maps are free and isomorphisms in dimensions $\leq k+1$ and $H_q(X') = H_q(Y') = 0$ for $q > k$.

If f is a k-equivalence, then as g, g' are so is f'. Thus f' is a weak equivalence, hence a homotopy equivalence since X' and Y' are free. Thus $F(g), F(f'), F(g')$ are k equivalences so $F(f)$ is. Q.E.D.

Corollary 7.5: If X is k-connected so is $\underline{\underline{L}}F(X)$.

7.6. For simplicial objects in an abelian category $\underline{A}$ the 0th homology functor $H_o\colon s\underline{A} \to \underline{A}$ is left adjoint to the functor $c\colon \underline{A} \to s\underline{A}$. This also holds for more general categories, such as categories of universal algebras having an underlying abelian group law, and in particular for the category of ring-modules. Hence given a simplicial ring and module (R,X), the canonical adjunction map $(R,X) \to (cH_oR, cH_oX)$ gives rise to a map $F(R,X) \to cF(H_oR, H_oX)$ and hence to a canonical map

$$H_o(F(R,X)) \to F(H_oR, H_oX) \tag{7.7}$$

We shall say that F is right exact if this map is always an isomorphism. $F = S, \Lambda$, and Γ are all right exact because they are left adjoint functors. For example if $F = S$ we have

$$\begin{aligned}
\mathrm{Hom}_{H_oR\text{-alg}}(S^{H_oR}(H_oX), A) &= \mathrm{Hom}_{H_oR\text{-mod}}(H_oX, A) \\
&= \mathrm{Hom}_{\underline{M}_R}(X, cA) = \mathrm{Hom}_{s\text{-}R\text{-alg}}(S^RX, cA) \\
&= \mathrm{Hom}_{H_oR\text{-alg}}(H_o(S^RX), A) .
\end{aligned}$$

7.8. If (B,M) is a ring-module let $L_qF(B,M) = H_q(\underline{\underline{L}}F(B,M))$. If F is right exact clearly

$$L_oF(B,M) \simeq F(B,M)$$

Proposition 7.9: There is a spectral sequence

$$E^2_{pq} = H_p\{(L_qF)(R,X)\} \Longrightarrow H_{p+q}(\underline{\underline{L}}F(R,X))$$

which when F is right exact has the edge homomorphism

$$(7.10) \qquad H_n(\underline{\underline{L}}F(R,X)) \to E^2_{no} = H_n\{F(R,X)\}$$

which is the map on homology induced by the canonical map $\underline{\underline{L}}F(X) \to F(X)$.

Proof: This spectral sequence is similar to the Kunneth spectral sequence th.6(b) of [HA], II and is constructed in pretty much the same way. We construct an exact sequence in $\underline{M}_R$

$$(7.11) \qquad \cdots \to P_{(2)} \to P_{(1)} \to P_{(0)} \to X \to 0$$

by recursion, letting $X_{(o)} = X$, $P_{(q)} \to X_{(q)}$ be a free resolution of $X_{(q)}$, and $X_{(q+1)} = \text{Ker } P_{(q)} \to X_{(q)}$. Let $Q_{(\cdot)} = N^{-1}_{(\cdot)}\{P_{(\cdot)}\}$ be the simplicial object in $\underline{M}_R$ obtained by applying the inverse of the normalization functor to the complex $P_{(\cdot)}$ ([ ], §3). Then

$$Q_{(k)} = \oplus_\eta P_{(t\eta)}$$

where η runs over all surjective monotone maps with source $[k]$ and target $[t\eta]$. From this we see that (i) $Q_{(k)}$ is a free

R module and (ii) the inclusion $P_{(o)} \to Q_{(k)}$, coming from η = unique map: $[k] \to [0]$, is a homotopy equivalence. Indeed by construction $H(P_{(k)}) = 0$ for $k > 0$ hence $P_{(k)}$ is contractible.

Now consider the bisimplicial abelian group $K_{pq} = F(R_q, Q_{(p)q})$ and the two associated spectral sequences having the homology of the diagonal simplicial abelian group K_{nn} for common abutment (see [ ] Satz. 2.15 or [ ]). Using the property (ii) we have

$$H_q^v(K_{p\cdot}) = H_q F(R, P_{(o)})$$

for all p hence

$$H_p^h H_q^v(K_{\cdot\cdot}) = \begin{cases} 0 & p > 0 \\ H_q F(R, P_{(o)}) & p = 0 \end{cases}$$

Thus the spectral sequence with this as E^2 degenerates showing that the map $F(R, P_{(o)}) \to \Delta K$ is a weak equivalence. For fixed m, the exactness of 7.11 together with property (i) imply that $Q_{(\cdot)n}$ is a free simplicial R_m module resolution of X_m, hence

$$H_q^h(K_{\cdot m}) = (L_q F)(R_m, X_m) \quad \text{and}$$

$$E_{pq}^2 = H_p^v H_q^h(K_{\cdot\cdot}) = H_p\{(L_q F)(R, X)\} \Longrightarrow H_{p+q}(\Delta K)$$

Combining this spectral sequence with the weak equivalence $\underline{\underline{L}}F(R,X) = F(R, P_{(o)}) \to \Delta K$ we obtain the desired spectral sequence 7.9. It remains to note that the edge homomorphism $H_n(\Delta K) \to E_{no}^2$ is induced by the map $\Delta K \to H_o^h(K_{\cdot\cdot}) = F(R,X)$ which when composed with $F(R, P_{(o)}) \to \Delta K$ gives the map $F(R, P_{(o)}) \to F(R,X)$, or alternatively the natural map $\underline{\underline{L}}F(R,X) \to F(R,X)$. This proves the assertion about the edge homomorphism so completes the proof of 7.9. Q.E.D.

<u>Corollary 7.12</u>: If F is right exact and $L_qF(R,X) = 0$ for $q > 0$, then $\underline{\underline{L}}F(R,X) \to F(R,X)$ is a weak equivalence.

<u>Proposition 7.13</u>: Suppose that filtered inductive limits in the target abelian category $\underline{A}$ of F are exact and that F commutes with filtered inductive limits. Then if M_i , $i \in I$ is a filtered inductive system of B modules

$$\lim L_qF(B,M_i) \simeq L_qF(B,\lim M_i)$$

<u>Proof</u>: Let $C(M)$ be the cotriple resolution of M with respect to the free B-module--underlying set pair of adjoint functors. As both functors commute with filtered inductive limits $\lim C(M_i) \simeq C(\lim M_i)$ hence $\lim L_qF(B,M_i) = H_q\{\lim F(B,C(M_i))\}$ (since filtered inductive limits are exact in $\underline{A}$) = $H_q\{F(B,\lim C(M_i)\} = H_q\{F(B,C(\lim M_i))\} = L_qF(B,\lim M_i)$. Q.E.D.

<u>Corollary 7.14</u>: If M is a flat B module & F satisfies the hypotheses of 7.13, then $L_qF(B,M) = 0$ for $q > 0$.

<u>Proof</u>: This is clearly true if M is a free B module, hence for any flat module since it is a filtered inductive limit of free modules by a theorem of Lazard.

7.15. A simplicial module X over R will be called <u>flat</u> if each X_q is a flat R_q module. Combining 7.12 and 7.14 we obtain

Corollary 7.16: Suppose F is right exact and satisfies the hypotheses of 7.13. Then if X is a flat R module the canonical homomorphism $\underline{\underline{L}}F(R,X) \to F(R,X)$ is a weak equivalence.

7.17. Recall that the symmetric and divided power algebras of a B module M are commutative graded algebras

$$S^B M = \bigoplus_{n\geq 0} S_n^B M \qquad \Gamma^B M = \bigoplus_{n\geq 0} \Gamma_n^B M$$

and the exterior algebra is a graded algebra

$$\Lambda^B M = \bigoplus_{n\geq 0} \Lambda_n^B M$$

which is skew-commutative with respect to the grading. Consequently if X is a simplicial module over a simplicial ring R, the bigraded algebras

$$H_q(S_n^R X)\ ,\ H_q(\Gamma_n^R X)\ ,\ H_q(\Lambda_n^R X) \qquad q \geq 0\ ,\ n \geq 0$$

are skew-commutative for the degrees q,q and n+q respectively.

7.18. Let C and Σ be the cone and suspension functors on $\underline{\underline{M}}_R$ given by

$$CX = X\otimes_{\mathbb{Z}}\mathbb{Z}\Delta(1)/X\otimes_{\mathbb{Z}}\mathbb{Z}\{0\}$$

$$\Sigma X = X\otimes_{\mathbb{Z}}\mathbb{Z}\Delta(1)/X\otimes_{\mathbb{Z}}\mathbb{Z}\dot{\Delta}(1)$$

so that there is a canonical exact sequence

(7.19) $$0 \to X \to CX \to \Sigma X \to 0$$

which splits in each dimension. Σ induces the suspension functor (again denoted by Σ) on $Ho(\underline{M}_R)$ ([HA], II, p.6.5), and as CX is contractible 7.19 gives rise to the suspension isomorphism

(7.20) $$H_q(X) \simeq H_{q+1}(\Sigma X) .$$

Proposition 7.21: There are canonical bigraded algebra isomorphisms

(7.22) $$H_q(\underline{\underline{L}} \wedge_n^R X) \simeq H_{q+n}(\underline{\underline{L}}S_n^R \Sigma X) \qquad q,n \geq 0$$

(7.23) $$H_q(\underline{\underline{L}}\Gamma_n^R X) \simeq H_{q+n}(\underline{\underline{L}}\wedge_n^R \Sigma X) \qquad q,n \geq 0$$

which reduce to the suspension isomorphism 7.20 when $n = 1$. Moreover

(7.24) $$H_q(\underline{\underline{L}}S_n^R \Sigma X) = 0 \qquad 0 \leq q < n$$

(7.25) $$H_q(\underline{\underline{L}}\wedge_n^R \Sigma X) = 0 \qquad 0 \leq q < n .$$

Before proving this we deduce some corollaries. Recall that $S, \wedge$, and Γ are left adjoint functors, hence right exact (7.7) and there are canonical isomorphisms of graded algebras

(7.26)
$$H_o(\underline{\underline{L}}S_n^R X) \simeq S_n^{H_oR}(H_oX)$$
$$H_o(\underline{\underline{L}}\wedge_n^R X) \simeq \wedge_n^{H_oR}(H_oX)$$
$$H_o(\underline{\underline{L}}\Gamma_n^R X) \simeq \Gamma_n^{H_oR}(H_oX)$$

Corollary 7.27: If $H_oX = 0$, then

$$H_q(\underline{\underline{L}}S_n^R X) = H_q(\underline{\underline{L}} \wedge_n^R X) = 0 \qquad 0 \leq q < n$$

and there are canonical graded algebra isomorphisms

(7.28) $$\Lambda_n^{H_oR}(H_1X) \simeq H_n(\underline{\underline{L}}S_n^R X)$$

(7.29) $$\Gamma_n^{H_oR}(H_1X) \simeq H_n(\underline{\underline{L}}\Lambda_n^R X)$$

Proof: As $H_oX = 0$, X is isomorphic in $Ho(\underline{M}_R)$ to ΣY for some Y by [HA], II, §6, prop.1. The corollary follows from the proposition using 7.26 and the suspension isomorphism $H_1X \simeq H_oY$.

In a similar way one may prove

Corollary 7.30: If $H_oX = H_1X = 0$, then

$$H_q(\underline{\underline{L}}S_n^R X) = 0 \qquad q < 2n$$

and there is a canonical graded algebra isomorphism

(7.31) $$\Gamma_n^{H_oR}(H_2X) \simeq H_{2n}(\underline{\underline{L}}S_n^R X)$$

7.32. 6.6 and 6.7 now follow from 7.27 and 7.28, using 7.16 to drop the $\underline{\underline{L}}$.

Remark 7.33: 7.28 is the unique algebra map which extends the canonical isomorphism for $n = 1$. Similarly by means of a suitable shuffle formula it is possible to define divided power operations on the right sides of 7.29 and 7.31 and then these maps are the unique homomorphisms of divided power algebras extending the canonical isomorphism for $n = 1$.

7.34. Proof of the proposition: If M is a B module let d be the Koszul differential on the bigraded algebra $\wedge_q M \otimes S_n M$ $q,n \geq 0$ where $\otimes, \wedge,$ and S are taken over B. d is the unique endomorphism of this bigraded algebra which is a skew-derivation with respect to the exterior degree q and is such that $d(m \otimes 1) = 1 \otimes m$, $d(1 \otimes m) = 0$. If M is a flat B module, then

$$(7.35) \qquad \cdots \xrightarrow{d} \wedge_2 M \otimes SM \xrightarrow{d} \wedge_1 M \otimes SM \xrightarrow{d} SM \to 0 \to \cdots$$

is a flat differential graded skew-commutative algebra which is a resolution of B considered as an SM algebra via the augmentation $SM \to B$. In effect one reduces by Lazard to the case where M is a finitely generated free B module, then to $M = B$ by the Kunneth formula, in which case the fact that 7.35 is a resolution is clear.

If

$$0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$$

is an exact sequence of flat B modules, we may define a differential d on the bigraded algebra $\wedge_q M' \otimes S_n M$ by requiring it to be a skew-derivation with respect to the exterior degree q such that $d(m' \otimes 1) = 1 \otimes im'$, $d(1 \otimes m) = 0$. Then

$$(7.36) \qquad \cdots \xrightarrow{d} \wedge_2 M' \otimes SM \xrightarrow{d} \wedge_1 M' \otimes SM \xrightarrow{d} SM \to 0 \cdots$$

is a flat differential graded skew-commutative algebra which is a resolution of SM'' considered as an SM algebra via $Sj\colon SM \to SM''$. In effect we may assume M'' free by Lazard in which case

$M \simeq M' \oplus M''$ so 7.36 is the tensor product $(\Lambda M' \otimes SM') \otimes SM''$, which by Kunneth has homology SM'' in dimension 0 .

To prove the proposition we may assume that X is a free simplicial R module and drop the $\underline{\underline{L}}$. Applying the sequence 7.35 dimension-wise to the exact sequence 7.19 we obtain exact sequences

$$(7.37) \quad 0 \to \Lambda_n^R X \to \dots \to \Lambda_1^R X \otimes_R S_{n-1}^R CX \to S_n^R CX \to S_n^R \Sigma X \to 0$$

of R modules. Using the fact that CX is contractible we obtain canonical isomorphisms

$$(7.38) \qquad H_q(S_n^R \Sigma X) \simeq H_{q-n}(\Lambda_n^R X) \qquad q \geq n$$

$$\simeq 0 \qquad q < n$$

To show these isomorphisms constitute an isomorphism of graded algebras, let $K_{pq}(n) = \Lambda_p^{R_q} X_q \otimes_{R_q} S_{n-p}^{R_q} X_q$ be considered a graded double complex, whose horizontal differential is the d in 7.37 and whose vertical differential comes from the simplicial structure. $K_{pq} = \oplus_n K_{pq}(n)$ is a bigraded ring for which the total differential is a derivation and hence the two spectral sequences of the double complex K_{pq} are algebra spectral sequences. But as 7.37 is exact and CX is contractible both spectral sequence collapse yielding algebra isomorphisms

$$H_m(S_n^R \Sigma X) \simeq H_m(K(n)) \simeq H_{m-n}(\Lambda_n^R X)$$

whose composition is nothing but 7.38. Thus 7.38 is an algebra isomorphism and we have proved half of the proposition.

The proof of the other half is similar where the sequence 7.36 is replaced by

$$(7.39) \qquad \to \Gamma_2 M' \otimes \Lambda M \xrightarrow{d} \Gamma_1 M' \otimes \Lambda M \xrightarrow{d} \Lambda M \to 0 \ldots$$

where d is the unique endomorphism of $\Gamma M' \otimes \Lambda M$ which is a skew-derivation for the exterior degree and is such that $d(\gamma_k(m')\otimes 1) = \gamma_{k-1}(m')\otimes im'$, $d(1\otimes m) = 0$. Q.E.D.

Corollary 7.40: If X is k-connected and $n > 0$, then

$$\underline{\underline{L}}S_n^R X \text{ is } 2(n-1)+k \text{ connected} \quad k \geq 1$$

$$\underline{\underline{L}}\Lambda_n^R X \text{ is } (n-1)+k \text{ connected} \quad k \geq 1$$

$$\underline{\underline{L}}\Gamma_n^R X \text{ is } k \text{ connected} \quad k \geq 0 .$$

Proof: The last assertion follows from 7.3, and the first two may be deduced from the last setting $X = \Sigma Y$ and using the proposition.

Remark 7.41: The connectivity assertions 7.40 are the best possible in general. In characteristic zero (i.e. when R is a simplicial Q algebra), then $\Gamma \simeq S$ hence iterating the proposition one finds that if X is k-1 connected, then $\underline{\underline{L}}\Lambda_n X$ and $\underline{\underline{L}}S_n X$ are nk-1 connected.

7.42. In characteristic zero the homological properties of the symmetric algebra are simpler because S_n is a canonical direct summand of the n-fold tensor product. For example, if U is a graded skew-commutative ring let $N \mapsto \tilde{S}^U N$ be the (skew-commutative) symmetric algebra functor, that is, the left adjoint of the forgetful functor from skew-commutative graded U-algebras to graded U-modules. Then

<u>Proposition 7.43</u>: Suppose that R is a simplicial ring of characteristic zero (i.e. $cQ \subset R$) and that X is a flat simplicial R-module such that HX is a flat HR module. Then there is a canonical isomorphism of bi-graded HR algebras

$$\tilde{S}_n^{HR}(HX) \simeq H(S_n^R X) \qquad n \geq 0$$

<u>Proof</u>: Let $T = \oplus T_n$ be the tensor algebra functor on ring-modules and let $\tilde{T} = \oplus \tilde{T}_n$ be the tensor algebra functor on graded ring-modules. The symmetric group Σ_n on n letters acts on T_n in the obvious way and on $\tilde{T}_n$ with the skew-commutative sign rule, and the shuffle map $\underline{\otimes}$ induces a Σ_n equivariant map of chain complexes

$$(7.44) \qquad \tilde{T}_n^R X \to T_n^R X$$

As X is R-flat and HX is HR flat the Kunneth spectral sequences show that 7.53 gives rise to Σ_n equivariant isomorphisms

$$\tilde{T}_n^{HR}(HX) \simeq H(\tilde{T}_n^R X) \simeq H(T_n^R X) \ .$$

In general the largest Σ_n-invariant quotient of $\tilde{T}_n$ (resp. T_n) is $\tilde{S}_n$ (resp. S_n) and in characteristic zero the symmetrization operator $\frac{1}{n!}\sum_{\sigma\in\Sigma_n}\sigma$ allows us to canonically identify this quotient as a direct summand of $\tilde{T}_n$ (resp. T_n). As homology commutes with direct sums we obtain isomorphisms

$$\tilde{S}_n^{HR}(HX) \simeq H(S_n^R X)$$

and it is easily seen that these isomorphisms are compatible with the algebra structures. Q.E.D.

Remark 7.45: This proposition enables one to compute $H(S^R X)$ in terms of HX when R is ck and k is a field of characteristic zero. There is a more complicated formula if k is of characteristic p which we now briefly describe. This formula is based on the Dold-Thom theorem which in its simplicial form asserts that if X is a reduced simplicial set then

(7.46) $$SP^\infty X \hookrightarrow \bar{\mathbb{Z}}X$$

is a weak homotopy equivalence of simplicial sets, where $SP^\infty X$ (resp. $\bar{\mathbb{Z}}X$) is the free abelian monoid (resp. group) generated by X with the basepoint put equal to the identity. Applying the free B module functor to both sides of 7.46 we obtain an isomorphism $BSp^\infty X \cong S^B \bar{B}X$.

(7.47) $$H(S^B(\bar{B}X)) \simeq H_*(\bar{\mathbb{Z}}X, B)$$

where $\bar{B}X = BX/B_*$ and where the right side denotes the homology of the generalize Eilenberg-MacLane space $\bar{\mathbb{Z}}X$ with coefficients in B. In particular if $X = \Delta(n)/\dot{\Delta}(n)$, then

(7.48) $$H(S^B K(B,n)) \simeq H_*(K(\mathbb{Z},n),B)$$

where $K(B,n)$ is the simplicial B module whose normalization is the complex with B located in dimension n and zero elsewhere. Now Cartan [ ] has given a formula for the right side of 7.48 when $B = \mathbb{Z}/p\mathbb{Z}$ which may be used by the Kunneth theorem to calculate $H(S^B X)$ when B is any field.

7.49. Finally we want to point out that the convergence theorem 8.8 can be used to give a proof of the Dold-Thom theorem (7.46). As this is a map of connected H-spaces it suffices to prove that 7.46 induces a map on homology ([ ], 1959-60, p.16-08) that is, that $\mathbb{Z}(SP^\infty X) \to \mathbb{Z}(\bar{\mathbb{Z}}X)$ is a weak equivalence. Filtering both sides by powers of the augmentation ideal we obtain a map of spectral sequences with the same E^1 term. But the augmentation ideals of both rings are regular so 8.8 implies these spectral sequences converge and hence the map is a weak equivalence.

§8. Regular ideals and the convergence theorem

Let I be an ideal in a ring A and let $B = A/I$. Let P be a flat differential graded or simplicial A-algebra resolution of B. Filtering P by I^nP $n \geq 0$ we obtain a spectral sequence of algebras whose E^1 term

$$(8.1) \qquad H_q(gr_n^I P) = \mathrm{Tor}_q^A(B, I^n/I^{n+1})$$

is a bigraded B-algebra anti-commutative for the homology degree q and whose first differential

$$(8.2) \qquad d: \mathrm{Tor}_q^A(B, I^n/I^{n+1}) \to \mathrm{Tor}_{q-1}^A(B, I^{n+1}/I^{n+2})$$

is a derivation of this algebra. It is clear that this differential bigraded B-algebra structure on $\mathrm{Tor}_*^A(B, gr_*^I A)$ is independent of the choice of the resolution P.

For $q = 1, n = 0$ the differential d yields isomorphisms

$$\mathrm{Tor}_1^A(B,B) \approx \mathrm{Tor}_0^A(B, I/I^2) \simeq I/I^2$$

which extend naturally to a canonical homomorphism of differential bigraded B-algebras

$$(8.3) \qquad \theta_{q,n}: \wedge_q(I/I^2) \otimes S_n(I/I^2) \to \mathrm{Tor}_q^A(B, I^n/I^{n+1})$$

where $\otimes, \wedge, S$ are over B and where the left side of 8.3 is endowed with the Koszul differential (7.34).

Definition 8.4: I is said to be quasi-regular (resp. regular) if I/I^2 is a flat (resp. projective) B module and if the canonical map

$$\theta_{q,o}: \wedge_q(I/I^2) \to \mathrm{Tor}_q^A(B,B) \qquad q \geq 0$$

is an isomorphism.

Example: Suppose M is a flat B module, $A = SM$, and $I = S_+M$. Then we may take P to be the Koszul complex and we find that I is quasi-regular and regular iff M is projective.

Proposition 8.5: If I is quasi-regular, then $\theta = \{\theta_{q,n}\}$ is an isomorphism. In particular

$$S_n(I/I^2) \simeq I^n/I^{n+1}$$

Proof: To simplify notation let $N = I/I^2$ and $T_q(\cdot) = \mathrm{Tor}_q^A(B,\cdot)$. We shall prove by induction on m, the following assertions

A_m: $S_kN \simeq I^k/I^{k+1}$ for $k \leq m$

B_m: $T_q(A/I^k) \to T_q(A/I^{k-1})$ is the 0 map for $k \leq m$, $q > 0$.

Note first that as N is flat, A_m implies that I^k/I^{k+1} for $k \leq m$ is flat B module, hence $\mathrm{Tor}_q^A(B, I^k/I^{k+1}) \simeq \mathrm{Tor}_q^A(B,B) \otimes (I^k/I^{k+1}) \simeq \wedge_q N \otimes S_k N$. Thus A_m implies that $\theta_{q,k}$ is an isomorphism for $k \leq m$.

A_o, B_o are trivial

$A_m, B_m \implies B_{m+1}$. Let ∂_m be the boundary operator for the T_* long exact sequence associated to

(8.6) $$0 \to I^{m-1}/I^m \xrightarrow{i_m} A/I^m \xrightarrow{j_m} A/I^{m-1} \to 0$$

and consider the diagram

(8.7)
$$\begin{array}{ccccccccc}
 & & T_{q+2}(A/I^{m-1}) & & & & T_{q+1}(A/I^{m}) & & \\
 & \overset{(i_{m-1})_*}{\nearrow} & & \overset{\partial_m}{\searrow} & & \overset{(i_m)_*}{\nearrow} & & \overset{(\partial_{m+1})_*}{\searrow} & \\
T_{q+2}(I^{m-2}/I^{m-1}) & & \xrightarrow{d} & & T_{q+1}(I^{m-1}/I^{m}) & & \xrightarrow{d} & & T_q(I^m/I^{m+1}) \\
\uparrow{\scriptstyle \theta_{q+2,m-2}} & & & & \uparrow{\scriptstyle \theta_{q+1,m-1}} & & & & \uparrow{\scriptstyle \theta_{q,m}} \\
\wedge_{q+2}N\otimes S_{m-2}N & & \xrightarrow{d} & & \wedge_{q+1}N\otimes S_{n-1}N & & \xrightarrow{d} & & \wedge_q N\otimes S_m N
\end{array}$$

The bottom row is exact since N is flat (see 7.35). A_m implies that the θ maps are isomorphisms, hence the middle row is exact. By B_m $(i_m)_*$ is surjective. Suppose $x \in \operatorname{Im}\{T_{q+1}(A/I^{m+1}) \to T_{q+1}(A/I^m)\}$, that is $(\partial_{m+1})_* x = 0$. If $x = (i_m)_* y$, then $dy = 0$ so $y = dz$. Then $x = (i_m)_* \partial_m (i_{m-1})_* y = 0$, and as x is arbitrary $T_{q+1}(A/I^{m+1}) \to T_q(A/I^m)$ is 0, and we have proved B_{m+1}.

$A_{m-1}, B_m \Longrightarrow A_m$. Consider the diagram above with $q = 0$. By B_m, $(i_{m-1})_*$ and $(i_m)_*$ are surjective and ∂_m is injective. But $(\partial_{m+1})_*: T_1(A/I^m) \xrightarrow{\sim} T_o(I^m/I^{m+1}) \simeq I^m/I^{m+1}$ which proves the middle row of the diagram is exact and the right hand d is surjective. As the same is true for the bottom row and as $\theta_{2,m-2}$ and $\theta_{1,m-1}$ are isomorphisms by A_{m-1}, we see $\theta_{o,m}$ is an isomorphism and hence have proved A_m.

Thus A_m is true for all m and 8.5 is proved. Moreover as the diagram 8.7 is functorial in the pair A,I we obtain from the proof of 8.5 the following.

Proposition 8.7: If I is quasi-regular there are exact sequences

$$\cdots \xrightarrow{d} \Lambda_{q+1}N \otimes S_{n-1}N \xrightarrow{d} \Lambda_q N \otimes S_n N \to \mathrm{Tor}_q^A(B, A/I^{n+1}) \to 0$$

where $N = I/I^2$, $q > 0$ which are functorial in the pair (A,I) .

Convergence theorem 8.8. Let R be a simplicial ring and J be a simplicial ideal in R such that $H_o(J) = 0$. If J_q is quasi-regular in R_q for each q , then $H_k(J^n) = 0$ for $k < n$.

Proof: We use induction on n , the case $n = 1$ being a hypothesis. Consider the exact and spectral sequences

(8.9) $$0 \to \mathrm{Tor}_1^R(R/J, J^n) \to J \otimes_R J^n \to J^{n+1} \to 0$$

(8.10) $$E^2_{pq} = H_p\{\mathrm{Tor}_q^R(J, J^n)\} \Longrightarrow H_{p+q}(J \overset{L}{\otimes}_R J^n)$$

(8.11) $$E^2_{pq} = \mathrm{Tor}_p^{H_*R}(H_*J, H_*J^n)_q \Longrightarrow H_{p+q}(J \overset{L}{\otimes}_R J^n)$$

(8.12)

$$\to \Lambda_{q+1}N \otimes S_{n-2}N \xrightarrow{d} \Lambda_q N \otimes S_{n-1}N \to \mathrm{Tor}_q^R(R/J, R/J^n) \to 0 \qquad q > 0$$

where 8.10 and 8.11 are Kunneth spectral sequences [HA], II, th.6 and where 8.12 follows from 8.7, the $\Lambda, S, \otimes$ being over R/J and $N = J/J^2$. As N is a quotient of J we have $H_o(N) = 0$, hence as N is flat over $\underline{\underline{L}}S_k N \simeq S_k N$ by 7.16 and so $S_k N$ is (k-1)-connected by 7.5. Similarly $\wedge_k N$ is (k-1)-connected. Again by flatness of N $\wedge_q N \otimes S_k N = \wedge_q N \overset{L}{\otimes} S_k N$, so $\wedge_q N \otimes S_k N$ is (q+k-1)-connected by the Kunneth spectral sequence analogous to 8.11. From 8.12 one therefore finds that

$$\mathrm{Tor}_q^R(R/J, R/J^n) = \begin{cases} \mathrm{Tor}_{q-1}^R(R/J, J^n) & \text{if } q > 1 \\ \mathrm{Tor}_{q-2}^R(J, J^n) & \text{if } q > 2 \end{cases}$$

is (q+n-2)-connected. Hence $E_{pq}^2 = 0$ for $q > 0$, $p < q+n$ in 8.10. By induction hypothesis J^n is (n-1)-connected so by 8.11 $J \overset{L}{\otimes}_R J^n$ is n-connected. Thus 8.10 shows that $J \otimes_R J^n$ is n-connected, whence by 8.9 and the fact that $\mathrm{Tor}_1^R(R/J, J^n)$ is (n+1)-connected, we see that J^{n+1} is n-connected which completes the induction. Q.E.D.

For noetherian rings there is a close relation between regular ideals and ideals generated by regular sequences.

Proposition 8.13: If A is noetherian the following conditions are equivalent:

(i) I is regular

(ii) I is quasi-regular

(iii) I/I^2 is a projective B module and $\wedge^2(I/I^2) \to \mathrm{Tor}_2^A(B,B)$ is surjective.

(iv) I/I^2 is a projective B module and $S(I/I^2) \simeq \mathrm{gr}^I A$.

(v) For each maximal ideal p in A containing I, the ideal IA_p in A_p is generated by a regular sequence.

Proof: (i) <==> (ii) I/I^2 is a finitely presented B module, hence it is flat iff projective iff locally free.

As Tor's, $\wedge$, S, gr are all compatible with localization we may suppose that A is a local noetherian ring and that I is contained in the maximal ideal of A. (i) ==> (iii) is obvious. (v) ==> (i). Let $\underline{f} = \{f_1, \ldots, f_n\}$ be a regular sequence generating I. Then the Koszul complex $K(\underline{f};A)$ is a free differential graded algebra resolution of B over A. Hence

$$\mathrm{Tor}^A_*(B,B) \simeq H_*(\underline{f};B) \simeq \wedge H_1(\underline{f};B)$$

and

$$I/I^2 \simeq \mathrm{Tor}^A_1(B,B) \simeq H_1(\underline{f};B) \simeq B^n .$$

Thus I is regular.

(iii) ==> (v). Let $\underline{f} = \{f_1, \ldots, f_n\}$ be a minimal system of generators for I. The Koszul complex $K(\underline{f};A)$ is a free differential graded algebra over A with an augmentation to $A/I = B$, hence there is a canonical homomorphism of graded B algebras

$$\theta_*: H_*(\underline{f};B) \to \mathrm{Tor}^A_*(B,B)$$

This homomorphism is an edge homomorphism in the spectral sequence

$$E^2_{pq} = \mathrm{Tor}^A_p(H_q(\underline{f};A),B) \Longrightarrow H_{p+q}(\underline{f};B)$$

whose five term exact sequence is

$$H_2(\underline{f};B) \xrightarrow{\theta_2} \mathrm{Tor}^A_2(B,B) \to H_1(\underline{f};A)\otimes_A B \to H_1(\underline{f};B) \xrightarrow{\theta_1} \mathrm{Tor}^A_1(B,B) \to 0$$

As I/I^2 is free over B , θ_1 is an isomorphism. The isomorphism $H_2(\underline{f};B) \approx \wedge^2 H_1(\underline{x},B)$ and the fact that θ is an algebra homomorphism show that θ_2 is isomorphic to the map $\wedge^2 I/I^2 \to \mathrm{Tor}_2^A(B,B)$ which is surjective by hypothesis. Hence $H_1(\underline{f};A)\otimes_A B = 0$ so by Nakyama $H_1(f;A) = 0$ and so $f_1,\ldots,f_n$ is a regular sequence ([ ], IV, prop.3).

(iv) <==> (v). See EGA, 0 IV, 15.1.11.

§9. Some applications of the spectral sequence.

In this section we give applications of the fundamental spectral sequence 6.8 of the vanishing or mod-$\underline{C}$ theory type.

9.1. If $B = S^{-1}A$, then $B \otimes_A B = B$ so the spectral sequence 6.8 may be applied. By 7.5, $E^2_{p1} = D_{p+1}(B/A) = 0$ for $p < n \Longrightarrow E^2_{pq} = 0$ for $p < n$, $q \geq 1$, and hence as $\mathrm{Tor}^A_+(B,B) = 0 \Longrightarrow D_{n+1}(B/A) = 0$. Thus by induction we find that $D_*(B/A) = 0$ which gives an alternative proof of 5.1.

Proposition 9.2: If A is noetherian and B is a localization of a finite type A-algebra, then $D_q(B/A)$ is a finitely generated B module for each q. Consequently if M is a finite generated B module, so are $D_q(B/A,M)$ and $D^q(B/A,M)$.

Proof: The second statement follows from the first by means of the spectral sequences 3.6, 3.7 and the fact that B is noetherian. The first statement reduces by 5.2 to the case where B is a finite type A algebra. Choosing a polynomial ring P over A with finitely many generators mapping onto B we are then reduced by 4.17 to the case where $A = P$, in which case $B = A/I$ and we can apply the spectral sequence. The abutment is a finitely generated B module in each dimension, hence working modulo the class of finitely generated B modules in the spectral sequence, it suffices to show that $D_k(B/A)$ finitely generated for $k \leq n \Longrightarrow H_p(S_q \underline{\underline{L}}D_{B/A})$ finite generated for $p \leq n$. But if a complex X of projective B modules, such as $\underline{\underline{L}}D_{B/A}$, has finitely generated homology in dimensions $\leq n$, it is homotopy equivalent to a complex F of free B modules which is finite type in dimensions $\leq n$. In effect construct inductively a q-equivalence $F^{(q)} \to X$ by attaching q-dimensional generators to $F^{(q-1)}$ to obtain a (q-1)-equivalence $F^{(q-1)'} \to X$ which is surjective on H_q; then add (q+1)-dimensional generators to $F^{(q-1)'}$ to obtain a q-equivalence $F^{(q)} \to X$. If $F^{(q-1)}$ and $H_q X$ are finitely generated, we may assume $F^{(q)}$ is also as B is noetherian; hence setting $F = \lim F^{(n)}$ we obtain a weak equivalence $F \to X$, where in dimensions $\leq n$, $F = F^{(n)}$ is finitely generated. As X is projective $F \to X$ is a homotopy equivalence. This shows that up to simplicial homotopy we may replace $\underline{\underline{L}}D_{B/A}$ by a free simplicial B module $N^{-1}F$ which is finitely generated in dimensions $\leq n$. Hence $E^2_{pq} = H_p(S_q \underline{\underline{L}}D_{B/A}) \simeq H_p(S_q N^{-1}F)$ is finitely generated for $p \leq n$ and the proof of 9.2 is complete. Q.E.D.

Remark 9.3: For a different proof of 9.2 see [ ], prop.17.2. That proof yields the stronger result that when B is a finite type A-algebra and A is noetherian, there is a free simplicial A-algebra resolution P of B with only finitely many generators in each dimension.

Theorem 9.4: (Nilpotent Extension Theorem) Suppose that A is noetherian, B is a localization of a finite type A-algebra, and M is a (not necessarily finitely generated) B module. If $u \in D^q(B/A,M)$ $q > 0$, then there is a surjective map $p: B' \to B$ of A-algebras where B' is a localization of a finite type A-algebra and the kernel of p is nilpotent.

<u>Proof</u>: Choose an A-algebra P and a surjection $P \to B$ where P is a localization of a finitely generated polynomial ring over A. Let I = Kernel of $P \to B$ and set $P_n = B/I^{n+1}$. We are going to show that for n sufficiently large we may take $B' = P_n$. First note that

$$D^q(P_n/P,M) \simeq D^q(P_n/A,M)$$

for $q > 0$ by 4.17 and 5.3, hence we may assume that $P = A$. Secondly

$$D^q(A_n/A,M) = H^q(\mathrm{Hom}_{A_n}(\underline{\underline{L}}D_{A_n/A},M))$$

$$= H^q(\mathrm{Hom}_B(\underline{\underline{L}}D_{A_n/A} \otimes_{A_n} B,M))$$

and hence there is an inverse system of spectral sequences

(9.5) $$E_2^{pq} = \mathrm{Ext}_B^p(H_q(\underline{\underline{L}}D_{A_n/A} \otimes_{A_n} B),M) \Longrightarrow D^{p+q}(A_n/A,M)$$

9.6. We shall say that an inverse system

$$\cdots \to M_{n+1} \to M_n \to \cdots$$

of objects in an abelian category with indexing set the integers ≥ 0 is a <u>strict-essentially-zero</u> inverse system if there is an integer N such that $M_n \to M_m$ is 0 for $n \geq m+N$. It is clear that the strict-essentially-zero inverse systems form a thick subcategory and are preserved by additive functors. Hence in virtue of the spectral sequence 9.5, theorem 9.4 will follow from

Theorem 9.7: Suppose A is a noetherian ring, I is an ideal in A, $A_n = A/I^{n+1}$, and $B = A_o$. Then for each q the inverse system $H_q(\underline{\mathbb{L}D}_{A_n/A} \otimes_{A_n} B)$ is strict-essentially-zero.

In virtue of the inverse system of spectral sequences

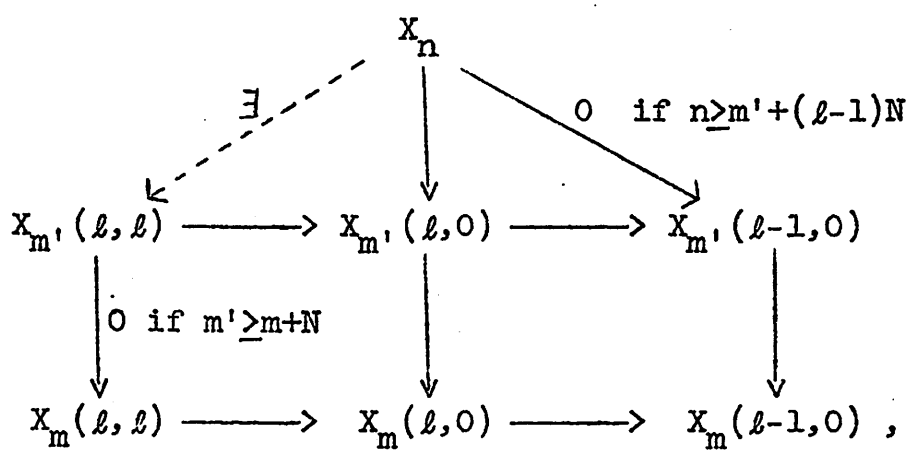
$$E^2_{pq} = H_{p+q}(S^B_q(\underline{\mathbb{L}D}_{A_n/A} \otimes_{A_n} B)) \Longrightarrow \mathrm{Tor}^A_{p+q}(A_n, B)$$

which results from 6.16, it suffices to prove the following two lemmas.

Lemma 9.8: If X_n is an inverse system in $Ho(\underline{M}_B)$ such that $H_k(X_n)$ is strict-essentially-zero for $k \leq r$, then $H_k(\underline{\mathbb{L}S}^B_q X_n)$ is strict-essentially-zero for $k \leq r$ and $q > 0$.

Lemma 9.9: If M is a finitely generated A module, then the inverse system $\mathrm{Tor}^A_q(A_n, M)$ is essentially zero for $q > 0$.

Proof of 9.8: By a step-by-step construction we may represent the inverse system X_n in $Ho(\underline{M}_B)$ by an inverse system in $\underline{M}_R$ of free simplicial R modules which we denote again by X_n. Let $X_n \to X_n(\ell,0)$ be the Postnikov quotient of X_n with $H_q(X_n) \simeq H_q(X_n(\ell,0))$ for $q \leq \ell$ and $H_q(X_n(\ell,0)) = 0$ for $q > \ell$. Suppose N is large enough so that $H_q(X_n) \to H_q(X_m)$ is zero for $n \geq m+N$, $q \leq r$. We show by induction on ℓ that the map $X_n \to X_m \to X_m(\ell,0)$ is homotopic to zero if $n \geq m + \ell N$, $\ell \leq r$. This is clear for $\ell = 0$ and if true for $\ell - 1$, then consider the diagram



where the rows are fibration sequences in $Ho(\underline{M}_B)$ and where $X_m(\ell,\ell)$ has only the homology group $H_\ell(X)$ in dimension ℓ and hence has the property that maps of $\ell - 1$ connected complexes to it are determined by the map on homology. The diagram completes the induction. Thus the composite $\underline{\underline{L}}S_q X_n \to \underline{\underline{L}}S_q X_m \to \underline{\underline{L}}S_q X_m(r,0)$ for $q > 0$ and $n \geq m+rN$ is zero in $Ho(\underline{M}_B)$. As the latter map is an r-equivalence by 7.3, the former induces the zero map on homology. Q.E.D.

<u>Proof of 9.9</u>: By dimension-shifting we may assume that $q = 1$, in which case choosing an exact sequence

$$0 \to K \to F \to M \to 0$$

with F a free finitely generated A module we have

$$\mathrm{Tor}_1^A(A/I^n, M) = \frac{K \cap I^n F}{I^n K}$$

By Artin-Rees there is an integer N with $K \cap I^n F \subset I^m K$ for $n \geq m+N$, hence $\mathrm{Tor}_1^A(A_n,M) \to \mathrm{Tor}_1^A(A_m,M)$ is zero for $n \geq m+N$.
Q.E.D.

Corollary 9.10: Suppose A noetherian, B is a finite type A-algebra and M is a B module. Let $\underline{T}_n$ be the Grothendieck topology on the category of finite type A-algebras over B in which the covering families are single maps $X \to Y$ which are surjective and have nilpotent kernel. Then

$$D^q(B/A,M) \simeq H^q_{\underline{T}_n}(B,\mathrm{Der}(\cdot/A,M))$$

Proof: With the notations of 2.1 there is an obvious map of topologies $f: \underline{T} \to \underline{T}_n$ which gives rise to a Leray spectral sequence

$$E_2^{pq} = H^p_{\underline{T}_n}(B,R^q f_*(\mathrm{Der}(\cdot/A,M))) \Longrightarrow D^{p+q}(B/A,M)$$

where $R^q f_*(\mathrm{Der}(\cdot/A,M))$ is the sheaf on $\underline{T}_n$ associated to the presheaf $X \to H^q_{\underline{T}}(X,\mathrm{Der}(\cdot/A,M)) = D^q(X/A,M)$. Theorem 9.7 thus shows this sheaf is zero for $q > 0$, hence the spectral sequence degenerates and the corollary follows.

Remark 9.11: In general for $q = 1$ any element $u \in D^1(B/A,M)$ may be killed by a nilpotent extension $X \to B$, namely take X to be the extension of B by M corresponding to u by 3.12. If the $D^q(B/A,\cdot)$ are derived functors of $D^0(B/A,\cdot)$, then dimension shifting would permit one to conclude the same for $q \geq 1$ (and this is the case for group and Lie algebra cohomology). But

9.7 for $q = 2$ is generally false. For example suppose $I = fA \subset A$ and M is an injective B module containing $D_2(B/A)$. Let $u \in D^2(B/A,M) \simeq \mathrm{Hom}_B(D_2(B/A),M)$ be the element corresponding to the inclusion. As the map $A_n = A/I^{n+1} \to B$ are cofinal in all nilpotent extensions of B as an A-algebra, to be able to kill u by a nilpotent extension means that

$$D_2(A_n/A,B) \to D_2(B/A)$$

is zero for n large. However as I is principal the 5 term exact sequence 6.12 shows that this map is isomorphic to

$$\mathrm{Tor}_2^A(A_n,B) \to \mathrm{Tor}_2^A(B,B) ,$$

which after some calculation is seen to be isomorphic to

$$\frac{Af \cap \mathrm{Ann}\ f^{n+1}}{f\ \mathrm{Ann}\ f^{n+1}} \xrightarrow{f^n} \frac{Af \cap \mathrm{Ann}\ f}{0} ,$$

which is zero if and only if $\mathrm{Ann}\ f^{n+1} = \mathrm{Ann}\ f^{n+2}$. But it is easy to produce examples of elements f in non-noetherian rings for which the increasing sequence $\mathrm{Ann}\ f^n$ does not stabilize.

§10. Local complete intersections

In this section we study when the cotangent complex $\underline{\underline{L}}D_{B/A}$ is of projective dimension ≤ 1. In the noetherian case we show this is equivalent to B being a local complete intersection over A. When A is a local noetherian ring with residue field k we show that A is regular (resp. a complete intersection) if and only if $D_2(k/A)$ (resp. $D_3(k/A)$) is zero. These results are not entirely new (compare [ ] and [ ]), but they are rather nice corollaries of the fundamental spectral sequence.

10.1. If M is a B module we let $K(M,q)$ $q \geq 0$ be the simplicial B module whose normalization is the chain complex $M[q]$ with M in dimension q and zero elsewhere. For any (q-1)-connected simplicial B module X we have

$$\mathrm{Hom}_{Ho(\underline{\underline{M}}_B)}(X,K(M,q)) \simeq \mathrm{Hom}_{D(B)}(NX,M[q])$$
$$\simeq \mathrm{Hom}_B(H_qX,M)$$

hence if $B = A/I$ there is a canonical map

$$\underline{\underline{L}}D_{B/A} \to K(I/I^2,1) \tag{10.2}$$

which is a 1-equivalence.

<u>Theorem 10.3</u>: The following assertions are equivalent when $B = A/I$:

(i) I is quasi-regular

(ii) $D_q(B/A,M) = 0$ for all B modules M and $q \geq 2$.

(iii) I/I^2 is a flat B module and $\underline{\underline{L}}D_{B/A} \simeq K(I/I^2,1)$.

Proof: (ii) and (iii) are equivalent by the universal coefficient spectral sequence 3.

(iii) <==> (i). $K(I/I^2,1)$ is homotopy equivalent to $\Sigma(c(I/I^2))$, hence by 7.22

$$H_{p+q}(\underline{\underline{L}}S_q^B K(I/I^2,1)) \simeq H_p(\underline{\underline{L}}\Lambda_q^B c(I/I^2)) \simeq \begin{cases} 0 & p > 0 \\ \Lambda_q(I/I^2) & p = 0 \end{cases}$$

As I/I^2 is flat both $\underline{\underline{L}}$'s may be dropped by 7.16. Therefore assuming 10.2 is an n-equivalence $n \geq 1$, we obtain from 7.3 the formula

$$E^2_{pq} = H_{p+q}(S_{\underline{\underline{q}}}^B \underline{\underline{L}}D_{B/A}) = H_{p+q}(S_q^B K(I/I^2,1)) = \begin{cases} 0 & p+q \leq n ,\ p > 0 \\ \Lambda_q(I/I^2) & p = 0 \end{cases}$$

If (iii) holds, then we may take $n = \infty$, so $E^2_{pq} = 0$ for $p > 0$ and the fundamental spectral sequence degenerates showing that the edge homomorphism $\Lambda(I/I^2) \to \mathrm{Tor}^A(B,B)$ is an isomorphism and hence I is quasi-regular. If I is quasi-regular and 10.2 is an n-equivalence, then as the edge homomorphism is an isomorphism the only possible non-zero differential issuing from $E^2_{n,1}$ namely $d_n\colon E^2_{n,1} \to E^2_{o,n}$ is zero, hence $E^2_{n,1} = D_{n+1}(B/A) = 0$ and so 10.2 is an (n+1)-equivalence. Thus induction yields (iii). Q.E.D.

Corollary 10.4: The following are equivalent when $B = A/I$:

(i) I is regular

(ii) $D^q(B/A,M) = 0$ for all B modules M and $q \geq 2$.

(iii) I/I^2 is a projective B module and $\underline{\underline{L}}D_{B/A} \simeq K(I/I^2,1)$.

Corollary 10.5: If A is noetherian and $B = A/I$, the following are equivalent:

(i) I is regular

(ii) $D_2(B/A,M) = 0$ for all finitely generated B modules M

(ii)' $D^2(B/A,M) = 0$ " " " " " " "

(iii) I/I^2 is a projective B module and $D_2(B/A) = 0$.

Proof: (ii) (ii)' and (iii) are equivalent by the universal coefficient spectral sequences. (i) ==> (ii) is clear from 10.3 while (iii) ==> (i) follows from 8.13 (iii) and the 5 term exact sequence 6.12.

10.6. Suppose A is noetherian and B is a finite type A-algebra. Choose a surjection of A-algebras $P \to B$ where P is a polynomial ring over A of finite type and I be the kernel of this map. B is said to be a local complete intersection over A if I is regular. By 5.4 we have $D_q(B/A,M) = D_q(B/P,M)$ for $q > 0$ hence by 10.3 this condition on B is independent of P and we obtain the following.

Theorem 10.7: Suppose A is noetherian and B is an A-algebra of finite type. The following assertions are equivalent:

(i) B is a local complete intersection over A

(ii) $D_q(B/A,M) = D^q(B/A,M) = 0$ for all B modules M and $q \geq 2$.

(iii) $D_2(B/A,M) = 0$ for all finitely generated B modules M

(iii)' $D^2(B/A,M) = 0$ " " " " " " "

(iv) $\underline{\underline{L}}D_{B/A}$ has projective dimension ≤ 1

Here a chain complex X of B modules is said to be of projective dimension $\leq r$ if it is isomorphic in the derived category to a chain complex of projective B modules which is zero in dimension $> r$, or equivalently $H^q\{Hom_B(X,M)\} = 0$ for all B modules M.

Corollary 10.8: If A is a local noetherian ring with residue class field k, the following are equivalent:

(i) A is regular

(ii) $D_q(k/A) = 0$ for $q \geq 2$

(iii) $D_2(k/A) = 0$

A local ring is regular if and only if its maximal ideal $\underline{m}$ is generated by regular sequence, i.e. iff $\underline{m}$ is regular, so this follows from 10.5.

10.9. If A is a local noetherian ring, with residue field k, then by Cohen $\hat{A} = P/I$ where P is a complete regular local ring. A is said to be a complete intersection if I is a regular ideal. By the change of rings exact sequence associated to $P \to \hat{A} \to k$ (4.17) and flat base extension (4.9)

$$\to D_q(\hat{A}/P,k) \to \underset{\substack{\| \\ 0 \text{ if } q\geq 2}}{D_q(k/P)} \to \underset{\substack{\wr\| \\ D_q(k/A)}}{D_q(k/\hat{A})} \to \cdots$$

we find that

$$(10.10) \qquad D_q(k/A) \simeq D_{q-1}(\hat{A}/P,k) \qquad q \geq 3$$

$$(10.11) \qquad 0 \to D_2(k/A) \to I/I^2 \otimes_{\hat{A}} k \overset{\alpha}{\to} \underline{m}_p/\underline{m}_p^2 \to D_1(k/A) \to 0$$

where α is induced by the inclusion $I \subset m$. Now as $\hat{A}$ is local noetherian, the projective dimension of $\underline{\underline{LD}}_{\hat{A}/P}$ is the largest q for which $H_q(\underline{\underline{LD}}_{\hat{A}/P} \otimes_{\hat{A}} k) = D_q(\hat{A}/P,k) \neq 0$. Thus from 10.10 and 10.5 we find

Corollary 10.12: If A is a local noetherian ring with residue field k , then the following conditions are equivalent:

(i) A is a complete intersection

(ii) $D_q(k/A) = 0$ for all $q \geq 3$

(iii) $D_3(k/A) = 0$

Remark 10.13: One may always arrange that the dimension of P is minimal such that $P/I = \hat{A}$. This is equivalent to $I \subset m_p^2$, whence the map α in 10.11 is zero, and so we obtain an interpretation of $\dim_k D_1(k/A)$ as the minimal number of generators for A and $\dim_k D_2(k/A)$ as the minimal number of relations. We will generalize this in the following chapter.

Remark 10.14: Suppose A noetherian and B finite type over A . Then 10.7 and 5.3 characterize when $\underline{\underline{LD}}_{B/A}$ has projective dimension 0 or 1 . In the next chapter we will give evidence for the following conjecture.

Conjecture 10.15: B is a local complete intersection over A iff B is finite Tor dimension over A and $\underline{\underline{L}}D_{B/A}$ has finite projective dimension.

The implication ==> is clear. One reduces to the case where $B = A/I$, whence if I is generated by r elements, Tor dim of A/I over A is $\leq r$ by means of the Koszul complex. For the converse, the hypothesis that B be of finite Tor dimension over A is necessary, as may be seen from the example of a local noetherian ring A which is a complete intersection but which is not regular; then the projective dimension of $\underline{\underline{L}}D_{k/A}$ is two. (Incidentally in all examples where $\underline{\underline{L}}D_{B/A}$ has finite projective dimension known to the author the projective dimension is ≤ 2 and one might conjecture this is always true.) By the argument used to prove 10.12 the conjecture 10.15 implies the following.

Conjecture 10.16: If A is a local noetherian ring with residue field k and $D_q(k/A) = 0$ for q sufficient large, then A is a complete intersection.

§11. Local rings in characteristic zero

A denotes a local noetherian ring with residue class field k . The fundamental spectral sequence when $B = k$ is

$$E^2_{pq} = H_{p+q}(S^k_q \underline{L}D_{k/A}) \Longrightarrow \mathrm{Tor}^A_{p+q}(k,k) \tag{11.1}$$

and it has an edge homomorphism $\mathrm{Tor}^A_n(k,k) \to D_n(k/A)$ which annihilates the decomposable elements of $\mathrm{Tor}^A_*(k,k)$. (6.15).

Theorem 11.2: If the characteristic of k is zero, then all the differentials in the spectral sequence 11.1 are zero, hence $D_*(k/A)$ is isomorphic to the indecomposable space of $\mathrm{Tor}^A_*(k,k)$.

Proof: By a result of Assmus [ ] $\mathrm{Tor}^A_*(k,k)$ is a commutative Hopf algebra, hence by a theorem of Borel (see [ ]) $\mathrm{Tor}^A_*(k,k)$ is isomorphic to a tensor product of an exterior algebra with odd dimensional generators and a polynomial ring with even dimensional generators. In other words, if $W_* \subset \mathrm{Tor}^A_*(k,k)$ is a complementary subspace to the decomposable elements, then $\tilde{S}W_* \simeq \mathrm{Tor}^A_*(k,k)$ where $\tilde{S}$ is the skew-commutative symmetric algebra functor (7.42). By 7.43 $E^2_{*q} \simeq \tilde{S}_q D_*(k/A)$. Abbreviating $\mathrm{Tor}^A_n(k,k)$ and $D_n(k/A)$ by T_n and D_n respectively we show by induction on n that the composition $W_q \to T_q \to D_q$ is an isomorphism for $q < n$ and surjective for $q = n$. For $n = 1$ this is clear since all three are m_A/m_A^2 . Assuming the assertion is true for n it follows that all differentials issuing from $E^2_{p1} = D_p$ are zero for $p < n$, hence as E^2 is generated by $E^2_{*,1}$ as an algebra and as the differentials are derivations, we have that all differentials issuing from E^r_{pq} are zero for $p < n$. Hence $E^\infty_{pq} = E^2_{pq}/B^\infty_{pq}$ for $p < n$, so

$$\operatorname{gr} T_n = \bigoplus_{p=0}^{n-1} E^\infty_{p,n-p} = \bigoplus_{p=0}^{n-2} E^2_{p,n-p}/B^\infty_{p,n-p} \oplus D_n .$$

As $W_q \simeq D_q$ for $q < n$, we have $\dim E^2_{pq} = \dim(\tilde{S}_q D_*)_{p+q} = \dim(\tilde{S}_q W_*)_{p+q}$ for $p \leq n-2$. But

$$\dim T_n = \dim \operatorname{gr} T_n = \sum_{p=0}^{n-2} \dim(E^2_{p,n-p}/B^\infty_{p,n-p}) + \dim D_n$$
$$\leq \sum_{p=0}^{n-2} \dim(\tilde{S}_{n-p} W_*)_n + \dim W_n = \dim T_n$$

which shows that $B^\infty_{p,n-p} = 0$ for $0 \leq p \leq n-2$ and $\dim W_n = \dim D_n$. The latter equality shows that the surjective map $W_n \to D_n$ is an isomorphism. The former equality shows that all differentials issuing from $E_{n,1}$ are zero, and hence the edge homomorphism $T_{n+1} \to D_{n+1}$ is surjective. But we know that the edge homomorphism kills decomposable elements, hence $W_{n+1} \to D_{n+1}$ is surjective and the induction is complete. Q.E.D.

11.3. If the characteristic of k is $p > 0$, then the above arguments work in dimensions $< 2p$. In effect $H_*(S^k_{q} \underline{L} D_{k/A}) = \tilde{S}_q(D_*)$ for $q < p$ because the proof of 7.43 used only the complete reducibility of the representation of the symmetric group Σ_q which holds in characteristic p if $q < p$. Also Borel's theorem in characteristic p shows that $\operatorname{Tor}^A_*(k,k)$ is a tensor product of monogenic algebras $k[x]/(x^m)$ where $m = 2$ if p and degree x are odd and $m = p^a, 1 \leq a \leq \infty$ otherwise.

(Actually only $m = p$ can occur when the degree of x is even because Tor_* is the homology of simplicial k-algebra, hence admits a canonical system of divided power operations for elements of degree ≥ 2, so that $x^2 = 0$ for all x of odd degree ≥ 1 and $x^p = 0$ for all x of even degree ≥ 2.) Thus $Tor_* \simeq \widetilde{S}W_*$ in dimensions $< 2p$. Therefore we have the following.

Theorem 11.4: If the characteristic of k is $p > 0$, then $D_q(k/A)$ is isomorphic to the indecomposable quotient space of $Tor_q^A(k,k)$ if $q < 2p$.

Remark 11.5: This is false for $q = 2p$ if A is not regular since the kernel of the map $Tor_{2p} \to D_{2p}$ will contain the p-th divided powers of elements of Tor_2 and these are not decomposable elements of Tor_{2p}.

11.6. For the rest of this section we suppose that k is of characteristic zero. The dual of the Hopf algebra $Tor_*^A(k,k)$ may be shown to be $Ext_A^*(k,k)$ with the Yoneda product. Ext^* is cocommutative hence by a theorem of Milnor and Moore [ ] it is the universal enveloping algebra of its (skew-)Lie algebra of primitive elements. Under duality the primitive subspace of Ext^* corresponds to the indecomposable quotient space of Tor_*. As $D^*(k/A)$ is the dual of $D_*(k/A)$ we have

Theorem 11.7: If k is of charactersitic zero, then $D^*(k/A)$ is canonically isomorphic to the subspace of primitive elements in $Ext_A^*(k,k)$. In particular $D^*(k/A)$ has a natural (skew-) Lie algebra structure and there is a canonical Hopf algebra isomorphism

$$U\{D^*(k/A)\} \simeq \mathrm{Ext}^*_A(k,k)$$

where U denotes the universal enveloping algebra. Moreover there is a canonical algebra isomorphism

$$\tilde{S}\{D_*(k/A)\} \simeq \mathrm{Tor}^A_*(k,k) .$$

The last assertion follows from the Poincare-Birkhoff-Witt theorem in characteristic 0 , which when $\underline{g}$ is a Lie algebra gives a canonical coalgebra isomorphism $\exp: \tilde{S}(\underline{g}) \to U(\underline{g})$ and hence a canonical algebra isomorphism ${}^t(\exp): U(\underline{g})' \to \tilde{S}(\underline{g})' \simeq \tilde{S}(\underline{g}')$.

11.8. The calculation of $D^*(k/A)$ with its Lie algebra structure is therefore equivalent to the calculation of $\mathrm{Ext}^*_A(k,k)$ with its Hopf algebra structure. If we assume A is complete, which doesn't change any of the groups involved, then by Cohen A has a field of representatives, so we may view A as a k-algebra and use a continuous form of the <u>cobar construction</u> to calculate Ext^* as follows.

Let $m = m_A$ be the maximal ideal of A . As A is complete it is a linearly compact vector space over k with topological dual

$$m' = \varinjlim_{n} \mathrm{Hom}_k(m/m^n,k) .$$

The topological dual of the complex $m[1]$, which is m in dimension 1 and zero elsewhere is $m'[-1]$. Let d be the unique degree -1 derivation of the tensor algebra $T(m'[-1])$ such that $d: m'[-1] \to m'[-1] \otimes m'[-1]$ is the transpose of the multiplication map $m \otimes m \to m$. The associativity of multiplication implies $d^2 = 0$, and the commutativity implies that d is a coderivation of $T(m'[-1])$ when this is endowed with the Hopf algebra structure in which $m'[-1]$ is primitive. Thus $T(m'[-1])$ together with d is a differential graded Hopf algebra. Its space of primitive elements is the differential graded Lie algebra $L(m'[-1])$ together with the induced differential, where L is the free (skew-) Lie algebra functor on graded vector spaces.

We recall the standard convention $H^q(K) = H_{-q}K$ for any complex K .

Proposition 11.9: There is a canonical Hopf algebra isomorphism

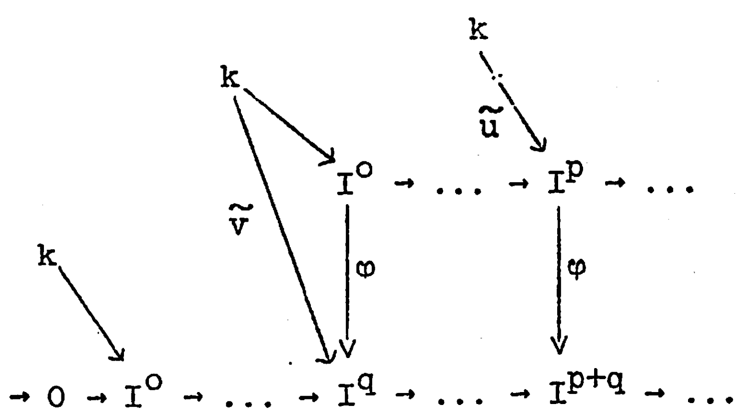
$$\mathrm{Ext}^*_A(k,k) \simeq H^*\{T(m'[-1]),d\}$$

and hence a canonical Lie algebra isomorphism

$$D^*(k/A) \simeq H^*\{L(m'[-1],d)\}$$

Proof: The second formula follows from the first using 11.7 and the fact that in characteristic zero the primitive element and homology functors for differential graded Hopf algebras commute (see [ ]). The first statement is presumably well known (compare [ ] 1958-59, p.15-09 for the case of graded algebras) so we shall present only an outline of its justification. Let A' be the continuous k-dual of A and make $A' \otimes T(m'[-1])$ into a differential graded right $T(m'[-1])$ module by defining $d\colon A' \to A'\otimes m'[-1]$ to be dual to the multiplication $A\otimes m \to A$. $I^{\cdot} = A'\otimes T(m'[-1])$ is a resolution of k, since it is the cobar construction of the coalgebra A'. Moreover A' is an injective A module, in fact, it is an injective hull of k, so one sees that $I^{\cdot}$ is an injective resolution of k as an A module, whence there is canonical k module isomorphism $\mathrm{Ext}_A^*(k,k) = H^*\{\mathrm{Hom}_A(k,I^{\cdot})\} = H^*(T(m'[-1]),d)$.

To see this isomorphism is compatible with products we recall that the Yondeda product of $u \in \mathrm{Ext}_A^p(k,k)$ and $v \in \mathrm{Ext}_A^q(k,k)$, denoted $u\cdot v \in \mathrm{Ext}_A^{p+q}(k,k)$, is the composition $v\circ u$ when u and v are thought of as degree p and q maps from k to k in the derived category $D(A)$. If we represent u and v by maps $\tilde{u}\colon k[p] \to I^{\cdot}$, $\tilde{v}\colon k[q] \to I^{\cdot}$ then to calculate $v\circ u$ we must choose a map φ in the diagram



whence $v\circ u$ is represented by $\varphi\circ\tilde{u}\colon k[p+q] \to I^{\cdot}$. In the situation at hand $\tilde{v}(\lambda) = \lambda\tilde{v}(1)$ where $\tilde{v}(1) \in T(m'[-1])^q$ so we may take φ to be $\varphi(\alpha) = \alpha\cdot\tilde{v}(1)$, where we use the right

$T(m'[-1])$ module structure of $I^{\cdot}$. Then $(\varphi\circ\tilde{u})(1) = \tilde{u}(1)\circ\tilde{v}(1)$, i.e. the Yoneda product $u \cdot v \in Ext^{p+q}$ corresponds to the product of u and v in the algebra $H^*(T(m'[-1]),d)$.

Finally we must show the isomorphism is compatible with the coalgebra structure on Ext^* , which we recall comes via duality from the algebra structure on Tor_* . The tensor product of the natural coalgebra structures on A' and $T(m'[-1])$ is a coalgebra structure on $I^{\cdot}$ for which d is a coderivation (see [ ] for details). In the special case where A is finite dimensional over k , it follows that the dual of $I^{\cdot}$ is a free differential graded algebra resolution of k (in fact its the bar resolution $B(A)$) , hence Tor_* is algebra-isomorphic to $H_*(k\otimes_A(I^{\cdot})')$ and Ext^* is coalgebra-isomorphic to $H^*(Hom_A(k,I^{\cdot})) = H^*(T(m'[-1]),d)$. The general case then follows from the special case by setting $A_n = A/m^n$ and letting $n \to \infty$. Q.E.D.

The map $A \mapsto D^*(k/A)$ is a contravariant functor from the category of local noetherian rings with residue field k to the category of (skew-graded) Lie algebras over k which transforms direct products into direct sums in virtue of the following Kunneth-type formula:

Theorem 11.10: If A and B are local noetherian rings with residue field k (of char. 0), then the canonical homomorphism of Lie algebras

$$D^*(k/A) \vee D^*(k/B) \simeq D^*(k/A\times_k B)$$

is an isomorphism, where $\vee$ denotes direct sum in the category of Lie algebras.

Proof: If R and S are two non-commutative augmented k-algebras, then their direct sum in this category is given by the formula

$$R \vee S = \oplus_\alpha (R \vee S)_\alpha \tag{11.11}$$

where α runs over all words in the free monoid generated by two letters a and b, where

$$(R \vee S)_{(\underbrace{a\ldots a}_{p}\ \underbrace{b\ldots b}_{q}\ldots)} = \underbrace{\bar{R}\otimes\ldots\otimes\bar{R}}_{p}\otimes\underbrace{\bar{S}\otimes\ldots\otimes\bar{S}}_{q}\otimes\ldots \quad , \tag{11.12}$$

and where $\bar{R}$ and $\bar{S}$ denote the augmentation ideals of R and S respectively. If R and S are differential graded algebras and $R \vee S$ is endowed with the differential which is the unique derivation of $R \vee S$ coinciding on $\bar{R} \simeq (R\vee S)_a$ and $\bar{S} \simeq (R \vee S)_b$ with the differentials of R and S respectively, then the decomposition 11.11 is compatible with the differentials and shows that

$$H(R \vee S) \simeq H(R) \vee H(S) \ .$$

If $C = A \times_k B$, then $m_C = m_A \oplus m_B$ and there is an isomorphism of differential graded algebras

$$T(m_A^![-1]) \vee T(m_B^![-1]) \simeq T(m_C^![-1])$$

and so an isomorphism of homology algebras, which by 11.9 gives isomorphisms

$$\begin{array}{ccc} \mathrm{Ext}^*_A(k,k) \vee \mathrm{Ext}^*_B(k,k) & \simeq & \mathrm{Ext}^*_C(k,k) \\ \uparrow\wr & & \uparrow\wr \\ U(D^*(k/A) \vee D^*(k/B)) & \simeq & U(D^*(k/C)) \end{array} .$$

Taking primitive elements, the theorem follows.

Remark 11.13: If we consider the category of local noetherian k algebras with residue field k which are essentially of finite type (resp. complete), then $C = (A \otimes_k B)_m$ (resp. $C = A \hat{\otimes}_k B$) is the direct sum in this category and we have the following formula

$$D^*(k/C) \simeq D^*(k/A) \times D^*(k/B) \quad ,$$

which follows easily from 4.9 and the compatibility of D^* with flat base change (4.9).

Remark 11.14: It is possible to write down a formula analogous to 11.11 and 11.12 for the direct sum of skew Lie algebras. For the formula see any account of the Hilton-Milnor theorem in algebraic topology.

Examples: 11.15. Suppose $A = k \oplus m$ where $m^2 = 0$. Then the cobar construction has zero differentials so $\mathrm{Ext}^*_A(k,k) = T(m'[-1])$ and $D^*(k/A) = L(m'[-1])$. If $\dim \underline{m} = 1$ and x is a basis for $\underline{m}'$, then $D^1(k/A) = kx$, $D^2(k/A) = kx^2$ where $x^2 = \frac{1}{2}[x,x]$ and all other $D^q = 0$. If $\dim \underline{m} = 2$ and x,y is a basis for m', then $D^1 = kx \oplus ky$, $D^2 = kx^2 \oplus k[x,y] \oplus ky^2$, $D^3 = k[x,[x,y]] \oplus k[y,[x,y]]$, etc.

11.16. Suppose C is the complete local ring at the intersection of a p plane and a q plane meeting transversally. Then $C = A \times_k B$ where A and B are power series rings in p and q variables. $D^1(k/A) = (m_C/m_C^2)' = (m_A/m_A^2)' \oplus (m_B/m_B^2)'$ so if $x_1,\ldots,x_p$ is a basis for m_A/m_A^2 and $y_1,\ldots,y_q$ is a basis for $(m_B/m_B^2)'$, then $D^*(k/C)$ is the quotient of the free skew Lie algebra generated by the degree 1 elements $x_1,\ldots,x_p,y_1,\ldots,y_q$ by the ideal generated by the relations

$$[x_i,x_j] = 0$$

$$[y_i,y_j] = 0$$