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A spectral sequence for the homology of a finite algebraic delooping

by

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Abstract

In the world of chain complexes E_n-algebras are the analogues of based n-fold loop spaces in the category of topological spaces. Fresse showed that operadic E_n-homology of an E_n-algebra computes the homology of an n-fold algebraic delooping. The aim of this paper is to construct two spectral sequences for calculating these homology groups and to treat some concrete classes of examples such as Hochschild cochains, graded polynomial algebras and chains on iterated loop spaces. In characteristic zero we gain an identification of the summands in Pirashvili's Hodge decomposition of higher order Hochschild homology in terms of derived functors of indecomposables of Gerstenhaber algebras and as the homology of exterior and symmetric powers of derived Kähler differentials.

Key Words: E_n-homology, Hochschild cohomology, André-Quillen homology, Π-Algebras, Grothendieck spectral sequence, Hodge decomposition.

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1. Introduction

The little n-cubes operad acts on and detects based n-fold loop spaces [Ma72]. Its algebraic counterpart, the operad that is given by (a cofibrant replacement of) its reduced chains, is the so-called E_n-operad and its algebras are E_n-algebras. In this sense, E_n-algebras are algebraic analogues of based n-fold loop spaces. Benoit Fresse constructed an n-fold bar construction for any E_n-algebra A_*, $B^n A_*$ [F11a] and showed that the E_n-homology of A_*, $H_*^{E_n}(A_*)$ is the homology of the n-fold desuspension of $B^n A_*$, thus E_n-homology calculates the homology of an algebraic n-fold delooping.

In characteristic zero the operad E_{n+1} is quasi-isomorphic to its homology, $H_*(E_{n+1})$. The homology of E_{n+1} codifies n-Gerstenhaber algebras. As one consequence the operations on the homology of any E_{n+1}-algebra are given by the n-Gerstenhaber algebra structure. However, in finite characteristic this does not hold any longer: If A_* is an algebra over the operad E_{n+1}, then $H_* A_*$ has more structure than an n-Gerstenhaber algebra. The homology of a free E_{n+1}-algebra on a chain complex C_* carries a restricted n-Gerstenhaber structure and in addition there are Dyer-Lashof operations to consider. Thus in these cases it is not enough to study the operad $H_*(E_{n+1})$ in order to understand all homology operations, but we have to understand the monad $A_* \mapsto H_*(E_{n+1}(A_*))$.

We start in Part 1 by developing a standard resolution spectral sequence in the cases of fields of characteristic two and zero. We identify its E^2-term as the derived functor of indecomposables with respect to a shifted (restricted) Gerstenhaber algebra structure (Theorem 3.9 and Theorem 4.3). The chain complex of an n-fold loop space carries an E_n-algebra structure. If the loop space is of the form $\Omega^n\Sigma^n X$ for $n \geqslant 2$ and connected X, then our spectral sequence gives an easy argument for the fact that E_n-homology of the chain algebra hands back the reduced homology of $\Sigma^n X$.

As we can express the indecomposables with respect to a Gerstenhaber structure as a composite of two functors we get a Grothendieck-type spectral sequence in the non-additive context by the work of Blanc and Stover [BS92]. This spectral sequence converges to the input of the E^2-page of the resolution spectral sequence.

In Part 2 we apply these spectral sequences for calculations of E_n-homology.

By forgetting structure every commutative algebra can be viewed as an E_n-algebra. In some cases classical work of Cartan [C54] can be used to identify E_n-homology groups of commutative algebras. We extend these classes of examples by calculating E_n-homology for free graded commutative algebras on one generator.

A different class of interesting examples of E_2-algebras is the class of reduced Hochschild cochain algebras of associative algebras: For any vector space V, the tensor algebra TV is the free associative algebra generated by V. Taking the composition with the reduced Hochschild cochains, $\bar{C}^*(-,-)$, we assign to any vector space V the E_2-algebra $\bar{C}^*(TV,TV)$. One can ask, how free this E_2-algebra is. For a free E_2-algebra on a vector space V, E_2-homology gives V back. Is the homology of the 2-fold delooping, *i.e.*, $H_*^{E_2}(\bar{C}^*(TV,TV))$, close to V? We give a positive answer for a one-dimensional vector space over the rationals (Theorem 8.3) and describe some partial results for the case when V is of dimension two.

In characteristic zero the resolution spectral sequence for calculating E_n-homology of commutative algebras has trivial differentials from the E^2-term onwards and we get a decomposition of E_n-homology. We identify the summands of the Hodge decomposition of higher order Hochschild homology in the sense of [P00] with derived functors of indecomposables of Gerstenhaber algebras. This recovers the identification of the Hodge summands of Hochschild homology of odd order as

$$HH_{m+1}^{(\ell)}(A;\mathbb{Q}) \cong H_{m-\ell+1}(\Lambda^{\ell}(\Omega^1_{P_*|\mathbb{Q}} \otimes_{P_*} \mathbb{Q}))$$

for a free simplicial resolution P_* of the commutative algebra A but we extend this result (see Theorem 9.5) to the Hodge summands of Hochschild homology of even order:

$$\begin{aligned}\mathrm{Tor}^{\Gamma}_{m+1-\ell}(\theta^{\ell},\mathcal{L}(A;\mathbb{Q})) &\cong (\mathbb{L}_m Q_{(2k-1)}\bar{A})_{(2k-1)(\ell-1)}\\ &\cong H_{m-\ell+1}(\mathrm{Sym}^{\ell}(\Omega^1_{P_*|\mathbb{Q}}\otimes_{P_*}\mathbb{Q})).\end{aligned}$$

Outline of the paper: In section 2 we construct a resolution spectral sequence and relate its E^1-term to Cohen's expression for the homology of free C_{n+1}-spaces. We explain the corresponding algebraic notions in section 3 (case of $\mathbb{F}_2$) and section 4 (rational case) and use these to identify the E^2-term of the resolution spectral sequence. Section 5 contains a reduction result for certain n-Gerstenhaber algebras that have an underlying free graded commutative algebra structure and we apply this to the example of the E_{n+1}-algebra of rational chains on a based $(n+1)$-fold iterated loop space. We set up a Blanc-Stover spectral sequence that converges to the E^2-term of our resolution spectral sequence in section 6.

As a first class of examples we present a calculation of E_n-homology of free graded commutative algebras in section 7. We discuss some calculations of E_2-homology of reduced Hochschild cochains for tensor algebras and group algebras in section 8. Section 9 contains our results about Pirashvili's Hodge decomposition for higher order Hochschild homology where we relate his description of the decomposition summands in terms of functor homology to homology groups of Gerstenhaber algebras and to the homology of symmetric (exterior) powers of derived Kähler differentials.

Notation: In the following we work relative to a field k, which is most of the time specified to be $\mathbb{Q}$ or $\mathbb{F}_2$. We denote by kx the k-vector space with basis element x.

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Part I. Spectral sequences for E_n-homology

2. A spectral sequence

We choose a Σ-cofibrant model of the operad E_{n+1} and as n will be fixed in the following, we call this model $E = (E(r))$. Usually $E_{n+1}(0)$ is used to keep track of base-points, but we use the reduced version with $E(0) = 0$. In the following we work with augmented E-algebras $\varepsilon\colon A_* \to k$ and E_{n+1}-homology is an invariant of non-unital E-algebras, so we will consider the augmentation ideal $\bar{A}_* = \ker(\varepsilon)$. However we will frequently switch to working with A_* when considering invariants of unital objects, *e.g.* André-Quillen homology.

We use a free simplicial resolution to establish a standard spectral sequence converging to the E_{n+1}-homology of any E-algebra.

Lemma 2.1 *For any E-algebra $\bar{A}_*$ there is a spectral sequence*

$$E^1_{s,t} \cong H_t(E^{\circ s}(\bar{A}_*)) \Rightarrow H^{E_{n+1}}_{s+t}(\bar{A}_*).$$

Proof: As $\bar{A}_*$ is an E-algebra, there is a simplicial resolution of $\bar{A}_*$ with augmentation $\epsilon\colon E(\bar{A}_*) \to \bar{A}_*$,

$$\cdots E^{\circ 3}(\bar{A}_*) \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} E^{\circ 2}(\bar{A}_*) \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} E(\bar{A}_*) \xrightarrow{\ \epsilon\ } \bar{A}_*.$$

The spectral sequence associated to the filtration by simplicial degree has

$$E^1_{s,t} = H^{E_{n+1}}_t(E^{s+1}(\bar{A}_*)) \Rightarrow H^{E_{n+1}}_*(\bar{A}_*).$$

But E_{n+1}-homology of a free E_{n+1}-algebra $E(B_*)$ is isomorphic to $H_*(B_*)$ [F09, 13.1.3, 4.4.2] and therefore the above E^1-term reduces to $H_t(E^{\circ s}(\bar{A}_*))$. $\square$

For topological spaces Cohen identified the homology of $C_{n+1}X$ for any space X. Here C_{n+1} denotes the operad of little $(n+1)$-cubes. He showed [CLM76, III, Theorem 3.1] that with $\mathbb{F}_p$-coefficients one gets

$$H_*(C_{n+1}X;\mathbb{F}_p) \cong W_n(H_*(X;\mathbb{F}_p)).$$

Here W_n is a free construction that takes the free restricted Lie algebra structure, the partial Dyer-Lashof structure and the commutativity of $H_*(C_{n+1}X;\mathbb{F}_p)$ into account.

A similar description holds for the monad of homology operations in our algebraic setting. In [F11b] Fresse describes the homology of $E(C_*)$ for any chain complex C_*. Note that

$$H_*(E(C_*)) = H_*(\bigoplus_{r\geqslant 1} E(r) \otimes_{k[\Sigma_r]} C_*^{\otimes r}) \cong \bigoplus_{r\geqslant 1} H_*(E(r) \otimes_{k[\Sigma_r]} C_*^{\otimes r}).$$

We can view the term $H_*(E(r) \otimes_{k[\Sigma_r]} C_*^{\otimes r})$ as the homology of a bicomplex and as we assumed that E is Σ-cofibrant, the associated spectral sequence has as vertical homology

$$E(r) \otimes_{k[\Sigma_r]} H_*(C_*^{\otimes r}).$$

As we are working over a field, there is a quasi-isomorphism from $H_*(C_*)$ to C_*. Therefore there are no higher differentials and we obtain the following result.

Lemma 2.2

$$H_*(E(C_*)) \cong \bigoplus_{r\geqslant 1} H_*(E(r) \otimes_{k[\Sigma_r]} (H_*C_*)^{\otimes r}).$$

For any space X we denote by $C_*(X,\mathbb{F}_p)$ the normalized chain complex of the simplicial $\mathbb{F}_p$-vector space of simplices in X. The following result is well-known; it is for instance used in [F11b], but for the reader's convenience we record it with a proof. Let $\bar{W}_n$ denote the nonunital variant of W_n determined by

$$\bar{W}_n(\bar{H}_*(X;\mathbb{F}_p)) = \bar{H}_*(C_{n+1}X;\mathbb{F}_p).$$

Proposition 2.3 *There is an isomorphism of monads on the category of nonnegatively graded chain complexes*

$$\tilde{W}_n(H_*(-)) \cong H_*(E(-)).$$

Proof: For any non-negatively graded chain complex C_* there exists a (non-unique) based space X^C such that the reduced homology of X^C is isomorphic to the homology of C_*. Let E_+ denote the unreduced version of E with $E_+(0) = \mathbb{F}_p$. By Lemma 2.2 we know that $H_*(E(C_*))$ is isomorphic to $H_*(\bigoplus_{r\geqslant 1} E(r) \otimes_{\mathbb{F}_p\Sigma_r} \tilde{H}_*(X^C;\mathbb{F}_p)^{\otimes r})$, which coincides with

$$H_*(\bigoplus_{r\geqslant 0} E_+(r) \otimes_{\mathbb{F}_p\Sigma_r} \tilde{H}_*(X^C;\mathbb{F}_p)^{\otimes r} / E_+(0) \otimes \tilde{H}_*(X^C;\mathbb{F}_p)^{\otimes 0}).$$

Switching to unreduced homology can be done by introducing the quotient by base point identification, so

$$H_*(\bigoplus_{r\geqslant 0} E_+(r) \otimes_{\mathbb{F}_p\Sigma_r} H_*(X^C;\mathbb{F}_p)^{\otimes r}/\sim)$$

where $\sim$ reduces occurrences of the class $[pt] \in H_0(X^C;\mathbb{F}_p)$ of the base point $pt \in X^C$ by contracting the elements in the operad by inserting the basis element of $E_+(0) = \mathbb{F}_p$ and divides out by $E_+(0) \otimes H_*(X^C;\mathbb{F}_p)^{\otimes 0} \cong \mathbb{F}_p$. As we are working over a field we can again replace the homology of X^C by its chain complex by picking representatives for cycles. Since Σ_r acts freely on $C_{n+1}(r)$, the normalized singular chains $C_*(C_{n+1}(r),\mathbb{F}_p)$ are free as an $\mathbb{F}_p\Sigma_r$-module. As E_+ is quasi-isomorphic to $C_*(C_{n+1},\mathbb{F}_p)$ as an operad, we can identify the term above with

$$H_*(\bigoplus_{r\geqslant 0} C_*(C_{n+1}(r),\mathbb{F}_p) \otimes_{\mathbb{F}_p\Sigma_r} C_*(X^C,\mathbb{F}_p)^{\otimes r}/\sim)$$

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via the Künneth spectral sequence. The fact that the shuffle transformation is lax symmetric monoidal yields that the latter is isomorphic to

$$H_*(C_*(\bigsqcup_{r\geqslant 0} C_{n+1}(r) \times_{\Sigma_r} (X^C)^r / \sim, \mathbb{F}_p)/C_*(C(0) \times (X^C)^0, \mathbb{F}_p))$$

where $\sim$ now denotes the usual reduction of base points, so we see that the above coincides with $\bar{H}_*(C_{n+1}X^C; \mathbb{F}_p)$.

Choosing a wedge of spheres for X^C and X^D and observing that every morphism $H_*(C_*; \mathbb{F}_p) \to H_*(D_*; \mathbb{F}_p)$ can be modelled via a map $X^C \to X^D$ one sees that this isomorphism is natural in C_*.

To show that this is indeed an isomorphism of monads we first note that the monad multiplication of $H_*(E(-))$ is induced by the composition in E, whereas the monad multiplication of $\bar{W}_n$ stems from the fact that W_n is left adjoint to the forgetful functor from what Cohen calls the category of allowable $AR_n\Lambda_n$-Hopf algebras to $\mathbb{F}_p$-modules that are unstable modules over the Steenrod algebra.

Iterating the isomorphism above yields

$$\begin{aligned} H_*(E(E(H_*(C_*)))) &\cong H_*(E(\tilde{H}_*(C_{n+1}X^C/\sim; \mathbb{F}_p))) \\ &\cong \tilde{H}_*(C_{n+1}(C_{n+1}X^C/\sim)/\sim; \mathbb{F}_p). \end{aligned}$$

Under this identification the multiplication of $H_*(E(-))$ corresponds to the map induced by the composition of C_{n+1} and hence to the monad multiplication of $\bar{W}_{n+1}$. □

Remark 2.4 In the following we also need to cover the case where we consider chain complexes that are concentrated in non-positive degrees, thus we have to modify the proof of Proposition 2.3. If C_{-*} is a chain complex concentrated in non-positive degrees, then the associated cochain complex, C^* with $C^n = C_{-n}$, is concentrated in non-negative degrees and it is quasi-isomorphic to its cohomology, $H^*(C^*)$. We choose a space X^C as above with $H_*(X^C; \mathbb{F}_p) \cong H^*(C^*)$. As above we get an isomorphism of monads between

$$C_{-*} \mapsto H_*(E(H^*(C^*))) \text{ and } C_{-*} \mapsto \bar{W}_n(H^*(C_*)). \tag{1}$$

3. The case $n = 1, p = 2$

Cohen showed in [CLM76, Theorem 3.1] that over a prime field the homology of a free C_{n+1}-algebra in spaces, $C_{n+1}X$, can be described as a free gadget on the homology of the underlying space. He also gave a description that allows to deduce the answer in characteristic zero.

For odd primes or for higher iterated loop spaces, the answer is pretty involved, but for $p = 2$ and $n = 1$ one is left with a 1-restricted Gerstenhaber structure.

Haynes Miller worked out the case $p = 2, n = 1$ in [Mi∞]: For any space, the homology of C_2X with coefficients in $\mathbb{F}_2$ is given by

$$S(1rL)(\tilde{H}_*(X;\mathbb{F}_2))$$

where $1rL(-)$ denotes the free 1-restricted Lie algebra and $S(-)$ is the free graded commutative algebra with the induced unique restricted Lie structure. We will recall the definitions and fix some notation.

Definition 3.1 A 1*-restricted Lie algebra over* $\mathbb{F}_2$ is a non-negatively graded (or non-positively graded) $\mathbb{F}_2$-vector space, $\mathfrak{g}_*$, together with two operations, a Lie bracket of degree one, $[-,-]$ and a restriction, ξ:

$$\begin{aligned} [-,-]\colon \quad & \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j+1}, \quad && i,j \geqslant 0, \\ \xi\colon \quad & \mathfrak{g}_i \to \mathfrak{g}_{2i+1} && i \geqslant 0. \end{aligned}$$

These satisfy the relations

(a) The bracket is bilinear, symmetric and satisfies the Jacobi relation

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0 \text{ for all } a,b,c \in \mathfrak{g}_*.$$

(b) The restriction interacts with the bracket as follows: $[\xi(a),b] = [a,[a,b]]$ and $\xi(a+b) = \xi(a) + \xi(b) + [a,b]$ for all homogeneous $a,b \in \mathfrak{g}_*$.

A *morphism of* 1*-restricted Lie-algebras* is a map of graded vector spaces of degree zero preserving the bracket and the restriction. We denote the category of 1-restricted Lie-algebras by $1rL$.

Remark 3.2 Note that these relations imply that $[a,a] = 0$ and $\xi(0) = 0$.

Definition 3.3 A 1*-restricted Gerstenhaber algebra over* $\mathbb{F}_2$ is a 1-restricted Lie algebra G_* together with an augmentation $\varepsilon\colon G_* \to \mathbb{F}_2$ and a graded commutative $\mathbb{F}_2$-algebra structure on G_* such that the multiplication in G_* interacts with the restricted Lie-structure as follows:

- (Poisson relation)

$$[a,bc] = [a,b]c + b[a,c] \text{ for all } a,b,c \in G_*.$$

- (multiplicativity of the restriction)

$$\xi(ab) = a^2\xi(b) + \xi(a)b^2 + ab[a,b] \text{ for all homogeneous } a,b \in G_*.$$

The augmentation is required to be multiplicative and to satisfy $\varepsilon[a,b] = 0$ for all $a,b \in G_*$ and $\varepsilon\xi(a) = 0$.

A *morphism of* 1*-restricted Gerstenhaber algebras* is a map of graded vector spaces of degree zero preserving the product, the augmentation, the bracket and the restriction. We denote the category of 1-restricted Gerstenhaber algebras by $1rG$.

In particular, the bracket and the restriction annihilate squares: $[a,b^2] = 2b[a,b] = 0$ and $\xi(a^2) = 2a^2\xi(a) + a^2[a,a] = 0$. Thus if 1 denotes the unit of the algebra structure in G_*, then $[a,1] = 0$ for all a and $\xi(1) = 0$.

Usually an augmentation is not part of the definition, but since we consider augmented E-algebras all 1-restricted Gerstenhaber algebras that we will encounter are naturally augmented. The requirements on ε are equivalent to ε being a morphism of 1-restricted Gerstenhaber algebras $\varepsilon\colon G_* \to \mathbb{F}_2$ where $\mathbb{F}_2$ is viewed as a commutative algebra with trivial 1-restricted Lie structure.

We denote by IG_* the augmentation ideal of G_*. This ideal carries a structure of a non-unital 1-restricted Gerstenhaber algebra and we will call both G_* and IG_* 1-restricted Gerstenhaber algebras.

For a 1-restricted Lie-algebra $\mathfrak{g}$, the free graded commutative algebra generated by $\mathfrak{g}$, $S(\mathfrak{g})$, carries a unique 1-restricted Gerstenhaber structure that is induced by the 1-restricted Lie algebra structure on $\mathfrak{g}$ and the relations in Definition 3.3.

Remark 3.4 The functor $S\colon 1rL \to 1rG$ is left adjoint to the augmentation ideal functor $I\colon 1rG \to 1rL$ and the forgetful functor $U\colon 1rL \to \mathrm{gr}\mathbb{F}_2$ from the category of 1-restricted Lie-algebras to the category of graded $\mathbb{F}_2$-vector spaces has the free 1-restricted Lie algebra functor $1rL\colon \mathrm{gr}\mathbb{F}_2 \to 1rL$ as a left adjoint.

For $p = 2, n = 1$ the structure on $H_*(C_2X;\mathbb{F}_2)$ looks so nice because

$$R_1(q) = \mathbb{F}_2\langle Q^I \mid I = (s_1,\dots,s_k) \text{ admissible, } e(I) \geqslant q, s_k \leqslant q\rangle$$

reduces to

$$R_1(q) = \mathbb{F}_2\langle Q^{(2^{k-1}q,\dots,2q,q)} \mid k \geqslant 1\rangle.$$

Therefore, in Cohen's identification of $H_*(C_2X;\mathbb{F}_2)$ the contribution of the Dyer-Lashof terms is absorbed into the free commutative algebra part: a term like

$$Q^{(2^{k-1}q,\dots,2q,q)} \otimes x_q$$

with x_q in $1rL(H_*X;\mathbb{F}_2)$ of degree q is identified (in what Cohen calls V_1) with $x_q^{2^k}$. Thus $H_*(C_2X;\mathbb{F}_2) \cong S(1rL)(\tilde{H}_*(X;\mathbb{F}_2))$.

As a corollary to Proposition 2.3 we get

Corollary 3.5 *For any non-negatively graded (or non-positively graded) chain complex over $\mathbb{F}_2$, C_*, we have*

$$H_*(E_2C_*;\mathbb{F}_2) \cong IS(1rL)(H_*C).$$

Definition 3.6 For an augmented 1-restricted Gerstenhaber algebra over $\mathbb{F}_2$ we denote by Q_{1rG} the $\mathbb{F}_2$-vector space of indecomposable elements with respect to the three operations, the product, the bracket and the restriction, *i.e.*, the quotient of IG_* by the ideal generated by these operations:

$$Q_{1rG}(G_*) = IG_*/\langle \xi(a), [a,b], ab, a,b \in IG_*\rangle.$$

We extend this notion to IG_*, so $Q_{1rG}(IG_*) = IG_*/\langle \xi(a), [a,b], ab, a,b \in IG_*\rangle$.

Similarly, we denote by $Q_a(-)$ the indecomposables with respect to the algebra structure and by $Q_{1rL}(-)$ the indecomposables with respect to the 1-restricted Lie algebra structure.

Lemma 3.7 *For any augmented* 1*-restricted Gerstenhaber algebra G_* over $\mathbb{F}_2$ the vector space of indecomposables Q_{1rG} of G_* can be computed as the composite*

$$Q_{1rG}(G_*) = Q_{1rL}(Q_a(G_*)).$$

Proof: As we demand that ε annihilates Lie brackets and restrictions, there is a well-defined 1-restricted Lie-structure on IG_* and the algebra indecomposables, $Q_a(G_*) = IG_*/J$, inherit a 1-restricted Lie algebra structure from G_*: For homogeneous $a,b \in \bar{G}_*$ we set

$$[a+J, b+J] := [a,b] + J \text{ and } \xi(a+J) := \xi(a) + J.$$

The relations from Definition 3.3 tell us that this gives a well-defined bracket and a well-defined restriction on $Q_a(G_*)$. Taking the 1-restricted Lie algebra indecomposables of $Q_a(G_*)$ kills expressions in $Q_a(G_*)$ that are of the form $\xi(a)$ with $a \in IG_*$ and $[a,b]$ with $a,b \in G_*$, so we kill everything in IG_* that is a product, a bracket or a restricted element. □

The algebraic indecomposables of a free commutative algebra on a (graded) vector space hand back the vector space and the indecomposables with respect to the 1-restricted Lie algebra structure of $1rL(V_*)$ have V_* as output, so we get:

Lemma 3.8

$$Q_{1rG}(S(1rL)(V_*)) = V_* \text{ and } Q_{1rG}(IS(1rL)(V_*)) = V_*.$$

We want to identify the E^2-term of the spectral sequence

$$E^1_{p,q} = H_q(E^p_2(\bar{A}_*)) \cong ((IS(1rL))^p(H_*(\bar{A}_*)))_q \Rightarrow H^{E_2}_{p+q}(\bar{A}_*).$$

The homology of $\bar{A}_*$ is a non-unital 1-restricted Gerstenhaber algebra over $\mathbb{F}_2$, so the free-forgetful adjunction identifies

$$\ldots \to (IS(1rL))^{p+1}(H_*(\bar{A}_*)) \to (IS(1rL))^p(H_*(\bar{A}_*)) \to \ldots \to IS(1rL)(H_*(\bar{A}_*))$$

as a resolution coming from a simplicial resolution of $H_*(\bar{A}_*)$. The term $(IS(1rL))^{p+1}(H_*(\bar{A}_*))$ is in resolution degree p. Applying the functor Q_{1rG} to this simplicial resolution gives

$$\ldots \to (IS(1rL))^p(H_*(\bar{A}_*)) \to (IS(1rL))^{p-1}(H_*(\bar{A}_*)) \to \ldots \to H_*(\bar{A}_*).$$

This shows:

Theorem 3.9 *The above E^1-term is isomorphic to*

$$E^1_{p,q} \cong (Q_{1rG}((IS(1rL))^{p+1}(H_*(\bar{A}_*)))_q$$

and the d^1-differential takes homology with respect to the resolution degree. Therefore the E^2-term calculates derived functors of indecomposables of the homology of $\bar{A}_*$,

$$E^2_{p,q} \cong (\mathbb{L}_p Q_{1rG}(H_*(\bar{A}_*)))_q.$$

4. The rational case

Most things are similar for the rational case with the difference that we consider n-Gerstenhaber algebras for all $n \geqslant 1$. In this section the ground field will always be $\mathbb{Q}$.

Definition 4.1 An *n-Lie algebra over* $\mathbb{Q}$ is a non-negatively graded (or non-positively graded) $\mathbb{Q}$-vector space, L_*, together with a Lie bracket of degree n:

$$[-,-]\colon L_i \times L_j \to L_{i+j+n}, i, j \geqslant 0,$$

such that the bracket is bilinear and satisfies a graded Jacobi relation

$$(-1)^{pr}[x,[y,z]] + (-1)^{qp}[y,[z,x]] + (-1)^{rq}[z,[x,y]] = 0,$$

and a graded antisymmetry relation

$$[x,y] = -(-1)^{pq}[y,x].$$

Here, x, y, z are homogenous elements in L and $p = |x| + n$, $q = |y| + n$ and $r = |z| + n$. A *morphism of n-Lie algebras* is a map of graded vector spaces of degree zero preserving the bracket. We denote the category of n-Lie algebras by nL.

Note that there is an operadic notion of n-Lie algebras involving n-ary Lie brackets. That is something different.

Definition 4.2 An *n-Gerstenhaber algebra over $\mathbb{Q}$* is an n-Lie algebra G_* together with a unital commutative $\mathbb{Q}$-algebra structure on G_* and an augmentation $\varepsilon\colon G_* \to \mathbb{Q}$ such that the Poisson relation holds:

$$[a,bc] = [a,b]c + (-1)^{p|b|}b[a,c], \text{ for all homogeneous } a,b,c \in G_* \text{ with } p = |a|+n,$$

and such that $\varepsilon[a,b] = 0$.

A *morphism of n-Gerstenhaber algebras* is a map of graded vector spaces of degree zero preserving the product, the augmentation and the bracket. We denote the category of (augmented) n-Gerstenhaber algebras by nG. As in the characteristic two case, we also consider IG_* as a non-unital n-Gerstenhaber algebra.

Let nG denote the free n-Gerstenhaber algebra functor from the category gr$Vct_{\mathbb{Q}}$ of graded rational vector spaces to the category of augmented n-Gerstenhaber algebras. Then this can be factored as $S \circ nL$ where nL denotes the free n-Lie algebra functor.

Similarly, we can factor the functor of n-Gerstenhaber indecomposables, Q_{nG}, as the algebraic indecomposables followed by the n-Lie indecomposables:

$$Q_{nG} = Q_{nL} \circ Q_a.$$

Theorem 4.3 *There is a spectral sequence with*

$$E^2_{p,q} \cong (\mathbb{L}_p Q_{nG}(H_*(\bar{A}_*)))_q \Rightarrow H_*^{E_{n+1}}(\bar{A}_*)$$

for every E_{n+1}-algebra $\bar{A}_*$ over the rationals.

Example 4.4 Let X be a connected and well-behaved topological space and let n be greater or equal to one. In characteristic zero, $H_*(C_{n+1}X;\mathbb{Q})$ is isomorphic to the free n-Gerstenhaber algebra generated by the reduced homology of X [CLM76]. Thus the above E^2-term for $A_* = C_*(C_{n+1}X;\mathbb{Q})$ is isomorphic to

$$E^2_{p,q} \cong (\mathbb{L}_p Q_{nG}(nG(\bar{H}_*(X;\mathbb{Q}))))_q$$

which is concentrated in bidegrees $(0,q)$ and thus the spectral sequence collapses and gives

$$H_q^{E_{n+1}}(C_*(C_{n+1}X;\mathbb{Q})) \cong \bar{H}_q(X;\mathbb{Q}).$$

Note that $H_*(C_{n+1}X;\mathbb{Q}) \cong H_*(\Omega^{n+1}\Sigma^{n+1}X,\mathbb{Q})$ if X is path-connected, thus in this case the algebraic delooping induced by E_{n+1}-homology corresponds to a geometric delooping.

Similar considerations hold for $H_*(C_2X;\mathbb{F}_2)$.

5. Some applications of the resolution spectral sequence

Let U be the forgetful functor from the category of n-Lie algebras to graded $\mathbb{Q}$-vector spaces.

Lemma 5.1 *Let V be an n-Lie algebra and let C be $S(V)$ where the n-Gerstenhaber algebra structure on C is induced by the n-Lie structure of V. Then a free resolution in the category of simplicial n-Gerstenhaber algebras of C is given by $Y_\bullet$ with*

$$Y_\ell = S(nL \circ U)^{\ell+1}(V), \ell \geqslant 0.$$

Proof: We use the adjunction (nL,U) to obtain the simplicial structure on $Y_\bullet$. The usual simplicial contraction for $(nL \circ U)^{\bullet+1}$ shows that $Y_\bullet$ is a resolution of $S(V)$. Note that the augmentation

$$Y_0 = S(nL \circ U)(V) \to S(V)$$

is a morphism of n-Gerstenhaber algebras.

The degeneracy maps send $nG(U(nL \circ U)^\ell(V))$ to $nG(U(nL \circ U)^{\ell+1}(V))$ via maps of the form $nG(f)$ with $f\colon U(nL \circ U)^\ell(V) \to U(nL \circ U)^{\ell+1}(V)$, thus $Y_\bullet$ is a free simplicial resolution of $S(V)$. □

Corollary 5.2 *For $S(V)$ as in Lemma 5.1 there is an isomorphism*

$$(\mathbb{L}_p Q_{nG}(S(V)))_q \cong (\mathbb{L}_p Q_{nL}(V))_q$$

for all $p \geqslant 0$ and all q. In particular, for any E_{n+1}-algebra A_* with $H_*(A_*) \cong S(V)$ with $S(V)$ as in Lemma 5.1, the E^2-term of Theorem 4.3 is isomorphic to

$$E^2_{p,q} \cong (\mathbb{L}_p Q_{nL}(V))_q.$$

Proof: Using the resolution $S(nL \circ U)^{\bullet+1}(V)$ of $S(V)$ we get that

$$Q_{nG}(S(nL \circ U)^{s+1}(V)) \cong U((nL \circ U)^s)(V) = Q_{nL}((nL \circ U)^{s+1}(V)).$$

As $(nL \circ U)^{\bullet+1}(V)$ is a simplicial resolution of V by free n-Lie algebras, the claim follows. □

Remark 5.3 There is an equivalence of categories between the category nL and the category of graded Lie algebras, L, where the latter category is nothing but the category of 0-Lie algebras [KM95, Proposition I.6.3]. The equivalence is given by the n-fold suspension, Σ^n, and desuspension, Σ^{-n}:

$$nL \underset{\Sigma^{-n}}{\overset{\Sigma^n}{\rightleftarrows}} L.$$

An analogous result holds in characteristic two [CLM76, III §15].

We can use the resolution exhibited in 5.2 to exploit the equivalences between the different n-Lie structures.

Corollary 5.4 *Suppose that $H_*(A) = S(V)$ as in 5.1. Then for every $\ell \in \mathbb{Z}$ the E^2-term of the spectral sequence calculating E_{n+1}-homology of A can be computed as*

$$E^2_{p,q} = (\mathbb{L}_p Q_{\ell L}(\Sigma^{n-\ell} V))_{q+n-\ell}.$$

Proof: Using the natural isomorphism $nL \cong \Sigma^{\ell-n} \ell L \Sigma^{n-\ell}$ and the fact that this is an isomorphism of monads we find

$$\begin{aligned} E^2_{p,q} &= (\pi_p U(nLU)^\bullet(V))_q \cong (\pi_p U(\Sigma^{\ell-n}\ell L\Sigma^{n-\ell}U)^\bullet(V))_q \\ &\cong (\pi_p U(\Sigma^{\ell-n}\ell LU\Sigma^{n-\ell})^\bullet(V))_q \cong (\pi_p \Sigma^{\ell-n} U(\ell LU)^\bullet(\Sigma^{n-\ell}V))_q \\ &\cong (\pi_p U(\ell LU)^\bullet(\Sigma^{n-\ell}V))_{q+n-\ell}, \end{aligned}$$

which proves the claim. □

Remark 5.5 Similar results hold in characteristic two, *i.e.*, if V is a 1-restricted Lie algebra over $\mathbb{F}_2$ and if $C = S(V)$ carries the induced 1-restricted Gerstenhaber algebra structure, then $S(1rL \circ U)^{\bullet+1}$ is a resolution of C by free 1-restricted Gerstenhaber algebras and the E^2-term of the resolution spectral sequence simplifies to

$$E^2_{p,q} \cong (\mathbb{L}_p Q_{1rL}(V))_q$$

and the suspension isomorphism yields that this in turn can be expressed as derived functors of indecomposables of restricted Lie algebras.

In the following we want to use the Tor interpretation of Lie-homology in our setting:

Remark 5.6 Let $\mathfrak{g}$ be a graded Lie-algebra, restricted if $k = \mathbb{F}_2$ and unrestricted over the rationals. The usual Tor interpretation (see for instance [Q70]) of $\mathbb{L}_s Q_L(\mathfrak{g})$ holds in the graded case. Indeed one easily identifies the indecomposables of the standard cofibrant replacement $X = L^{\bullet+1}(\mathfrak{g})$ (or $X = rL^{\bullet+1}(\mathfrak{g})$ in the restricted case) with $\overline{\mathfrak{U}(X)} \otimes_{\mathfrak{U}(X)} k$. Here $\mathfrak{U}(\mathfrak{g})$ denotes the (restricted) universal enveloping algebra of the bigraded Lie algebra $\mathfrak{g}$ while $\overline{\mathfrak{U}(\mathfrak{g})}$ denotes its augmentation ideal. The Künneth spectral sequence constructed by Quillen [Q67, II.6] can be generalized to the graded setting, and since X consists of free graded (restricted) Lie algebras, $\overline{\mathfrak{U}(X)}$ is a cofibrant $\mathfrak{U}(X)$-module. Hence we get a spectral sequence of internally bigraded vector spaces

$$E^2_{p,q} = \mathrm{Tor}_p^{\pi_*(\mathfrak{U}(X))}(\pi_*(\overline{\mathfrak{U}(X)}), k)_q \Rightarrow \pi_{p+q}(\overline{\mathfrak{U}(X)} \otimes_{\mathfrak{U}(X)} k),$$

where q is the degree originating from taking homotopy groups. Filtering $\mathfrak{U}(X)$ and $\mathfrak{U}(\mathfrak{g})$ by the standard filtration for enveloping algebras and considering the associated spectral sequences, a bigraded version of the Poincaré-Birkhoff-Witt theorem shows that the augmentation $\mathfrak{U}(X) \to \mathfrak{U}(\mathfrak{g})$ induces an isomorphism on E^1 (see [Pr70] for the case of characteristic two). Hence the above E^2-term equals $\mathrm{Tor}_p^{\mathfrak{U}(\mathfrak{g})}(\overline{\mathfrak{U}(\mathfrak{g})}, k)$ concentrated in degree $q = 0$. Finally the short exact sequence

$$0 \longrightarrow \overline{\mathfrak{U}(\mathfrak{g})} \longrightarrow \mathfrak{U}(\mathfrak{g}) \longrightarrow k \longrightarrow 0$$

yields

$$\mathbb{L}_s Q_L(\mathfrak{g}) \cong \mathrm{Tor}_{s+1}^{\mathfrak{U}(\mathfrak{g})}(k, k).$$

An example of how this simplifies our spectral sequence is given by the chains on an iterated loop space on a highly connected space. For an $(n+1)$-connected space X the space $\Omega^{n+1}X$ is path-connected. A classical result expresses $H_*(\Omega^{n+1}X;\mathbb{Q})$ as a free graded commutative algebra: The connectivity assumptions ensure that due to the Milnor-Moore result [MM65, p.263] the Hurewicz map

$$\pi_*(\Omega^{n+1}X) \otimes \mathbb{Q} \to H_*(\Omega^{n+1}X;\mathbb{Q})$$

induces an isomorphism of Hopf algebras between the enveloping algebra of the Lie-algebra $\pi_*(\Omega^{n+1}X) \otimes \mathbb{Q}$ and $H_*(\Omega^{n+1}X;\mathbb{Q})$. Here, the Lie-structure on the source is given by the Samelson product. For $n \geqslant 1$, this Lie-structure is trivial and thus the enveloping algebra is isomorphic to the free graded commutative algebra generated by $\pi_*(\Omega^{n+1}X) \otimes \mathbb{Q}$:

$$S(\pi_*(\Omega^{n+1}X) \otimes \mathbb{Q}) \cong H_*(\Omega^{n+1}X;\mathbb{Q}).$$

Cohen showed [CLM76, p. 215] that the Whitehead product on $\Sigma^{-n-1}\pi_*(X) \otimes \mathbb{Q}$ corresponds to the Browder-bracket λ_n on $H_*(\Omega^{n+1}X;\mathbb{Q})$. Gaudens and Menichi observed [GM07, Theorem 4.1] that this leads to an isomorphism of nG-algebras

$$S(\Sigma^{-n}\pi_*(\Omega X) \otimes \mathbb{Q}) \cong H_*(\Omega^{n+1}X;\mathbb{Q}) \tag{2}$$

where the n-Lie structure on the left-hand side is induced by the Samelson bracket on $\pi_*(\Omega X)$.

Proposition 5.7 *For every $(n+1)$-connected space X*

$$\mathbb{L}_s(Q_{nG})(H_*(\Omega^{n+1}X;\mathbb{Q}))_q \cong \mathrm{Tor}^{H_*(\Omega X;\mathbb{Q})}_{s+1,q+n}(\mathbb{Q},\mathbb{Q}).$$

Proof: Corollary 5.2 implies that the E^2-term of the resolution spectral sequence in this case is isomorphic to

$$(\mathbb{L}_s Q_{nL})_t(\Sigma^{-n}(\pi_*(\Omega X)\otimes\mathbb{Q}))).$$

Corollary 5.4 together with the Tor-description of Lie-homology and the Milnor-Moore Theorem show the claim. □

Remark 5.8 Up to a shift in degrees the above E^2-term is isomorphic to the E^2-term of the Rothenberg-Steenrod spectral sequence [RS65]. The latter converges to the homology of the space X. We conjecture that there is an isomorphism of spectral sequences between our resolution spectral sequence and the (shifted) Rothenberg-Steenrod spectral sequence.

Remark 5.9 Anderson constructed a spectral sequence [An71] whose E^2-page is

$$E^2_{p,q} \cong HH^{[n+1]}_p(H_*(\Omega^{n+1}X;\mathbb{Q}))_q$$

and which converges to $H_{p+q}(X;\mathbb{Q})$. Here $HH^{[n+1]}_*$ denotes Hochschild homology of order $n+1$ in the sense of [P00].

However, in his setting $H_*(\Omega^{n+1}X;\mathbb{Q})$ is considered as a graded commutative algebra, whereas the n-Lie structure is ignored. In this situation Hochschild homology of order $n+1$ is isomorphic to E_{n+1}-homology,

$$HH^{[n+1]}_p(H_*(\Omega^{n+1}X;\mathbb{Q}))_q \cong H^{E_{n+1}}_{p-n-1}(\bar{H}_*(\Omega^{n+1}X;\mathbb{Q}))_q.$$

Thus his spectral sequence starts off with E_{n+1}-homology of the underlying graded commutative algebra of $H_*(\Omega^{n+1}X;\mathbb{Q})$ and converges to $H_*(X;\mathbb{Q})$.

6. The Blanc-Stover composite functor spectral sequence

We know that working relative to $\mathbb{F}_2$ we can factor Q_{1rG} as $Q_{1rL}\circ Q_a$ and similarly in the rational setting we have $Q_{nG} = Q_{nL}\circ Q_a$. Therefore we want to use the composite functor spectral sequence of Blanc and Stover in order to approximate the E^2-term of our resolution spectral sequence (as in Theorem 3.9 and Theorem 4.3). Let $\mathcal{C}$ and $\mathcal{B}$ denote categories of universal graded algebras such as the category of graded commutative algebras (over $\mathbb{Q}$ or $\mathbb{F}_2$), the category of (restricted) n-Lie algebras or of (restricted) n-Gerstenhaber algebras. Let $\mathcal{A}$ denote a concrete category such as the category of graded vector spaces over a field. Moreover let $T\colon \mathcal{C} \to \mathcal{B}$ and $Z\colon \mathcal{B} \to \mathcal{A}$ be functors, then Blanc and Stover prove the existence of the following spectral sequence.

Theorem 6.1 *[BS92, Theorem 4.4] Suppose that TF is Z-acyclic for every free F in $\mathcal{C}$. Then for every C in $\mathcal{C}$ there is a Grothendieck spectral sequence with*

$$E^2_{s,t} = (\mathbb{L}_s \bar{Z}_t)(\mathbb{L}_* T)C \Rightarrow (\mathbb{L}_{s+t}(Z \circ T))C.$$

The condition that TF is Z-acyclic means that the left derived functors $\mathbb{L}_* Z$ applied to TF are trivial but in degree zero where they are isomorphic to ZTF. The terms $\mathbb{L}_* \bar{Z}$ are a certain extension of the derived functors of Z to the category of Π-$\mathcal{B}$-algebras, *i.e.*, $\bar{Z}$ takes the homotopy operations into account that live on the homotopy groups of every simpicial $\mathcal{B}$-algebra.

If we unravel the notation in [BS92, Theorem 4.4] then the E^2-term gives

$$E^2_{s,t} = \pi_s \pi^i_t Z(B_{\bullet,\bullet})$$

where the notation is as follows: Let $Y_\bullet \to C$ be a cofibrant resolution of C in the Quillen model category of simplicial objects in $\mathcal{C}$. Then $B_{\bullet,\bullet}$ is a free resolution of $TY_\bullet$ in the E^2-model category structure on the category of bisimplicial objects in $\mathcal{B}$ [DKS93, 5.10], [BS92, 4.1]. As we have various E^2-terms floating around, we will call this model structure the DKS-model structure. If $B_{\bullet,\bullet}$ in bidegree (t,s) is $B_{t,s}$ then π^i_t is the tth homotopy group with respect to the first simplicial direction and then $\pi^i_t B_{\bullet,\bullet}$ is a free simplicial resolution of $\pi_t TY_\bullet$ by Π-$\mathcal{B}$-algebras.

Proposition 6.2

- *If the ground field is $\mathbb{F}_2$ and if we consider the sequence of functors*

$$1rG \xrightarrow{Q_a} 1rL \xrightarrow{Q_{1rL}} \mathrm{gr}\mathbb{F}_2,$$

then for any $C \in 1rG$ the E^2-term of the composite functor spectral sequence simplifies to

$$E^2_{s,t} = (\mathbb{L}_s(\bar{Q}_{1rL})_t)(AQ_*(C|\mathbb{F}_2,\mathbb{F}_2)).$$

- *For the sequence*

$$nG \xrightarrow{Q_a} nL \xrightarrow{Q_{nL}} \mathrm{gr}Vct_{\mathbb{Q}}$$

over the rationals and for $C \in nG$, the spectral sequence has E^2-term isomorphic to

$$E^2_{s,t} = (\mathbb{L}_s(\bar{Q}_{nL})_t)(AQ_*(C|\mathbb{Q},\mathbb{Q})).$$

Proof: There are two adjoint pairs, $(1rG, V)$ and (nG, V), where V denotes the forgetful functor to the underlying category of graded vector spaces. Associated to these are standard simplicial resolutions for calculating $\mathbb{L}_* Q_a(C)$, namely $(1rG \circ V)^{\bullet+1}(C)$ for characteristic two and $(nG \circ V)^{\bullet+1}(C)$ for characteristic zero. The free 1-restricted Gerstenhaber algebra generated by a graded vector space W_* is $S(1rL(W_*))$, the free graded commutative algebra generated by the free 1-restricted Lie algebra on W_*. In particular the above mentioned resolutions consist of free graded commutative algebras and $Q_a(S(1rL(W_*)))$ is $1rL(W_*)$, which is Q_{1rL}-acyclic. Since the derived functors of Q_a compute André-Quillen homology we get a spectral sequence of the form above. Similar arguments hold for n-Gerstenhaber algebras. $\square$

Remark 6.3 How can one calculate these E^2-terms? First one resolves C simplicially by free Gerstenhaber algebras, $P_\bullet \to C$, and takes indecomposables, $Q_a(P_\bullet)$. This is now a simplicial object in some category of Lie algebras (n-Lie or restricted 1-Lie), thus one has to find a free resolution of this object in the DKS-model structure as explained in [BS92, 4.1.1].

Over the rationals we want to compare E_n-homology of a commutative algebra with E_m-homology for $n \neq m$. To this end, we first compare the model category structures on the corresponding categories of simplicial shifted Lie algebras.

Let us briefly recall the model category structure on simplicial n-Lie algebras, snL [Q67, II, §4]:

- A map $f : \mathfrak{g}_\bullet \to \mathfrak{g}'_\bullet$ is a *weak equivalence* if $U(f)$ is a weak equivalence of simplicial $\mathbb{Q}$-vector spaces.
- Such a map is a *fibration*, if the induced map $\bar{f}$

$$\begin{tikzcd}
\mathfrak{g}_\bullet \arrow[dr, dotted, "\bar{f}"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right] & & \\
& \pi_0(\mathfrak{g}_\bullet) \times_{\pi_0(\mathfrak{g}'_\bullet)} \mathfrak{g}'_\bullet \arrow[r] \arrow[d] & \mathfrak{g}'_\bullet \arrow[d] \\
& \pi_0(\mathfrak{g}_\bullet) \arrow[r, "\pi_0(f)"] & \pi_0(\mathfrak{g}'_\bullet)
\end{tikzcd}$$

  is surjective.
- A map is a *cofibration* if it has the left lifting property with respect to acyclic fibrations.

Proposition 6.4 *The model categories of simplicial n-Lie algebras and of simplicial graded Lie algebras are Quillen equivalent.*

Proof: The equivalence (Σ^n, Σ^{-n}) between n-Lie algebras and graded Lie algebras that we mention in Remark 5.3 preserves and detects weak equivalences as these are given by reference to the underlying simplicial graded vector spaces, hence a morphism $f\colon \mathfrak{g} \to \Sigma^{-n}\mathfrak{g}'$ of n-Lie algebras is a weak equivalence if and only if $\Sigma^n(f)\colon \Sigma^n\mathfrak{g} \to \mathfrak{g}'$ is one. Similarly, suspension does not affect surjectivity of maps. □

What we actually need is to extend this Quillen equivalence to the corresponding model categories of simplicial Lie Π-algebras.

Theorem 6.5 *The model categories of simplicial n-Lie-Π algebras and of simplicial graded Lie Π-algebras are Quillen equivalent.*

Proof: We first show that the n-fold suspension and desuspension functors pass to functors between (shifted)-Π-Lie algebras. Let $\mathfrak{g}_*$ be an n-Lie algebra with n-Lie homotopy operations. These operations are parametrized by elements in

$$[nL(\mathbb{Q}\mathbb{S}^m(\ell)_\bullet), nL(\bigoplus_{i=1}^{N}\mathbb{Q}\mathbb{S}^{m_i}(\ell_i)_\bullet)]_{snL}$$

where $\mathbb{S}^m(\ell)$ is the graded simplicial set that has the simplicial m-sphere in degree ℓ. If we consider $\Sigma^n\mathfrak{g}_*$, then this inherits operations parametrized by

$$\begin{aligned}
&[\Sigma^n nL(\mathbb{Q}\mathbb{S}^m(\ell)_\bullet), \Sigma^n nL(\bigoplus_{i=1}^{N}\mathbb{Q}\mathbb{S}^{m_i}(\ell_i)_\bullet)]_{sL}\\
&\qquad = [L(\Sigma^n\mathbb{Q}\mathbb{S}^m(\ell)_\bullet), L(\bigoplus_{i=1}^{N}\Sigma^n\mathbb{Q}\mathbb{S}^{m_i}(\ell_i)_\bullet)]_{sL}\\
&\qquad = [L(\mathbb{Q}\mathbb{S}^m(\ell-n)_\bullet), L(\bigoplus_{i=1}^{N}\mathbb{Q}\mathbb{S}^{m_i}(\ell_i-n)_\bullet)]_{sL}.
\end{aligned}$$

Vice versa the n-fold desuspension of a graded Lie Π-algebra inherits Lie homotopy operations.

Weak equivalences and fibrations are again determined by the underlying simplicial vector spaces and on this level (de)suspensions just shift the internal grading. □

Lemma 6.6 *Let A be an augmented E-algebra and $\ell \in \mathbb{Z}$. Then there is a natural isomorphism*

$$\Sigma^{n-\ell}(\mathbb{L}_s\bar{Q}_{nL})_t(AQ_*(H_*A|\mathbb{Q};\mathbb{Q})) \cong (\mathbb{L}_s\bar{Q}_{\ell L})_t(\Sigma^{n-\ell}AQ_*(H_*A|\mathbb{Q};\mathbb{Q})).$$

Proof: Let $Y_\bullet$ be a resolution of H_*A as a simplicial augmented Gerstenhaber algebra such that $Y_\bullet$ is cofibrant as a simplicial commutative algebra. As we explained above, we have

$$(\mathbb{L}_s \bar{Q}_{nL})_t(AQ_*(H_*A|\mathbb{Q};\mathbb{Q})) \cong \pi_s \pi_t^i Q_{nL}(B_{\bullet,\bullet})$$

where $B_{\bullet,\bullet}$ is a cofibrant replacement of $Q_a Y_\bullet$ in the category of bisimplicial n-Lie algebras with respect to the DKS-model structure. According to [BS92, 4.1] the following conditions are sufficient for $B_{\bullet,\bullet}$ to be cofibrant:

- For fixed external degree s each $B_{\bullet,s}$ is homotopy equivalent to $nL(X[s]_\bullet)$ as a simplicial n-Lie algebra, where $X[s]_\bullet$ is weakly equivalent to a sum of spheres $\bigoplus_i \mathbb{Q}S^{m_i}(r_i)_\bullet$.
- The external degeneracies are induced by maps $X[s] \to X[s+1]$ which are inclusions of summands up to homotopy.

Blanc and Stover show as well that such a $B_{\bullet,\bullet}$ can always be constructed.

Suppose that $B_{\bullet,\bullet}$ is a cofibrant replacement fulfilling these conditions. Now consider $\Sigma^{n-\ell} B_{\bullet,\bullet}$. Regrading the internal nonsimplicial degree does not affect the homotopy groups, so

$$\pi_s \pi_t^i \Sigma^{n-\ell} B_{\bullet,\bullet} = \delta_{s,0} \Sigma^{n-\ell} \pi_t Q_a Y_\bullet = \delta_{s,0} \pi_t \Sigma^{n-\ell} Q_a Y_\bullet$$

and we find that $\Sigma^{n-\ell} B_{\bullet,\bullet}$ is a resolution of $\Sigma^{n-\ell} Q_a Y_\bullet$. It is easy to see that $\Sigma^{n-\ell} B_{\bullet,\bullet}$ is cofibrant as well: We know that $\Sigma^{n-\ell} B_{\bullet,s} = \ell L(\Sigma^{n-\ell} X[s]_\bullet)$ and that suspending a simplicial sphere $\mathbb{Q}S^{m_i}(r_i)$ internally just shifts r_i. Hence we can compute $(\mathbb{L}_s \bar{Q}_{\ell L})_t(\Sigma^{n-\ell} AQ_*(H_*A|\mathbb{Q};\mathbb{Q}))$ as the homotopy groups of $Q_{\ell L} \Sigma^{n-\ell} B_{\bullet,\bullet}$. Exploiting the adjunction between $Q_{\ell L}$ and the functor which endows a graded $\mathbb{Q}$-vector space with a trivial ℓ-Lie algebra structure we obtain

$$Q_{\ell L} \circ \Sigma^{n-\ell} \cong \Sigma^{n-\ell} Q_{nL}.$$

Therefore the homotopy groups in question are the homotopy groups of

$$\Sigma^{n-\ell} Q_{nL} B_{\bullet,\bullet}$$

and the result follows. □

Since the composite functor spectral sequence is the spectral sequence associated to the bisimplicial object given by applying the indecomposables functor to a resolution with respect to the DKS model structure it is clear that an isomorphism of resolutions yields a morphism of spectral sequences. Note that deriving a functor followed by a suspension equals suspending the derived functor.

Corollary 6.7 *The above isomorphism is part of a morphism between the suspension of the Grothendieck spectral sequence*

$$\Sigma^{n-\ell}(\mathbb{L}_s \bar{Q}_{nL})_t(AQ_*(H_*A|\mathbb{Q};\mathbb{Q})) \Rightarrow \Sigma^{n-\ell}\mathbb{L}_{s+t} Q_{nG} H_*(\bar{A})$$

associated to $Q_{nL} \circ Q_a$ and the Grothendieck spectral sequence

$$(\mathbb{L}_s \bar{Q}_{\ell L})_t(\Sigma^{n-\ell} AQ_*(H_*A|\mathbb{Q};\mathbb{Q})) \Rightarrow \mathbb{L}_{s+t}(Q_{\ell L} \circ \Sigma^{n-\ell} \circ Q_a) H_*(\bar{A}).$$

Remark 6.8 We could deduce that the E^2-term of the spectral sequence calculating E_{n+1}-homology is

$$\mathbb{L}_p Q_{nG}(H_*(\bar{A})) \cong \Sigma^{\ell-n} \mathbb{L}_p(Q_{\ell L} \circ \Sigma^{n-\ell} \circ Q_a)(H_*(\bar{A})),$$

but this is clear since

$$Q_{\ell L} \circ \Sigma^{n-\ell} \cong \Sigma^{n-\ell} Q_{nL}.$$

Identifying n-Lie-Π algebra structures is hard. Sometimes we can reduce the complexity of that task. In characteristic zero, the Blanc-Stover spectral sequence simplifies due to the following well-known result.

Lemma 6.9 *Let $k = \mathbb{Q}$. A Lie-Π algebra $\pi_*(\mathfrak{g}_\bullet)$ is a bigraded Lie algebra.*

Proof: The Lie-Π structure on the homotopy groups of a simplicial graded Lie algebra is the structure induced by elements in

$$[L(\mathbb{Q}S^n(k)), L(\bigoplus_{i=1}^N \mathbb{Q}S^{n_i}(k_i))]_{sL} = \pi_n(L(\bigoplus_{i=1}^N \mathbb{Q}S^{n_i}(k_i)))_k.$$

Set $X = \bigoplus_{i=1}^N \mathbb{Q}S^{n_i}(k_i)$. Interpreting the Lie operad as a constant simplicial operad in graded vector spaces we find that we need to calculate $\pi_n(\bigoplus_{j\geqslant 0} Lie(j) \otimes_{\Sigma_j} (X)^{\otimes j})_k$.

Since over $\mathbb{Q}$ every Σ_j-module is projective we see that this is isomorphic to

$$\bigoplus_{j\geqslant 0} \bigoplus_{a+b=k} Lie(j)_a \otimes_{\Sigma_j} \bigoplus_{\substack{n_1+\ldots+n_j=n \\ b_1+\ldots+b_j=b}} \pi_{n_1}(X)_{b_1} \otimes \ldots \otimes \pi_{n_j}(X)_{b_j},$$

i.e., the free bigraded Lie algebra on N generators of degree (k_i, n_i), where we now consider the Lie operad as an operad in bigraded modules concentrated in bidegree (0,0). This yields that all homotopy operations on $\pi_*(\mathfrak{g}_\bullet)$ are the ones induced by the Lie structure of $\mathfrak{g}_\bullet$ via the Eilenberg-Zilber map. □

We obtain an analogous result in finite characteristic.

Lemma 6.10 *Let k be $\mathbb{F}_p$. If $\mathfrak{g} = \pi_* X_\bullet$ is a restricted Π-Lie-algebra that is concentrated in $\pi_0 X_\bullet$, then the Π-Lie-algebra structure on $\mathfrak{g}$ reduces to a restricted Lie-algebra structure.*

Proof: According to [BS92, §3] the operations on $\pi_0 X_\bullet$ are parametrized by elements in the set of homotopy classes of simplicial restricted Lie algebras

$$[rL(\mathbb{S}^0(k)), rL(\bigoplus_{i=1}^{N} \mathbb{S}^0(k_i)]_{srL}.$$

Here, $\mathbb{S}^0(r)$ is the simplicial graded $\mathbb{F}_p$-vector space that is $\mathbb{F}_p[r]$ in every simplicial degree and $\mathbb{F}_p[r]$ is the graded $\mathbb{F}_p$-vector space that is $\mathbb{F}_p$ concentrated in degree r. As the simplicial direction is constant in this case, the above set of homotopy classes reduces to the set of homomorphisms of restricted Lie-algebras

$$rL(rL(\mathbb{F}_p[k]), rL(\bigoplus_{i=1}^{N} \mathbb{F}_p[k_i])) \cong rL(\bigoplus_{i=1}^{N} \mathbb{F}_p[k_i])_k.$$

Thus we get the free restricted Lie-algebra on N generators in degree k and the operations reduce to a restricted Lie-structure on $\mathfrak{g}$. □

Part II. Examples

7. E_{n+1}-homology of free graded commutative algebras

In the following, we will consider free graded commutative algebras on one generator. For the general case note that working over a field ensures that E_n-homology of a tensor product of graded commutative algebras can be computed from the E_n-homology of the tensor factors: if A_*, B_* are two graded commutative algebras, then E_n-homology of $\overline{A_* \otimes B_*}$ can be identified with Hochschild homology of order n with coefficients in the ground field k:

$$H_*^{E_n}(\overline{A_* \otimes B_*}) \cong HH_{*+n}^{[n]}(A_* \otimes B_*; k)$$

which is defined as the homotopy groups of some simplicial set arising as the evaluation of a certain Γ-module $\mathcal{L}(A_* \otimes B_*; k)$ on the simplicial n-sphere,

$$HH_{*+n}^{[n]}(A_* \otimes B_*; k) \cong \pi_{*+n}\mathcal{L}(A_* \otimes B_*; k)(\mathbb{S}^n).$$

The latter is isomorphic to

$$\pi_{*+n}(\mathcal{L}(A_*; k)(\mathbb{S}^n) \otimes \mathcal{L}(B_*; k)(\mathbb{S}^n))$$

and hence the Künneth theorem expresses this in terms of tensor products of Hochschild homology groups of order n. For more background on Hochschild homology of order n see [P00] or [LR11, p. 207].

7.1. Characteristic zero, $n \geqslant 1$

Let $A = \mathbb{Q}[x]$ with the generator x being of degree zero, thus $H_*A = A$.

We know that E_{n+1}-homology of the non-unital algebra $\overline{\mathbb{Q}[x]}$ is isomorphic to the shifted $\mathbb{Q}$-homology of $K(\mathbb{Z},n+1)$ (see [C54] or [LR11]):

$$H_*^{E_{n+1}}(\overline{\mathbb{Q}[x]}) \cong H_{*+n+1}(K(\mathbb{Z},n+1),\mathbb{Q}).$$

We know that rationally the cohomology of $K(\mathbb{Z},n+1)$ is an exterior algebra on a generator in degree $n+1$ if $n+1$ is odd and is a polynomial algebra on a generator of degree $n+1$ for even $n+1$ and thus by dualizing we get the answer for $H_{*+n+1}(K(\mathbb{Z},n+1),\mathbb{Q})$.

For odd $n+1$ the polynomial algebra $\mathbb{Q}[x]$ is actually a free n-Gerstenhaber algebra because the bracket $[x,x]$ has to be trivial. Therefore the derived functors of n-Gerstenhaber indecomposables are trivial but in degree zero where we obtain the $\mathbb{Q}$-span of x and thus

$$H_*^{E_{n+1}}(\overline{\mathbb{Q}[x]}) \cong \begin{cases} \mathbb{Q}, & * = 0, \\ 0, & * > 0, \end{cases}$$

and this agrees with the above result.

For arbitrary degrees the result is essentially the same:

Proposition 7.1 *The free graded commutative algebra $S(x_j)$ on a generator x_j of degree $j \in \mathbb{Z}$ has as E_{n+1}-homology:*

$$H_*^{E_{n+1}}(\overline{S(x_j)}) \cong \Sigma^{-(n+1)}\overline{S(x_{j+n+1})}.$$

In particular we can identify E_{n+1}-homology of $\overline{S(x_j)}$ with the shifted homology of the Eilenberg-MacLane space $K(\mathbb{Z}, j+n+1)$ if $j+n+1>0$.

Proof: The E^2-term of the resolution spectral sequence can be identified with

$$E^2_{s,q} \cong \mathrm{Tor}_{s+1}^{S(x_{n+j})}(\mathbb{Q},\mathbb{Q})_{q+n} \cong S^{s+1}(x_{n+j}[1])_{q+n}.$$

The internal degree of $x_{n+j}[1]$ is still $n+j$ but for forming the free graded commutative algebra, $S(x_{n+j}[1])$, $x_{n+j}[1]$ is viewed as a generator in degree $n+j+1$. By S^{s+1} we denote the monomials of length $s+1$. Powers of $x_{n+j}[1]$ are trivial if $n+j$ is even thus we get a single contribution from $s=0$ of internal degree $n+j=q+n$.

In the odd case, we obtain the condition that the internal degree of $x_{n+j}[1]^{s+1}$ has to be $q+n$, thus $(s+1)(n+j)=q+n$. Therefore we get that $s+q$ has to be $sn+sj+s+j$ as claimed. □

Remark 7.2 The isomorphism

$$H_*^{E_{n+1}}(\overline{S(x_j)}) \cong \Sigma^{-(n+1)}\overline{S(x_{j+n+1})}$$

is not only an additive isomorphism but an isomorphism of algebras if we consider $H_*^{E_{n+1}}(\overline{S(x_j)})$ equipped with the multiplicative structure arising for example from computing $H_*^{E_{n+1}}(\overline{S(x_j)})$ via the iterated bar construction B^{n+1} as in [F11a]: If $j+n+1$ is even and we denote the suspension of an element x by sx, generating cycles in $B^{n+1}(\overline{S(x_j)})$ are of the form

$$s(s^n x_j) \otimes \ldots \otimes s(s^n x_j) \in (\Sigma B^n(\overline{S(x_j)}))^{\otimes r}$$

with $s^n x_j \in \Sigma^n \overline{S(x_j)} \subset B^n(\overline{S(x_j)})$. One easily calculates that

$$(s(s^n x_j))^{\otimes r} \cdot (s(s^n x_j))^{\otimes s} = \binom{r+s}{r} (s(s^n x_j))^{\otimes s+r}$$

with respect to the shuffle product on $B^{n+1}(\overline{S(x_j)})$, hence $H_*^{E_{n+1}}(\overline{S(x_j)})$ is, up to suspension, isomorphic to a polynomial algebra over $\mathbb{Q}$.

7.2. *Characteristic two*

For a generator, x_0, in degree zero, we have that E_2-homology of the non-unital polynomial algebra on x_0 over a field k is (up to a 2-shift in degree) the homology with k-coefficients of $K(\mathbb{Z},2) = \mathbb{C}P^\infty$. It turns out that shifting the degree of the polynomial generator down to degree minus one, trades the complex numbers for the reals. More generally, we compute E_2-homology for every polynomial algebra $\mathbb{F}_2[x_n]$ with a generator of degree $n \in \mathbb{Z}$. We always assume that the 1-restricted Lie structure on the polynomial algebra is trivial. Note that the suspension of a 1-restricted Lie algebra is a restricted Lie algebra, similar to 5.3.

Proposition 7.3 *Up to a shift, E_2-homology of a polynomial algebra $\mathbb{F}_2[x_{-1}]$ is isomorphic to the homology of $\mathbb{R}P^\infty$:*

$$H_s^{E_2}(\overline{\mathbb{F}_2[x_{-1}]}) \cong H_{s+1}(K(\mathbb{Z}/2\mathbb{Z},1);\mathbb{F}_2).$$

Proof: According to 5.5 our E^2-term is given by

$$E^2_{p,q} = ((\mathbb{L}_p Q_{1rL})(\mathbb{F}_2 x_{-1}))_q.$$

Using the suspension, we obtain from the $\mathbb{F}_2$-analogue of 5.4 that

$$\Sigma(\mathbb{L}_p Q_{1rL})(\mathbb{F}_2 x_{-1}) \cong (\mathbb{L}_p Q_{rL})(\Sigma\mathbb{F}_2 x_{-1}) \cong (\mathbb{L}_p Q_{rL})(\mathbb{F}_2 x_0)$$

where x_0 is a generator in degree zero.

As the restricted Lie-structure on $\mathbb{F}_2 x_0$ is trivial, its restricted enveloping algebra is

$$\mathfrak{U}_r(\mathbb{F}_2 x_0) \cong \mathbb{F}_2[x_0]/x_0^2 \cong \mathbb{F}_2[C_2].$$

Therefore

$$(\mathbb{L}_s Q_{rL})(\mathbb{F}_2 x_0) \cong \mathrm{Tor}_{s+1}^{\mathbb{F}_2[C_2]}(\mathbb{F}_2,\mathbb{F}_2) \cong H_{s+1}(C_2;\mathbb{F}_2) = H_{s+1}(\mathbb{R}P^\infty;\mathbb{F}_2).$$

Hence the E^2-term is concentrated in bidegrees $(s+1,-1)$, thus $H_{s+1}(\mathbb{R}P^\infty;\mathbb{F}_2)$ is isomorphic to $H_s^{E_2}(\overline{\mathbb{F}_2[x_{-1}]})$. □

In broader generality we can determine E_2-homology of $\mathbb{F}[x_j]$ for arbitrary degree $j \in \mathbb{Z}$. Again varying the degree j of x_j results in a shift.

Proposition 7.4 *With respect to the multiplicative structure induced by the shuffle product on the twofold bar construction we have*

$$H_*^{E_2}(\overline{\mathbb{F}_2[x_j]}) = \Sigma^{-2}\overline{\Gamma_{\mathbb{F}_2}(x_{j+2})}.$$

where $\Gamma_{\mathbb{F}_2}(x_{j+2})$ is a divided power algebra on a generator x_{j+2} in degree $j+2$.

Proof: As above we get that

$$\Sigma(\mathbb{L}_s Q_{1rL})(\mathbb{F}_2 x_j) \cong (\mathbb{L}_s Q_{rL})(\mathbb{F}_2 x_{j+1}) = \mathrm{Tor}_{s+1}^{\mathfrak{U}(\mathbb{F}_2 x_{j+1})}(\mathbb{F}_2,\mathbb{F}_2).$$

As the Lie-structure on $\mathbb{F}_2 x_{j+1}$ is trivial, the enveloping algebra is again a truncated polynomial algebra. We take the periodic free resolution of $\mathbb{F}_2$ over $\mathbb{F}_2[x_{j+1}]/x_{j+1}^2$

$$\ldots \to \Sigma^{\ell(j+1)}\mathbb{F}_2[x_{j+1}]/x_{j+1}^2 \to \ldots \to \Sigma^{j+1}\mathbb{F}_2[x_{n+1}]/x_{j+1}^2 \to \mathbb{F}_2[x_{j+1}]/x_{j+1}^2$$

where the maps are given by multiplication by x_{j+1}. Tensoring this with $\mathbb{F}_2$ over $\mathbb{F}_2[x_{j+1}]/x_{j+1}^2$ gives a chain complex with $\Sigma^{\ell(j+1)}\mathbb{F}_2$ in degree ℓ and trivial differential, and therefore

$$\mathrm{Tor}_{s+1}^{\mathfrak{U}(\mathbb{F}_2 x_{j+1})}(\mathbb{F}_2,\mathbb{F}_2) \cong \Sigma^{(s+1)(j+1)}\mathbb{F}_2.$$

Note that this is concentrated in bidegree $(p,q) = (s,(s+1)(j+1)-1)$ and contributes to total degree $p+q = (s+1)(j+2)-2$.

In order to determine the multiplicative structure on

$$H_*^{E_2}(\overline{\mathbb{F}_2[x_j]}) \cong H_*(\Sigma^{-2}B^2(\overline{\mathbb{F}_2[x_j]})),$$

we note that just as in 7.2 generating cycles in $B^2(\overline{\mathbb{F}_2[x_j]})$ are of the form

$$s(sx_j) \otimes \ldots \otimes s(sx_j).$$

Again we see that

$$(s(sx_j))^{\otimes r} \cdot (s(sx_j))^{\otimes s} = \binom{r+s}{r} (s(sx_j))^{\otimes s+r},$$

hence $\Sigma^2 H_*^{E_2}(\overline{\mathbb{F}_2[x_j]})$ is a divided power algebra with $\gamma_r(s(sx_j)) = (s(sx_j))^{\otimes r}$. $\square$

8. Reduced Hochschild cochains

Normalized Hochschild cochains $C^*(A,A)$ of an associative k-algebra A constitute an algebra over an unreduced E_2-operad [MS02]. The induced 1-Gerstenhaber structure on Hochschild cohomology was already described in [Ge63]. Unfortunately the augmentation inherited from $C^0(A,A) \cong A \to k$ is not compatible with the E_2-structure: Applying the brace operations as described by McClure and Smith to cochains in the augmentation kernel might yield the unit element in $C^0(A,A)$. Hence we are led to consider the following variant: Denote by $\bar{C}^*(A,A)$ the cochain complex with

$$\bar{C}^k(A,A) = \begin{cases} C^*(\bar{A},\bar{A}), k > 0, \\ C^0(\bar{A},A), k = 0, \end{cases}$$

and set the braces to be zero whenever one of the arguments is in $k \subset \bar{C}^0(A,A)$. One easily checks that this is an E_2-algebra and that the augmentation respects the E_2-structure.

The augmentation ideal of these *reduced Hochschild cochains* is $C^*(\bar{A},\bar{A})$. *i.e.*, the normalized cochain complex computing Hochschild cohomology of A with coefficients in $\bar{A}$.

Note that we consider Hochschild cochains, so with respect to cohomological grading the Lie bracket on cohomology is of degree -1. In the following we consider Hochschild cochains as a non-positively graded chain complex, so that we get an ordinary 1-restricted Gerstenhaber structure on the homology.

8.1. Hochschild cochains for $k[x]$

The following is a standard result.

Lemma 8.1 *For $A = k[x]$ reduced Hochschild cohomology is*

$$HH^0(k[x],\overline{k[x]}) = HH^1(k[x],\overline{k[x]}) \cong \overline{k[x]}.$$

In order to exploit our spectral sequence, we have to understand the induced structure on Hochschild cohomology. Hochschild cohomology is concentrated in degrees zero and one, so the multiplication (cup-product) gives rise to a square-zero extension.

A derivation $f \in \mathrm{Der}(k[x],\overline{k[x]})$ can be identified with $f(x)$. We denote by p' the formal derivative of a polynomial $p \in k[x]$.

Lemma 8.2 *The Lie bracket*

$$HH^1(k[x],\overline{k[x]}) \otimes HH^1(k[x],\overline{k[x]}) \to HH^1(k[x],\overline{k[x]})$$

is given by the usual Lie bracket of derivations, i.e.,

$$[f,g] = g \circ f - f \circ g = g' f - f' g,$$

whereas for $\alpha \in \overline{k[x]} \cong HH^0(k[x],\overline{k[x]})$ *and* $f \in HH^1(k[x],\overline{k[x]})$ *the bracket is given by*

$$[f,\alpha] := f(\alpha) = \alpha' f.$$

For $k = \mathbb{F}_2$ *the restriction* $\xi\colon HH^1(\mathbb{F}_2[x],\overline{\mathbb{F}_2[x]}) \to HH^1(\mathbb{F}_2[x],\overline{\mathbb{F}_2[x]})$ *sends* f *to* $f' f$.

Proof: The first facts can be found in [Ge63]. A direct calculation shows that ξ satisfies the properties of a restriction. To this end note that second derivatives of polynomials in characteristic two vanish. We also know that we actually have a (1-restricted) Gerstenhaber structure on $HH^*(\mathbb{F}_2[x],\overline{\mathbb{F}_2[x]})$, so the restriction is determined by $\xi(x)$. Then

$$\xi(x)' g + \xi(x) g' = [\xi(x), g] = [x,[x,g]] = g + x g', \text{ for all } g$$

implies that $\xi(x) = x$, so our choice of ξ is a unique restriction on $HH^*(\mathbb{F}_2[x],\overline{\mathbb{F}_2[x]})$. $\square$

Note that the Lie-bracket is trivial on $HH^0 \times HH^0$ and so is the restriction on HH^0. In particular, HH^* is far from being a free (restricted) 1-Gerstenhaber algebra.

Let $k = \mathbb{Q}$. The Hochschild cohomology groups in question are

$$HH^*(\mathbb{Q}[x],\overline{\mathbb{Q}[x]})_+ = \mathbb{Q}[x_0] \otimes \Lambda[y_{-1}],$$

where x_0 is identified with $x \in HH^0(\mathbb{Q}[x],\overline{\mathbb{Q}[x]})$ and y_{-1} with $x \in HH^1(\mathbb{Q}[x],\overline{\mathbb{Q}[x]})$. The 1-Lie structure described in Lemma 8.2 corresponds to setting $[y_{-1},x_0] = x_0$, all other brackets can then be determined by using the Poisson relation. Hence applying 5.1 yields the following.

Theorem 8.3 *For* $\mathbb{Q}[x]$ *the* E_2*-homology of the reduced Hochschild cochains on* $\mathbb{Q}[x]$ *is concentrated in degree* -1 *with*

$$H^{E_2}_{-1}(\bar{C}^*(\mathbb{Q}[x],\mathbb{Q}[x])) \cong \mathbb{Q}.$$

Proof: According to 5.1 the E^2-page of the spectral sequence we consider is given by
$$E^2_{p,q} \cong (\mathbb{L}_p Q_{1L}(W))_q$$
with 1-Lie structure on $W = \mathbb{Q}\langle x_0, y_{-1}\rangle$ given by $[y_{-1}, x_0] = x_0$. Using the equivalence between the category of 1-Lie algebras and the category of Lie algebras this yields
$$E^2_{p,q} \cong (\Sigma^{-1}\mathrm{Tor}^{\mathfrak{U}(\Sigma W)}_{p+1}(\mathbb{Q},\mathbb{Q}))_q,$$
where now $\Sigma W = \mathbb{Q}\langle x_1, y_0\rangle$. We use the resolution for graded Lie algebras given by May in [Ma66]: Set

$$P_i := \begin{cases} \mathfrak{U}(\Sigma W), & i = 0, \\ \mathbb{Q}\langle a^{(i)}_{i-1}, b^{(i)}_i\rangle \otimes \mathfrak{U}(\Sigma W), & i > 0, \end{cases}$$

with lower indices indicating the internal degree of the elements. Define $P_1 \to P_0$ by $a^{(1)}_0 \otimes 1 \mapsto y_0$ and $b^{(1)}_1 \otimes 1 \mapsto x_1$. Define $P_i \to P_{i-1}$ by $a^{(i)}_{i-1} \otimes 1 \mapsto b^{(i-1)}_{i-1} \otimes y_0 + (i-1)b^{(i-1)}_{i-1} \otimes 1 - a^{(i-1)}_{i-2} \otimes x_1$ and $b^{(i)}_i \otimes 1 \mapsto b^{(i-1)}_{i-1} \otimes x_1$ for $i > 1$. This is a $\mathfrak{U}(\Sigma W)$-free resolution of $\mathbb{Q}$, and hence $E^2_{p,q}$ vanishes expect for $p = 0$ and $q = -1$. □

8.2. *$\bar{C}^*(TV, TV)$ for V a $\mathbb{Q}$-vector space of dimension at least two.*

Let V be a fixed $\mathbb{Q}$-vector space of dimension at least 2. After adding a unit element, Hochschild cohomology of TV with coefficients in $\bar{T}V$ can be identified as the square-zero extension

$$HH^*(TV, \bar{T}V)_+ \cong \mathbb{Q} \rtimes M(-1)$$

where $M(-1) = HH^1(TV, \bar{T}V) = \mathrm{Der}(\bar{T}V)/\{\text{inner derivations}\}$ is concentrated in degree minus one and $\mathbb{Q}$ is in degree zero. The first summand in the Hodge decomposition of Hochschild homology is isomorphic to Harrison homology which is André-Quillen homology up to a shift in degree. This allows us to compute the input for the Blanc-Stover spectral sequence:

Proposition 8.4

$$AQ_*(HH^*(TV, \bar{T}V)_+|\mathbb{Q};\mathbb{Q}) \cong HH^{(1)}_{*+1}(\mathbb{Q} \rtimes M(-1);\mathbb{Q})$$

and $HH^{(1)}_*(\mathbb{Q} \rtimes M(-1);\mathbb{Q})$ is additively isomorphic to the free graded Lie-algebra generated by the graded vector space $M(-1)$.

Proof: The first statement is a general fact about the relationship between André-Quillen homology, Harrison homology and Hochschild homology in characteristic zero (see for instance [Lo97, 4.5.13]). Hochschild homology of the graded square zero extension $\mathbb{Q} \rtimes M(-1)$ with coefficients in $\mathbb{Q}$ is the homology of the complex

$$\ldots \xrightarrow{b} \mathbb{Q} \otimes_{\mathbb{Q}} (\mathbb{Q} \rtimes M(-1))^{\otimes n} \xrightarrow{b} \mathbb{Q} \otimes_{\mathbb{Q}} (\mathbb{Q} \rtimes M(-1))^{\otimes(n-1)} \xrightarrow{b} \ldots$$

where $\mathbb{Q} \otimes_{\mathbb{Q}} (\mathbb{Q} \rtimes M(-1))^{\otimes n}$ is in degree n and the boundary map b is an alternating sum of multiplication and augmentation maps. This chain complex is the associated chain complex of a simplicial graded $\mathbb{Q}$-vector space and its non-degenerate part in degree n is isomorphic to $M(-1)^{\otimes n}$. On that part, the boundary is trivial and thus we get that

$$HH_*(\mathbb{Q} \rtimes M(-1); \mathbb{Q}) \cong T(M(-1))$$

where the nth homology group corresponds to tensors of length n.

The Hodge decomposition is given by an action of Eulerian idempotents on the Hochschild chains and homology groups. The first idempotent, $e_n^{(1)}\colon HH_n(\mathbb{Q} \rtimes M(-1); \mathbb{Q}) \to HH_n(\mathbb{Q} \rtimes M(-1); \mathbb{Q})$ splits off Harrison homology. Reutenauer showed in [Re86, (2.2),(2.4)] that the image of this idempotent applied to a tensor algebra, TW, is precisely the free Lie algebra, LW. □

Note that in addition to the free graded Lie structure on André-Quillen homology we have the internal non-trivial (and non-free) 1-Lie structure coming from the first Hochschild cohomology group $M(-1)$.

With respect to this 1-Lie structure $AQ_*(HH^*(TV, \bar{T}V)_+|\mathbb{Q};\mathbb{Q})$ consists of the Lie subalgebra $AQ_0(HH^*(TV, \bar{T}V)_+|\mathbb{Q};\mathbb{Q}) = HH^*(TV, \bar{T}V)$ and the ideal $AQ_{*\geqslant 1}(HH^*(TV, \bar{T}V)_+|\mathbb{Q};\mathbb{Q})$. In particular

$$\mathbb{L}_s Q_{1L} HH^*(TV, \bar{T}V) = (\mathbb{L}_s Q_{1L})_0 AQ_*(HH^*(TV, \bar{T}V)_+|\mathbb{Q};\mathbb{Q}).$$

Hence we can identify certain elements in the Blanc-Stover spectral sequence in the case $V = \mathbb{Q}\{x, y\}$.

Lemma 8.5 *The Lie homology of $HH^*(T\mathbb{Q}\{x,y\}, \bar{T}\mathbb{Q}\{x,y\})$ is*

$$\mathbb{L}_s Q_{1L} HH^*(T\mathbb{Q}\{x,y\}, \bar{T}\mathbb{Q}\{x,y\}) = \begin{cases} \mathbb{Q}, & s = 0,1,3, \\ 0 & else. \end{cases}$$

Proof: We know that

$$HH^*(T\mathbb{Q}\{x,y\}, \bar{T}\mathbb{Q}\{x,y\}) \cong \mathrm{Der}(\bar{T}\mathbb{Q}\{x,y\}))/\{\text{inner derivations}\}$$

as a 1-Lie algebra concentrated in internal degree -1. Consider the derivations $D_{x,v}$ and $D_{y,w}$ defined by

$$D_{x,v}(x) = v, D_{x,v}(y) = 0, \quad D_{y,w}(x) = 0, D_{y,w}(y) = w.$$

for $v, w \in \bar{T}\mathbb{Q}\{x,y\}$. These form a basis of $\mathrm{Der}(\bar{T}\mathbb{Q}\{x,y\})$ as a vector space and are eigenvectors with respect to $[-, D_{x,x}]$ as well as $[-, D_{y,y}]$. Observe also that a typical inner derivation is of the form $D_{x,vx} - D_{x,xv} + D_{y,vy} - D_{y,yv}$ and hence an eigenvector as well. In particular $\mathrm{Der}(\bar{T}\mathbb{Q}\{x,y\}))/\{\text{inner derivations}\}$ splits into eigenspaces with respect to $[-, D_{x,x}]$ and $[-, D_{y,y}]$. Hence we can apply [Fu86, 1.5.2]. Since possible eigenvalues are limited we see that the Lie homology of $\mathrm{Der}(\bar{T}\mathbb{Q}\{x,y\}))/\{\text{inner derivations}\}$ is the homology of the complex

$$0 \longrightarrow \mathbb{Q}\{D_{x,x}\wedge D_{y,y}\wedge D_{x,y}\wedge D_{y,x}\} \longrightarrow \mathbb{Q}\{D_{x,x}\wedge D_{x,y}\wedge D_{y,x}, D_{y,y}\wedge D_{x,y}\wedge D_{y,x}\} \longrightarrow \mathbb{Q}\{D_{x,x}\wedge D_{y,y}, D_{x,y}\wedge D_{y,x}\} \longrightarrow \mathbb{Q}\{D_{x,x}, D_{y,y}\} \longrightarrow \mathbb{Q}\{D_{x,x}, D_{y,y}\} \longrightarrow 0$$

endowed with the usual differential of the Chevalley-Eilenberg complex. Hence the claim follows. □

Proposition 8.6 *The E^2-page of the Blanc-Stover spectral sequence has*

$$E^2_{s,0} = \begin{cases} \mathbb{Q} & \text{in internal degree } -1 \text{ for } s=0, \\ \mathbb{Q} & \text{in internal degree } -2 \text{ for } s=1, \\ \mathbb{Q} & \text{in internal degree } -4 \text{ for } s=3, \\ 0, & \text{else.} \end{cases}$$

Remark 8.7 The generators for $s = 0$ and $s = 1$ in the Blanc-Stover spectral sequence are permanent cycles and they cannot be boundaries for degree reasons. Therefore they give rise to permanent cycles $x_{0,-1}$ and $x_{1,-2}$ in the resolution spectral sequence. If the reduced Hochschild cochains on $T\mathbb{Q}\{x,y\}$ were free as an E_2-algebra, then we would get something of rank 2 as E_2-homology and this could correspond to these two survivors, but a priori $x_{0,-1}$ and $x_{1,-2}$ could be hit by differentials starting on elements in bidegree $(r,-r)$ for some $r \geqslant 2$.

8.3. Group algebras

Let G be a discrete group and k be a field. There is an identification of Hochschild cohomology of the group algebra $k[G]$ with group cohomology

$$HH^*(k[G], \overline{k[G]}) \cong H^*(G; \overline{k[G]}^c),$$

where $\overline{k[G]}^c$ denotes $\overline{k[G]}$ with the $k[G]$-action being induced by the conjugation action of G on G. We will consider cases where Hochschild cohomology results in an étale algebra, so we need the following result.

Lemma 8.8 *Let k be a field of characteristic zero or two and let A be an augmented étale k-algebra. Then $\bar{A}$ has trivial derived 1-(restricted) Gerstenhaber indecomposables and trivial E_2-homology.*

Proof: As explained in 6.3, let $P_\bullet \to A$ be a simplicial resolution of A by free 1-(restricted) Gerstenhaber algebras. As A is étale, it has trivial indecomposables and $Q_a(P_\bullet)$ has trivial homotopy groups in all degrees. Therefore the constant bisimplicial 1-(restricted) Lie algebra which is zero in all bidegrees is a valid resolution of $Q_a(P_\bullet)$. Application of the composite functor spectral sequence yields the result. □

Hochschild cochains on some group algebras have trivial 2-fold algebraic delooping:

Proposition 8.9 *Let G be a finite group. If*

(a) either G is abelian, the order of G is odd and $k = \mathbb{F}_2$,

(b) or if k is algebraically closed and of characteristic two and the order of G is odd,

(c) or if k is algebraically closed and of characteristic zero,

then

$$H_*^{E_2}(\bar{C}^*(k[G],k[G])) = 0, \textit{ for all } * \geqslant 0.$$

Proof: If G is finite and if the characteristic of k is prime to $|G|$ or the characteristic is zero, then $H^*(G,\overline{k[G]}^c) \cong H^0(G,\overline{k[G]}^c) = (\overline{k[G]}^c)^G \cong Z(\overline{k[G]})$. The multiplication induced by the E_2-action is the usual one. In the first case this center is $\overline{\mathbb{F}_2[G]}$. As G is finite abelian, it suffices to consider the case C_{p^r} for an odd prime p. But $\mathbb{F}_2[C_{p^r}]$ is étale over $\mathbb{F}_2$.

In the last two cases $k[G]$ is isomorphic to a product of matrix rings (Wedderburn) and hence the center is $Z(k[G]) \cong \prod_r k$, where r is the number of conjugacy classes of G. This is again an étale k-algebra. □

9. On the Hodge decomposition for higher order Hochschild homology

Over the rationals the operad E_n is formal, *i.e.*, there is a quasi-isomorphism between E_n and the operad of $(n-1)$-Gerstenhaber algebras (see [LV∞] for a nice overview on formality). As every $\mathbb{Q}[\Sigma_r]$-module $G_{n-1}(r)$ is projective, this quasi-isomorphism induces an isomorphism of operadic homology theories between E_n-homology and G_{n-1}-homology. As a consequence, our resolution spectral sequence has to collapse at the E^2-term and we obtain

$$\bigoplus_{p+q=\ell} (\mathbb{L}_p Q_{(n-1)G}(\bar{A}))_q \cong H_\ell^{E_n}(\bar{A}).$$

For an augmented commutative $\mathbb{Q}$-algebra A, we can identify E_n-homology with Hochschild homology of order n:

$$H_*^{E_n}(\bar{A}) \cong HH_{*+n}^{[n]}(A,\mathbb{Q}).$$

The latter groups possess a Hodge decomposition [P00, Proposition 5.2]. For odd n the Hodge summands of Hochschild homology of order n are a re-indexed version of the Hodge summands for ordinary Hochschild homology:

$$HH_{\ell+n}^{[n]}(A;\mathbb{Q}) = \bigoplus_{i+nj=\ell+n} HH_{i+j}^{(j)}(A;\mathbb{Q}).$$

However, for even n the summands are only described in terms of functor homology:

$$HH_{\ell+n}^{[n]}(A;\mathbb{Q}) = \bigoplus_{i+nj=\ell+n} \mathrm{Tor}_i^{\Gamma}(\theta^j, \mathcal{L}(A,\mathbb{Q})).$$

For $j = 1$ the terms consist of André-Quillen homology:

$$\mathrm{Tor}_i^{\Gamma}(\theta^1, \mathcal{L}(A,\mathbb{Q})) \cong AQ_i(A|\mathbb{Q};\mathbb{Q}).$$

For $i = 0$ one obtains $\theta^j \otimes_\Gamma \mathcal{L}(A,\mathbb{Q}) \cong \mathbb{Q} \otimes_A \mathrm{Sym}_A^j(\Omega^1_{A|\mathbb{Q}})$, the j-th symmetric power generated by the module of Kähler differentials.

Theorem 9.1 *Let A be a commutative augmented $\mathbb{Q}$-algebra. For all $\ell, k \geqslant 1$ and $m \geqslant 0$:*

- $$HH_{m+1}^{(\ell)}(A;\mathbb{Q}) \cong (\mathbb{L}_m Q_{2kG}\bar{A})_{(\ell-1)2k}.$$

- $$\mathrm{Tor}_{m-\ell+1}^{\Gamma}(\theta^\ell, \mathcal{L}(A;\mathbb{Q})) \cong (\mathbb{L}_m Q_{(2k-1)G}\bar{A})_{(\ell-1)(2k-1)}.$$

Thus the Hodge summands of higher order Hochschild homology can be identified with Gerstenhaber homology groups.

Of course, the convention is that negatively indexed Tor-groups vanish.

Note, that the case $\ell = 1$ comes for free: The first Hodge summand is André-Quillen homology,

$$HH^{(1)}_{m+1}(A;\mathbb{Q}) \cong AQ_m(A|\mathbb{Q};\mathbb{Q}) \cong \mathrm{Tor}^{\Gamma}_m(\theta^1,\mathcal{L}(A;\mathbb{Q}))$$

and this in turn is $\mathbb{L}_m Q_{kG}(\bar{A})_0$ for all $m \geqslant 0$, $k \geqslant 1$.

For alternative approaches to the Hodge decomposition of higher order Hochschild homology see [B∞, Gi08].

In order to prove Theorem 9.1 we need a stability result. For the remainder of this section $A \to \mathbb{Q}$ is an augmented commutative $\mathbb{Q}$-algebra.

Lemma 9.2 *The derived functors of Gerstenhaber indecomposables are stable in the following sense:*

$$(\mathbb{L}_m Q_{nG}\bar{A})_{qn} \cong (\mathbb{L}_m Q_{(n+2)G}\bar{A})_{q(n+2)}.$$

Proof: We consider the standard resolution that calculates $(\mathbb{L}_m Q_{nG}\bar{A})$. In simplicial degree ℓ and internal degree r this is $(nG)^{\ell+1}(\bar{A})_r$. This resolution is concentrated in degrees of the form $r = qn$ because iterated n-Lie brackets on degree zero elements are concentrated in these degrees. We can identify the terms $(nG)^{\ell+1}(\bar{A})_{qn}$ with the terms $((n+2)G)^{\ell+1}(\bar{A})_{q(n+2)}$ where we just exchange n-Lie brackets by $(n+2)$-Lie brackets and adjust the internal degrees.

This yields an isomorphism of resolutions and hence an isomorphism on the corresponding homology groups. □

Remark 9.3 Note that there is no stability result when one passes from n to $n+1$: Take for instance $A = \mathbb{Q}[x]$. For even n this is a free n-Gerstenhaber algebra but for odd n it is not.

Proof of Theorem 9.1: As the claim is clear for $\ell = 1$, we do an induction on the label of the Hodge summands. We start with the $2k$-Gerstenhaber case. Thus assume that we know the claim for all Hodge summands $HH^{(j)}_p$ for all $1 \leqslant j \leqslant \ell$ and all $p \geqslant 1$. Lemma 9.2 allows us to choose k such that $1 \leqslant m < 2k$ and to consider $H^{E_{2k+1}}_{m+2k\ell}(\bar{A})$:

$$\bigoplus_{p+q=m+2k\ell} \mathbb{L}_p Q_{2kG}(\bar{A})_q \cong H^{E_{2k+1}}_{m+2k\ell}(\bar{A}) \cong \bigoplus_{i+j(2k+1)=m+2k(\ell+1)+1} HH^{(j)}_{i+j}(A;\mathbb{Q}).$$

The summands $\mathbb{L}_{m+2k(\ell-r)} Q_{2kG}(\bar{A})_{2kr}$ for $r < \ell$ are already identified with Hodge summands. The remaining non-trivial summand in $H^{E_{2k+1}}_{m+2k\ell}(\bar{A})$ is $\mathbb{L}_m Q_{2kG}(\bar{A})_{2k\ell}$ and in the Hodge decomposition we still have the summand for $j = \ell + 1$. In this case

$$i + (\ell+1)(2k+1) = m + 2k(\ell+1) + 1.$$

Hence $i = m + 1 - (\ell + 1)$ and $i + j = m + 1$.

For the $(2k-1)$-Gerstenhaber case the argument is similar, but the degree count is different: For $j = \ell + 1$ we get

$$i + (\ell+1)2(k-2) = m - \ell + 2(k-2)(\ell+1)$$

and thus $i = m - \ell$. □

Remark 9.4 *A posteriori* Theorem 9.1 yields a description of derived functors of $2k$-Gerstenhaber algebras in terms of (higher) André-Quillen homology: A classical spectral sequence argument allows an identification of $HH^{(\ell)}_{m+1}(A,\mathbb{Q})$ with $D^{(\ell)}_{m+1-\ell}(A;\mathbb{Q})$ [Lo97, 3.5.8,4.5.13] which in turn is $H_{m+1-\ell}((\Lambda^{\ell}_{P_*}\Omega^1_{P_*|\mathbb{Q}}) \otimes_{P_*} \mathbb{Q})) \cong H_{m+1-\ell}(\Lambda^{\ell}(\Omega^1_{P_*|\mathbb{Q}} \otimes_{P_*} \mathbb{Q}))$. Here P_* is a free simplicial resolution of A in commutative $\mathbb{Q}$-algebras, for instance $P_t = (SI)^{\circ(t+1)}(A)$. Thus we obtain

$$H_{m+1-\ell}(\Lambda^{\ell}(\Omega^1_{P_*|\mathbb{Q}} \otimes_{P_*} \mathbb{Q})) \cong (\mathbb{L}_m Q_{2kG}\bar{A})_{(\ell-1)2k}.$$

We show in the following that the identification of the Hodge summands follows independently from an easy spectral sequence argument. We are also able to prove an analogous result for the even case:

Theorem 9.5 *For every augmented commutative $\mathbb{Q}$-algebra A we can identify the Hodge summands of Hochschild homology of order $2k$ for $k \geqslant 1$ as*

$$\begin{aligned}\mathrm{Tor}^{\Gamma}_{m+1-\ell}(\theta^{\ell},\mathcal{L}(A;\mathbb{Q})) &\cong (\mathbb{L}_m Q_{(2k-1)}\bar{A})_{(2k-1)(\ell-1)}\\ &\cong H_{m-\ell+1}(\mathrm{Sym}^{\ell}(\Omega^1_{P_*|\mathbb{Q}} \otimes_{P_*} \mathbb{Q})).\end{aligned}$$

We also recover the identification for Hodge summands of Hochschild homology of odd order:

$$HH^{(\ell)}_{m+1}(A;\mathbb{Q}) \cong \mathbb{L}_m Q_{2kG}(\bar{A})_{2k(\ell-1)} \cong H_{m-\ell+1}(\Lambda^{\ell}(\Omega^1_{P_*|\mathbb{Q}} \otimes_{P_*} \mathbb{Q})).$$

Remark 9.6 The functor homology terms $\mathrm{Tor}_*(\theta^{\ell},\mathcal{L}(A;\mathbb{Q}))$ also describe the homology of the ℓth homogeneous layer in the Taylor tower of the Γ-module $\mathcal{L}(A;\mathbb{Q})$, $D_{\ell}(\mathcal{L}(A;\mathbb{Q}))[1]$, [Ri01, Proposition 4.7]:

$$H_*(D_{\ell}(\mathcal{L}(A;\mathbb{Q}))[1]) \cong \mathrm{Tor}_*(\theta^{\ell},\mathcal{L}(A;\mathbb{Q})).$$

Thus our results identifies these homology groups with derived functors of n-Gerstenhaber indecomposables for odd n and with the homology of the ℓth symmetric power of the module of derived Kähler differentials.

Proof of Theorem 9.5: Let $D_{*,*}$ be the bicomplex with $D_{r,s} = (nG)^{\circ(r+1)}((SI)^{\circ(s+1)}(A))$. Taking n-Gerstenhaber indecomposables yields another bicomplex $C_{*,*}$ with $C_{r,s} = Q_{nG}(D_{r,s}) \cong (nG)^{\circ(r)}((SI)^{\circ(s+1)}(A))$:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
(SI)^{\circ(3)}(A) \arrow[d] & (nG)((SI)^{\circ(3)}(A)) \arrow[l] \arrow[d] & (nG)^{\circ(2)}((SI)^{\circ(3)}(A)) \arrow[l] \arrow[d] & \ldots \arrow[l] \\
(SI)^{\circ(2)}(A) \arrow[d] & (nG)((SI)^{\circ(2)}(A)) \arrow[l] \arrow[d] & (nG)^{\circ(2)}((SI)^{\circ(2)}(A)) \arrow[l] \arrow[d] & \ldots \arrow[l] \\
(SI)(A) & (nG)((SI)(A)) \arrow[l] & (nG)^{\circ(2)}((SI)(A)) \arrow[l] & \ldots \arrow[l]
\end{tikzcd}$$

Taking vertical homology, H^v_*, first and then horizontal homology, H^h_*, gives

$$H^h_r(H^v_s(C_{*,*})) \cong \mathbb{L}_r Q_{nG}(\bar{A})$$

concentrated in the $(s = 0)$-line: the vertical homology groups are trivial but for $s = 0$ because $(SI)^{\circ(\bullet+1)}(A)$ is a resolution of A.

Switching the roles of vertical and horizontal homology gives

$$H^v_r(H^h_s(C_{*,*})) \cong H_r \mathbb{L}_s Q_{nG}(SI)^{\circ(\bullet+1)}(A).$$

We know by Corollary 5.2 that $\mathbb{L}_s Q_{nG}(SI)^{\circ(\bullet+1)}(A)$ is isomorphic to $\mathbb{L}_s Q_{nL}(SI)^{\circ(\bullet)}(A)$ and using the suspension correspondence between n-Lie algebras and graded Lie algebras we get

$$\mathbb{L}_s Q_{nG}(SI)^{\circ(\bullet+1)}(A) \cong \Sigma^{-n}\mathbb{L}_s Q_L \Sigma^n (SI)^{\circ(\bullet)}(A).$$

Since $\Sigma^n(SI)^{\circ(\bullet)}(A)$ carries a trivial Lie structure we can identify these groups as

$$\Sigma^{-n}\mathrm{Tor}^{\mathfrak{U}(\Sigma^n(SI)^{\circ(\bullet)}(A))}_{s+1}(\mathbb{Q},\mathbb{Q}) \cong \Sigma^{-n} S^{s+1}(\Sigma^n(SI)^{\circ(\bullet)}A[1]).$$

Recall that $\Sigma^n(SI)^{\circ(\bullet)}A[1]$ is still concentrated in internal degree n, but for the free graded commutative algebra generated by it, $S(\Sigma^n(SI)^{\circ(\bullet)}A[1])$, we consider its elements as being of degree $n+1$, thus the total internal degree of elements in $\Sigma^{-n}S^{s+1}(\Sigma^n(SI)^{\circ(\bullet)}A[1])$ is sn.

For $n = 2k$ we therefore obtain

$$HH^{(\ell)}_{m+1}(A;\mathbb{Q}) \cong \mathbb{L}_m Q_{2kG}(\bar{A})_{2k(\ell-1)} \cong H_{m-\ell+1}(\Lambda^\ell(\Omega^1_{P_*|\mathbb{Q}} \otimes_{P_*} \mathbb{Q}))$$

because

$$(SI)^{\circ(t)}(A) \cong Q_a((SI)^{\circ(t+1)}(A)) \cong \Omega^1_{P_t|\mathbb{Q}} \otimes_{P_t} \mathbb{Q}$$

with $P_t = (SI)^{\circ(t+1)}(A)$.

For $n = 2k-1$ however, we get symmetric powers of the Kähler differentials and have

$$\begin{aligned}\mathrm{Tor}^{\Gamma}_{m+1-\ell}(\theta^{\ell}, \mathcal{L}(A;\mathbb{Q})) &\cong (\mathbb{L}_m Q_{(2k-1)G}\bar{A})_{(2k-1)(\ell-1)} \\ &\cong H_{m-\ell+1}(\mathrm{Sym}^{\ell}(\Omega^1_{P_*|\mathbb{Q}} \otimes_{P_*} \mathbb{Q}))\end{aligned}$$

again with $P_t = (SI)^{\circ(t+1)}(A)$. □

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