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Level Number Sequences of Trees and the Lambda Algebra

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We enumerate ordered partitions of positive integers, restricted by the condition that the $(j+1)$th entry is not greater than d times the jth entry for all j. We also consider restrictions on the first and/or last entries, and on congruence classes modulo $2d-2$ of all the entries. Such sequences arise as level number sequences of d-ary trees, but we are motivated by constructions arising in algebraic topology.

1. INTRODUCTION

In their paper 'Level number sequences for trees' [4], Flajolet and Prodinger attack the enumeration problem for certain integer sequences; namely, finite sequences of positive integers in which each integer is no greater than twice the preceding one. Their paper has caught the attention of algebraic topologists because similar collections of integer sequences are often found in that field. This paper is a sequel to Flajolet–Prodinger, motivated by the topological constructions.

In Section 2, we give a direct generalization of their theorem: If H_n is the number of sequences of positive integers of length l, sum n, final integer j, initial integer $\leqslant m$, and satisfying the restriction $i_{k+1} \leqslant d i_k$ for each k, then the generating function $H(q) = 1 + H_1 q + H_2 q^2 + \cdots + H_n q^n + \cdots$ for the number with sum n is

$$H(q) = \frac{a(q)}{1-b(q)},$$

where

$$a(q) = \sum_{k\geqslant 0} \left(\frac{q^{e(k)}(1-q^{me(k)})}{1-q^{e(k)}} \prod_{j=0}^{k-1} \frac{-q^{e(j)}}{1-q^{e(j)}} \right)$$

and

$$b(q) = \sum_{k\geqslant 0} \left(\frac{q^{e(k)}}{1-q^{e(k)}} \prod_{j=0}^{k-1} \frac{-q^{e(j)}}{1-q^{e(j)}} \right),$$

where $e(k) = 1 + d + d^2 + d^3 + \cdots + d^k$.

Flajolet and Prodinger treated the case $d=2$, $m=1$. From their point of view, we have generalized from the binary tree to the d-ary forest of up to m trees.

In Section 3 and 4, we take $d=2$ and consider the case in which the final integer j must be odd. We also solve the analogous problem where we count sequences according to $n+l$ instead of n. These sections answer questions in our memoir on the lambda algebra [10].

In Section 5 we show that the method can be stretched to handle the case in which d is odd, and in which all the integers are required to lie in two congruence classes modulo $2d-2$. This solves the problem for the odd-prime lambda algebra and enables us to compute Henn's constants [6] for odd primes.

Finally, in Section 6, we give a quite different derivation of the generating functions. The method is perhaps 'more combinatorial' or 'more elementary'.

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In addition to the topological motivation, the enumeration problems discussed here have application in a variety of places. In two unpublished research reports [2], [11] Clowes, Mitrani and Wilson indicated connections with Minc's entropic cyclic groupoids [8], partitions of unity into powers of 1/2 [9], partitions of a power of 2 by smaller powers of 2 [9], Even and Lempel's proper words [3], and certain prefix codes or Huffman codes [1].

We are indebted to Hans-Werner Henn for calling our attention to the work of Flajolet and Prodinger, to Phil Hanlon for valuable guidance to the literature, and to Philippe Flajolet for his interest, his advice and his encouragement.

2. Direct Generalization

In this section we generalize the Flajolet–Prodinger result from binary to d-ary trees, and allow the initial integer to be $\leq m$. This presents no difficulty, and the best reason for repeating an outline of the proof here is to focus attention on certain parameters in the results.

Let $\mathbf{H_n}$ denote the set of finite sequences $I=(i_1, i_2, \ldots, i_l)$ of positive integers satisfying the conditions

C1: $i_1 \leq m$,

C2: $i_{j+1} \leq di_j \quad (1 \leq j \leq l-1)$,

C3: $i_1 + i_2 + \cdots + i_l = n$.

We say that such a sequence has sum n, length l, initial i_1, final i_l, maximum initial m and multiplier d.

Let $H_n = \text{card}(\mathbf{H_n})$, and let $H(q)$ denote the generating function $1 + H_1 q + H_2 q^2 + \cdots + H_n q^n + \cdots$.

Theorem.

$$H(q) = \frac{a(q)}{1-b(q)}$$

where

$$a(q) = \sum_{k\geq 0} \left(\frac{q^{e(k)}(1-q^{me(k)})}{1-q^{e(k)}} \prod_{j=0}^{k-1} \frac{-q^{e(j)}}{1-q^{e(j)}} \right)$$

and

$$b(q) = \sum_{k\geq 0} \left(\frac{q^{e(k)}}{1-q^{e(k)}} \prod_{j=0}^{k-1} \frac{-q^{e(j)}}{1-q^{e(j)}} \right),$$

where $e(0)=1$ *and*

$$e(k) = e_d(k) = 1 + d + d^2 + d^3 + \cdots + d^k = \frac{d^{k+1}-1}{d-1}.$$

Proof. It is convenient to rewrite $a(q)$ in the form

$$a(q) = \sum_{k\geq 0} (-1)^k (1-q^{me(k)}) \frac{q^{E(k)}}{(1-q)(1-q^{e(1)})(1-q^{e(2)}) \cdots (1-q^{e(k)})}$$

and, similarly,

$$b(q) = \sum_{k\geq 0} (-1)^k \frac{q^{E(k)}}{(1-q)(1-q^{e(1)})(1-q^{e(2)}) \cdots (1-q^{e(k)})},$$

where

$$E(k) = E_d(k) = e(0) + e(1) + \cdots + e(k)$$
$$= \sum_{k=0}^{k} (k+1-i)d^i = \frac{1}{(d-1)^2}[d^{k+2} - (k+2)(d-1) - 1].$$

Putting $m = 1$ and $d = 2$ we recover the Flajolet–Prodinger result [4].

The case $m = \infty$ is of particular interest. In that case we make the convention that $q^\infty = 0$ and so we obtain $a(q) = b(q)$ and therefore

$$H(q) = \frac{b(q)}{1 - b(q)} = b(q) + b(q)^2 + b(q)^3 + \cdots,$$

which is one of the heuristics that led us to the new proof given below in Section 6.

Putting $d = \infty$ and using the same convention we recover the well-known generating function $q(1 - q^m)/(1 - 2q)$ for compositions of n with initial $\leqslant m$ (unrestricted as to the multiplier d).

The proof is a direct generalization of the original one. Introduce $H_{n,j}^{[l]}$ for the number of sequences with sum n, length l and *final* integer $= j$. (We suppress the maximum initial m and the multiplier d from the notation.) Using the bivariate generating functions

$$H^{[k]}(q, u) = \sum_{n \geqslant 1} \sum_{j \geqslant 1} H_{n,j}^{[l]} q^n u^j$$
$$H(q, u) = \sum_{k \geqslant 1} H^{[k]}(q, u)$$

we have $H(q, 1) = H(q)$, the function required in the theorem.

The proof begins with a functional equation.

Lemma.

$$H(q, u) = \frac{qu(1 - q^m u^m)}{1 - qu} + \frac{qu}{1 - qu}[H(q, 1) - H(q, q^d u^d)]$$

with the convention $q^\infty = 0$.

The first term in the above expression represents $H^{[1]}$ since $1 \leqslant i_1 \leqslant m$.

Proof. The proof of the lemma follows Flajolet–Prodinger. As $l \to l + 1$, the generating functions $H^{[l]}(q, u)$ satisfy the substitution relation

$$q^n u^j \to \sum_{k=1}^{d_j} q^{n+k} u^k = q^n \frac{qu}{1 - qu}(1 - q^{dj} u^{dj}). \tag{2.1}$$

Summing over the length l leads to the functional equation of the lemma. □

To finish the proof of the theorem, we parse the lemma as

$$\phi(u) = \lambda(u) + \nu(u)\phi(1) + \mu(u)\phi(\sigma(u)), \tag{2.2}$$

(cf. [4] or [5]) which is solved by iteration to obtain

$$\phi(u) = \sum_{k=0}^{\infty} \left([\lambda(\sigma^k(u)) + \nu(\sigma^k(u))\phi(1)] \prod_{j=0}^{k-1} \mu(\sigma^j(u)) \right) \tag{2.3}$$

and, therefore, putting $u=1$ and solving for $\phi(1)$, and using an abbreviated notation,

$$\phi(1)=\frac{\sum \lambda^k(\prod \mu^j)}{1-\sum \nu^k(\prod \mu^j)}. \tag{2.4}$$

We are following Flajolet and Prodinger except that in (2.2) we have introduced a distinction between λ and ν which will be pertinent in distinguishing cases.

We have $\sigma(u)=q^d u^d$, and hence the kth iterate is

$$\sigma^k(u)=q^{e(k)-1}u^{d^k}. \tag{2.5}$$

Solving for $H(q, 1)=\phi(1)$, we obtain the theorem. □

Note that $b(q)$ is independent of m, because $H^{[1]}$ enters in (2.2)–(2.4) only through $\lambda(u)$. Moreover, the reason that $a(q)=b(q)$ in the case $m=\infty$ is simply that in that case we have $\lambda(u)=\nu(u)$.

We now come to the asymptotics.

Our first observation is that the denominator of the generating function $H(q)$ is independent of m, and that as $d\to\infty$ the numerator $a(q)$ increasingly resembles $b(q)$, and hence behaves well in the neighborhood of the principal pole of $H(q)$ (where $b(q)=1$). Therefore the Flajolet–Prodinger analysis of the case $d=2$ is valid *a fortiori* for $d>2$.

It is evident that the principal pole, which for $d=2$ is at $q=0.557\cdots$, by Flajolet–Prodinger, will move toward $q=0.5$ as d gets larger, since we are approaching the case of unrestricted compositions, where $b(q)$ degenerates into $q/(1-q)$.

Hence we need only compute this pole, i.e. solve $b(q)=1$ for its real root between 0.5 and 0.557.

Note also that the pole is independent of m, since $b(q)$ is.

The following table gives some numerical results:

d	Root q of $b(q)=1$	Growth rate $\gamma=1/q$
2	0.5573678720	1.7941471875
3	0.5206401166	1.9207125384
5	0.5042116835	1.9832939868
7	0.5009982120	1.9960151077
11	0.5000611472	1.9997554411

When $d>11$ it suffices for practical purposes to approximate this root by Newton's method, applied to just the first two terms of $b(q)$, since other terms contain powers of q greater than the d^2 power. Using 0.5 as a first approximation, we obtain for the second approximation the expression

$$q=\frac{1}{2}+\frac{1}{2^{d+3}-2d-8}.$$

For $d=11$ this gives $q=0.5000611471$, and for larger d the approximation will be even better.

The constant factor K in $f_d(n)\approx K(1/q)^n$ (where $f_d(n)=\sum_l \sum_j f_l(n, j)$) is given by

$$K=a(\gamma)/\gamma b'(\gamma),$$

where $\gamma=1/q$. Numerically this constant is less interesting and perhaps less precise than the growth rate, so we will not present any computations. We advise the interested reader to differentiate symbolically, since the numerical differentiation routines that we tried did not give very precise results.

3. The Lambda Algebra

The agreement to several decimal places between the Flajolet–Prodinger principal pole at 0.55737, or its reciprocal of 1.79415, and the lambda-algebra growth rate of 1.79415, which is implicit in the memoir [10, p. 27], requires some explanation.

The *lambda algebra* Λ_2 for the prime $p = 2$ has a basis (as a vector space over the field of two elements) consisting of sequences $I = (i_1, i_2, \ldots, i_l)$ of *non-negative* integers satisfying C1 with $m = \infty$ and C2 with $d = 2$. It is a differential algebra with bi-grading (n, l), where as before n denotes the sum $\sum i_k$.

We are also interested in sub-algebras in which the maximum initial integer m is finite. Topologically, condition C1 for finite m gives us an algebra derived from the $(m-1)$-sphere; when m is infinite we have a limiting object called the 'stable' sphere.

The fact that 0 is allowed in the sequences makes a substantial change. Of course 0 can only occur at the *end* of the sequence I, because of the multiplier condition C2; but at the end of I we may find an arbitrarily long string of zeroes. (This contrasts with the formulation of Flajolet–Prodinger, who actually identify a finite string I of positive integers with the unique infinite string in which I is followed by *infinitely* many zeroes.) Clearly, for a fixed sum n we have infinitely many elements in Λ.

However, we are motivated by the computation of the homology of Λ, and it has been shown that we obtain the same homology if we restrict our sequences to have *odd final integer* ('odd ending', [10]). This restores the finiteness of H_n but it raises the question of how the odd-final condition changes the enumeration problem.

In our Memoir we casually asserted that about half the 2-sequences would have odd endings. This is true if we *fix the length* l and let the sum n go to infinity. But is is not true if we sum over l.

We pose the following question: Of all the compositions of the integer n, what proportion have an odd integer at the end? The reader is invited to test his intuition. For positive sequences satisfying C2 with $d = 2$, the answer is about 64.2%, as we will show below.

Suppose that we want to solve the enumeration problem for positive sequences with odd endings, $d = 2$, by the Flajolet–Prodinger method. The difficulty is that it is not immediately clear how to 'add a slice' in this case: a sequence with an even penultimate, such as the sequence $(2, 4, 7)$, does not arise directly by adding the final 7 to the sequence $(2, 4)$, in that the latter is *not allowed* because of its even ending.

Lemma. *The number of positive sequences with $d = 2$, sum n, length l and odd final $j = 2i - 1$ is the same as the number of positive sequences with $d = 2$, sum $n + 1$, length l and even final $2i$.*

Proof. This is obvious from the correspondence

$$(i_1, \ldots, i_{l-1}, 2i - 1) \rightarrow (i_1, \ldots, i_{l-1}, 2i). \qquad \square$$

This allows us to consider $(2, 4, 7)$ as arising, not from $(2, 4)$, but from $(2, 3)$. We can now follow the previous line. A sequence of length l ending $j = 2i - 1$ gives rise to sequences of length $l + 1$ with an odd integer in the range $[1, 2j - 1]$ appended, *and also* to sequences of that length where the j is replaced by $j + 1$ and an odd integer in the range $[1, 2j + 1]$ appended.

We therefore have for the generating functions the substitution

$$q^n u^j \to \sum_{\substack{k=1 \\ k \text{ odd}}}^{2j-1} q^{n+k} u^k + \sum_{\substack{k=1 \\ k \text{ odd}}}^{2j+1} q^{n+1+k} u^k, \tag{3.1}$$

leading to the functional equation

$$H(q, u) = H^{[1]}(q, u) + \frac{qu(q+1)}{1-q^2u^2} H(q, 1) - \frac{qu(1+q^3u^2)}{1-q^2u^2} H(q, q^2u^2). \tag{3.2}$$

For length 1, $m = \infty$, we have $H^{[1]}(q, u) = qu/(1-q^2u^2)$.

THEOREM. *In the case $d = 2$, final i_l odd, $m = \infty$, we obtain $H(q, 1) = a(q)/(1 - b(q))$ where*

$$b(q) = (1+q) \sum_{k \geq 0} \left(\frac{q^{2^{k+1}-1}}{1-q^{2^{k+2}-2}} \prod_{j=0}^{k-1} \frac{-q^{2^{j+1}-1}(1+q^{2^{j+2}-1})}{1-q^{2^{j+2}-2}} \right), \tag{3.3}$$

while $a(q) = b(q)/(1+q)$.

This $b(q)$ looks rather more complicated than the original one (Section 2), but in fact it is the *same function*, as can be seen upon cancellation of the redundant factors. Similar results may be obtained for other cases of m (or of d).

Therefore the proportion of sequences with odd endings (for each fixed sum n) is simply $1/(1+q)$, where q is the root 0.557367872013993, giving the result 64.2109046918245%. This of course, is the limit as $n \to \infty$. When $n = 10$, for example, there are 248 positive sequences, of which 158 have odd endings; the above percentage predicts 159.

The formula $1/(1+q)$ is of course quite enlightening: it gives an immediate answer to the exercise posed above about unrestricted compositions with odd endings; it also tells us that for compositions with a given 'total degree' = sum plus length (see Section 4 below), the proportion with odd endings should be $1/(1 + 0.674697263873469)$, or about 0.5971228481540975.

Algebraic topologists who are interested in the 'unstable' lambda algebras, which give information about unstable spheres, will have noticed that they are characterized by the initial limit m, which does not affect the growth rate. In other words, the growth rate of the lambda algebra for the n-sphere is the same as that for the stable sphere.

4. Sum Plus Length

For the lambda algebra, we also want the asymptotics in terms of $n + l$, i.e.sum plus length. This is because our homology calculations proceed in that fashion [10]; the differential decreases n by 1 while increasing l by 1 and thus leaves fixed the 'internal degree' or 'total degree' $t = n + l$.

There is no difficulty following the example of Flajolet and Prodinger, replacing the sum n and its dummy variable q by the internal degree t and the dummy variable v. We will only do this for $d = 2$ since the generalization is not suited for the mod p lambda algebra, which will be treated in Section 5 below.

As $l \to l + 1$ the generating functions

$$F_l(v, u) = \sum_t \sum_j f_l(t, j) v^t u^j,$$

where $f_l(t, j)$ counts sequences of total degree t and length l with final component j, satisfy the substitution relation

$$v^t u^j \to \sum_{k=1}^{2j} v^{t+k+1} u^k = v^t \frac{v^2 u}{1 - vu} (1 - v^{2j} u^{2j}). \tag{4.1}$$

Summing over the length l leads to the functional equation

$$F(v, u) = F_1(v, u) + \frac{v^2 u}{1 - vu} [F(v, 1) - F(v, v^2 u^2)], \tag{4.2}$$

where we have written F for $\sum_{l=1}^{\infty} F_l$. For length 1 we have $F_1(v, u) = v^2 u/(1 - vu)$ in the case of interest for our application.

We have $\sigma(u) = v^2 u^2$, $\lambda(u) = v - 2u/(1 - vu) = \nu(u) = -\mu(u)$.

THEOREM. $F(v, 1) = b(v)/(1 - b(v))$, *where*

$$b(v) = \sum_{k \geq 0} \left(\frac{v^{2^{k+1}}}{1 - v^{2^{k+1}-1}} \prod_{j=0}^{k-1} \frac{-v^{2^{j+1}}}{1 - v^{2^{j+1}-1}} \right). \tag{4.3}$$

To obtain crude bounds on the asymptotic growth rate or, equivalently, on the location of the principal pole, we observe first that the number of unrestricted compositions of total degree t is a Fibonacci number: they divide into those obtained from $t - 1$ by increasing the final component by 1, and those obtained from $t - 2$ by appending a new final component of 1. Thus we have an upper bound for the growth rate of 1.618 or, equivalently, a lower bound for the pole of 0.618. By a similar argument, a lower bound for our growth rate is given by the positive root of $x^3 - x - 1 = 0$, which is about 1.325.

It is amusing that the Fibonacci numbers give an upper bound for this problem, and a lower bound for the original problem [4]. One can easily write down a bijection between compositions of the integer n (i.e. summing to n) consisting entirely of 1's and 2's, and unrestricted compositions of total degree $n + 2$.

Our generating function turns out to have a dominant pole at the reciprocal of 0.674697263873469, or about 1.482146221045796. We have thus explained the 1.48 of our Memoir on the lambda algebra [10, p. 27], just as Flajolet and Prodinger explained the 1.79.

5. The Lambda Algebra for Odd Primes

In Section 2 we showed that when the multiplier d increases, the growth constant increases toward 2. However, topologists familiar with the lambda algebras of odd primes know that they become more sparse with increasing p. In this section we will clarify the matter.

Historically the odd-prime lambda algebra has been presented as follows. Let p be an odd prime. Let $q = 2p - 2$. Then Λ_p is generated by two types of generators: λ_i $(i \geq 1)$ in dimension $qi - 1$, and μ_i $(i \geq 0)$ in dimension qi. These are subject to relations which differ slightly for the two types of generators. However, it is easily seen that the various relations become a single relation if we express them in terms of the dimensions of the generators. In fact, Λ_p may be regarded as additively generated (over the field of p elements) by sequences of non-negative integers congruent to -1 or $0 \pmod q$ and satisfying C2 with $d = p$.

For our application [10] we consider the sub-differential-algebra consisting of sequences in which the final is of type λ, i.e. the last integer is $\equiv -1 \pmod q$. This means that 0 will never occur. Our enumeration problem is therefore similar to that of Section 3, and can be solved by an extension of the same method.

LEMMA. *The number of positive sequences satisfying* C2 *with* $d=p$, *sum* n, *length* l *and final* $\equiv -1$ $(\bmod\, q=2p-2)$ *is the same as the number of positive sequences satisfying* C2 *with* $d=p$, *sum* $n+1$, *length* l *and final* $\equiv 0$ $(\bmod\, q=2p-2)$.

PROOF. We use the correspondence

$$(i_1, \ldots, i_{l-1}, aq-1) \to (i_1, \ldots, i_{l-1}, aq)$$

and a little arithmetic modulo q is enough to show that this respects condition C2, i.e. $aq-1 \leqslant di_{l-1}$ if and only if $aq \leqslant di_{l-1}$.

To see what the substitution relation is among the generating functions $F_l(v, u)$ (we use v as the variable associated with dimension, because q is now reserved for $2p-2$), we observe that if the final of a sequence of length l is $j=iq-1$ (representing λ_i), then the largest integer that can be appended, in the correct congruence class, is $dj-d-1$ $(d=p)$, while if we replace $j=iq-1$ by $j+1=iq$ (i.e. μ_i), then the largest admissible integer is $dj+d+1$. The substitution relation $F_l \to F_{l+1}$ is thus

$$v^r u^j \to \frac{v^r(vu)^{q-1}}{1-(vu)^q}(1-(vu)^{dj-d+2}) + \frac{v^{r+1}(vu)^{q-1}}{1-(vu)^q}(1-(vu)^{dj+d}). \tag{5.1}$$

The rest is as before. □

THEOREM. *In the case* $d=p$, $m=\infty$, *all integers* $\equiv -1$ *or* $\equiv 0$ $(\bmod\, 2d-2)$, *and final integer* $i_l \equiv -1$ $(\bmod\, 2d-2)$, *the generating function is*

$$F(v, 1) = \left(\frac{1}{1+v}\right)\left(\frac{b(v)}{1-b(v)}\right),$$

where

$$b(v) = (1+v)\sum_{k=0}^{\infty}\left(\frac{v^{(q-1)e(k)}}{1-v^{qe(k)}}\prod_{j=0}^{k-1}\frac{-v^{(p-1)e(j)}}{1-v^{qe(j)}}(1+v^{1+qe(j)})\right) \tag{5.2}$$

with $e(\ )$ *as in Section* 2. *The total exponent of* v *in the numerator of the* kth *term can be seen to be* $2p^{k+1}-k-3$.

If we drop the special restriction on the final i_l, the effect is to drop the factor $1/(1+v)$ from $F(v, 1)$.

The sub-algebra of λ's only, i.e. the case in which *all* the integers in the sequence are $\equiv -1$ $(\bmod\, q)$, can be done in the same way, and in fact the result in that case can easily be extracted from the above: we retain only the first term of (5.1), and therefore $F(v, 1) = b(v)/(1-b(v))$, where

$$b(v) = \sum_{k=0}^{\infty}\left(\frac{v^{(q-1)e(k)}}{1-v^{qe(k)}}\prod_{j=0}^{k-1}\frac{-v^{(p-1)e(j)}}{1-v^{qe(j)}}\right). \tag{5.3}$$

Numerically we obtain the following results:

p	$\gamma = 1/q$ (Section 2)	Growth rate of A	Growth rate for λ's only
3	1.9207125384	1.3399397350160394	1.1894783298340625
5	1.9832939868	1.1539266395493675	1.0946855342920447
7	1.9960151077	1.0988036752877630	1.0618402938949412
11	1.9997554411	1.0574699059979176	1.0361824498692680

The numbers called 'Growth rate of Λ' in the above table provide a precise evaluation of the constants C_n that occur in Henn's article on the growth rate of homotopy groups [6]. Apparently, the growth rate found there for homotopy is the same as the one we have found for Adams E_1 term because both are derived from an EHP-type sequence.

There appears to be no reason why other combinations of congruence classes could not also be handled in the above manner (or, for that matter, by the new method in Section 6 below). However, the congruence arithmetic underlying the above lemma would be difficult to write out in full generality. Lacking the motivation to attempt this generalization, I leave it to the reader.

6. An Alternative Derivation of the Generating Function

Flajolet and Prodinger [4] give for the case $d = 2$ and $m = 1$ the generating function $H(q) = a(q)/(1 - b(q))$ that we generalized in Section 2. Here (specializing and re-indexing)

$$b(q) = \sum_{k\geq 1} \frac{(-1)^{k+1} q^{2^{k+1}-k-2}}{\prod_{j=1}^{k}(1-q^{2^j-1})} = \frac{q}{1-q} - \frac{q^4}{(1-q)(1-q^3)} + \frac{q^{11}}{(1-q)(1-q^3)(1-q^7)} \pm \cdots$$

We will give another derivation of this formula. Our derivation goes direct to the generating function, by a counting argument, without using a functional equation.

We must first interpret the individual terms of $b(q)$. We observe that the alternating signs in $b(q)$ suggest the operation of some kind of inclusion–exclusion, and then that the term for each fixed k in $b(q)$ represents 'totally bad' sequences of length k.

Define $b_k(q)$ by writing $b(q) = \sum_{k\geq 1} (-1)^{k+1} b_k(q)$ and consider, for example, $b_3(q)$. We interpret this term as the generating function for all sequences of length 3 that violate the condition $i_{j+1} \leq 2i_j$ in both places. The sequence $(1, 4, 12)$, for example, with sum 17, corresponds to the partition of 17 by one 7, two 3's and four 1's, being the sum (in the sense of vector addition)

$$(1, 2, 4) + 2(0, 1, 2) + 4(0, 0, 1).$$

This explains $b_3(q)$ as the generating function for (unordered) partitions using 1, 3 and 7, and using each at least once.

Our method is clearest and most natural in the case $m = \infty$, because in that case, as we have seen, that $a(q) = b(q)$.

Lemma. *Let $B_{n,l}$ denote the number of* 'totally bad' *sequences of positive integers with sum n and length l, i.e. that* violate *condition* C2, $i_{j+1} \leq 2i_j$, *for* every j. *Then $B_{n,l}$ is the coefficient of q^n in $b_l(q)$.*

Proof. Each sequence $I = (i_1, i_2, \ldots, i_l)$ totally bad in the above sense may be written as the (vector) sum

$$a_1(1, 2, \ldots, 2^{l-1}) + a_2(0, 1, 2, \ldots, 2^{l-2}) + a_3(0, 1, 2, 4, \ldots, 2^{l-3}) \\ \cdots + a_{l-1}(0, 0, \ldots, 0, 1, 2) + a_l(0, 0, \ldots, 0, 1),$$

where $a_j = i_j - 2i_{j-1}$ $(1 \leq j \leq l)$, all a_j being strictly positive. (The sequence (a) is the *excess* sequence of I.) This gives a bijection between $B_{n,l}$ and the set of all (unordered) partitions of the sum n by the integers $\{1, 3, 7, \ldots, 2^l - 1\}$ using each of these integers at least once. It is well known that $b_l(q)$ is the generating function for such partitions. This completes the proof of the lemma. □

Now we must interpret $H(q)$. We have from Section 2 for the case $m=\infty$ the generating function

$$H(q)=\frac{b(q)}{1-b(q)}=b(q)+b(q)^2+b(q)^3+\cdots,$$

where $b(q)=b_1(q)-b_2(q)+b_3(q)\pm\cdots$. If we were to replace $b(q)$ by $b_1(q)$, $H(q)$ would become the standard generating function for unrestricted compositions [7]. In that simpler situation, the term $b_1(q)^l$ counts compositions of length l. Our case is different in that, while each of our $b_k(q)$ is a positive series (by the lemma), $b(q)$ is not (e.g. the coefficient of q^7 is -1), so that $b(q)$ cannot count anything. Moreover, in our interpretation the term $b(q)^l$ does not correspond to compositions of length l. It is therefore not quite straightforward to see our $H(q)$ directly as a generating function.

We proceed as follows.

THEOREM. *$H(q)$ is the generating function for sequences satisfying* C1 *with* $m=\infty$, C2 *with* $d=2$, *and* C3.

PROOF. We must show that every sequence I with length l and sum n appears the right number of times in the q^n term of $H(q)$. This means that we must see how many ways we can write I as the concatenation of totally bad subsequences. (Here 'subsequences' are understood to be consecutive.)

It must be kept in mind that negative signs are associated with $b_k(q)$ for k even.

Given an arbitrary sequence, e.g. $(1,2,3,8,1,4,9)$, we take account of the 'bad places,' i.e. pairs (i_j, i_{j+1}) violating condition C2. In the example there are three bad places: $(3,8)$, $(1,4)$ and $(4,9)$. We claim therefore that this sequence will be represented $2^3=8$ times in the expansion of $H(q)$, i.e. in the coefficient of q^{28}. In fact, at each bad place the sequence may or may not be divided. At one extreme, we find the sequence in the form $(1)(2)(3,8)(1,4,9)$ in the term $b_1(q)^2b_2(q)b_3(q)$, which carries a negative sign. At the other extreme, we find this sequence (or any other of length 7) in $b_1(q)^7$, which carries a positive sign. When a division point is introduced, the number of subsequences of even length increases or decreases by 1. Thus, if there are any bad places, the signs add out to give a total coefficient of zero. If there are no bad places, there is a single representation as above, of the form b_1^l, with a positive sign, and thus a total coefficient of unity.

This completes the proof for the 'natural' case $d=2$, $m=\infty$. □

It is not difficult to adapt this argument to the other cases that we have discussed, but it becomes more complicated and perhaps less natural.

Take, for example, the case $m=1$, $d=2$ of Flajolet–Prodinger. In the proof of the lemma, this means that $a_1=1$, so that the integer 2^l-1 must be used just once in the partition. Thus in the lemma we must restrict $a_1=1$: the unordered partition uses 2^l-1 exactly once. However, this affects only the *first* bad sequence, and so the factor of $(1-q^{2^k-1})$ is removed from only one factor in each term of the expansion of $H(q)$. We thus arrive at $H(q)=\sum_k(-1)^k a(q)b(q)^k=a(q)/(1-b(q))$, as previously shown.

In the case of Section 5, taking for example $d=3$, $q=2d-2=4$, the term

$$b_3(v)=v^{49}\frac{(1+v)(1+v^5)(1+v^{17})}{(1-v^4)(1-v^{16})(1-v^{52})}$$

is explained as the generating function for (unordered) partitions using 49 once, 4, 16 and 52 any number of times, and 1, 5 and 17 either 0 or 1 times each. Each such partition corresponds to a 'totally bad' sequence of length 3:

$$(3, 11, 35) + n_1(4, 12, 36) + n_2(0, 4, 12) + n_3(0, 0, 4) + \varepsilon_1(0, 0, 1) + \varepsilon_2(0, 1, 4) + \varepsilon_3(1, 4, 12),$$

where to meet the condition of an odd final we set $\varepsilon_1 = 0$, which explains the $1/(1+v)$ in $F(v, 1)$. This, of course, is the same $1/(1+q)$ that we came upon in Section 3.

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Postscript

At the Adams Memorial Symposium (Manchester, July 1990) two additional references were brought to my attention:

G. E. Andrews, The Rogers–Ramanujan reciprocal and Minc's partition function, *Pac. J. Math.,* **95** (1981), 251–256.

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