ has enough projectives, and $F: \mathcal{A} \to \mathcal{B}$ is a left adjoint, then for every set $\{A_i\}$ of objects in $\mathcal{A}$:*

$$L_*F\left(\bigoplus_{i \in I} A_i\right) \cong \bigoplus_{i \in I} L_*F(A_i).$$

Proof If $P_i \to A_i$ are projective resolutions, then so is $\oplus P_i \to \oplus A_i$. Hence

$$L_*F(\oplus A_i) = H_*(F(\oplus P_i)) \cong H_*(\oplus F(P_i)) \cong \oplus H_*(F(P_i)) = \oplus L_*F(A_i). \ \diamond$$

Corollary 2.6.12 $\operatorname{Tor}_*(A, \oplus_{i\in I} B_i) = \oplus_{i\in I} \operatorname{Tor}_*(A, B_i)$.

Proof If $P \to A$ is a projective resolution, then

$$\begin{aligned}\operatorname{Tor}_*(A, \oplus B_i) = H_*(P \otimes (\oplus B_i)) &\cong H_*(\oplus(P \otimes B_i)) \cong \oplus H_*(P \otimes B_i)\\ &= \oplus Tor_*(A, B_i).\end{aligned}$$ ◇

Definition 2.6.13 A nonempty category I is called *filtered* if

1. For every $i, j \in I$ there are arrows $\begin{smallmatrix} i \searrow \\ j \nearrow \end{smallmatrix} k$ to some $k \in I$.
2. For every two parallel arrows $u, v: i \rightrightarrows j$ there is an arrow $w: j \to k$ such that $wu = wv$.

A *filtered colimit* in $\mathcal{A}$ is just the colimit of a functor $A: I \to \mathcal{A}$ in which I is a filtered category. We shall use the notation $\underrightarrow{\operatorname{colim}}(A_i)$ for such a filtered colimit.

If I is a partially ordered set (poset), considered as a category, then condition (1) always holds, and (2) just requires that every pair of elements has an upper bound in I. A filtered poset is often called *directed;* filtered colimits over directed posets are often called *direct limits* and are often written $\underrightarrow{\lim} A_i$.

We are going to show that direct limits and filtered colimits of modules are exact. First we obtain a more concrete description of the elements of $\underrightarrow{\operatorname{colim}}(A_i)$.

Lemma 2.6.14 *Let I be a filtered category and $A: I \to$ **mod**–R a functor. Then*

1. *Every element $a \in \underrightarrow{\operatorname{colim}}(A_i)$ is the image of some element $a_i \in A_i$ (for some $i \in I$) under the canonical map $A_i \to \underrightarrow{\operatorname{colim}}(A_i)$.*
2. *For every i, the kernel of the canonical map $A_i \to \underrightarrow{\operatorname{colim}}(A_i)$ is the union of the kernels of the maps $\varphi: A_i \to A_j$ (where $\varphi: i \to j$ is a map in I).*

Proof We shall use the explicit construction of $\underrightarrow{\operatorname{colim}}(A_i)$. Let $\lambda_i: A_i \to \oplus_{i\in I} A_i$ be the canonical maps. Every element a of $\underrightarrow{\operatorname{colim}} A_i$ is the image of

$$\sum_{j\in J} \lambda_j(a_j)$$

for some finite set $J = \{i_1, \cdots, i_n\}$. There is an upper bound i in I for

$\{i_1, \cdots, i_n\}$; using the maps $A_j \to A_i$ we can represent each a_j as an element in A_i and take a_i to be their sum. Evidently, a is the image of a_i, so (1) holds.

Now suppose that $a_i \in A_i$ vanishes in $\varinjlim(A_i)$. Then there are $\varphi_{jk}: j \to k$ in I and $a_{jk} \in A_j$ so that $\lambda_i(a_i) = \sum \lambda_k(\varphi_{jk}(a_j)) - \lambda_j(a_j)$ in $\oplus A_i$. Choose an upper bound t in I for all the i, j, k in this expression. Adding $\lambda_t(\varphi_{it}a_i) - \lambda_i(a_i)$ to both sides we may assume that $i = t$. Adding zero terms of the form

$$[\lambda_t\varphi_{jt}(a_j) - \lambda_k\varphi_{jk}(a_j)] + [\lambda_t\varphi_{jt}(-a_j) - \lambda_k\varphi_{jk}(-a_j)],$$

we can assume that all the k's are t. If any φ_{jt} are parallel arrows in I, then by changing t we can equalize them. Therefore we have

$$\lambda_t(a_t) = \lambda_t(\sum \varphi_{jt}(a_j)) - \sum \lambda_j(a_j)$$

with all the j's distinct and none equal to t. Since the λ_j are injections, all the a_j must be zero. Hence $\varphi_{it}(a_i) = a_t = 0$, that is, $a_i \in \ker(\varphi_{it})$. ◇

Theorem 2.6.15 *Filtered colimits (and direct limits) of R-modules are exact, considered as functors from* $(\mathbf{mod}\text{–}R)^I$ *to* $\mathbf{mod}\text{–}R$.

Proof Set $\mathcal{A} = \mathbf{mod}\text{–}R$. We have to show that if I is a filtered category (*e.g.*, a directed poset), then $\varinjlim: \mathcal{A}^I \to \mathcal{A}$ is exact. Exercise 2.6.4 showed that $\varinjlim$ is right exact, so we need only prove that if $t: A \to B$ is monic in $\mathcal{A}^I$(*i.e.*, each t_i is monic), then $\varinjlim(A_i) \to \varinjlim(B_i)$ is monic in $\mathcal{A}$. Let $a \in \varinjlim(A_i)$ be an element that vanishes in $\varinjlim(B_i)$. By the lemma above, a is the image of some $a_i \in A_i$. Therefore $t_i(a_i) \in B_i$ vanishes in $\varinjlim(B_i)$, so there is some $\varphi: i \to j$ so that

$$0 = \varphi(t_i(a_i)) = t_j(\varphi(a_i)) \text{ in } B_j.$$

Since t_j is monic, $\varphi(a_i) = 0$ in A_j. Hence $a = 0$ in $\varinjlim(A_i)$. ◇

Exercise 2.6.5 (AB5) The above theorem does not hold for every cocomplete abelian category $\mathcal{A}$. Show that if $\mathcal{A}$ is the opposite category $\mathbf{Ab}^{op}$ of abelian groups, then the functor $\varinjlim: \mathcal{A}^I \to \mathcal{A}$ need not be exact when I is filtered.

An abelian category $\mathcal{A}$ is said to satisfy axiom (AB5) if it is cocomplete and filtered colimits are exact. Thus the above theorem states that **mod**–R and R–**mod** satisfy axiom (AB5), and this exercise shows that $\mathbf{Ab}^{op}$ does not.

Exercise 2.6.6 Let $f: X \to Y$ be a continuous map. Show that the inverse image sheaf functor f^{-1}: Sheaves(Y) $\to$ Sheaves(X) is exact. (See 2.6.6.)

The following consequences are proven in the same manner as their counterparts for direct sum. Note that in categories like R–**mod** for which filtered colimits are exact, homology commutes with filtered colimits.

Corollary 2.6.16 *If $\mathcal{A} = R$–**mod** (or $\mathcal{A}$ is any abelian category with enough projectives, satisfying axiom (AB5)), and $F: \mathcal{A} \to \mathcal{B}$ is a left adjoint, then for every $A: I \to \mathcal{A}$ with I filtered*

$$L_*F(\varinjlim(A_i)) \cong \varinjlim L_*F(A).$$

Corollary 2.6.17 *For every filtered $B: I \to R$–**mod** and every $A \in$ **mod**–R,*

$$\mathrm{Tor}_*(A, \varinjlim(B_i)) \cong \varinjlim \mathrm{Tor}_*(A, B_i).$$

2.7 Balancing Tor and Ext

In earlier sections we promised to show that the two left derived functors of $A \otimes_R B$ gave the same result and that the two right derived functors of $\mathrm{Hom}(A, B)$ gave the same result. It is time to deliver on these promises.

Tensor Product of Complexes 2.7.1 Suppose that P and Q are chain complexes of right and left R-modules, respectively. Form the double complex $P \otimes_R Q = \{P_p \otimes_R Q_q\}$ using the sign trick, that is, with horizontal differentials $d \otimes 1$ and vertical differentials $(-1)^p \otimes d$. $P \otimes_R Q$ is called the *tensor product double complex*, and $\mathrm{Tot}^{\oplus}(P \otimes_R Q)$ is called the *(total) tensor product chain complex* of P and Q.

Theorem 2.7.2 $L_n(A \otimes_R)(B) \cong L_n(\otimes_R B)(A) = \mathrm{Tor}_n^R(A, B)$ *for all n.*

Proof Choose a projective resolution $P \xrightarrow{\epsilon} A$ in **mod**–R and a projective resolution $Q \xrightarrow{\eta} B$ in R–**mod**. Thinking of A and B as complexes concentrated in degree zero, we can form the three tensor product double complexes $P \otimes Q$, $A \otimes Q$, and $P \otimes B$. The augmentations ϵ and η induce maps from $P \otimes Q$ to $A \otimes Q$ and $P \otimes B$.

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
A \otimes Q_2 & \vdots & P_0 \otimes Q_2 & \xleftarrow{d} & P_1 \otimes Q_2 & \xleftarrow{d} & \cdots & & \\
\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle -d}\downarrow & & \downarrow & & \\
A \otimes Q_1 & \vdots & P_0 \otimes Q_1 & \xleftarrow{d} & P_1 \otimes Q_1 & \xleftarrow{d} & P_2 \otimes Q_1 & \longleftarrow & \cdots \\
\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle -d}\downarrow & & {\scriptstyle d}\downarrow & & \\
A \otimes Q_0 & \vdots & P_0 \otimes Q_0 & \xleftarrow{d} & P_1 \otimes Q_0 & \xleftarrow{d} & P_2 \otimes Q_0 & \longleftarrow & \cdots \\
\hline
 & & P_0 \otimes B & \xleftarrow{d} & P_1 \otimes B & \xleftarrow{d} & P_2 \otimes B & \longleftarrow & \cdots
\end{array}
$$

Using the Acyclic Assembly Lemma 2.7.3, we will show that the maps

$$A \otimes Q = \mathrm{Tot}(A \otimes Q) \xleftarrow{\epsilon \otimes Q} \mathrm{Tot}(P \otimes Q) \xrightarrow{P \otimes \eta} \mathrm{Tot}(P \otimes B) = P \otimes B$$

are quasi-isomorphisms, inducing the promised isomorphisms on homology:

$$L_*(A \otimes_R)(B) \xleftarrow{\cong} H_*(\mathrm{Tot}(P \otimes Q)) \xrightarrow{\cong} L_*(\otimes_R B)(A).$$

Consider the double complex C obtained from $P \otimes Q$ by adding $A \otimes Q[-1]$ in the column $p = -1$. The translate $\mathrm{Tot}(C)[1]$ is the mapping cone of the map $\epsilon \otimes Q$ from $\mathrm{Tot}(P \otimes Q)$ to $A \otimes Q$ (see 1.2.8 and 1.5.1), so in order to show that $\epsilon \otimes Q$ is a quasi-isomorphism, it suffices to show that $\mathrm{Tot}(C)$ is acyclic. Since each $\otimes Q_q$ is an exact functor, every row of C is exact, so $\mathrm{Tot}(C)$ is exact by the Acyclic Assembly Lemma.

Similarly, the mapping cone of $P \otimes \eta \colon \mathrm{Tot}(P \otimes Q) \to P \otimes B$ is the translate $\mathrm{Tot}(D)[1]$, where D is the double complex obtained from $P \otimes Q$ by adding $P \otimes B[-1]$ in the row $q = -1$. Since each $P_p \otimes$ is an exact functor, every column of D is exact, so $\mathrm{Tot}(D)$ is exact by the Acyclic Assembly Lemma 2.7.3. Hence $\mathrm{cone}(P \otimes \eta)$ is acyclic, and $P \otimes \eta$ is also a quasi-isomorphism.

◇

Acyclic Assembly Lemma 2.7.3 *Let C be a double complex in* **mod**–R. *Then*

- $\operatorname{Tot}^{\Pi}(C)$ *is an acyclic chain complex, assuming either of the following:*
  1. *C is an upper half-plane complex with exact columns.*
  2. *C is a right half-plane complex with exact rows.*
- $\operatorname{Tot}^{\oplus}(C)$ *is an acyclic chain complex, assuming either of the following:*
  3. *C is an upper half-plane complex with exact rows.*
  4. *C is a right half-plane complex with exact columns.*

Remark The proof will show that in (1) and (3) it suffices to have every diagonal bounded on the lower right, and in (2) and (4) it suffices to have every diagonal bounded on the upper left. See 5.5.1 and 5.5.10.

Proof We first show that it suffices to establish case (1). Interchanging rows and columns also interchanges (1) and (2), and (3) and (4), so (1) implies (2) and (4) implies (3). Suppose we are in case (4), and let $\tau_n C$ be the double subcomplex of C obtained by truncating each column at level n:

$$(\tau_n C)_{pq} = \begin{cases} C_{pq} & \text{if } q > n \\ \ker(d^v\colon C_{pn} \to C_{p,n-1}) & \text{if } q = n \\ 0 & \text{if } q < n \end{cases}.$$

Each $\tau_n C$ is, up to vertical translation, a first quadrant double complex with exact columns, so (1) implies that $\operatorname{Tot}^{\oplus}(\tau_n C) = \operatorname{Tot}^{\Pi}(\tau_n C)$ is acyclic. This implies that $\operatorname{Tot}^{\oplus}(C)$ is acyclic, because every cycle of $\operatorname{Tot}^{\oplus}(C)$ is a cycle (hence a boundary) in some subcomplex $\operatorname{Tot}^{\oplus}(\tau_n C)$. Therefore (1) implies (4) as well.

In case (1), translating C left and right, suffices to prove that $H_0(\operatorname{Tot}(C))$ is zero. Let

$$c = (\cdots, c_{-p,p}, \cdots, c_{-2,2}, c_{-1,1}, c_{0,0}) \in \prod C_{-p,p} = \operatorname{Tot}(C)_0$$

be a 0-cycle; we will find elements $b_{-p,p+1}$ by induction on p so that

$$d^v(b_{-p,p+1}) + d^h(b_{-p+1,p}) = c_{-p,p}.$$

Assembling the b's will yield an element b of $\prod C_{-p,p+1}$ such that $d(b) = c$, proving that $H_0(\operatorname{Tot}(C)) = 0$. The following schematic should help give the idea.

$$
\begin{array}{ccccccccc}
b_{-p,p+1} & & & & & & & & \\
\downarrow & & & & & & & & \\
c_{-p,p} & \longleftarrow & b_{-p+1,p} & & & & & & \\
 & & \downarrow & & & & & & \\
 & & c_{-p+1,p-1} & \longleftarrow & b_{-p+2,p-1} & & & & \\
 & & & & \vdots & & & & \\
 & & & & \downarrow & & & & \\
 & & & & c_{-2,2} & \longleftarrow & b_{-1,2} & & \\
 & & & & & & \downarrow & & \\
 & & & & & & c_{-1,1} & \longleftarrow & b_{01} \\
 & & & & & & & & \downarrow \\
 & & & & & & & & c_{00} \longleftarrow 0(=b_{10}) \\
 & & & & & & & & \downarrow \\
 & & & & & & & & 0
\end{array}
$$

We begin the induction by choosing $b_{10} = 0$ for $p = -1$. Since $C_{0,-1} = 0$, $d^v(c_{00}) = 0$; since the 0^{th} column is exact, there is a $b_{01} \in C_{01}$ so that $d^v(b_{01}) = c_{00}$. Inductively, we compute that

$$
\begin{aligned}
d^v(c_{-p,p} - d^h(b_{-p+1,p})) &= d^v(c_{-pp}) + d^h d^v(b_{-p+1,p}) \\
&= d^v(c_{-pp}) + d^h(c_{-p+1,p-1}) - d^h d^h(b_{-p+2,p-1}) \\
&= 0.
\end{aligned}
$$

Since the $-p^{th}$ column is exact, there is a $b_{-p,p+1}$ so that

$$d^v(b_{-p,p+1}) = c_{-p,p} - d^h(b_{-p+1,p})$$

as desired. ◇

Exercise 2.7.1 Let C be the periodic upper half-plane complex with $C_{pq} = \mathbb{Z}/4$ for all p and $q \geq 0$, all differentials being multiplication by 2.

$$\begin{array}{ccccccccc}
 & & \downarrow 2 & & \downarrow 2 & & \downarrow 2 & & \\
\cdots & \xleftarrow{2} & \mathbb{Z}/4 & \xleftarrow{2} & \mathbb{Z}/4 & \xleftarrow{2} & \mathbb{Z}/4 & \xleftarrow{2} & \cdots \\
 & & \downarrow 2 & & \downarrow 2 & & \downarrow 2 & & \\
\cdots & \xleftarrow{2} & \mathbb{Z}/4 & \xleftarrow{2} & \mathbb{Z}/4 & \xleftarrow{2} & \mathbb{Z}/4 & \xleftarrow{2} & \cdots
\end{array}$$

1. Show that $H_0(\mathrm{Tot}^{\Pi}(C)) \cong \mathbb{Z}/2$ on the cycle $(\ldots, 1, 1, 1) \in \prod C_{-p,p}$ even though the rows of C are exact. *Hint:* First show that the 0-boundaries are $\prod 2\mathbb{Z}/4$.
2. Show that $\mathrm{Tot}^{\oplus}(C)$ is acyclic.
3. Now extend C downward to form a doubly periodic plane double complex D with $D_{pq} = \mathbb{Z}/4$ for all $p, q \in \mathbb{Z}$. Show that $H_0(\mathrm{Tot}^{\Pi}(D))$ maps onto $H_0(\mathrm{Tot}^{\Pi} C) \cong \mathbb{Z}/2$. Hence $\mathrm{Tot}^{\Pi}(D)$ is not acyclic, even though every row and column of D is exact. Finally, show that $\mathrm{Tot}^{\oplus}(D)$ is acyclic.

Exercise 2.7.2

1. Give an example of a 2^{nd} quadrant double chain complex C with exact columns for which $\mathrm{Tot}^{\oplus}(C)$ is not an acyclic chain complex.
2. Give an example of a 4^{th} quadrant double complex C with exact columns for which $\mathrm{Tot}^{\Pi}(C)$ is not acyclic.

Hom Cochain Complex 2.7.4 Given a chain complex P and a cochain complex I, form the double cochain complex $\mathrm{Hom}(P, I) = \{\mathrm{Hom}(P_p, I^q)\}$ using a variant of the sign trick. That is, if $f: P_p \to I^q$, then $d^h f: P_{p+1} \to I^q$ by $(d^h f)(p) = f(dp)$, while we define $d^v f: P_p \to I^{q+1}$ by

$$(d^v f)(p) = (-1)^{p+q+1} d(fp) \quad \text{for } p \in P_p.$$

$\mathrm{Hom}(P, I)$ is called the *Hom double complex*, and $\mathrm{Tot}^{\Pi}(\mathrm{Hom}(P, I))$ is called the *(total) Hom cochain complex*. *Warning:* Different conventions abound in the literature. Bourbaki [BX] converts $\mathrm{Hom}(P, I)$ into a double chain complex and obtains a total Hom chain complex. Others convert I into a chain complex Q with $Q_q = I^{-q}$ and form $\mathrm{Hom}(P, Q)$ as a chain complex, and so on.

Morphisms and Hom 2.7.5 To explain our sign convention, suppose that C and D are two chain complexes. If we reindex D as a cochain complex, then an n-cycle f of $\mathrm{Hom}(C, D)$ is a sequence of maps $f_p: C_p \to D^{n-p} = D_{p-n}$

such that $f_p d = (-1)^n d f_{p+1}$, that is, a morphism of chain complexes from C to the translate $D[-n]$ of D. An n-boundary is a morphism f that is null homotopic. Thus $H^n \operatorname{Hom}(C, D)$ is the group of chain homotopy equivalence classes of morphisms $C \to D[-n]$, the morphisms in the quotient category $\mathbf{K}$ of the category of chain complexes discussed in exercise 1.4.5.

Similarly, if X and Y are cochain complexes, we may form $\operatorname{Hom}(X, Y)$ by reindexing X. Our conventions about reindexing and translation ensure that once again an n-cycle of $\operatorname{Hom}(X, Y)$ is a morphism $X \to Y[-n]$ and that $H^n \operatorname{Hom}(X, Y)$ is the group of chain homotopy equivalence classes of such morphisms. We will return to this point in Chapter 10 when we discuss $\mathbf{R}\operatorname{Hom}$ in the derived category $\mathbf{D}(\mathcal{A})$.

Exercise 2.7.3 To see why $\operatorname{Tot}^{\oplus}$ is used for the tensor product $P \otimes_R Q$ of right and left R-module complexes, while Tot^{Π} is used for Hom, let I be a cochain complex of abelian groups. Show that there is a natural isomorphism of double complexes:

$$\operatorname{Hom}_{\mathbf{Ab}}(\operatorname{Tot}^{\oplus}(P \otimes_R Q), I) \cong \operatorname{Hom}_R(P, \operatorname{Tot}^{\Pi}(\operatorname{Hom}_{\mathbf{Ab}}(Q, I)).$$

Theorem 2.7.6 *For every pair of R-modules A and B, and all n,*

$$\operatorname{Ext}^n_R(A, B) = R^n \operatorname{Hom}_R(A, -)(B) \cong R^n \operatorname{Hom}_R(-, B)(A).$$

Proof Choose a projective resolution P of A and an injective resolution I of B. Form the first quadrant double cochain complex $\operatorname{Hom}(P, I)$. The augmentations induce maps from $\operatorname{Hom}(A, I)$ and $\operatorname{Hom}(P, B)$ to $\operatorname{Hom}(P, I)$. As in the proof of 2.7.2, the mapping cones of $\operatorname{Hom}(A, I) \to \operatorname{Tot}(\operatorname{Hom}(P, I))$ and $\operatorname{Hom}(P, B) \to \operatorname{Tot}(\operatorname{Hom}(P, I))$ are translates of the total complexes obtained from $\operatorname{Hom}(P, I)$ by adding $\operatorname{Hom}(A, I)[-1]$ and $\operatorname{Hom}(P, B)[-1]$, respectively. By the Acyclic Assembly Lemma 2.7.3 (or rather its dual), both mapping cones are exact. Therefore the maps

$$\operatorname{Hom}(A, I) \to \operatorname{Tot}(\operatorname{Hom}(P, I)) \leftarrow \operatorname{Hom}(P, B)$$

are quasi-isomorphisms. Taking cohomology yields the result:

$$\begin{aligned} R^* \operatorname{Hom}(A, -)(B) &= H^* \operatorname{Hom}(A, I) \\ &\cong H^* \operatorname{Tot}(\operatorname{Hom}(P, I)) \\ &\cong H^* \operatorname{Hom}(P, B) = R^* \operatorname{Hom}(-, B)(A). \end{aligned}$$ ◇

$$
\begin{array}{c|ccccccc}
\vdots & \vdots & & \vdots & & & & \\
\uparrow & \uparrow & & \uparrow & & & & \\
\mathrm{Hom}(A,I^2) & \mathrm{Hom}(P_0,I^2) & \longrightarrow & \mathrm{Hom}(P_1,I^2) & \longrightarrow & \cdots & & \\
\uparrow & \uparrow & & \uparrow & & \uparrow & & \\
\mathrm{Hom}(A,I^1) & \mathrm{Hom}(P_0,I^1) & \longrightarrow & \mathrm{Hom}(P_1,I^1) & \longrightarrow & \mathrm{Hom}(P_2,I^1) & \longrightarrow & \cdots \\
\uparrow & \uparrow & & \uparrow & & \uparrow & & \\
\mathrm{Hom}(A,I^0) & \mathrm{Hom}(P_0,I^0) & \longrightarrow & \mathrm{Hom}(P_1,I^0) & \longrightarrow & \mathrm{Hom}(P_2,I^0) & \longrightarrow & \cdots \\
\hline
 & \mathrm{Hom}(P_0,B) & \longrightarrow & \mathrm{Hom}(P_1,B) & \longrightarrow & \mathrm{Hom}(P_2,B) & \longrightarrow & \cdots
\end{array}
$$

Definition 2.7.7 ([CE]) In view of the two above theorems, the following definition seems natural. Let T be a left exact functor of p "variable" modules, some covariant and some contravariant. T will be called *right balanced* under the following conditions:

1. When any one of the covariant variables of T is replaced by an injective module, T becomes an exact functor in each of the remaining variables.
2. When any one of the contravariant variables of T is replaced by a projective module, T becomes an exact functor in each of the remaining variables. The functor Hom is an example of a right balanced functor, as is $\mathrm{Hom}(A \otimes B, C)$.

Exercise 2.7.4 Show that all p of the right derived functors $R^*T(A_1, \cdots, \hat{A}_i, \cdots, A_p)(A_i)$ of T are naturally isomorphic.

A similar discussion applies to right exact functors T which are *left balanced*. The prototype left balanced functor is $A \otimes B$. In particular, all of the left derived functors associated to a left balanced functor are isomorphic.

Application 2.7.8 (External product for Tor) Suppose that R is a commutative ring and that A, A', B, B' are R-modules. The *external product* is the map

$$\mathrm{Tor}_i(A, B) \otimes_R \mathrm{Tor}_j(A', B') \to \mathrm{Tor}_{i+j}(A \otimes_R A', B \otimes_R B')$$

constructed for every i and j in the following manner. Choose projective resolutions $P \to A$, $P' \to A'$, and $P'' \to A \otimes A'$. The Comparison Theorem 2.2.6 gives a chain map $\text{Tot}(P \otimes P') \to P''$ which is unique up to chain homotopy equivalence. (We saw above that $H_i \,\text{Tot}(P \otimes P') = \text{Tor}_i(A, A')$, so we actually need the version of the Comparison Theorem contained in the porism 2.2.7.) This yields a natural map

$$H_n(P \otimes B \otimes P' \otimes B') \cong H_n(P \otimes P' \otimes B \otimes B') \to H_n(P'' \otimes B \otimes B') \\ = \text{Tor}_n(A \otimes A', B \otimes B').$$

On the other hand, there are natural maps $H_i(C) \otimes H_j(C') \to H_{i+j}\,\text{Tot}(C \otimes C')$ for every pair of complexes C, C'; one maps the tensor product $c \otimes c'$ of cycles $c \in C_i$ and $c' \in C'_j$ to $c \otimes c' \in C_i \otimes C'_j$. (Check this!) The external product is obtained by composing the special case $C = P \otimes B$, $C' = P' \otimes B'$:

$$\text{Tor}_i(A, B) \otimes \text{Tor}_j(A', B') = H_i(P \otimes B) \otimes H_j(P' \otimes B') \to H_{i+j}(P \otimes B \otimes P' \otimes B')$$

with the above map.

Exercise 2.7.5

1. Show that the external product is independent of the choices of P, P', P'' and that it is natural in all four modules A, A', B, B'.
2. Show that the product is associative as a map to $\text{Tor}_*(A \otimes A' \otimes A'', B \otimes B' \otimes B'')$.
3. Show that the external product commutes with the connecting homomorphism δ in the long exact Tor sequences associated to $0 \to B_0 \to B \to B_1 \to 0$.
4. (Internal product) Suppose that A and B are R-algebras. Use (1) and (2) to show that $\text{Tor}_*^R(A, B)$ is a graded R-algebra.